Sheet 4 - Geometrical Hahn-Banach theorem, convexity, duality and weak topologies

Exercise 1. Let E be a l.c.t.v.s. One says that H is a closed half-space if there exists a $\varphi \in E^*$ and $a \in \mathbb{R}$ such that $H = \{x \in E \mid \varphi(x) \leqslant a\}$ (why is it closed ?).

1. If C is a convex subset of E, show that its closure $\overline{C}$ is convex.
2. Let A be a closed convex subset of E. Show that A is the intersection of the closed half-spaces containing A.
3. Deduce that $\overline{co(A)}$ is the intersection of the closed half-spaces containing A for any subset A of E.

Exercise 2. Let H be the Hilbert space $L^2([-1,1])$. For every $\alpha \in \mathbb{R}$, let $C_\alpha \subset H$ be the subset of continuous functions $f : [-1,1] \to \mathbb{R}$ such that $f(0) = \alpha$. Prove that C_α is a convex dense subset of H. Deduce that, if $\alpha \neq \beta$, then C_α and C_β are convex disjoint subsets that cannot be separated by a continuous linear form.

Exercise 3 (Amenability *(Moyennabilité)* of $\mathbb{Z}$). A group G is *amenable* if there exists a function $\mu : \mathcal{P}(G) \to \mathbb{R}_+$, called *mean*, such that

- $\mu(G) = 1$,
- μ is finitely additive: $\forall A, B \subset G$ with $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$,
- μ is left-invariant: $\forall g \in G$ and $A \subset G$, $\mu(gA) = \mu(A)$.

1. Let $\mathbb{1}$ be the constant sequence equal to 1 and $s : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ be the shift operator, defined by $s(x)_i = x_{i+1}$ for all $i \in \mathbb{Z}$ and $x \in \ell^\infty(\mathbb{Z})$. If F denotes the range of $s - \mathrm{id}$, prove that $\|y - \mathbb{1}\|_\infty \geq 1$ for all y in F.
2. Prove that there exists a continuous linear form ϕ on $\ell^\infty(\mathbb{Z})$ such that $\phi(\mathbb{1}) = 1$ and $\phi \circ s = \phi$.
3. Deduce that $\mathbb{Z}$ is amenable.

Exercise 4 (Hahn-Banach theorems for complex spaces). Let E be a l.c.t.v.s. over $\mathbb{C}$.

1. Let $f : E \to \mathbb{C}$ be a $\mathbb{C}$-linear form. Prove that its real part is $\mathbb{R}$-linear. Conversely, show that for every $\mathbb{R}$-linear form $g : E \to \mathbb{R}$, there exists a unique $\mathbb{C}$-linear form $f : E \to \mathbb{C}$ such that $\mathrm{Re} f = g$.
2. **(Analytic form)** Let F be a subspace of E and let $f : F \to \mathbb{C}$ be a $\mathbb{C}$-linear form. Suppose that there is a semi-norm $p : E \to [0, \infty)$ such that
$$|f(x)| \leq p(x), \quad \forall x \in F.$$
Prove that there there exists a linear form $\tilde{f} : E \to \mathbb{C}$ extending f, and such that $|\tilde{f}| \leq p$.
3. **(Geometric form)** Let $A, B \subset E$ two disjoint convex sets with A open. Prove that there exists $f \in E^*$ such that
$$\sup_{x \in A} \mathrm{Re} f(x) \leq \inf_{x \in B} \mathrm{Re} f(x).$$

Exercise 5. Recall that a point a in a convex set C is extreme if, whenever $a = \theta b + (1-\theta)c$ with $\theta \in (0,1)$ and $b, c \in C$, one has $b = c$.

1. In a Hilbert space, what are the extreme points of the unit closed ball ? What about the open ball ?
2. Let c_0 denote the space of real sequences $(a_n)_{n \in \mathbb{N}}$ which converge to zero. We endow c_0 with the norm $\|\cdot\|_\infty$.
   (a) Show that $(c_0, \|\cdot\|_\infty)$ is a Banach space.
   (b) Show that the closed unit ball of c_0 does not admit extreme points.
   (c) Is it compact ?
3. Let $I \subset \mathbb{R}$ be an interval. Show that the unit closed ball of $L^1(I)$ does not admit extreme points.

Exercise 6. An $n \times n$ matrix with real entries is bi-stochastic if its entries are non-negative, and the sum of the entries of either rows or columns equals 1. One denotes $SM_n(\mathbb{R})$ the set of bistochastic matrices. Show that every matrix in $SM_n(\mathbb{R})$ is actually a convex combination of permutation matrices.

Exercise 7. Let E be a Banach space, (x_n) a sequence in E and (f_n) a sequence in E^*. Suppose that x_n converges weakly to $x \in E$ and f_n converges $*$-weakly to $f \in E^*$. Is it true that $f_n(x_n)$ converges to $f(x)$?
Hint: Consider $E = \ell^2$ and $x_n = e_n$ the canonical basis.

Exercise 8. Let E be a Banach space. Recall that if $f, f_1, \cdots, f_n$ are linear forms on E such that $\bigcap_{i=1}^{n} \ker f_i \subset \ker f$, there exist $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ such that $f = \sum_{i=1}^{n} \lambda_i f_i$.

1. Show that if E is finite-dimensional, then the weak topology $\sigma(E, E^*)$ and the strong topology coincide.
2. We assume that E is infinite-dimensional.
   (a) Show that every weak open subset of E contains a straight line.
   (b) Deduce that $B = \{x \in E \mid \|x\| < 1\}$ is not open for the weak topology.
   (c) Show that $S = \{x \in E \mid \|x\| = 1\}$ is not closed for the weak topology.

Exercise 9. *See Exercise 3, Sheet 3.* Let E be a Banach space and (x_n) be a sequence of E converging towards x in the weak topology. Show that (x_n) is bounded and that

$$\|x\| \leqslant \liminf_{n \to +\infty} \|x_n\|.$$

Exercise 10. Let E, F be two Banach spaces and $T : E \to F$ be a linear map. Show that T is strongly continuous (*i.e.* continuous from $(E, \|\cdot\|_E)$ to $(F, \|\cdot\|_F)$) if and only if T is weakly continuous (*i.e.* continuous from $(E, \sigma(E, E^*))$ to $(F, \sigma(F, F^*))$).

Exercise 11. Let $p, q \in [1, +\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. We introduce the map

$$J_p : \begin{array}{ccc} \ell^q & \to & (\ell^p)^* \\ (a_n) & \mapsto & \left((x_n) \mapsto \sum_{n=0}^{+\infty} a_n x_n \right) \end{array}$$

and the canonical family of sequences e^k of ℓ^p, for which every term is zero, except the k^{th} which is 1.

1. For $p < +\infty$, show that J_p is a surjective isometry.
2. Show that J_∞ is a non-surjective isometry.
3. For $1 < p < +\infty$, deduce that e^k converges weakly but not strongly towards the null sequence when $k \to +\infty$.
4. We still assume that $1 < p < +\infty$, and consider the following subset of ℓ^p :
$$E = \{e^n + ne^m \mid n, m \in \mathbb{N},\ m > n\}.$$
   (a) Show that E is a closed subset for the strong topology.
   (b) Show that 0 is in the weak closure of E.
   (c) Show that a sequence of E cannot converge weakly towards 0.

Exercise 12 (The weak topology is not metrizable). Let E be an infinite dimensional Banach space. The purpose of this exercise is to prove that there does not exist a distance on E that generates the weak topology.

1. Suppose first that every weakly convergent sequence is strongly convergent (i.e. for the norm). Prove that if such a distance existed, then the weak topology and the norm topology would be the same.
2. Now we assume that there exists a weakly convergent sequence that does not converge in norm.
   (a) Prove that there exists a sequence $(e_n) \subset E$ such that e_n converges weakly to 0, and $\|e_n\| = 1$ for all $n \in \mathbb{N}$.
   (b) Define $y_{n,m} = e_n + ne_m$ and prove that the set $F = \{y_{n,m} : m > n\}$ is closed for the norm.
   (c) Prove that 0 lies in the weak closure of F, but there is no sequence in F converging weakly to 0.
   (d) Conclude that the weak topology is not metrizable.