Sheet 5 - Duality and weak topologies

Exercise 1. Let H be a Hilbert space with an orthonormal basis $(e_n)_{n\in\mathbb{N}}$. Show that (e_n) converges weakly towards 0.

Exercise 2. Let E be a Banach space, (x_n) a sequence in E and (f_n) a sequence in E^*. In each one of the following cases, is it true that $f_n(x_n)$ converges to $f(x)$?

1. $\begin{cases} x_n \to x \\ f_n \to f \end{cases}$
2. $\begin{cases} x_n \to x \\ f_n \overset{*}{\rightharpoonup} f \end{cases}$
3. $\begin{cases} x_n \rightharpoonup x \\ f_n \to f \end{cases}$
4. $\begin{cases} x_n \rightharpoonup x \\ f_n \overset{*}{\rightharpoonup} f \end{cases}$

Exercise 3 (Weak but not strong convergence: three crucial examples). Let ϕ be a nonzero function of $\mathcal{D}(\mathbb{R})$ (which is the set of compactly supported C^∞ functions from $\mathbb{R}$ to $\mathbb{R}$).

1. (Evanescence) Show that $u_n(x) = \phi(x-n) \rightharpoonup 0$ in $L^2(\mathbb{R})$, but does not converge strongly.
2. (Concentration) Show that $v_n(x) = \sqrt{n}\phi(nx) \rightharpoonup 0$ in $L^2(\mathbb{R})$, but does not converge strongly.
3. (Oscillation) Let $w \in L^2(0,2\pi)$ be a 2π-periodic non constant function and define $w_n(x) = w(nx)$. Show that $w_n \rightharpoonup \frac{1}{2\pi}\int_0^{2\pi} w$ in $L^2(0,2\pi)$, but does not converge strongly.

Exercise 4. Let E be a Banach space. We recall that H is an affine hyperplane of E^* if and only if there exists a nonzero linear form φ on E^* and $\alpha \in \mathbb{R}$ such that

$$H = \{f \in E^* \mid \varphi(f) = \alpha\}.$$

1. (a) Show that φ is weakly-$*$ continuous if and only if there exists $x \in E$ such that

$$\varphi = ev_x \ \text{(} i.e.\ \forall f \in E^*, \varphi(f) = f(x)).$$

   (b) Show that φ is weakly continuous if and only if φ is strongly continuous.
2. (a) Show that an affine hyperplane H of E^* is weakly-$*$ closed if and only if it is of the form

$$H = \{f \in E^* \mid f(x) = \alpha\},$$

   for some $x \in E$ and some $\alpha \in \mathbb{R}$.

   (b) Show that an affine hyperplane H of E^* is weakly closed if and only if it is of the form

$$H = \{f \in E^* \mid \varphi(f) = \alpha\},$$

   for some (strongly) continuous linear form φ and some $\alpha \in \mathbb{R}$.

Exercise 5. Let $p, q \in [1, +\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. We introduce the map

$$\begin{array}{rccc} J_p: & \ell^q & \to & (\ell^p)^* \\ & (a_n) & \mapsto & \left((x_n) \mapsto \displaystyle\sum_{n=0}^{+\infty} a_n x_n \right) \end{array}$$

and the canonical family of sequences e^k of ℓ^p, for which every term is zero, except the k^{th} which is 1.

1. For $p < +\infty$, show that J_p is a surjective isometry.
2. Show that J_∞ is a non-surjective isometry.
3. For $1 < p < +\infty$, deduce that e^k converges weakly but not strongly towards the null sequence when $k \to +\infty$.
4. We still assume that $1 < p < +\infty$, and consider the following subset of ℓ^p :

$$E = \{e^n + ne^m \mid n, m \in \mathbb{N},\ m > n\}.$$

   (a) Show that E is a closed subset for the strong topology.
   (b) Show that 0 is in the weak closure of E.
   (c) Show that a sequence of E cannot converge weakly towards 0.

Exercise 6 (**The weak topology is not metrizable).** Let E be an infinite dimensional Banach space. The purpose of this exercise is to prove that there does not exist a distance on E that generates the weak topology.

1. Suppose first that every weakly convergent sequence is strongly convergent (i.e. for the norm). Prove that if such a distance existed, then the weak topology and the norm topology would be the same.
2. Now we assume that there exists a weakly convergent sequence that does not converge in norm.
   (a) Prove that there exists a sequence $(e_n) \subset E$ such that e_n converges weakly to 0, and $\|e_n\| = 1$ for all $n \in \mathbb{N}$.
   (b) Define $y_{n,m} = e_n + ne_m$ and prove that the set $F = \{y_{n,m} \,:\, m > n\}$ is closed for the norm.
   (c) Prove that 0 lies in the weak closure of F, but there is no sequence in F converging weakly to 0.
   (d) Conclude that the weak topology is not metrizable.

Exercise 7. We denote by c_0 the space of real sequences converging to 0 endowed with the uniform norm $\|\cdot\|_\infty$ and by e^k the sequence for which every term is zero, except the k^{th} which is 1, and

$$S = \left\{ \varphi \in c_0^* \;\middle|\; \sum_{k=1}^{+\infty} \varphi(e^k) = 0 \right\}.$$

1. Show that S is strongly closed (*i.e.* closed for the topology induced by the norm).
2. Show that S is weakly closed (*i.e.* closed for the topology $\sigma(c_0^*, c_0^{**})$).
3. Show that S is not weakly-$*$ closed (*i.e.* not closed for the topology $\sigma(c_0^*, c_0)$).

Exercise 8 (**Schur's property for ℓ^1).**

1. Recall why weak and strong topologies always differ in an infinite dimensional norm vector space.

The aim is to prove that a sequence of ℓ^1 converges weakly if and only if it converges strongly. Take $u^n = (u_k^n)_{k\in\mathbb{N}}$ a sequence of ℓ^1 weakly converging to 0.

2. Show that for all k, $\lim_{n\to\infty} u_k^n \to 0$.
3. Show that if $u_n \not\to 0$ in ℓ^1, one can additionnaly assume that $\|u^n\|_{\ell^1} = 1$.
4. Define via a recursive argument two increasing sequences of $\mathbb{N}$, (a_k) and (n_k), such that

$$\forall k, \qquad \sum_{j=a_k}^{a_{k+1}-1} |u_j^{n_k}| \geq \frac{3}{4}.$$

5. Show that there exists $v \in \ell^\infty$ such that $(v, u^{n_k}) \geq \frac{1}{2}$ for all k. Conclude.

Out of topic:

Exercise 9. Let E be a $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) topological vector space and F be a finite dimensional subspace of E. Pick a basis (e_i) on F, and consider the linear injection $f : \mathbb{K}^n \to E$ mapping the canonical basis of $\mathbb{K}^n$ on (e_i).

1. Why is f continuous? Give its range.
2. Show that there exists a balanced neighbourhood V of 0_E such that $f^{-1}(V) \subset B$ the open unit ball of $\mathbb{K}^n$.
3. Deduce that F is closed in E.