Sheet 6 - Reflexivity

Exercise 1. Let $p, q \in [1, +\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. We introduce the map

$$J_p : \begin{array}{ccc} \ell^q & \to & (\ell^p)^* \\ (a_n) & \mapsto & \left((x_n) \mapsto \sum_{n=0}^{+\infty} a_n x_n \right) \end{array}$$

and the canonical family of sequences e^k of ℓ^p, for which every term is zero, except the k^{th} which is 1.

1. For $p < +\infty$, show that J_p is a surjective isometry.
2. Show that J_∞ is a non-surjective isometry.
3. For $1 < p < +\infty$, deduce that e^k converges weakly but not strongly towards the null sequence when $k \to +\infty$.
4. We still assume that $1 < p < +\infty$, and consider the following subset of ℓ^p :

$$E = \{e^n + ne^m \mid n, m \in \mathbb{N},\ m > n\}.$$

   (a) Show that E is a closed subset for the strong topology.
   (b) Show that 0 is in the weak closure of E.
   (c) Show that a sequence of E cannot converge weakly towards 0.

Exercise 2. Let E, F be two Banach spaces and $T : E \to F$ be a linear map.

1. Show that T is strongly continuous (*i.e.* continuous from $(E, \|\cdot\|_E)$ to $(F, \|\cdot\|_F)$) if and only if T is weakly continuous (*i.e.* continuous from $(E, \sigma(E, E^*))$ to $(F, \sigma(F, F^*))$).
   Hint: Show that T is weakly continuous if and only if $f \circ T : (E, \sigma(E, E^*)) \to \mathbb{R}$ is continuous for all $f \in F^\star$.
2. If T is a continuous isomorphism, show that E is reflexive if and only if F is reflexive.

Exercise 3. Let E be a Banach space.

1. Assume that E is separable. Show that every bounded sequence of E^* admits a subsequence which converges in the weak-$*$ topology.
2. Assume that E is reflexive and separable. Show that every bounded sequence of E admits a subsequence which converges in the weak topology.

Exercise 4.

1. Let $I \subset \mathbb{R}$ be an interval. Show that the unit closed ball of $L^1(I)$ does not admit extreme points.
2. Deduce that $L^1(I)$ is not the dual of a Banach space.
3. Is $L^1(I)$ reflexive?

Exercise 5. The aim of this exercise is to prove by two different methods that the space $(C([0,1]), \|\cdot\|_\infty)$ of continuous real-valued functions on $[0,1]$ is not reflexive.

1. By compactness.

   (a) Prove that if E is a reflexive Banach space and $\varphi \in E^*$, then there exists $x \in E$ with $\|x\| = 1$ such that $\langle \varphi, x \rangle = \|\varphi\|$.
   (b) Define $\varphi \in C([0,1])^*$ by

$$\langle \varphi, f \rangle = \int_0^{\frac{1}{2}} f(t)\, dt - \int_{\frac{1}{2}}^1 f(t)\, dt, \quad \forall f \in C([0,1]),$$

   and show that $\|\varphi\| = 1$.
   (c) Prove that $\langle \varphi, f \rangle < 1$ for all $f \in C([0,1])$ such that $\|f\|_\infty \leq 1$.
   (d) Conclude that $C([0,1])$ is not reflexive.

2. By separability.
   (a) Show that if E is a Banach space and its dual E^* is separable, then E is separable.
   Hint: Take a dense subset (φ_n) of E^* and chose $x_n \in E$ such that $\|x_n\| = 1$ and $\langle \varphi_n, x_n \rangle \geq \frac{1}{2}\|\varphi_n\|$.
   (b) Show that $C([0,1])$ is separable.
   (c) Show that $C([0,1])^*$ is not separable.
   Hint: Consider the functions $\delta_t : C([0,1]) \to \mathbb{R}$ defined by $\langle \delta_t, f \rangle = f(t)$ for each $t \in [0,1]$.
   (d) Conclude that $C([0,1])$ is not isomorphic to $C([0,1])^{**}$ as Banach spaces. *Remark: This is stronger than not being reflexive.*

Definition. A Banach space is called *uniformly convex* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(x, y \in E, \|x\| \leq 1, \|y\| \leq 1, \|x - y\| > \varepsilon) \implies \left(\|\frac{x+y}{2}\| < 1 - \delta. \right)$$

Theorem (Milman-Pettis)**.** Let E be a Banach space. If it is uniformly convex, then it is reflexive.

Proof. See *Analyse fonctionnelle*, Brézis. □

Exercise 6. Let (X, μ) be a measure space, $p, q \in]1, +\infty[$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\begin{array}{rccc} \varphi_p : & L^p(X,\mu) & \to & (L^q(X,\mu))^* \\ & f & \mapsto & \left(g \mapsto \int fg \mathrm{d}\mu \right). \end{array}$$

1. Show that φ_p is a well-defined and continuous isometry. Deduce that $\varphi_p(L^p)$ is a closed subset of $(L^q)^*$.
2. Assume that $p \geqslant 2$.
   (a) Show that $\forall \alpha, \beta \geqslant 0$, $\alpha^p + \beta^p \leqslant (\alpha^2 + \beta^2)^{\frac{p}{2}}$. Deduce that $\forall a, b \in \mathbb{R}$,
   $$\left| \frac{a+b}{2} \right|^p + \left| \frac{a-b}{2} \right|^p \leqslant \frac{1}{2}|a|^p + \frac{1}{2}|b|^p.$$
   (b) Show that L^p is uniformly convex. Deduce that L^p is reflexive.
3. Show that L^p is reflexive for every $p \in]1, +\infty[$.
4. Show that φ_p is onto. Deduce that L^p is isometrically isomorphic to $(L^q)^*$.