Sheet 7 - Distributions - Episode 1: first handlings

Exercise 1. Some basic facts.

1. Let $H : \mathbb{R} \to \mathbb{R}$ be the Heaviside step function

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0, \end{cases}$$

   and δ_0 the distribution on $\mathbb{R}$ given by $\langle \delta_0, \varphi \rangle = \varphi(0)$. Show that $H' = \delta_0$ in the sense of distributions.

2. Give an example of distribution of order n for all $n \in \mathbb{N}$.

3. Show that the formula

$$\langle \Lambda, \varphi \rangle = \sum_{n \geq 0} \varphi^{(n)}(n), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}),$$

   defines a distribution $\Lambda \in \mathcal{D}'(\mathbb{R})$ and compute its order.

4. Let $\theta \in C^\infty(\mathbb{R})$. Compute $\theta \delta_0'$.

Exercise 2. Study the convergence of the following sequences in the sense of distributions. Give the order of the distributions of the sequence and of the limit if it exists.

1. $f_n(x) = n^d f(nx)$ with $f \in L^1(\mathbb{R}^d)$
2. $T_n = (-1)^n \delta_{1/n}$
3. $T_n = \frac{n}{2}(\delta_{1/n} - \delta_{-1/n})$
4. $f_n(x) = \cos(nx)$
5. $f_n(x) = n^p \cos(nx)$ with $p > 0$

Exercise 3 (Jump formula)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 on $\mathbb{R}^*$. We say that f has a jump at 0 if the limits $f(0^\pm) = \lim_{x \to 0^\pm} f(x)$ exist and denote by $[[f(0)]] = f(0^+) - f(0^-)$ the height of the jump. We denote by $\{f'\}$ the derivative of the regular part of f, *i.e.*

$$\{f'\}(x) = \begin{cases} f'(x) & \text{if } f \text{ is differentiable at } x \\ 0 & \text{otherwise} \end{cases}$$

1. Show that in the sense of distributions:

$$f' = \{f'\} + [[f(0)]]\delta_0.$$

2. Let (x_n) be an increasing sequence indexed by $\mathbb{Z}$ such that

$$\lim_{n \to -\infty} x_n = -\infty \quad \text{and} \quad \lim_{n \to +\infty} x_n = +\infty.$$

   Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function presenting jumps at every x_n. Show that in the sense of distributions

$$f' = \{f'\} + \sum_{n \in \mathbb{Z}} [[f(x_n)]]\delta_{x_n}.$$

3. Let $P = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + a\frac{\mathrm{d}}{\mathrm{d}x} + b$ be a differential operator with $a, b \in \mathbb{R}$.
   (a) Let f and g be two C^2 functions on $\mathbb{R}$ such that $Pf = 0$ and $Pg = 0$, and define $h = f\mathbb{1}_{\mathbb{R}^+} + g\mathbb{1}_{\mathbb{R}^-}$. Compute Ph in the sense of distributions.
   (b) Deduce a solution of the equation $T'' + aT' + bT = \delta_0$.

Exercise 4. Let I be an interval of $\mathbb{R}$.

1. Solve $T' = 0$ in $\mathcal{D}'(I)$.

2. Let $f, g \in C^\infty(I)$, show that every solution T of $T' + fT = g$ in $\mathcal{D}'(I)$ is a smooth function and that the set of solutions is a one-dimensional affine subspace of $C^\infty(I) \subset \mathcal{D}'(I)$.

3. Let $P = \sum_{0 \leq k \leq N} a_k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k$ be a differential operator with $a_N = 1$ and a_k complex or real values. Use the previous questions to prove that the set of distributions solving $PT = 0$ is a N-dimensional subspace of $C^\infty(I) \subset \mathcal{D}'(I)$.

For the next questions, you might be inspired by the third part of Exercise 3.

4. For $f \in C^\infty(\mathbb{R})$, solve $T' + fT = \delta_0$ in $\mathcal{D}'(\mathbb{R})$.
5. Solve $T'' - T' - 2T = \delta_0$ in $\mathcal{D}'(\mathbb{R})$.

Exercise 5 (Rankine-Hugoniot condition)**.** Let Ω, Ω_- and Ω_+ be open subsets of $\mathbb{R}^d$ such that $\Sigma = \partial\Omega_- \cap \partial\Omega_+$ is a smooth hypersurface of Ω and $\Omega = \Omega_- \cup \Sigma \cup \Omega_+$. We denote by $\vec{\nu}$ the normal vector on Σ pointing from Ω_- to Ω_+.

Let $\vec{q}$ be a vector field on Ω, such that $\vec{q}\big|_{\Omega_+}$ and $\vec{q}\big|_{\Omega_-}$ are C^1 and $\vec{q}(x)$ admits a limit when x tends to Σ in Ω_- or Ω_+, respectively denoted by $\vec{q}_-(x)$ and $\vec{q}_+(x)$. We assume furthermore that $\operatorname{div}(\vec{q}) = 0$ on $\Omega_- \cup \Omega_+$.

Prove the Rankine-Hugoniot condition:

$$\operatorname{div}(\vec{q}) = 0 \text{ in the sense of distributions} \iff (\vec{q}_+ - \vec{q}_-) \cdot \vec{\nu} = 0 \text{ on } \Sigma.$$

Remark: for $d = 1$, compare with the first question of Exercise 3.
Differential calculus handout:

- $\operatorname{div}(\vec{q}) = \vec{\nabla} \cdot \vec{q} = \sum_i \frac{\partial q_i}{\partial x_i}$, $\quad \operatorname{div}(\varphi \vec{q}) = \varphi \operatorname{div}(\vec{q}) + \vec{q} \cdot \vec{\nabla}\varphi$ for φ a C^1 function,
- Green-Ostrogradski formula: if Ω is a C^1 regular open set, $\int_\Omega \operatorname{div} \vec{v}\, d\mu = \int_{\partial\Omega} \vec{v} \cdot \vec{n}\, d\sigma$ where $\vec{n}$ is the outward pointing unit normal field on $\partial\Omega$.

Exercise 6. We denote by $\arg(z)$ the principal value argument of the complex number z, *i.e.* the argument in $]-\pi, \pi]$. For all $\varepsilon > 0$, we define

$$f_\varepsilon(x) = \ln(x + i\varepsilon) = \ln|x + i\varepsilon| + i \arg(x + i\varepsilon).$$

1. Compute $f_{0^+} = \lim_{\varepsilon \to 0^+} f_\varepsilon$ and $f_{0^-} = \lim_{\varepsilon \to 0^-} f_\varepsilon$ in $\mathcal{D}'(\mathbb{R})$.
2. Compute f'_{0^+} and f'_{0^-} in the sense of distributions.
   Hint: you may use the *principal value* function $\mathrm{p.v.}(1/x)$ defined in the homework sheet.
3. Deduce the following limits in $\mathcal{D}'(\mathbb{R})$:

$$\lim_{\varepsilon \to 0^+} \frac{1}{x + i\varepsilon} = -i\pi\delta_0 + \mathrm{p.v.}(1/x) \qquad \text{et} \qquad \lim_{\varepsilon \to 0^+} \frac{1}{x - i\varepsilon} = i\pi\delta_0 + \mathrm{p.v.}(1/x).$$

Exercise 7. Show that $f(x, y) = \frac{1}{x+iy}$ defines an element of $\mathcal{D}'(\mathbb{R}^2)$ and compute $\partial_x f + i\partial_y f$ in the sense of distributions.