Sheet 8 - Distributions - Episode 2: the return of supports, convolution and the Fourier transform

Exercise 1. Let $T \in \mathcal{D}'(\mathbb{R}^d)$.

1. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$, show that if $\operatorname{supp} T \cap \operatorname{supp} \varphi = \emptyset$ then $\langle T, \varphi\rangle = 0$. Is the converse true?
2. Suppose that T has compact support and take $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that $\psi = 1$ in a neighbourhood of $\operatorname{supp} T$. Show that $\psi T = T$.
3. Let $\varphi \in C^\infty(\mathbb{R}^d)$ and assume that $\operatorname{supp} T \cap \operatorname{supp} \varphi$ is compact. Show that $\langle T, \varphi\rangle$ can be defined in a meaningful way.
4. Let $T, S \in D'(\mathbb{R}^d)$ and suppose T and S satisfy the following: for every compact K in $\mathbb{R}^d$,
$$\left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \;\middle|\; x \in \operatorname{supp} T,\ y \in \operatorname{supp} S,\ x + y \in K\right\}$$
is compact. Show that in this case, $T * S$ and $S * T$ are well-defined.

Exercise 2. Show that the convolution product is not associative without assumptions on the supports by considering the distributions 1, δ_0' and H in $\mathcal{D}'(\mathbb{R})$, where H is the Heaviside function (see Sheet 7).

Exercise 3. Compute the following convolutions:

1. $\delta_a * \delta_b$ in $\mathbb{R}^d$,
2. $T * \delta_a$ with $T \in \mathcal{D}'(\mathbb{R}^d)$,
3. $H * H$ (Heaviside),
4. $(x^p \delta_0^{[q]}) * (x^m \delta_0^{[n]})$,
5. $\delta_0^{[k]} * (x^m H)$,
6. $(x^p H) * (x^q H)$,
7. $\mathbb{1}_{[a,b]} * \mathbb{1}_{[c,d]}$,
8. $\mathbb{1}_{[0,1]} * (xH)$,
9. $\delta_{S(0,r)} * |x|^2$ in $\mathbb{R}^3$.

Exercise 4.

1. If $d \geq 3$, show that $u_0(x) = \left(-d(d-2)\mathrm{Vol}(B(0,1))\|x\|^{d-2}\right)^{-1}$ is a fundamental solution for the Laplacian, *i.e.* $\Delta u_0 = \delta_0$ in the sense of distributions.
2. Give a solution of $\Delta u = f$ in the sense of distributions for f in $\mathcal{E}'(\mathbb{R}^d)$.
3. What can you say about the regularity of u if f is a function in $\mathcal{S}(\mathbb{R}^d)$?
4. Consider the linear PDE $u - \Delta u = f$ for $f \in \mathcal{S}(\mathbb{R}^d)$. Express in terms of the Bessel kernel $B = \mathcal{F}^{-1}\left((1 + |\xi|^2)^{-1}\right)$ a solution in $\mathcal{S}(\mathbb{R}^d)$.

Exercise 5. Define $\mathcal{D}'_+(\mathbb{R}) = \{T \in \mathcal{D}'(\mathbb{R}) \mid \operatorname{supp} T \subset \mathbb{R}^+\}$.

1. Show that the convolution of two elements of $\mathcal{D}'_+(\mathbb{R})$ is well-defined and gives an element of $\mathcal{D}'_+(\mathbb{R})$. *See Exercise 1. In the following we will assume that the convolution is associative and commutative in* $\mathcal{D}'_+(\mathbb{R})$. What is the identity element for the convolution on $\mathcal{D}'_+(\mathbb{R})$?
2. Show that for all $a \in \mathbb{R}$ and all $T, S \in \mathcal{D}'_+(\mathbb{R})$, we have $(e^{ax}T) * (e^{ax}S) = e^{ax}(T * S)$.
3. For $T \in \mathcal{D}'_+(\mathbb{R})$, let T^{-1} denote the inverse of T in $\mathcal{D}'_+(\mathbb{R})$ for the convolution whenever it exists. Show that T^{-1} is indeed unique and compute $(\delta_0')^{-1}$, $(H)^{-1}$ and $(\delta_0' - \lambda\delta_0)^{-1}$ for $\lambda \in \mathbb{R}$.
4. Let P a polynomial that splits in $\mathbb{R}$, compute $[P\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\delta_0]^{-1}$.
5. Solve the following system in $\mathcal{D}'_+(\mathbb{R})$
$$\begin{cases} \delta_0'' * X + \delta_0' * Y &= \delta_0 \\ \delta_0' * X + \delta_0'' * Y &= 0. \end{cases}$$

Exercise 6. We will study the behaviour of the convergence of distributions with respect to the convolution product.

1. Let $T \in \mathcal{E}'(\mathbb{R}^d)$, $V \in \mathcal{D}'(\mathbb{R}^d)$ and (V_n) be a sequence of distributions in $\mathcal{D}'(\mathbb{R}^d)$. Prove that if $V_n \to V$ in $\mathcal{D}'(\mathbb{R}^d)$ then $V_n * T \to V * T$ in $\mathcal{D}'(\mathbb{R}^d)$.
2. Let $T \in \mathcal{D}'(\mathbb{R}^d)$, $V \in \mathcal{E}'(\mathbb{R}^d)$ and (V_n) be a sequence of distributions in $\mathcal{E}'(\mathbb{R}^d)$. Prove that if $V_n \to V$ in $\mathcal{E}'(\mathbb{R}^d)$ then $V_n * T \to V * T$ in $\mathcal{D}'(\mathbb{R}^d)$.
3. Show that there exist two sequences of distributions T_n and V_n tending to 0 in $\mathcal{D}'(\mathbb{R})$ and such that $T_n * V_n \to \delta_0$.

Exercise 7. Let $F : \mathcal{D}(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ be a continuous linear map. We say that F commutes with translations if, for all $x \in \mathbb{R}^d$, $\tau_x \circ F = F \circ \tau_x$, i.e.

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^d), \forall y \in \mathbb{R}^d, \quad F(\varphi)(x+y) = \tau_x(F(\varphi))(y) = F(\tau_x \varphi)(y) = F(z \mapsto \varphi(x+z))(y).$$

1. Show that if there exists $T \in \mathcal{D}'(\mathbb{R}^d)$ such that, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $F(\varphi) = T * \varphi$, then F commutes with translations.
2. Show that for all $T \in \mathcal{D}'(\mathbb{R}^d)$, and all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we have $\langle T, \varphi \rangle = T * \check{\varphi}(0)$ where $\check{\varphi}(x) = \varphi(-x)$.
3. Show that if F commutes with translations, there exists $T \in \mathcal{D}'(\mathbb{R}^d)$ such that, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $F(\varphi) = T * \varphi$.

Exercise 8.

1. Let $p \in [1, +\infty]$ and $f \in L^p(\mathbb{R}^d)$. Prove that f defines a tempered distribution.
2. Let $P \in \mathbb{R}[X_1, \ldots, X_d]$ and $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that $|f| \leqslant |P|$. Prove that f defines a tempered distribution.

Exercise 9. Prove that the following distributions are tempered and compute their Fourier transform, with the following convention: for $f \in L^1(\mathbb{R}^d)$,

$$\mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} \mathrm{d}x.$$

1. δ_0 in $\mathbb{R}^d$,
2. $e^{-\frac{|x|^2}{2\sigma}}$ in $\mathbb{R}$ with $\sigma > 0$,
3. H (Heaviside),
4. $\mathrm{p.v.}(1/x)$,
5. $\delta_0^{[n]}$ in $\mathbb{R}$,
6. $|x|$ in $\mathbb{R}$,

Exercise 10. Let $k > 0$ and $T \in \mathcal{S}'(\mathbb{R})$ such that $T^{[4]} + kT \in L^2(\mathbb{R})$. Show that for every $j \in [\![0, 4]\!]$, $T^{[j]} \in L^2(\mathbb{R})$.

Exercise 11. Let $U \in \mathcal{S}'(\mathbb{R}^d)$. Prove that if $\Delta U = 0$ then U is a polynomial.

Exercise 12. We define 4 notions of support:

- for $T \in \mathcal{D}'(\mathbb{R}^d)$,
  $\mathrm{supp}_{dist}\, T =$ complement of the biggest open set Ω_1 on which: $\forall \varphi \in \mathcal{D}(\Omega_1),\ \langle T, \varphi \rangle = 0$;
- for $\mu \in \mathcal{M}(\mathbb{R}^d)$ (the set of signed Radon measures, i.e., Borel measures on $\mathbb{R}^d$, with values in $\overline{\mathbb{R}}$ which are finite on compact sets),
  $\mathrm{supp}_{mes}\, \mu =$ complement of the biggest open set Ω_2 on which: $\forall A \subset \Omega_2,\ \mu(A) = 0$;
- for $f \in L^1_{loc}(\mathbb{R}^d)$,
  $\mathrm{supp}_{int}\, f =$ complement of the biggest open set Ω_3 on which: $f_{|\Omega_3} = 0$ almost everywhere;
- for $g \in C^0(\mathbb{R}^d)$,
  $\mathrm{supp}_{cont}\, g = \overline{\{x \in \mathbb{R}^d \mid f(x) \neq 0\}}$.

1. Let $g \in C^0(\mathbb{R}^d)$, prove that $\mathrm{supp}_{cont}\, g = \mathrm{supp}_{int}\, g$.
2. Let $f \in L^1_{loc}(\mathbb{R}^d)$, we define a Radon measure by means of a density function $\mu = f \mathrm{d}x$, prove that $\mathrm{supp}_{int}\, f = \mathrm{supp}_{mes}\, \mu$.
3. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and let T be the distribution associated with μ, prove that $\mathrm{supp}_{dist}\, T \subset \mathrm{supp}_{mes}\, \mu$.
4. Let $g \in C^0(\mathbb{R}^d)$ and let T be the distribution associated with g, prove that $\mathrm{supp}_{dist}\, T = \mathrm{supp}_{cont}\, g$.