Sheet 9 - Sobolev spaces

Exercise 1.

1. Show that $u(x) = |x|$ belongs to $W^{1,2}(]-1,1[)$ but not to $W^{2,2}(]-1,1[)$.
2. Show that $v(x) = \frac{\sin(x^2)}{\sqrt{1+x^2}}$ belongs to $L^2(\mathbb{R})$ but not to $W^{1,2}(\mathbb{R})$.

Exercise 2. Let Ω_1, Ω_2 be two open subsets of $\mathbb{R}^d$ and $p \in [1,+\infty]$. Let $f : \Omega_1 \to \Omega_2$ be a C^1-diffeomorphism such that the differential of both f and f^{-1} is bounded. We define

$$F: \begin{array}{ccc} W^{1,p}(\Omega_1) & \to & W^{1,p}(\Omega_2) \\ u & \mapsto & u \circ f^{-1}. \end{array}$$

Prove that F is well-defined and that there exist $C_1, C_2 > 0$ such that, for all $u \in W^{1,p}(\Omega_1)$,

$$C_2 \|u\|_{W^{1,p}(\Omega_1)} \leqslant \|F(u)\|_{W^{1,p}(\Omega_2)} \leqslant C_1 \|u\|_{W^{1,p}(\Omega_1)}.$$

Exercise 3. Let $p \in [1,+\infty[$ and let Ω be an open subset of $\mathbb{R}^d$. Assume that Ω is bounded in one direction, meaning that Ω is contained in the region between two parallel hyperplanes. Prove Poincaré's inequality: there exists $c > 0$ such that for every $f \in W_0^{1,p}(\Omega)$,

$$\|f\|_{L^p} \leqslant c \|\nabla f\|_{L^p},$$

where $\|\nabla f\|_{L^p} = \left(\sum_{i=1}^{d} \|\partial_i f\|_{L^p}^p \right)^{1/p}$. *Hint:* consider first the case $\Omega \subset [-M, M] \times \mathbb{R}^{d-1}$.

Remark: This shows that $f \mapsto \|\nabla f\|_{L^p}$ defines a norm on $W_0^{1,p}(\Omega)$, which is equivalent to $\|\cdot\|_{W^{1,p}(\Omega)}$.

Exercise 4. Let Ω be an open subset of $\mathbb{R}^d$, $p \in [1,+\infty]$ and $k \in \mathbb{N}^*$. Prove the following integration by parts formula: for all $f \in W^{k,p}(\Omega)$, $g \in W_0^{k,q}(\Omega)$ (with $\frac{1}{p} + \frac{1}{q} = 1$) and all multi-index α such that $|\alpha| \leqslant k$,

$$\int_\Omega g D^\alpha f \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega f D^\alpha g \mathrm{d}x.$$

Exercise 5. Show that the spaces $W^{k,p}$, $C_b(\bar{\Omega})$, $C^\alpha(\bar{\Omega})$ defined in the lesson are Banach spaces.

Exercise 6. Let Ω be an open subset of $\mathbb{R}^d$ and let $p \in]1,+\infty[$.

1. Prove that for all $F \in \left(W_0^{1,p}(\Omega)\right)^*$, there exist $f_0, f_1, \ldots, f_d \in L^q(\Omega)$ (with $\frac{1}{p} + \frac{1}{q} = 1$) such that for all $g \in W_0^{1,p}(\Omega)$,

$$\langle F, g \rangle = \int_\Omega f_0 g \mathrm{d}x + \sum_{i=1}^{d} \int_\Omega f_i \partial_i g \mathrm{d}x.$$

2. Prove that we also have

$$\|F\| \leqslant \left(\sum_{i=0}^{d} \|f_i\|_{L^q}^q \right)^{\frac{1}{q}}.$$

3. Assuming that Ω is bounded, prove that we may take $f_0 = 0$.

Exercise 7. Let I be an interval of $\mathbb{R}$, $p \in (1, +\infty]$ and $f \in W^{1,p}(I)$.

1. Let $a \in I$, show that the function T, given by
$$T(x) = \int_a^x f'(t)\mathrm{d}t,$$
is well-defined, continuous and differentiable almost everywhere. *You may use Lebesgue's differentiation theorem: if* $u \in L^1_{loc}(\mathbb{R}^d)$ *then for almost every* $x \in \mathbb{R}^d$,
$$\frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\mathrm{d}y \quad \overset{r\to 0}{\longrightarrow} \quad u(x).$$

2. Prove that $T' = f'$ almost everywhere and in the sense of distributions.

3. Deduce that there exists $c \in \mathbb{R}$ such that for almost every $x \in \mathbb{R}$,
$$f(x) = c + \int_a^x f'(t)\mathrm{d}t.$$
In particular, f admits a continuous and almost everywhere differentiable representative.

4. Show that f is Hölder-continuous, and Lipschitz-continuous if $p = +\infty$.

5. Show that $W^{1,p}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, meaning that there exists a constant $c > 0$ such that for all $f \in W^{1,p}(\mathbb{R})$, f belongs to $L^\infty(\mathbb{R})$ and
$$\|f\|_{L^\infty} \leqslant c\|f\|_{W^{1,p}}.$$
We say that $W^{1,p}(I)$ embeds continuously into $L^\infty(I)$.
Hint: For $p < \infty$, consider $G(t) = t|t|^{p-1}$ and study the function $G \circ g$ for $g \in \mathcal{D}(\mathbb{R})$.

6. Prove that, more generally, $W^{1,p}(I) \hookrightarrow L^\infty(I)$. *You may use the fact that there exists a bounded linear map $P : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$ ($p < \infty$) such that $Pf|_I = f$ for all $f \in W^{1,p}(I)$.*

7. Suppose that I is bounded.
   (a) Show that $W^{1,p}(I)$ is closed under multiplication.
   (b) Show that the embedding $W^{1,p}(I) \hookrightarrow C(\overline{I})$ is compact, meaning that every element of $W^{1,p}(I)$ is in $C(\overline{I})$, and that every bounded sequence in $(W^{1,p}(I), \|\cdot\|_{W^{1,p}})$, possesses a subsequence which converges uniformly on $\overline{I}$ to a limit in $C(\overline{I})$.

8. If I is unbounded, prove that
$$\lim_{\substack{|x|\to+\infty \\ x\in I}} f(x) = 0.$$

Exercise 8. Let $f \in L^2([0,1])$. Our aim is to solve the following equation (Dirichlet problem)
$$\begin{cases} -u'' + u &= f \quad \text{in }]0,1[, \\ u(0) &= 0 \\ u(1) &= 0. \end{cases}$$
In what follows, we will write $H^k = W^{k,2}(]0,1[)$ and $H^1_0 = W^{1,2}_0(]0,1[)$.

1. Show that the Dirichlet problem admits a weak solution, *i.e.* there exists $u \in H^1_0$ such that for all $v \in H^1_0$,
$$\int_0^1 u'v'\mathrm{d}t + \int_0^1 uv\mathrm{d}t = \int_0^1 fv\mathrm{d}t.$$

2. Show that if $u \in C^2(]0,1[) \cap C([0,1])$ is a classical solution, then u is also a weak solution.

3. Show that if u is a weak solution, then $u \in H^2$.

4. Show that if $f \in H^k$, then every weak solution u is in H^{k+2}.

5. Assume that $f \in C([0,1])$. Show that if u is a weak solution, then u is C^2 and is a solution in the classical sense.