Optimization with constraint(s)

Exercise 1: qualification
Check that each admissible point for the following constraints are qualified.

1. $2x + y \leq 2, x \geq 0, y \geq 0,$
2. $x^2 + y^2 \leq 1, x \geq 0, y \geq 0,$
3. $2x - y = -5, 5x + 2y \geq 37, x \geq 0, y \geq 0,$
4. $x^2 + y^2 + z \leq 6, x \geq 0, y \geq 0, z \geq 0,$
5. $4x + 3y \leq 10, y - 4x^2 \geq -1, x \geq 0, y \geq 0$

Exercice 2: let's optimize !
Determine the following quantities:

1. $$\max_{\substack{2x+y\leq 2\\ x\geq 0\\ y\geq 0}} x^2 + y^2,$$
2. $$\min_{\substack{x^2+y^2\leq 1\\ x\geq 0\\ y\geq 0}} x^2 - 2y,$$
3. $$\max_{\substack{2x-y=-5\\ 5x+2y\geq 37\\ x\geq 0\\ y\geq 0}} 3xy - x^3,$$
4. $$\max_{\substack{2x+2y\leq 1\\ x\geq 0\\ y\geq 0}} x^2 + x + 4y^2.$$

Exercise 3: perturbed problems
For the following optimization problems, compare the value obtained by the direct approach with the value estimated by applying the envelope theorem to the previous exercise (you may assume that the conditions of the envelope theorem are met).

1. $$\min_{\substack{x^2+y^2\leq 0.9\\ x\geq 0\\ y\geq 0}} x^2 - 2.2y,$$
2. $$\max_{\substack{2x+2.1y\leq 1\\ x\geq 0\\ y\geq 0}} x^2 + x + 4y^2.$$

Exercise 4: linear programming and duality
Optimizing a linear function under linear constraints is called *linear programming*. We will consider two coupled problems, called *primal* and *dual* problem.

Let $p = (p_1, \ldots, p_n)$ be a price vector, $c = (c_1, \ldots, c_n)$ be the constraint vector, B be the constraint matrix.

We will always denote $u \leq v$ if u and v are same-dimensional vectors such that $u_i \leq v_i$ for all i.

$$\text{(primal)}\quad \max_{\substack{Bx\leq c,\\ x\geq 0.}} p\cdot x \qquad\Bigg|\qquad \min_{\substack{{}^tB\lambda\geq p,\\ \lambda\geq 0.}} \lambda\cdot c \quad\text{(dual)}$$

where $\cdot$ designs the scalar product. We assume that the constraints are qualified at each admissible point.

1. Assume that the solution of the primal problem is reached at some point a. Let us denote by $\hat{\pi}$ the vector Lagrange multiplier associated with the constraints $Bx \leq c$ and by μ the multiplier associated with the non-negativity constraints $x \geq 0$. Write the system of (in)equalities satisfied by these vectors.

2. Show that $p \cdot a = \hat{\pi} \cdot c$. *Hint:* You will use the symmetry of the scalar product, and the fact that ${}^tAx \cdot y = x \cdot Ay$ if A is a matrix and (x, y) a couple of vectors.

3. Assume that the solution of the dual problem is reached at some point π. Les us denote by $\hat{a}$ the vector Lagrange multiplier associated with the constraints ${}^tB\lambda \geq p$ and by γ the multiplier associated with the non-negativity constraints $\lambda \geq 0$. Write the system of (in)equalities satisfied by these vectors.

4. Show that $\pi \cdot c = p \cdot \hat{a}$.

5. We aim to show that $p \cdot a = p \cdot \hat{a}$.

   (a) Starting from the dual problem, show that $\hat{a}$ is admissible for the primal problem.

   (b) Show that for all $x \geq 0$ such that $Bx \leq c$, $p \cdot x \leq p \cdot \hat{a}$. *Hint:* Use the system of (in)equalities given by the dual problem.

   (c) Deduce that $\hat{a}$ solves the primal problem.

6. Likewise, show that $\hat{\pi}$ solves the dual problem.

7. Write the Kuhn & Tucker Lagrangians associated with both problems. How can you understand the result of this exercise with these expressions?

Exercise 5: let's optimize with Kuhn & Tucker !
Solve exercise 2 with the Kuhn & Tucker conditions.

Exercise 6: optimization under inequality constraints, convex case
Let f, g_1, ..., g_p be $\mathcal{C}^1$ convex functions from $\mathbb{R}^d$ to $\mathbb{R}$, and let us denote $g = (g_1, \cdots, g_p)$. We are looking for the minimum of f under the constrains $g_i \leq 0$ for all i.

We assume in the all exercise that there exists $x^\star$ and $\pi \in \mathbb{R}^p$ such that

$$\begin{cases} \pi_i \geq 0 \ \forall i, \\ g_i(x^\star) \leq 0 \ \forall i, \\ \pi \cdot g(x^\star) = 0, \\ \nabla f(x^\star) + \sum_{i=1}^p \pi_i \nabla g_i(x^\star) = 0. \end{cases}$$

1. **Sufficient condition of minimization** Show that $x^\star$ reaches the minimum of f under the constraints $g_i(x) \leq 0$. *Hint:* use that $x \mapsto L(x, \pi) = f(x) + \pi \cdot g(x)$ is convex.

2. **Envelope theorem**
   For $c \in \mathbb{R}^p$, let us consider the following perturbed problem: find the infimum of f under the constraints $g_i(x) \leq c_i$ for all i.

   (a) If $m^*(0)$ et $m^*(c)$ denotes respectively the values obtained for the initial and perturbed problems, show that
   $$m^*(c) \geq m^*(0) - \pi \cdot c,$$
   where $\pi \in \mathbb{R}^p$ is the Lagrange multiplier associated with the unperturbed problem. (You may assume that the infimum of the perturbed problem is reached at an admissible -for the perturbed constraint- point v, or work directly.)

   (b) Deduce that if $c \mapsto m^*(c)$ is differentiable, then
   $$\nabla m^*(0) = -\pi.$$