Optimization with equality constraint(s)

Exercise 1: let's optimize !
For each function, determine the local and global extrema under equality constraints. Pictures are required.

1. $f(x, y, z) = (x-2)^2 + y^2 + z^2$ under constraint $x^2 + 2y^2 + 3z^2 = 1$.
2. $f(x, y) = 3x - y$ under constraint $x^2 + y^2 = 5$.
3. $f(x, y) = x^2 + y^2$ under constraint $x + 2y = 5$.
4. $f(x, y) = (xy)^a$ under constraint $2x + 3y = 12$, with $a > 0$.
5. $f(x, y) = xy^2$ under constraint $x^2 + 4y^2 = 6$.
6. $f(x, y, z) = \frac{1}{3}x^3 + y + z^2$ under constraints $\begin{cases} x + y + z = 0, \\ x + y - z = 0. \end{cases}$
7. $f(x, y, z, t) = x^2 + y^2 + z^2 + t^2$ under constraints $\begin{cases} x + y = 2, \\ z + t = 0. \end{cases}$
8. $f(x, y, z) = x^2 + (y-1)^2 + z^2$ under constraints $\begin{cases} x + y = \sqrt{2}, \\ x^2 + y^2 = 1. \end{cases}$

Exercise 2: economical wrapping
What is the minimal surface of a right-angled parallelepipoid wrapping a volume of $12m^3$?

Exercise 3: spectral theorem
Let $A \in \mathcal{S}_n(\mathbb{R})$ and $F : \mathbb{R}^n \to \mathbb{R}$ be the *quadratic form associated with the matrix* A, *i.e.*

$$F(x) = {}^t x A x.$$

Let us denote by $G : \mathbb{R}^n \to \mathbb{R}$ the squared Euclidean norm, *i.e.*

$$G(x) = \|x\|_2^2 = \sum_{i=1}^{n} x_i^2 = {}^t x x.$$

Let us denote by $\mathbb{S}$ the unit sphere associated with this norm:

$$\mathbb{S} = \{x \in \mathbb{R}^n,\, \|x\|_2 = 1\} = \{x \in \mathbb{R}^n,\, G(x) = 1\}.$$

1. Calculate $\nabla F(x)$ and $\nabla G(x)$ for $x \in \mathbb{R}^n$.
2. Show by a compacity argument that F attains a maximum on $\mathbb{S}$.
3. Deduce that A has a real eigenvalue:

$$\exists \lambda \in \mathbb{R}, \exists x \in \mathbb{S},\ Ax = \lambda x.$$

Remark: to show that A is diagonalizable, proceed by induction on the dimension.

Exercise 4: entropy
Let $a_1, \ldots, a_n$, a be $n+1$ different real values, with $n \geq 3$. The aim is to maximize the function H defined by

$$H(p) = -\sum_{k=1}^{n} p_k \ln p_k$$

on the space E defined by

$$E = \left\{ (p_1, \ldots p_n) \in (\mathbb{R}_+^\star)^n \mid \sum_{k=1}^{n} p_k = 1 \text{ and } \sum_{k=1}^{n} a_k p_k = a \right\}.$$

E is assumed to be nonempty, which implies that some a_k are larger than a and some others are smaller.

1. Show that $-H$ is convex on $(\mathbb{R}_+^\star)^n$, hence on the convex subset E.
2. Show that
$$f(x) = \sum_{k=1}^{n} (a_k - a) e^{(a_k - a)x},\ x \in \mathbb{R},$$
defines an increasing bijection from $\mathbb{R}$ into itself.
3. Justify that the Lagrange multipliers method can be applied. Express the Lagrange multipliers in terms of $f^{-1}(0)$ and a_k.
4. Conclude.

Exercise 5: inequality of arithmetic and geometric means

1. Optimize the function $f : \mathbb{R}_+^n \to \mathbb{R}$ defined by $f(x_1, \cdots, x_n) = x_1 \cdots x_n$ under the constraint $x_1 + \cdots x_n = 1$.
2. Deduce the inequality of arithmetic and geometric means:
$$\forall (x_1, \cdots, x_n), \quad \prod_{i=1}^{n} x_i^{1/n} \leq \frac{\sum_{i=1}^{n} x_i}{n}.$$

Exercise 6: standard utility maximization problem Let $x = (x_1, \cdots, x_n)$ represent a commodity vector and $p = (p_1, \cdots, p_n)$ be the corresponding price. Let $U : \mathbb{R}_+^n \to \mathbb{R}$ denotes a $\mathcal{C}^1$ utility function for the consumer. We study the standard utility maximization problem

$$V(p, m) = \max_{p \cdot x = m} U(x),$$

where $m > 0$ is the total amount of money owned by the consumer.

1. Write the associated Lagrangian function.
2. We assume that the optimum $V(p, m)$ is attained at a point $x(p, m)$ with every components positive. Justify the existence of a Lagrange multiplier $\lambda(p, m)$.
3. We assume furthermore that $(p, m) \mapsto x(p, m)$ and $(p, m) \mapsto \lambda(p, m)$ are $\mathcal{C}^1$. Using the envelope theorem, express in terms of $x(p, m)$ and $\lambda(p, m)$ the derivatives of V with respect to p and m.
4. Justify why the multiplier is called *marginal utility of money.*
5. Give an economical interpretation for $\frac{\partial V}{\partial p_i}$. The expression of the marginal utility of price with respect to the multiplier and x is called *Roy's identity.*
6. In this question, we take $U(x) = \sum_i A_i \ln(x_i - a_i)$, with A_i and a_i positive real values such that $\sum_i A_i = 1$ and $R = m - p \cdot a > 0$. Precise the domain of definition of U. Show that U attains its maximum under the constraint $p \cdot x = m$ at the point $x(p, m)$ defined by
$$x_i = a_i + \frac{R A_i}{p_i}$$
and verify Roy's identity.