Variational and viscosity solutions of the evolutionary Hamilton-Jacobi equation

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New trends in Hamilton-Jacobi equations
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Hamilton-Jacobi equation: dynamical point of view

- $C^2$ Hamiltonian function $H : (t, q, p) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$.
- Evolutionary Hamilton-Jacobi equation:

$$\partial_t u(t, q) + H(t, q, \partial_q u(t, q)) = 0, \quad (HJ)$$

coupled with a Lipschitz initial condition $u(0, \cdot) = u_0$. 
Hamilton-Jacobi equation: dynamical point of view

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coupled with a Lipschitz initial condition $u(0, \cdot) = u_0$.
- Hamiltonian action along a $C^1$ path $\gamma = (q, p) : \mathbb{R} \to \mathbb{R}^d \times \mathbb{R}^d$

$$A_s^t(\gamma) = \int_s^t p(\tau) \cdot \dot{q}(\tau) - H(\tau, q(\tau), p(\tau)) d\tau.$$
- Hamiltonian system

$$\begin{cases} \dot{q} = \partial_p H(t, q, p) \\ \dot{p} = -\partial_q H(t, q, p) \end{cases} \quad \leadsto \quad \text{Hamiltonian flow } \phi^t_s.$$
Method of characteristics for classical solutions

Let $u$ be a $C^2$ solution on $[0, T] \times \mathbb{R}^d$. Then for all $0 \leq t \leq T$

$$u(t, q) = u(0, q_0) + A^t_0 (\phi^T_0(q_0, d_{q_0}u_0)).$$
Method of characteristics for classical solutions

Consequence: no $C^2$ solution even for smooth data.

After the characteristics cross, the method defines a \textit{multivalued solution}.
A **multivalued solution** is a multivalued function defined on $[0, T] \times \mathbb{R}^d$ with multigraph matching at each time the **wavefront** $\mathcal{F}_0^u u_0 \subset \mathbb{R}^{d+1}$

$$\mathcal{F}_0^t u_0 := \{(q, u_0(q_0) + A_0^t (\phi_0^T(q_0, d_{q_0} u_0))) \mid \phi_0^t(q_0, d_{q_0} u_0) = (q, \star)\}$$
The convex case: Lax-Oleinik semigroup

- $H$ is Tonelli (i.e. $C^2$, superlinear and strictly convex in $p$)
  \[ \iff L(t, q, v) = \sup_{p \in \mathbb{R}^d} p \cdot v - H(t, q, p) \text{ is Tonelli.} \]
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- $t \mapsto q(t)$ solves the Euler-Lagrange equation
  \[ t \mapsto (q(t), p(t)) \text{ solves the Hamiltonian system.} \]
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  \text{for } p(t) = \partial_v L(t, q(t), \dot{q}(t)),
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The viscosity solution is given by the *Lax-Oleinik semigroup*

\[
T^t_s u(q) := \inf_{c : [s, t] \to \mathbb{R}^d} u(c(s)) + \int_s^t L(\tau, c(\tau), \dot{c}(\tau)) \, d\tau.
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\[ T_s^t u(q) := \inf_{c: [s, t] \to \mathbb{R}^d, c(t) = q} \left( u(c(s)) + \int_s^t L(\tau, c(\tau), \dot{c}(\tau)) \, d\tau \right). \]

**Tonelli:** the infimum is attained by a $C^2$ solution of the Euler-Lagrange equation.

**Consequence:** Existence of backward characteristics for the viscosity solution, i.e. the viscosity solution is part of the multivalued solution.
Nonconvex case: the viscosity solution is not necessarily part of the multivalued solution


In these examples, the wavefront has a unique continuous section with a shock denying the entropy condition. Hence the viscosity solution cannot be part of the multivalued solution.
Viscosity solutions: an axiomatic characterisation

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Viscosity operator

- If \( \|d^2H(t, q, p)\| \leq C \), and \((V^t_s)_{s \leq t} : \text{Lip}(\mathbb{R}^d) \rightarrow \text{Lip}(\mathbb{R}^d)\) is s.t.
  1. Consistency: if \( u \) is a \( C^2 \) solution of HJ, then \( V^t_s u_s = u_t \),
  2. Monotonicity: \( u \leq v \Rightarrow V^t_s u \leq V^t_s v \) for \( s \leq t \),
  3. Additivity: for \( c \in \mathbb{R} \), \( V^t_s (c + u) = c + V^t_s u \),
  4. Regularity: \( (t, q) \mapsto V^t_\tau u(q) \) is locally Lipschitz and \( q \mapsto V^t_\tau u(q) \) Lipschitz uniformly w.r.t. \( t \in [\tau, T] \),
  5. Markov: \( V^t_s = V^t_\tau \circ V^\tau_s \) for \( s \leq \tau \leq t \).

then \( (t, q) \mapsto V^t_s u(q) \) solves (HJ) in the viscosity sense, with initial condition \( u \) at time \( s \).
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- If furthermore \( \|dH(t, q, p)\| \leq C(1 + \|p\|) \), then there exists such an operator and it is unique.
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Variational operator

It is a family of operators \((R^t_s)_{s \leq t} : \text{Lip}(\mathbb{R}^d) \rightarrow \text{Lip}(\mathbb{R}^d)\) s.t.

1. Variational property: if \(u\) is \(C^1\), for all \(s < t\), the graph of \(R^t_s u\) is contained in the wavefront \(\mathcal{F}^t_s u\).
   
   \[\implies \text{Consistency}\]

2. Monotonicity: \(u \leq v \implies R^t_s u \leq R^t_s v\) when \(s \leq t\),

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Variational operator: requirements & first consequences

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3. Additivity: if $c \in \mathbb{R}$, $R_s^t (c + u) = c + R_s^t u$.

- 2.+3. \implies \| R_s^t u - R_s^t v \|_\infty \leq \| u - v \|_\infty.
- 1.+2. \implies$ if $u$ is semiconcave, a variational operator gives for small time the minimal section of the wavefront, which is $C^0$.
- $u \in C^2 \implies (t, q) \mapsto R_0^t u(q)$ solves (HJ) a.e. on $[0, +\infty) \times \mathbb{R}^d$. 
Variational operator: requirements & first consequences

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- global notion (no definition of "to be a variational solution of (HJ) at \((t, q)\)"")
- a priori no Markov property.
- \(u\) Lipschitz \(\implies (t, q) \mapsto R^t_0 u(q)\) solves (HJ) a.e.
Variational operator: Chaperon-Sikorav method

Generating family of the geometric solution

Find a $C^1$ function $S^t_0 u_0 : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ \textit{n.d. quadratic at infinity} s.t.

$$\phi^t_0(\text{gr } du_0) = \{ (q, \partial_q S^t_0 u_0(q, \chi)), \partial_\chi S^t_0 u_0(q, \chi) = 0 \} ,$$

$$\mathcal{F}^t_0 u_0 = \{ (q, S^t_0 u_0(q, \chi)), \partial_\chi S^t_0 u_0(q, \chi) = 0 \} ,$$

$$\partial_t S^t_0 u_0(q, \chi) = -H(t, q, \partial_q S^t_0 u_0(q, \chi)) \text{ if } \partial_\chi S^t_0 u_0(q, \chi) = 0.$$
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\]

Remark: if $(t, q) \mapsto \chi(t, q)$ is $C^1$ and satisfies $\partial_\chi S^t_0 u_0(q, \chi(t, q)) = 0$, then $(t, q) \mapsto S^t_0 u_0(q, \chi(t, q))$ solves Hamilton-Jacobi.
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$$\{\text{Crit. points of } S^t_0 u_0(q, \cdot)\} \overset{1:1}{\longleftrightarrow} \{\text{Ham. traj. with } |\gamma(0) \in \text{gr}(du_0), \gamma(t) \in T^*_q \mathbb{R}^d\}$$

with associated critical value corresponding to the Hamiltonian action of the trajectory plus the initial cost $u_0(q_0)$. 

Critical value selector $\sigma$ (ex: minmax selector)

It selects for any smooth and n.d. quadratic at infinity function on $\mathbb{R}^k$ a critical value, with

- $\sigma(c + f) = c + \sigma(f)$ for $c \in \mathbb{R}$,
- if $\phi$ is a $C^1$-diffeo., $\sigma(f \circ \phi) = \sigma(f)$,
- $\sigma(f \oplus Q) = \sigma(f)$,
- $\sigma(f) \leq \sigma(g)$ when $f \leq g$ and $f - g$ Lipschitz,
- if $\mu \mapsto f_\mu$ is such that critical points and values of $f_\mu$ do not depend on $\mu$, $\mu \mapsto \sigma(f_\mu)$ is constant.
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Proposition

If $H$ is a $C^2$ Hamiltonian satisfying

$$\|d^2H(t, q, p)\| \leq C, \quad \|dH(t, q, p)\| \leq C(1 + \|p\|),$$

then $R^t_0 u_0(q) := \sigma(S^t_0 u_0(q, \cdot))$ defines a variational operator.
References on variational solutions


References on variational solutions


Graph selector with more sophisticated symplectic tools


References on variational solutions


- existence of variational solutions for (HJ) multi-time equations for commuting Hamiltonians, whereas


- Appendix C: existence of variational solutions for (multi-time) HJ contact equation
- Appendix B: extension to the non-compact case for Hamiltonians flow with *finite propagation speed*
Properties of the variational operator

Let $u$ and $v$ be Lipschitz functions and $H$ (or $K$) be a Hamiltonian s.t.

$$\|d^2H(t, q, p)\| \leq C, \quad \|dH(t, q, p)\| \leq C(1 + \|p\|).$$

- $R^t_s u$ is Lipschitz and $1 + Lip(R^t_s u) \leq e^{C(t-s)}(1 + Lip(u))$,
- $R^t_s u \leq R^t_s v$ if $u \leq v$,
- $\|R^t_s u - R^t_s v\|_\infty \leq \|u - v\|_\infty$,
- $R^t_{s,H} \leq R^t_{s,K}$ if $H \geq K$,
- $\|R^t_{s,H} u - R^t_{s,K} u\|_\infty \leq (t - s)\|H - K\|_\infty$. 

Consequence:
extension to Lipschitz initial conditions and Hamiltonians ($C^0$-variational solutions).

Remark:
the non-expansion and monotonicity properties can be localized with explicit bounds on the Hamiltonian trajectories.
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- \( \| R^t_s u - R^t_s v \|_\infty \leq \| u - v \|_\infty \),
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- $\|R^t_s u - R^t_s v\|_\infty \leq \|u - v\|_\infty$,
- $R^t_s,H \leq R^t_s,K$ if $H \geq K$,
- $\|R^t_s,H u - R^t_s,K u\|_\infty \leq (t - s)\|H - K\|_\infty$.

**Consequence:** extension to Lipschitz initial conditions and Hamiltonians ($C^0$-variational solutions).

**Remark:** the non-expansion and monotonicity properties can be localized with explicit bounds on the Hamiltonian trajectories.
Convergence of the iterated variational operator

Conjectured by Chaperon and Viterbo:

**Theorem (Wei, R.)**

If $s \leq t_1 \leq \cdots \leq t_N \leq t$ is a $N$-step subdivision with maximal step tending to 0,

$$R_{t_N}^t \circ R_{t_{N-1}}^{t_N} \circ \cdots \circ R_{s}^{t_1} u(q) \xrightarrow{N \to \infty} V_s^t u(q) \ (\text{loc. uniform in } s \leq t, q).$$
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**Sketch of the proof:**

- Local semi-group type Lipschitz estimates w.r.t. \( s, t, u, q \),
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- Local semi-group type Lipschitz estimates w.r.t. $s, t, u, q$,
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- Diagonal extraction,
- Check that any accumulation point satisfies the 5 axioms of the (unique) viscosity operator.
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\[
H(t, q, p) = H_1(t, q_1, p_1) + H_2(t, q_2, p_2) \quad \text{and} \quad u(q) = u_1(q_1) + u_2(q_2)
\]

with \( p_1 \mapsto H(t, q_1, p_1) \) convex and \( p_2 \mapsto H(t, q_2, p_2) \) concave.
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**Theorem (R. ’18)**

*If $H(p)$ is neither convex nor concave, there exists a smooth $u$ such that $R^t_0 u \neq V^t_0 u$.***

For this initial condition, the graph of the viscosity solution is not contained in the wavefront.
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Semi-Lagrangian numerical scheme for nonconvex Hamiltonians (j.w. in progress with H. Hivert)

Input:

- An integrable $C^2$ 1D Hamiltonian $H(p)$, entered as a function.

\[ \phi^t_0(q, p) = (q + tH'(p), p), \quad A^t_0(\gamma) = t \left( pH'(p) - H(p) \right). \]

- Spatial step $\Delta x$, time step $\Delta t$.

- A semiconcave initial condition $u$ with a finite number of shocks, processed as a list of 3-tuples $(x_i, u_i, p_i)$ ("Clarke 1-jet").
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  \[ (x_i, u_i, p_i) \rightarrow (x_i + \Delta tH'(p_i), u_i + \Delta t(p_iH'(p_i) - H(p_i)), p_i). \]

- "Follow" and select the minimal section.

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Then iterate.
Semi-Lagrangian scheme: comparison with Lax-Friedrichs

\[ H(p) = p^4 - p^2 \quad u_0(q) = \begin{cases} 
q & \text{if } q < 0 \\
-\frac{3}{4}q + q^2/2 & \text{if } q > 0
\end{cases} \]
Semi-Lagrangian scheme: precision test

\[ T = 50, \quad \Delta t = 0.05, \quad \Delta x = 0.01 \]

\[ H(p) = \begin{cases} 
    p + p^2 & \text{if } p < 0 \\
    p - p^2 & \text{if } p > \delta 
\end{cases} \]

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Explicit example where the solutions differ

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with \( H \in C^2 \) on \( \mathbb{R} \) and such that \( H'' \) cancels exactly once in \((0, \delta)\).
Explicit example where the solutions differ
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Characteristics for the variational solution

In the blue domain, the variational solution coincides with the classical solution $f_r$ for $H(p) = p + p^2$ with smooth initial condition $f(q) = -q + q^2/2$. 
Explicit example where the solutions differ

\[ u_0(q) = \begin{cases} q & \text{if } q < 0 \\ -q + q^2/2 & \text{if } q > 0 \end{cases} \quad H(p) = \begin{cases} p + p^2 & \text{if } p < 0 \\ p - p^2 & \text{if } p > \delta \end{cases} \]

Oleinik’s guess

If \( x(t) \) denotes the position of the presumed viscosity shock, then

\[ x'(t) = \frac{H(p_+(t)) - H(p_-(t))}{p_+(t) - p_-(t)} = H'(p_+(t)) \]

where \( p_-(t) = \partial_x f_r(t, x(t)) \) and \( p_+(t) = \psi(p_-(t)) \).
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Characteristics for the viscosity solution are tangentially issued from the shock.
Explicit example where the solutions differ

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\[ T = 50, \quad \Delta t = 0.05, \quad \Delta x = 0.01 \]

- Initial data
- Explicit variational solution
- Explicit viscosity solution
- Approximated variational solution
- Approximated viscosity solution