Variational and viscosity solutions of the evolutionary Hamilton-Jacobi equation

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New trends in Hamilton-Jacobi equations
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Hamilton-Jacobi equation: dynamical point of view

- $\mathcal{C}^2$ Hamiltonian function $H : (t, q, p) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$.
- Evolutionary Hamilton-Jacobi equation:

$$\partial_t u(t, q) + H(t, q, \partial_q u(t, q)) = 0, \tag{HJ}$$

coupled with a Lipschitz initial condition $u(0, \cdot) = u_0$.

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  coupled with a Lipschitz initial condition $u(0,\cdot) = u_0$.
- Hamiltonian action along a $\mathcal{C}^1$ path $\gamma = (q,p) : \mathbb{R} \to \mathbb{R}^d \times \mathbb{R}^d$

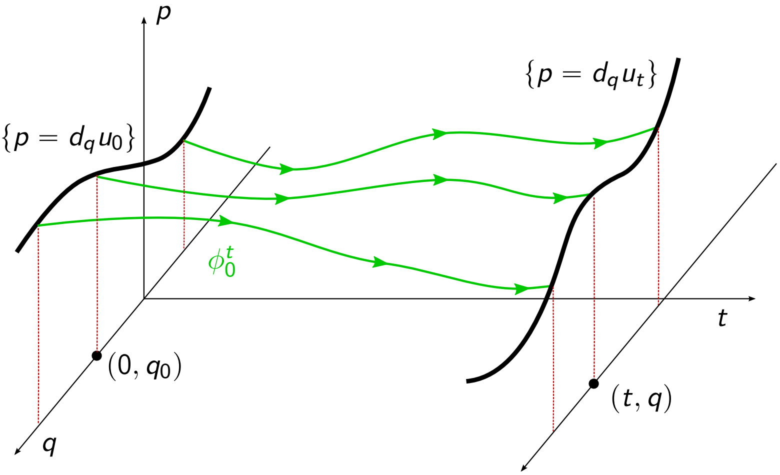
$$\mathcal{A}_s^t(\gamma) = \int_s^t p(\tau)\cdot \dot{q}(\tau) - H(\tau, q(\tau), p(\tau))d\tau.$$

- Hamiltonian system

$$\begin{cases} \dot{q} = \partial_p H(t,q,p) \\ \dot{p} = -\partial_q H(t,q,p) \end{cases} \rightsquigarrow \text{ Hamiltonian flow } \phi_s^t.$$

Method of characteristics for classical solutions

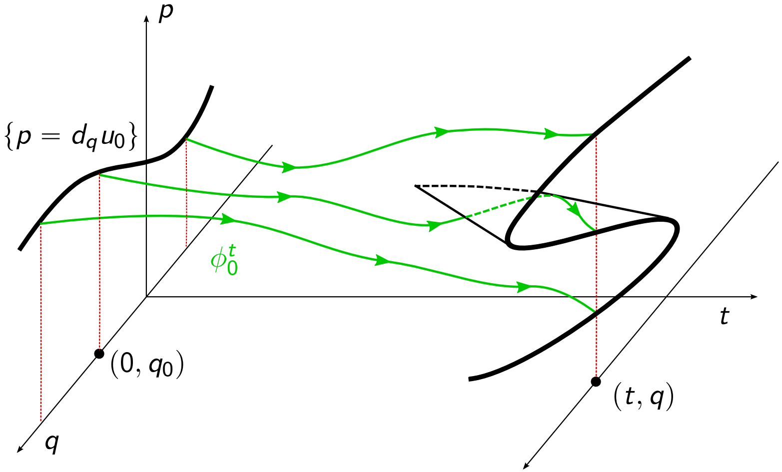
Let u be a $\mathcal{C}^2$ solution on $[0, T] \times \mathbb{R}^d$. Then for all $0 \leqslant t \leqslant T$



and $u(t, q) = u(0, q_0) + \mathcal{A}_0^t\left(\phi_0^\tau(q_0, d_{q_0} u_0)\right)$.

Method of characteristics for classical solutions

Consequence: no $\mathcal{C}^2$ solution even for smooth data.

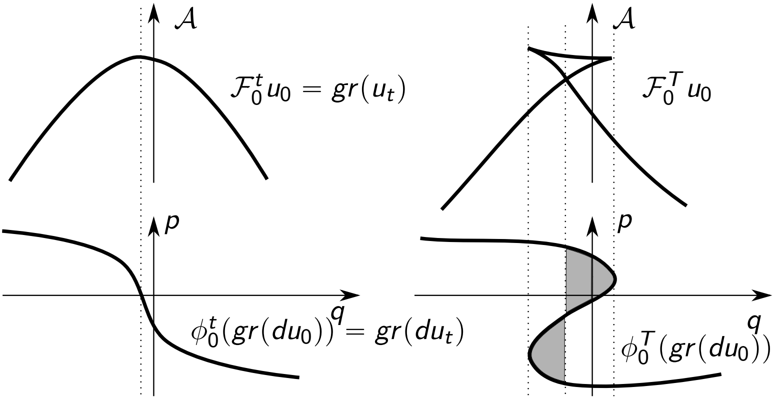


After the characteristics cross, the method defines a *multivalued solution*.

Multivalued solution

A *multivalued solution* is a multivalued function defined on $[0, T] \times \mathbb{R}^d$ with multigraph matching at each time the *wavefront* $\mathcal{F}_0^t u_0 \subset \mathbb{R}^{d+1}$

$$\mathcal{F}_0^t u_0 := \left\{ \left(q, u_0(q_0) + \mathcal{A}_0^t\left(\phi_0^\tau(q_0, d_{q_0}u_0)\right)\right) \,\middle|\, \phi_0^t(q_0, d_{q_0}u_0) = (q, \star) \right\}$$



The convex case: Lax-Oleinik semigroup

- H is Tonelli (*i.e.* $\mathcal{C}^2$, superlinear and strictly convex in p) $\iff L(t,q,v) = \sup_{p\in\mathbb{R}^d} p\cdot v - H(t,q,p)$ is Tonelli.

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- $t \mapsto q(t)$ solves the Euler-Lagrange equation
  $\iff$ for $p(t) = \partial_v L\left(t, q(t), \dot{q}(t)\right)$,
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The viscosity solution is given by the *Lax-Oleinik semigroup*

$$T_s^t u(q) := \inf_{\substack{c\,:\,[s,t]\to\mathbb{R}^d \\ c(t)=q}} u(c(s)) + \int_s^t L\left(\tau, c(\tau), \dot{c}(\tau)\right) d\tau.$$

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Tonelli: the infimum is attained by a $\mathcal{C}^2$ solution of the Euler-Lagrange equation.
Consequence: Existence of backward characteristics for the viscosity solution, *i.e.* the viscosity solution is part of the multivalued solution.

Nonconvex case: the viscosity solution is not necessarily part of the multivalued solution

- Chenciner, Aspects géométriques de l'études des chocs dans les lois de conservation *Problèmes d'évolution non linéaires, Séminaire de Nice*, 1975.
- Viterbo, Solutions of Hamilton-Jacobi equations and symplectic geometry *Séminaire sur les Équations aux Dérivées Partielles*, 1994-1995.
- Bernardi, Cardin, On C^0-variational solutions for Hamilton-Jacobi equations *Discrete Contin. Dyn. Syst.*, 2011.
- Wei, Viscosity solution of the Hamilton-Jacobi equation by a limiting minimax method *Nonlinearity*, 2014.

In these examples, the wavefront has a unique continuous section with a shock denying the entropy condition. Hence the viscosity solution cannot be part of the multivalued solution.

Viscosity solutions: an axiomatic characterisation

Alvarez, Guichard, Lions, Morel, Axioms and fundamental equations of image processing *Arch. Rational Mech. Anal.*, 1993.

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Viscosity operator

- If $\|d^2 H(t,q,p)\| \leqslant C$, and $(V_s^t)_{s \leqslant t} : \mathrm{Lip}(\mathbb{R}^d) \to \mathrm{Lip}(\mathbb{R}^d)$ is s.t.
  1. Consistency: if u is a $\mathcal{C}^2$ solution of HJ, then $V_s^t u_s = u_t$,
  2. Monotonicity: $u \leqslant v \Rightarrow V_s^t u \leqslant V_s^t v$ for $s \leqslant t$,
  3. Additivity: for $c \in \mathbb{R}$, $V_s^t(c+u) = c + V_s^t u$,
  4. Regularity: $(t,q) \mapsto V_\tau^t u(q)$ is locally Lipschitz and $q \mapsto V_\tau^t u(q)$ Lipschitz uniformly w.r.t. $t \in [\tau, T]$,
  5. Markov: $V_s^t = V_\tau^t \circ V_s^\tau$ for $s \leqslant \tau \leqslant t$.

  then $(t,q) \mapsto V_s^t u(q)$ solves (HJ) in the viscosity sense, with initial condition u at time s.

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Bernard, The Lax-Oleinik semi-group: a Hamiltonian point of view *Proc. Roy. Soc. Edinburgh*, 2012.

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- If furthermore $\|dH(t,q,p)\| \leqslant C(1+\|p\|)$, then there exists such an operator and it is unique.

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Ishii, Uniqueness of unbounded viscosity solution of Hamilton-Jacobi equations *Indiana Univ. Math. J.*, 1984.

Variational operator: requirements & first consequences

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1. Variational property: if u is $\mathcal{C}^1$, for all $s < t$, the graph of $R_s^t u$ is contained in the wavefront $\mathcal{F}_s^t u$.
   ($\implies$ Consistency)
2. Monotonicity: $u \leqslant v \Rightarrow R_s^t u \leqslant R_s^t v$ when $s \leqslant t$,
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- 2.+3. $\implies$ $\|R_s^t u - R_s^t v\|_\infty \leqslant \|u - v\|_\infty$.
- 1.+2. $\implies$ if u is semiconcave, a variational operator gives for small time the minimal section of the wavefront, which is $\mathcal{C}^0$.
- u $\mathcal{C}^2$ $\implies$ $(t, q) \mapsto R_0^t u(q)$ solves (HJ) a.e. on $[0, +\infty) \times \mathbb{R}^d$.

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- global notion (no definition of "to be a variational solution of (HJ) at (t,q)")
- a priori no Markov property.
- u Lipschitz $\overset{???}{\implies}$ $(t,q) \mapsto R_0^t u(q)$ solves (HJ) a.e.

Variational operator: Chaperon-Sikorav method

Generating family of the geometric solution

Find a $\mathcal{C}^1$ function $S_0^t u_0 : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ *n.d.quadratic at infinity* s.t.

$$\begin{aligned}
\phi_0^t(\mathrm{gr}\ du_0) &= \left\{(q, \partial_q S_0^t u_0(q,\chi)), \partial_\chi S_0^t u_0(q,\chi) = 0\right\},\\
\mathcal{F}_0^t u_0 &= \left\{(q, S_0^t u_0(q,\chi)), \partial_\chi S_0^t u_0(q,\chi) = 0\right\},\\
\partial_t S_0^t u_0(q,\chi) &= -H(t, q, \partial_q S_0^t u_0(q,\chi)) \text{ if } \partial_\chi S_0^t u_0(q,\chi) = 0.
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\end{aligned}$$

Remark: if $(t,q) \mapsto \chi(t,q)$ is $\mathcal{C}^1$ and satisfies $\partial_\chi S_0^t u_0(q, \chi(t,q)) = 0$, then $(t,q) \mapsto S_0^t u_0(q,\chi(t,q))$ solves Hamilton-Jacobi.

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$$\left\{\text{Crit. points of } S_0^t u_0(q,\cdot)\right\} \overset{1:1}{\longleftrightarrow} \left\{\text{Ham. traj. with } \left| \begin{array}{l} \gamma(0) \in gr(du_0), \\ \gamma(t) \in T_q^\star \mathbb{R}^d \end{array} \right.\right\}$$

with associated critical value corresponding to the Hamiltonian action of the trajectory plus the initial cost $u_0(q_0)$.

Variational operator: Chaperon-Sikorav method

Critical value selector σ (ex: minmax selector)

It selects for any smooth and n.d.quadratic at infinity function on $\mathbb{R}^k$ a critical value, with

- $\sigma(c+f) = c + \sigma(f)$ for $c \in \mathbb{R}$,
- if ϕ is a $\mathcal{C}^1$-diffeo., $\sigma(f \circ \phi) = \sigma(f)$,
- $\sigma(f \oplus Q) = \sigma(f)$,
- $\sigma(f) \leqslant \sigma(g)$ when $f \leqslant g$ and $f - g$ Lipschitz,
- if $\mu \mapsto f_\mu$ is such that critical points and values of f_μ do not depend on μ, $\mu \mapsto \sigma(f_\mu)$ is constant.

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Proposition

If H is a $\mathcal{C}^2$ Hamiltonian satisfying

$$\|d^2 H(t,q,p)\| \leqslant C, \ \|dH(t,q,p)\| \leqslant C(1+\|p\|),$$

then $R_0^t u_0(q) := \sigma\left(S_0^t u_0(q,\cdot)\right)$ defines a variational operator.

References on variational solutions

- Chaperon, Une idée du type "géodésiques brisées" pour les systèmes hamiltoniens *C. R. Acad. Sci.*, 1984.
- Sikorav, Sur les immersions lagrangiennes dans un fibré cotangent admettant une phase génératrice globale *C. R. Acad. Sci.*, 1986.
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Graph selector with more sophisticated symplectic tools

- Floer, Morse theory for Lagrangian intersections *J. Differential Geom.*, 1988.
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- Appendix C: existence of variational solutions for (multi-time) HJ contact equation
- Appendix B: extension to the non-compact case for Hamiltonians flow with *finite propagation speed*

Properties of the variational operator

Let u and v be Lipschitz functions and H (or K) be a Hamiltonian s.t.

$$\|d^2H(t,q,p)\| \leqslant C, \ \|dH(t,q,p)\| \leqslant C(1+\|p\|).$$

- $R_s^t u$ is Lipschitz and $1 + Lip(R_s^t u) \leqslant e^{C(t-s)}(1 + Lip(u))$,
- $R_s^t u \leqslant R_s^t v$ if $u \leqslant v$,
- $\|R_s^t u - R_s^t v\|_\infty \leqslant \|u - v\|_\infty$,
- $R_{s,H}^t \leqslant R_{s,K}^t$ if $H \geqslant K$,
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Consequence: extension to Lipschitz initial conditions and Hamiltonians ($\mathcal{C}^0$*-variational solutions*).

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Convergence of the iterated variational operator

Conjectured by Chaperon and Viterbo:

Theorem (Wei, R.)

If $s \leqslant t_1 \leqslant \cdots \leqslant t_N \leqslant t$ is a N-step subdivision with maximal step tending to 0,

$$R_{t_N}^{t} \circ R_{t_{N-1}}^{t_N} \circ \cdots \circ R_s^{t_1} u(q) \underset{N \to \infty}{\longrightarrow} V_s^t u(q) \ (\textit{loc. uniform in } s \leqslant t, q).$$

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- Check that any accumulation point satisfies the 5 axioms of the (unique) viscosity operator.

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Semi-Lagrangian numerical scheme for nonconvex Hamiltonians (j.w. in progress with H. Hivert)

Input:

- An integrable $\mathcal{C}^2$ 1D Hamiltonian $H(p)$, entered as a function.

  $$\rightsquigarrow \quad \phi_0^t(q,p) = \left(q + tH'(p), p\right), \quad \mathcal{A}_0^t(\gamma) = t\left(pH'(p) - H(p)\right).$$

- Spatial step Δx, time step Δt.
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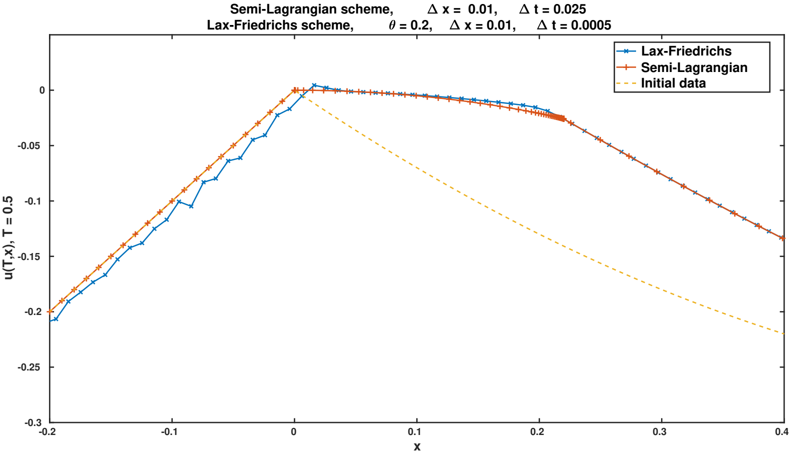
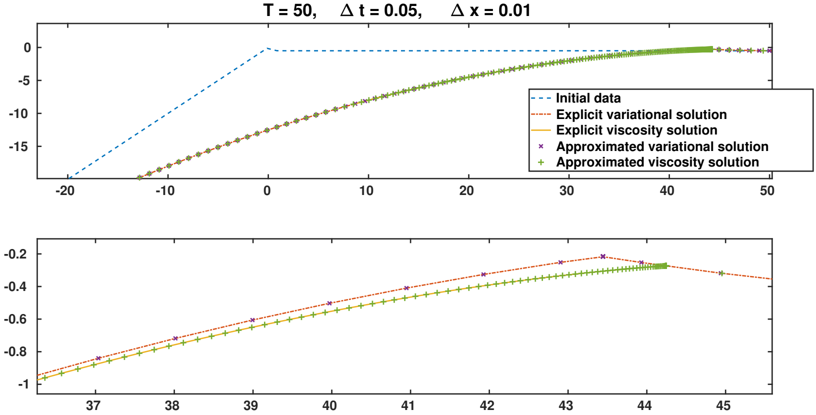
Then iterate.

Semi-Lagrangian scheme: comparison with Lax-Friedrichs



$$H(p) = p^4 - p^2 \qquad u_0(q) = \begin{cases} q & \text{if } q < 0 \\ -\frac{3}{4}q + q^2/2 & \text{if } q > 0 \end{cases}$$

Semi-Lagrangian scheme: precision test



$$H(p) = \begin{cases} p + p^2 & \text{if } p < 0 \\ p - p^2 & \text{if } p > \delta \end{cases} \qquad u_0(q) = \begin{cases} q & \text{if } q < 0 \\ -q + q^2/2 & \text{if } q > 0 \end{cases}$$

Explicit example where the solutions differ

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with H $\mathcal{C}^2$ on $\mathbb{R}$ and such that H'' cancels exactly once in $(0, \delta)$.

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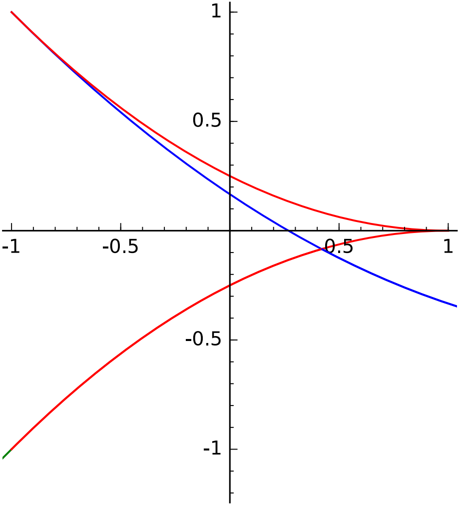

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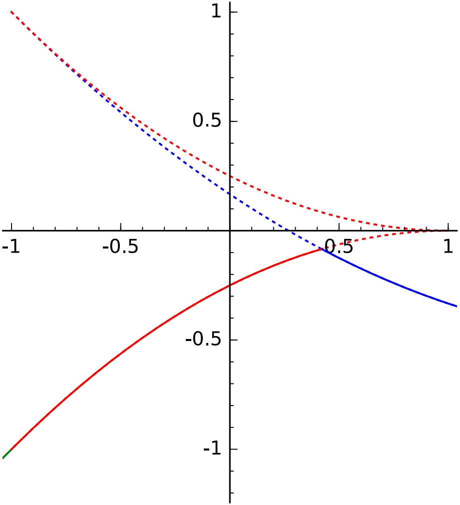

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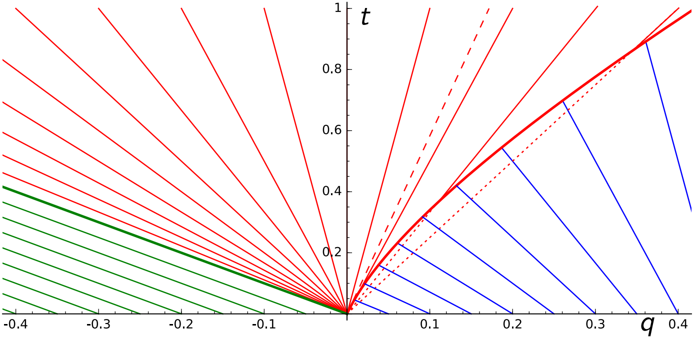
Characteristics for the variational solution



In the blue domain, the variational solution coincides with the classical solution f_r for $H(p) = p + p^2$ with smooth initial condition $f(q) = -q + q^2/2$.

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Oleinik's guess

If $x(t)$ denotes the position of the presumed viscosity shock, then

$$x'(t) = \frac{H(p_+(t)) - H(p_-(t))}{p_+(t) - p_-(t)} = H'(p_+(t))$$

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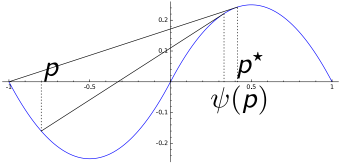

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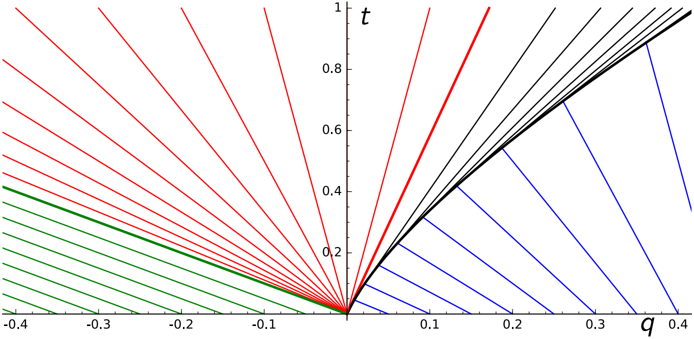

Characteristics for the viscosity solution are tangentially issued from the shock.

Explicit example where the solutions differ

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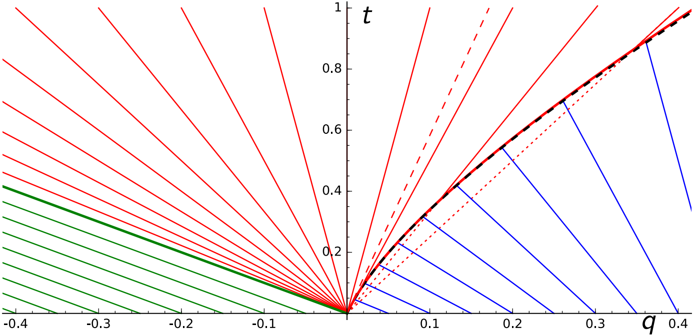

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