Online Appendix A

Equations of the U-Pb chronometer with three common lead components.

We assume that at age t_0, the system incorporated U and three common Pb components, labelled α_0^1, α_0^2 and α_0^3, in the proportions f_1, f_2 and f_3, with $f_1+f_2+f_3=1$. For the ^{238}U-^{206}Pb system, mass balance after time t reads:

$$\left(\frac{^{206}\mathrm{Pb}}{^{204}\mathrm{Pb}}\right)_t = f_1\alpha_0^1 + f_2\alpha_0^2 + f_3\alpha_0^3 + \left(\frac{^{238}\mathrm{U}}{^{204}\mathrm{Pb}}\right)_t \left(e^{\lambda_{238\mathrm{U}}t} - 1\right) + \frac{^{234}\mathrm{U}_{xs,0}}{^{204}\mathrm{Pb}} - \frac{^{230}\mathrm{Th}_{sec}}{^{204}\mathrm{Pb}} \quad (1)$$

Similarly, for the ^{235}U-^{207}Pb system:

$$\left(\frac{^{207}\mathrm{Pb}}{^{204}\mathrm{Pb}}\right)_t = f_1\beta_0^1 + f_2\beta_0^2 + f_3\beta_0^3 + \left(\frac{^{235}\mathrm{U}}{^{204}\mathrm{Pb}}\right)_t \left(e^{\lambda_{235\mathrm{U}}t} - 1\right) + \frac{^{231}\mathrm{Pa}_{xs,0}}{^{204}\mathrm{Pb}} \quad (2)$$

and for the ^{232}Th-^{208}Pb system:

$$\left(\frac{^{208}\mathrm{Pb}}{^{204}\mathrm{Pb}}\right)_t = f_1\gamma_0^1 + f_2\gamma_0^2 + f_3\gamma_0^3 + \left(\frac{^{232}\mathrm{Th}}{^{204}\mathrm{Pb}}\right)_t \left(e^{\lambda_{232\mathrm{Th}}t} - 1\right) \quad (3)$$

Parameters f_2 and f_3 are obtained from Eqs.1-3 as:

$$f_2\left(\alpha_0^2 - \alpha_0^1\right) + f_3\left(\alpha_0^3 - \alpha_0^1\right) = \left(\frac{^{206}\mathrm{Pb}}{^{204}\mathrm{Pb}}\right)_t - \alpha_0^1 - \left(\frac{^{238}\mathrm{U}}{^{204}\mathrm{Pb}}\right)_t \left(e^{\lambda_{238\mathrm{U}}t} - 1 + \frac{^{234}\mathrm{U}_{xs,0}}{^{238}\mathrm{U}} - \frac{\lambda_{238\mathrm{U}}}{\lambda_{230\mathrm{Th}}}\right) \quad (4)$$

$$f_2\left(\beta_0^2 - \beta_0^1\right) + f_3\left(\beta_0^3 - \beta_0^1\right) = \left(\frac{^{207}\mathrm{Pb}}{^{204}\mathrm{Pb}}\right)_t - \beta_0^1 - \frac{1}{137.88}\left(\frac{^{238}\mathrm{U}}{^{204}\mathrm{Pb}}\right)_t \left(e^{\lambda_{235\mathrm{U}}t} - 1 + \frac{^{231}\mathrm{Pa}_{xs,0}}{^{238}\mathrm{U}}\right) \quad (5)$$

$$f_2\left(\gamma_0^2 - \gamma_0^1\right) + f_3\left(\gamma_0^3 - \gamma_0^1\right) = \left(\frac{^{208}\mathrm{Pb}}{^{204}\mathrm{Pb}}\right)_t - \gamma_0^1 - \left(\frac{^{232}\mathrm{Th}}{^{238}\mathrm{U}}\right)_t \left(\frac{^{238}\mathrm{U}}{^{204}\mathrm{Pb}}\right)_t \left(e^{\lambda_{232\mathrm{Th}}t} - 1\right) \quad (6)$$

In Eq.4 and Eq.6, we define D as:

$$D = \left(\alpha_0^2 - \alpha_0^1\right)\left(\gamma_0^3 - \gamma_0^1\right) - \left(\alpha_0^3 - \alpha_0^1\right)\left(\gamma_0^2 - \gamma_0^1\right) \quad (7)$$

Therefore:

$$\begin{aligned} f_2 = {} & \frac{1}{D}\left(\gamma_0^3 - \gamma_0^1\right)\left[\left(\frac{^{206}\mathrm{Pb}}{^{204}\mathrm{Pb}}\right)_t - \alpha_0^1 - \left(\frac{^{238}\mathrm{U}}{^{204}\mathrm{Pb}}\right)_t \left(e^{\lambda_{238\mathrm{U}}t} - 1 + \frac{^{234}\mathrm{U}_{xs,0}}{^{238}\mathrm{U}} - \frac{\lambda_{238\mathrm{U}}}{\lambda_{230\mathrm{Th}}}\right)\right] \\ & - \frac{1}{D}\left(\alpha_0^3 - \alpha_0^1\right)\left[\left(\frac{^{208}\mathrm{Pb}}{^{204}\mathrm{Pb}}\right)_t - \gamma_0^1 - \left(\frac{^{232}\mathrm{Th}}{^{238}\mathrm{U}}\right)_t \left(\frac{^{238}\mathrm{U}}{^{204}\mathrm{Pb}}\right)_t \left(e^{\lambda_{232\mathrm{Th}}t} - 1\right)\right] \quad (8) \end{aligned}$$

and:

$$f_3 = \frac{1}{D}\left(\alpha_0^2-\alpha_0^1\right)\left[\left(\frac{^{208}\mathrm{Pb}}{^{204}\mathrm{Pb}}\right)_t - \gamma_0^1 - \left(\frac{^{232}\mathrm{Th}}{^{238}\mathrm{U}}\right)_t\left(\frac{^{238}\mathrm{U}}{^{204}\mathrm{Pb}}\right)_t\left(e^{\lambda_{232\mathrm{Th}}t}-1\right)\right]$$
$$-\frac{1}{D}\left(\gamma_0^2-\gamma_0^1\right)\left[\left(\frac{^{206}\mathrm{Pb}}{^{204}\mathrm{Pb}}\right)_t - \alpha_0^1 - \left(\frac{^{238}\mathrm{U}}{^{204}\mathrm{Pb}}\right)_t\left(e^{\lambda_{238\mathrm{U}}t}-1+\frac{^{234}\mathrm{U}_{xs,0}}{^{238}\mathrm{U}}-\frac{\lambda_{238\mathrm{U}}}{\lambda_{230\mathrm{Th}}}\right)\right] \quad (9)$$

Introducing Eq.8 and Eq.9 into Eq.5 and neglecting the $^{231}\mathrm{Pa}_{\mathrm{xs},0}/^{204}\mathrm{Pb}$ term gives:

$$\frac{\left(\beta_0^2-\beta_0^1\right)\left(\gamma_0^3-\gamma_0^1\right)-\left(\gamma_0^2-\gamma_0^1\right)\left(\beta_0^3-\beta_0^1\right)}{D}\left[\left(\frac{^{206}\mathrm{Pb}}{^{204}\mathrm{Pb}}\right)_t-\alpha_0^1\right]$$
$$+\frac{\left(\alpha_0^2-\alpha_0^1\right)\left(\beta_0^3-\beta_0^1\right)-\left(\alpha_0^3-\alpha_0^1\right)\left(\beta_0^2-\beta_0^1\right)}{D}\left[\left(\frac{^{208}\mathrm{Pb}}{^{204}\mathrm{Pb}}\right)_t-\gamma_0^1\right]$$
$$-\left[\frac{\left(\beta_0^2-\beta_0^1\right)\left(\gamma_0^3-\gamma_0^1\right)-\left(\gamma_0^2-\gamma_0^1\right)\left(\beta_0^3-\beta_0^1\right)}{D}\left(e^{\lambda_{238\mathrm{U}}t}-1+\frac{^{234}\mathrm{U}_{xs,0}}{^{238}\mathrm{U}}-\frac{\lambda_{238\mathrm{U}}}{\lambda_{230\mathrm{Th}}}\right)\right.$$
$$+\frac{\left(\alpha_0^2-\alpha_0^1\right)\left(\beta_0^3-\beta_0^1\right)-\left(\alpha_0^3-\alpha_0^1\right)\left(\beta_0^2-\beta_0^1\right)}{D}\left(\frac{^{232}\mathrm{Th}}{^{238}\mathrm{U}}\right)_t\left(e^{\lambda_{232\mathrm{Th}}t}-1\right)$$
$$\left.-\frac{1}{137.88}\left(e^{\lambda_{235\mathrm{U}}t}-1\right)\right]\left(\frac{^{238}\mathrm{U}}{^{204}\mathrm{Pb}}\right)_t$$
$$=\left(\frac{^{207}\mathrm{Pb}}{^{204}\mathrm{Pb}}\right)_t-\beta_0^1 \quad (10)$$

Setting:

$$a_4 = \frac{\left(\beta_0^2-\beta_0^1\right)\left(\gamma_0^3-\gamma_0^1\right)-\left(\gamma_0^2-\gamma_0^1\right)\left(\beta_0^3-\beta_0^1\right)}{D} \quad (11)$$

and:

$$a_2 = \frac{\left(\alpha_0^3-\alpha_0^1\right)\left(\beta_0^2-\beta_0^1\right)-\left(\alpha_0^2-\alpha_0^1\right)\left(\beta_0^3-\beta_0^1\right)}{D} \quad (12)$$

allows Eq.10 to be reduced to:

$$a_4\left[\left(\frac{^{206}\mathrm{Pb}}{^{204}\mathrm{Pb}}\right)_t-\alpha_0^1\right]-a_2\left[\left(\frac{^{208}\mathrm{Pb}}{^{204}\mathrm{Pb}}\right)_t-\gamma_0^1\right]$$
$$-a_4\left(e^{\lambda_{238\mathrm{U}}t}-1+\frac{^{234}\mathrm{U}_{xs,0}}{^{238}\mathrm{U}}-\frac{\lambda_{238\mathrm{U}}}{\lambda_{230\mathrm{Th}}}\right)+a_2\left(\frac{^{232}\mathrm{Th}}{^{238}\mathrm{U}}\right)_t\left(e^{\lambda_{232\mathrm{Th}}t}-1\right)$$
$$+\frac{1}{137.88}\left(e^{\lambda_{235\mathrm{U}}t}-1\right)\left(\frac{^{238}\mathrm{U}}{^{204}\mathrm{Pb}}\right)_t$$
$$=\left(\frac{^{207}\mathrm{Pb}}{^{204}\mathrm{Pb}}\right)_t-\beta_0^1 \quad (13)$$

Now, we set:

$$a_1 = a_4\alpha_0^1 - a_2\gamma_0^1 - \beta_0^1 \tag{14}$$

and:

$$a_3 = a_4\left(e^{\lambda_{238\text{U}}t} - 1 + \frac{^{234}\text{U}_{xs,0}}{^{238}\text{U}} - \frac{\lambda_{238\text{U}}}{\lambda_{230\text{Th}}}\right) - a_2\left(\frac{^{232}\text{Th}}{^{238}\text{U}}\right)_t\left(e^{\lambda_{232\text{Th}}t} - 1\right) \\ - \frac{1}{137.88}\left(e^{\lambda_{235\text{U}}t} - 1\right) \tag{15}$$

Multiplying Eq.13 by $^{204}\text{Pb}/^{206}\text{Pb}$ for normalization to ^{206}Pb leads to the general expression:

$$\left(\frac{^{207}\text{Pb}}{^{206}\text{Pb}}\right)_t = -a_1\left(\frac{^{204}\text{Pb}}{^{206}\text{Pb}}\right)_t - a_2\left(\frac{^{208}\text{Pb}}{^{206}\text{Pb}}\right)_t - a_3\left(\frac{^{238}\text{U}}{^{206}\text{Pb}}\right)_t + a_4 \tag{16}$$

which is equivalent to:

$$x_1\left(\frac{^{204}\text{Pb}}{^{206}\text{Pb}}\right)_t + x_2\left(\frac{^{207}\text{Pb}}{^{206}\text{Pb}}\right)_t + x_3\left(\frac{^{208}\text{Pb}}{^{206}\text{Pb}}\right)_t + x_4\left(\frac{^{238}\text{U}}{^{206}\text{Pb}}\right)_t = 1 \tag{17}$$

with $x_1 = a_1/a_4$, $x_2 = 1/a_4$, $x_3 = a_2/a_4$ and $x_4 = a_3/a_4$. Dividing Eq.15 by a_4 and using the approximation $e^{\lambda t} - 1 \approx \lambda t$ for both ^{232}Th and ^{235}U for which $\lambda t \ll 1$, the expression of x_4 reads:

$$x_4 = \lambda_{238\text{U}}t + \frac{^{234}\text{U}_{ex}}{^{238}\text{U}} - \frac{\lambda_{238\text{U}}}{\lambda_{230\text{Th}}} - x_3\left(\frac{^{232}\text{Th}}{^{238}\text{U}}\right)_t\lambda_{232\text{Th}}t \\ - x_2\frac{1}{137.88}(\lambda_{235\text{U}}t) \tag{18}$$

Finally, t is obtained from:

$$t = \frac{x_4 - \frac{^{234}\text{U}_{xs,0}}{^{238}\text{U}} + \frac{\lambda_{238\text{U}}}{\lambda_{230\text{Th}}}}{\lambda_{238\text{U}} - x_3\left(\frac{^{232}\text{Th}}{^{238}\text{U}}\right)_t\lambda_{232\text{Th}} - \frac{x_2}{137.88}\lambda_{235\text{U}}} \tag{19}$$