

The self-dual point of the two-dimensional random-cluster model is critical above $q = 4$

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Abstract

We prove the statement in the title of the paper, in the case of the square lattice, and derive the sharpness of the phase transition.

Introduction

Since random-cluster models were introduced by Fortuin and Kasteleyn in 1969 [4], they have become an important tool in the study of phase transitions. The spin correlations of Potts models get rephrased as cluster connectivity properties of their random-cluster representations, which allows for the use of geometric techniques, thus leading to several important applications. Nevertheless, only few aspects of the random-cluster models are known in full generality.

The *random-cluster model* on a finite connected graph is a model on bonds of this graph, each one being either closed or open. The probability of a configuration is proportional to

$$p^{\#\text{ open bonds}}(1-p)^{\#\text{ closed bonds}}q^{\#\text{ clusters}},$$

where the *bond-weight* $p \in [0, 1]$ and the *cluster-weight* $q \in (0, \infty)$ are the parameters of the model. For $q \geq 1$, these models can be extended to infinite-volume lattices where they exhibit a phase transition at some critical parameter $p_c(q)$ (depending on the lattice), yet there are no general conjectures for the value of the critical point.

However, in the case of planar graphs, there is a connection (related to the Kramers-Wannier duality [7] for the Ising model) between random-cluster models on a graph and on its dual with the same cluster-weight q and appropriately related bond-weights p and $p^* = p^*(p)$. This relation leads in the particular case of \mathbb{Z}^2 (which is isomorphic to its dual) to a natural conjecture: the critical point is the same as the so-called *self-dual point* satisfying $p_{sd} = p^*(p_{sd})$, which has a known value:

$$p_c(q) = p_{sd}(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$

In the present article, we prove this result for $q \geq 4$ and we derive an even stronger result (Theorem 2): The random-cluster model exhibits a sharp transition at its self-dual point.

Theorem 1 *Let $q \geq 4$. The critical point $p_c = p_c(q)$ for the random-cluster model with parameter q on the square lattice satisfies*

$$p_c = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$

Theorem 2 *Let $q \geq 4$. For every $p < p_c$, there exist $0 < c, C < \infty$ such that*

$$\mathbb{P}_{p,q}(0 \text{ is connected to } a) \leq Ce^{-c|a|}$$

for any $a \in \mathbb{Z}^2$ (where $\mathbb{P}_{p,q}$ is the unique infinite-volume measure and $|\cdot|$ is the Euclidean norm).

The critical point was previously known in three famous cases. For $q = 1$, the model is simply Bernoulli bond-percolation, proved by Kesten [6] to be critical at $p_c(1) = 1/2$. For $q = 2$, the self-dual value corresponds to the critical temperature of the Ising model, as first derived by Onsager [10]; one can actually couple realizations of the Ising and FK models to relate the critical points of each, see [5] and references therein for details. Finally, for $q \geq 25.72$, a proof is known based on the fact that the random-cluster model exhibits a first order phase transition (see [8, 9]).

It is conjectured that among all random-cluster models, the phase transition is of first order if and only if q is greater than 4. Theorem 1 is therefore optimal in this class of models. Nevertheless, we were not able to prove that the transition is of first order: it remains a challenging open problem.

Extensive literature exists concerning these models in statistical physics; we refer the interested reader to the monograph of Grimmett [5] and references therein. Exact computations can be performed in many cases (see [1]), and despite the fact that they do not lead to fully rigorous mathematical proofs, they do provide insight and further conjectures on the behavior of these models at and near criticality.

The current article belongs to a series devoted to the implications and the extensions of the so-called *parafermionic observables* for two-dimensional lattice models [2, 3]. These observables were first introduced in [11] for random-cluster models with parameter $q \in [0, 4]$, as (anti-)holomorphic parafermions of fractional spin $\sigma \in [0, 1]$, given by certain vertex operators. So far holomorphicity was rigorously proved only for $q = 2$, and probably holds exactly only for this value. Our proof uses an appropriate generalization of these vertex operators to random-cluster models with $q \geq 4$. Interestingly, the spin variable becomes purely imaginary, therefore it does not have an immediate physical interpretation. However, this allows us to write better estimates even in the absence of exact holomorphicity and relates our observables to the connectivity properties of the model. For $p \neq p_{sd}$ we prove that observables behave like massive harmonic functions and decay exponentially fast with respect to the distance to the boundary of the domain. Translated into connectivity properties, this implies the sharpness of the phase transition at p_{sd} .

Section 1 give an overview of probabilistic properties of the random-cluster model. Section 2 introduces the observable. In Section 3, we derive a representation formula, similar to the formula for massive harmonic functions. Section 4 then contains the proofs of Theorems 1 and 2.

1 Basic features of the model

We start with an introduction to the basic features of random-cluster models; more details and proofs can be found in the Grimmett's book [5].

Definition of the random-cluster model. The random-cluster measure can be defined on any graph. However, we will restrict ourselves to the square lattice, denoted by $\mathbb{L} = (\mathbb{Z}^2, \mathbb{E})$ with \mathbb{Z}^2 denoting the set of *sites* and \mathbb{E} the set of *bonds*. In this paper, G will always denote a connected subgraph of \mathbb{L} , *i.e.* a subset of vertices of \mathbb{Z}^2 together with all the bonds between them. We denote by ∂G the boundary of G , *i.e.* the set of sites of G linked by a bond to a site $\mathbb{Z}^2 \setminus G$.

A *configuration* ω on G is a random subgraph of G , having the same sites and a subset of its bonds. We will call the bonds belonging to ω *open*, the others *closed*. Two sites a and b are said to be *connected*, if there is an *open path* — a path composed of open bonds only — connecting them. The maximal connected components will be called *clusters*.

A set ξ of *boundary conditions* is a set of “abstract” bonds (*i.e.* in addition to those in \mathbb{L}) in between the sites of the boundary, they encode the way in which these sites are connected outside of G . We denote by $\omega \cup \xi$ the graph obtained by adding the new bonds in ξ to the configuration ω . Let $o(\omega)$ (resp. $c(\omega)$) denote the number of open (resp. closed) bonds of ω and $k(\omega, \xi)$ the number of connected components of $\omega \cup \xi$. The probability measure $\mathbb{P}_{p,q,G}^\xi$ of the random-cluster model on G with parameters p and q and boundary conditions ξ is defined by

$$\mathbb{P}_{p,q,G}^\xi(\{\omega\}) = \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega,\xi)}}{Z_{p,q,G}^\xi}, \quad (1.1)$$

for any subgraph ω of G , where $Z_{p,q,G}^\xi$ is a normalizing constant referred to as the *partition function*. When there is no possible confusion, we will drop the reference to parameters in the notation.

The domain Markov property. Another property that will be instrumental in our study is the so-called *domain Markov property*: One can encode, using appropriate boundary conditions ξ , the influence of the configuration outside G . Consider G a graph and E the set of sites of G . For $F \subset E$, consider the graph G' with F as the set of sites and the bonds of G in between sites of F as the set of bonds. Then, for any boundary conditions ϕ , \mathbb{P}_G^ϕ conditioned to match some configuration ω outside of the box G' is equal to $\mathbb{P}_{G'}^\xi$, where ξ describes the connections inherited from ω .

The FKG inequality and comparison between boundary conditions. Random-cluster models with parameter $q \geq 1$ are *positively correlated*. This property has two important consequences: The first one is the *Fortuin-Kasteleyn-Ginibre inequality*:

$$\mathbb{P}_{p,q,G}^\xi(A \cap B) \geq \mathbb{P}_{p,q,G}^\xi(A)\mathbb{P}_{p,q,G}^\xi(B), \quad (1.2)$$

which holds for every pair of *increasing* events A and B and any boundary conditions ξ . (An event is called increasing if it is preserved by addition of open bonds, see [5].) This correlation inequality is extremely important to the study of random-cluster models, allowing the combination of several increasing events such as the existence of open paths in a configuration.

The second property is a *comparison between boundary conditions*: More precisely, for any boundary conditions $\phi \leq \xi$ (meaning that the abstract bonds existing in ϕ exist in ξ as well), we have

$$\mathbb{P}_{p,q,G}^\phi(A) \leq \mathbb{P}_{p,q,G}^\xi(A) \quad (1.3)$$

for any increasing event A .

Dobrushin boundary conditions. The following definition is deliberately not as general as would be possible, in order to limit the introduction of notation. Let G be a finite subgraph of \mathbb{L} ; assume that its boundary is a self-avoiding polygon in \mathbb{L} , and let a and b be two sites of ∂G . The triple (G, a, b) is called a *Dobrushin domain*. Orienting its boundary counterclockwise defines two oriented boundary arcs ab and ba ; the *Dobrushin boundary conditions* are defined to be free on ab (there are no extra bonds between boundary sites) and wired on ba (all the boundary sites are pairwise connected). We will refer to those arcs as the free and the wired arc, respectively. The measure associated to these boundary conditions will be denoted by $\mathbb{P}_{p,q,G,a,b}$ or simply $\mathbb{P}_{G,a,b}$.

Planar duality for Dobrushin domains. In two dimensions, one can associate to any random-cluster measure with parameters p and q on a Dobrushin domain (G, a, b) a dual measure. First, define the *dual graph* G^* as follows: Place a site in the center of every face of G

and every face of \mathbb{L} adjacent to the free arc. Bonds of the dual graph correspond to bonds of the primal graph and link nearest neighbors. Construct a bond model on G^* by declaring any bond of the dual graph to be open (resp. closed) if the corresponding bond of the primal lattice is closed (resp. open) for the initial random-cluster model. The new model on the dual graph is then a random-cluster measure with Dobrushin boundary conditions and parameters $p^* = p^*(p, q)$ and q satisfying

$$p^*(p, q) = \frac{(1-p)q}{(1-p)q+p}, \text{ or equivalently } \frac{p^*p}{(1-p^*)(1-p)} = q.$$

This relation is known as the *planar duality*. It is then natural to define the self-dual point fixed by solving the equation $p^*(p_{sd}, q) = p_{sd}$, which gives

$$p_{sd} = \sqrt{q}/(1 + \sqrt{q}).$$

Infinite-volume measures and the critical point. The domain Markov property and the comparison between boundary conditions allow us to define infinite-volume measures as the limit of a sequence of random-cluster measures in finite boxes (with well-chosen boundary conditions).

The infinite-volume measure is not necessarily unique since the definition can depend on the sequence of random-cluster measures one considers. Nevertheless, it can be shown that for a fixed $q \geq 1$, uniqueness can fail only on a countable set \mathcal{D}_q (see Theorem ??? of [5]).

Since the infinite-volume measure is unique for almost every p (at fixed q), there exists a *critical point* p_c such that for any infinite-volume measure with $p < p_c$ (resp. $p > p_c$), there is almost surely no infinite component of connected sites (resp. one infinite component).

An elegant argument (due to Zhang in the case of percolation) provides the inequality $p_c \geq p_{sd}$ (see Theorem 6.17 of [5]): the hard part of the proof is to show the reverse inequality. Together with general arguments on uniqueness of infinite-volume measures, it can be shown that uniqueness can fail only when $p = \sqrt{q}/(1 + \sqrt{q})$ (see Chapter 6 of [5]). Hence, the infinite-volume measure is unique whenever $p \neq p_{sd}$. We denote this measure by $\mathbb{P}_{p,q}$. Interestingly, it is conjectured that uniqueness does fail at the self-dual point if and only if $q > 4$, which is our case.

2 Definition of the observable

The medial lattice and the loop representation. Let (G, a, b) be a Dobrushin domain. In this paragraph, we aim for the construction of the loop representation of the random-cluster model, defined on the so-called medial graph. In order to do that, consider \mathbb{L} together with its dual \mathbb{L}^* ; declare *black* the sites of the primal lattice \mathbb{L} and *white* the sites of the dual lattice \mathbb{L}^* . Replace every site by a colored diamond, as on Figure 1. The *medial graph* $G_\diamond = (V_\diamond, E_\diamond)$ is defined as follows (see Figure 1 again): E_\diamond is the set of edges of \mathbb{L}_\diamond which belong to both a diamond corresponding to a site of G and a diamond corresponding to a site of G^* ; V_\diamond is the set of all the endpoints of the edges in E_\diamond . We obtain a subgraph of a rotated (and rescaled) version of the square lattice. We give G_\diamond an additional structure as an oriented graph by orienting its edges clockwise around white faces.

The random-cluster measure with Dobrushin boundary conditions has a rather convenient representation in this setting. Consider a configuration ω , it defines clusters in G and dual clusters in G^* . Through every vertex of the medial graph passes either an open bond of G or a dual open bond of G^* , hence there is a unique way to draw Eulerian (*i.e.* using every edge exactly once) loops on the medial lattice — *interfaces*, separating clusters from dual clusters. Namely, a loop arriving at a vertex of the medial lattice, always makes a $\pi/2$ turn so as not to cross the open or dual open bond through this vertex, see Figure 1. Besides loops, the

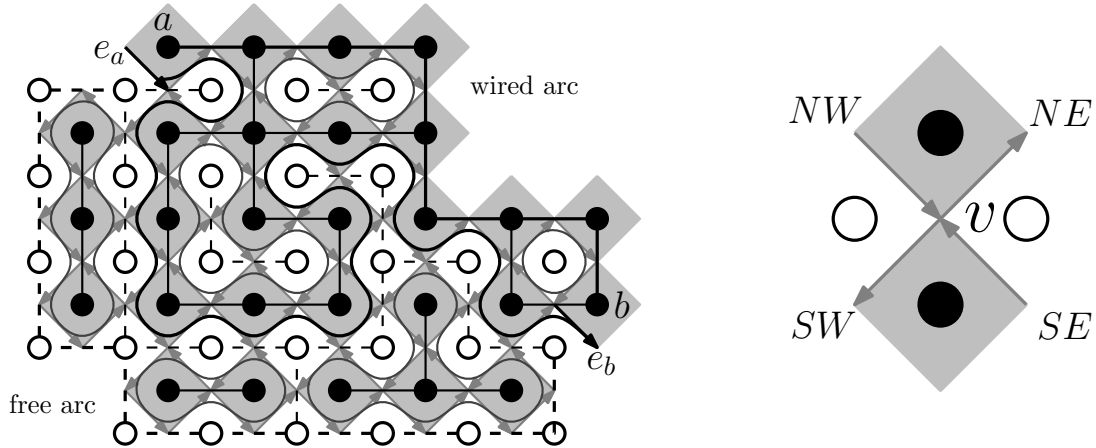


Figure 1: Left: Construction of the medial lattice and the loop representation. The black (resp. white) sites are the sites of G (resp. G^*). The open bonds of G (resp. G^*) are represented by solid (resp. dashed) bonds. **Right:** Orientation around a vertex v .

configuration will have a single curve joining the vertices adjacent to a and b , which are the only vertices in V_\diamond with three adjacent edges (the fourth edges emanating from them will be denoted by e_a and e_b , respectively). This curve is called the *exploration path*; we will denote it by γ . It corresponds to the interface between the cluster connected to the wired arc and the dual cluster connected to the free arc.

This gives a bijection between random-cluster configurations on G and Eulerian loop configurations on G_\diamond . The probability measure can be nicely rewritten (using Euler's formula) in terms of the loop picture:

$$\mathbb{P}_{G,a,b}(\omega) = \frac{x^{\# \text{ open bonds}} \sqrt{q}^{\# \text{ loops}}}{\tilde{Z}(p, q, G)}, \quad \text{where } x = \frac{p}{(1-p)\sqrt{q}}$$

and $\tilde{Z}(p, q, G)$ is a normalizing constant. Notice that $p = p_{sd}$ if and only if $x = 1$. This bijection is called the *loop representation* of the random-cluster model. Note that the orientation of the medial graph gives a natural orientation to the interfaces in the loop representation.

Observables for Dobrushin domains. Fix a Dobrushin domain (G, a, b) . Following [11], we now define an observable F on the edges of its medial graph, *i.e.* a function $F : E_\diamond \rightarrow \mathbb{R}_+$. Roughly speaking, F is a modification of the probability that the exploration path passes through an edge. First, we introduce the following definition: The *winding* $W_\Gamma(z, z')$ of a curve Γ between two edges z and z' of the medial graph is the total rotation (in radians) that the curve makes from the mid-point of the edge z to the that of the edge z' .

We define the observable F_{e_a} for any edge $e \in E_\diamond$ by

$$F_{e_a}(e) = \mathbb{E}_{G,a,b} \left(e^{-i\sigma W_\gamma(e_a, e)} \mathbb{1}_{e \in \gamma} \right), \quad (2.1)$$

where γ is the exploration path and σ is given by the relation

$$\cos(\sigma\pi/2) = \frac{\sqrt{q}}{2} \quad (2.2)$$

(remember that we are limiting ourselves to the case $q > 4$ in most of what follows, hence σ will be pure imaginary). We define the function \tilde{F}_{e_a} associated to the value $-\sigma$ similarly.

Remark 3 For $q \in [0, 4]$, the observables \tilde{F} and F are holomorphic and anti-holomorphic parafermions of spin σ , which is a real number in $[0, 1]$. For $q \geq 4$, σ is pure imaginary and doesn't have an obvious physical meaning; it would nonetheless be amusing to find one.

Remark 4 For $q \geq 4$, the winding weight in the definition of the observable is always real positive, and so are F and \tilde{F} . Moreover, the positive weight allows an easy estimate of the probability of passing through an edge in terms of the observables:

$$F_{e_a}(e) + \tilde{F}_{e_a}(e) = 2 \mathbb{E}_{G,a,b}[\cos(\sigma W_\gamma(e_a, e)) \mathbb{1}_{e \in \gamma}] \geq 2 \mathbb{P}_{G,a,b}(e \in \gamma). \quad (2.3)$$

Observables in infinite strips. We shall consider the loop representation and the observable for random-cluster measures in a slightly more general setting. More precisely, we will consider the random-cluster measure on a *horizontal strip* \mathcal{S}_L with free boundary conditions on the bottom side and wired boundary conditions on the top side. Formally, define the following sets (for $m, L \geq 0$):

$$\begin{aligned} \mathcal{S}_L &= \mathbb{Z} \times \llbracket 0, L \rrbracket, \\ \mathcal{B}(m, L) &= \llbracket -m, m \rrbracket \times \llbracket 0, L \rrbracket \end{aligned}$$

and the points $a_m = (-m, 0)$ and $b_m = (m, 0)$. Here and in the following, $\llbracket \cdot, \cdot \rrbracket$ denotes the integer interval between the two (real) end-points, *i.e.* the interval $[\cdot, \cdot] \cap \mathbb{Z}$. By extension, we still call these horizontal strips Dobrushin domains.

The decreasing sequence of measures $(\mathbb{P}_{\mathcal{B}(m,L),a_m,b_m})_{m \geq 0}$ converges weakly when m goes to infinity (for a fixed L). The limit is called the *random-cluster measure on the infinite strip with Dobrushin boundary conditions* and is denoted by $\mathbb{P}_{\mathcal{S}_L, \infty, -\infty}$. When $p \neq p_{sd}$, this measure should be seen as the (unique) infinite-volume measure under the conditioning that all of the bonds below \mathbb{Z} are open and all of those strictly above $iL + \mathbb{Z}$ are closed.

The $\mathbb{P}_{\mathcal{S}_L, \infty, -\infty}$ -probability of having an infinite cluster is 0 (for fixed L , the model is essentially one-dimensional). Applying the same construction as in the finite volume case, we define a family of loops together with a single interface going from $-\infty$ to ∞ (the exploration path). Since this curve cannot turn infinitely many times around an edge of the medial lattice, the winding W_∞ is well-defined (up to an additive constant). We set the winding to be equal to 0 for edges of the bottom side which point inside the domain.

The observables are defined exactly as in the case of the finite Dobrushin domains and are the limits of the observables in domains $(\mathcal{B}(m, L), a_m, b_m)$. Indeed, the exploration path in \mathcal{S}_L coincides with the exploration paths γ_m in $\mathcal{B}(m, L)$ for m large enough almost surely since there is no infinite cluster. Moreover, in this case, the windings $W_\infty(e)$ and $W_m(e)$ in \mathcal{S}_L and $\mathcal{B}(m, L)$ are equal, so that a.s.

$$e^{-i\sigma W_m(e)} \mathbb{1}_{e \in \gamma_m} \longrightarrow e^{-i\sigma W_\infty(e)} \mathbb{1}_{e \in \gamma} \text{ when } m \rightarrow \infty. \quad (2.4)$$

Lebesgue's dominated convergence theorem implies that the observable in \mathcal{S}_L is the limit of the observables in finite volume.

3 A representation formula for the observable

In this section, we study the observable F in a Dobrushin domain (we drop the dependency on e_a in the notation). More precisely, we bound the sum of F over a set $A \subset E_\diamond$ in terms of the sum over the boundary edges of A . Let (G, a, b) be a Dobrushin domain, we set $\partial_v V_\diamond$ to be the boundary of G_\diamond - *i.e.* vertices which have less than four neighboring edges. For a set A of edges of E_\diamond , $\partial_e A$ denotes the set of edges of $E_\diamond \setminus A$ sharing a vertex with an edge of A (also called the external boundary of the set).

Proposition 5 For any $x \neq 1$ and $q \neq 4$, there exists $C_1 = C_1(p, q) < \infty$ such that for any set of edges $A \subset E_\diamond$ not containing any edge adjacent to a vertex of $\partial_v V_\diamond$, there exists a function $\alpha : \partial_e A \rightarrow [-C_1, C_1]$ such that

$$\sum_{e \in A} F(e) = \sum_{e \in \partial_e A} \alpha_e F(e).$$

The proposition is obtained by summing a local relation which is a massive version of one in [11]. Consider a vertex $v \in V_\diamond \setminus \partial_v V_\diamond$. It has four neighboring edges in G_\diamond , which we label counterclockwise NW , SW , SE and NE , so that the SW and NE edges point away from v (see Figure 1). There are a priori two ways to do so, but the choice will be irrelevant.

Lemma 6 For $x > 0$, $q \neq 4$ and every vertex $v \in V_\diamond \setminus \partial_v V_\diamond$,

$$F(NW) + F(SE) - \Lambda(x) (F(SW) + F(NE)) = 0, \quad (3.1)$$

where $\Lambda(x)$ is given by

$$\Lambda(x) = \frac{e^{i\sigma\pi/2} + x}{e^{i\sigma\pi/2}x + 1}.$$

Observe that $\Lambda(x) = 1$ if and only if $x = 1$.

Proof: We handle the case of finite Dobrushin domains. The case of the infinite strip \mathcal{S}_L follows harnessing the convergence of observables in finite boxes to the observable in the strip.

Let v be a vertex of $V_\diamond \setminus \partial_v V_\diamond$ (which corresponds to a bond of the primal lattice). It has four adjacent edges, oriented as explained above. We consider the involution s on the space of configurations which switches the state (open or closed) of the bond of the primal lattice corresponding to v .

Let e be an edge of the medial graph and denote by $e_\omega = e^{-i\sigma W_\gamma(e_\omega, e)} \mathbb{1}_{e \in \gamma} p(\omega)$ the contribution of ω to $F(e)$ (here $p(\omega)$ is the probability of the configuration ω). Since s is an involution, the following relation holds:

$$F(e) = \sum_{\omega} e_\omega = \frac{1}{2} \sum_{\omega} [e_\omega + e_{s(\omega)}].$$

In order to prove (3.1), it suffices to prove the following for any configuration ω :

$$NW_\omega + NW_{s(\omega)} + SE_\omega + SE_{s(\omega)} = \Lambda(x) (SW_\omega + SW_{s(\omega)} + NE_\omega + NE_{s(\omega)}). \quad (3.2)$$

When $\gamma(\omega)$ does not go through any of the edges adjacent to v , it is easy to see that neither does $\gamma(s(\omega))$. All the contributions then vanish and identity (3.2) trivially holds. Thus we can assume that $\gamma(\omega)$ passes through at least one edge adjacent to v . The interface follows the orientation of the medial graph, and thus can enter v through either NW or SE and leave through NE or SW . Without loss of generality we assume that it enters first through the edge NW and leaves last through the edge SW ; the other cases are treated similarly.

Two cases can occur (see Figure 2): Either the exploration curve, after arriving through NW , leaves through NE and then returns a second time through SE , leaving through SW ; or the exploration curve arrives through NW and leaves through SW , with NE and SE belonging to a loop. Since the involution exchanges the two cases, we can assume that ω corresponds to the first case. Knowing the term NW_ω , it is possible to compute the contributions of ω and $s(\omega)$ to all of the edges adjacent to v . Indeed,

- the probability of $s(\omega)$ is equal to $x\sqrt{q}$ times the probability of ω (due to the fact that there is one additional open edge and one additional loop);

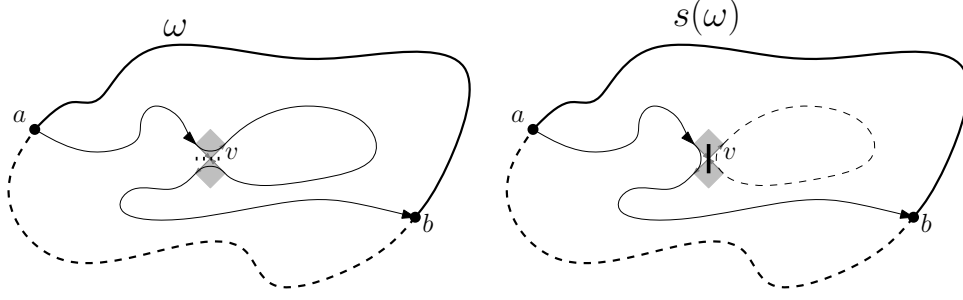


Figure 2: Two associated configurations ω and $s(\omega)$

configuration	NW	SE	NE	SW
ω	NW_ω	$e^{i\sigma\pi}NW_\omega$	$e^{-i\sigma\pi/2}NW_\omega$	$e^{i\sigma\pi/2}NW_\omega$
$s(\omega)$	$x\sqrt{q}NW_\omega$	0	0	$e^{i\sigma\pi/2}x\sqrt{q}NW_\omega$

Figure 3: The contributions of the different edges.

- windings of the curve can be expressed using the winding of the north-west edge. For instance, the winding of the north-east edge in the configuration ω is equal to the winding of the north-west edge plus an additional $\pi/2$ turn.

Contributions are computed in Figure 3. Using the identity $e^{i\sigma\pi/2} + e^{-i\sigma\pi/2} = \sqrt{q}$, we deduce (3.2) by summing the contributions of all the edges around v . \square

Proof of Proposition 5: Recall that $\Lambda(x) \neq 1$ since $x \neq 1$ and $q \neq 4$. Sum Identity (3.1) over all vertices adjacent to edges of A , and divide by $(1 - \Lambda(x))$. It provides a weighted sum of $F(e)$ (with coefficients denoted by $c(e)$) identical to zero:

$$\sum_{e \in A} c(e)F(e) + \sum_{e \in \partial A} c(e)F(e) = 0.$$

For an edge $e \in A$, $F(e)$ will appear in two identities, corresponding to its endpoints. Since e is oriented away from one of its ends and towards the other one, the coefficients will be 1 and $-\Lambda(x)$. Thus $F(e)$ for $e \in A$ will enter the sum with a coefficient $c(e) = (1 - \Lambda(x))/(1 - \Lambda(x)) = 1$.

For an edge $e \in \partial_e A$, $F(e)$ will appear in one identity, corresponding to its endpoint belonging to A . The coefficient will be 1 or $-\Lambda(x)$, depending on the orientation of e with respect to this endpoint. Thus $F(e)$ will enter the sum with a coefficient $c(e)$ equal to either $1/(1 - \Lambda(x))$ or $-\Lambda(x)/(1 - \Lambda(x))$. The proposition follows immediately by setting $\alpha_e := -c(e)$ and $C_1 := \max\{1, |\Lambda(x)|\}/|1 - \Lambda(x)|$. \square

4 Proof of Theorems 1 and 2

The main step of the proof is to show that, whenever $p < p_{sd}$, there is a very low probability of having vertical crossings of an extremely large rectangle. This statement is sufficient to prove Theorem 1: On the one hand, when $p < p_{sd}$, the previous estimate provides bounds on the size of the cluster of the origin (in particular, it is almost surely finite): It implies that p is sub-critical and $p_c \geq p_{sd}$. As explained earlier, this statement is already proven (see Chapter 6 of [5]), nevertheless, these bounds will be instrumental for the proof of Theorem 2. On the other hand, duality implies that the probability of having a horizontal crossing of an extremely

large rectangle is very close to 1, whenever $p > p_{sd}$. It is an easy step to deduce that there is an infinite cluster when $p > p_{sd}$, thus $p_c \leq p_{sd}$. Theorem 2 is derived using Theorems (5.64) and (5.66) of [5] and the estimate on the size of the cluster at the origin.

For $L \geq 0$, consider an infinite horizontal strip $\mathcal{S}_L = \mathbb{Z} \times \llbracket 0, L \rrbracket$ together with its medial lattice. We define two families of sets, the former ones being subsets of the strip and the latter of the set E_\diamond of edges of its medial graph. More precisely, write $e \sim a$ if the edge $e \in E_\diamond$ is adjacent to the site $a \in \mathbb{L}$. For every $n \geq 0$, we define the following (possibly empty) sets, as depicted in Figure 4:

$$\begin{aligned} \mathcal{R}(m, n) &:= \llbracket 0, m \rrbracket \times \llbracket n, L - n \rrbracket, & \mathcal{R}_\diamond(m, n) &:= \{e \in E_\diamond : \exists a \in \mathcal{R}(m, n), e \sim a\}, \\ \mathcal{R}^-(m, n) &:= \llbracket 0, m \rrbracket \times \llbracket 0, n - 1 \rrbracket, & \mathcal{R}_\diamond^-(m, n) &:= \{e \in E_\diamond : \exists a \in \mathcal{R}^-(m, n), e \sim a\}, \\ \mathcal{R}^+(m, n) &:= \llbracket 0, m \rrbracket \times \llbracket L - n + 1, L \rrbracket, & \mathcal{R}_\diamond^+(m, n) &:= \{e \in E_\diamond : \exists a \in \mathcal{R}^+(m, n), e \sim a\}. \end{aligned}$$

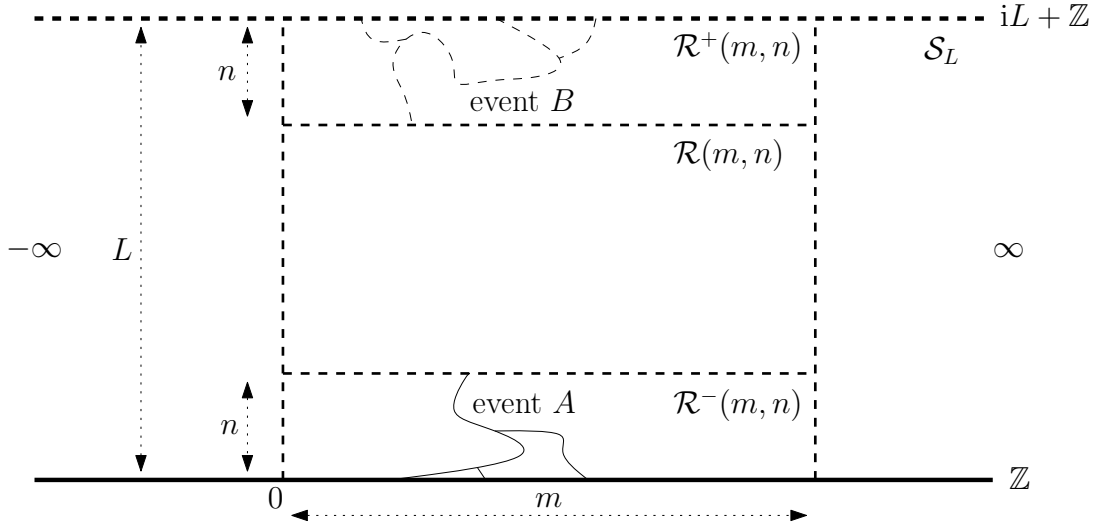


Figure 4: Definition of the different rectangles and events A and B.

Recall that in a Dobrushin domain, γ denotes the exploration path, *i.e.* the interface between the open cluster connected to the wired arc and the dual open cluster connected to the free arc. The following lemma bounds the probability that the exploration path passes through the rectangle $\mathcal{R}(m, n)$:

Lemma 7 *Let $q > 4$ and $p \neq p_{sd}$, then there exist positive constants $c_2 = c_2(p, q)$ and $C_2 = C_2(p, q)$ such that for any $n < L/2$ and $m \geq C_2$,*

$$\mathbb{P}_{\mathcal{S}_L, \infty, -\infty}(\gamma \cap \mathcal{R}_\diamond(m, n) \neq \emptyset) \leq C_2 m e^{-c_2 n}.$$

Proof: Consider the observable F defined in the strip \mathcal{S}_L , and recall that in our setting $x \neq 1$ and F is non-negative by Remark 4. Set $c_2 := -\log(2C_1/(2C_1 + 1))$ and $C_2 := \max\{4C_1, 8 \exp(|\sigma|2\pi)\}$ where C_1 is defined in Proposition 5.

Fix $m \geq C_2$ and consider some $n < L/2$. Denote

$$U_n := \sum_{e \in \mathcal{R}_\diamond(m, n)} F(e), \quad V_n := \sum_{e \in \mathcal{R}_\diamond(m, n)} \tilde{F}(e).$$

Proposition 5 along with the non-negativity of F implies the following estimate:

$$U_n = \sum_{e \in \partial_e \mathcal{R}_\diamond(m, n)} \alpha_e F(e) \leq C_1 \sum_{e \in \partial_e \mathcal{R}_\diamond(m, n)} F(e). \quad (4.1)$$

Divide the boundary $\partial_e \mathcal{R}_\diamond(m, n)$ into four parts: the bottom A_{bot} , the top A_{top} and both sides A_{left} and A_{right} .

On the one hand, since F is invariant under horizontal translations, the sums over the left and right sides are the same as over any vertical cross-section of $\mathcal{R}_\diamond(m, n)$ and we conclude that

$$\sum_{e \in A_{\text{left}} \cup A_{\text{right}}} F(e) = \frac{2}{m} U_n. \quad (4.2)$$

On the other hand, the top and the bottom are contained inside $U_{n-1} \setminus U_n$, and therefore

$$\sum_{e \in A_{\text{top}} \cup A_{\text{bottom}}} F(e) \leq U_{n-1} - U_n. \quad (4.3)$$

Combining Equations (4.1), (4.2) and (4.3) and using the inequality $m \geq C_2 \geq 4C_1$, we obtain that

$$U_n \leq \frac{2C_1}{m} U_n + C_1(U_{n-1} - U_n) \leq \left(\frac{1}{2} - C_1\right) U_n + C_1 U_{n-1},$$

hence

$$U_n \leq \frac{2C_1}{2C_1 + 1} U_{n-1} = e^{-c_2} U_{n-1}. \quad (4.4)$$

Take now $n = 0$, A_{top} and A_{bottom} are thus at a distance one to the boundary of the strip: an interface arriving there must have a winding bounded by $\pm 2\pi$. Thus for $e \in A_{\text{top}} \cup A_{\text{bottom}}$ we have

$$F(e) = \mathbb{E}_{\mathcal{S}_{l, \infty, -\infty}}[e^{-i\sigma W(e)} \mathbb{1}_{e \in \gamma}] \leq e^{|\sigma|2\pi} \mathbb{P}_{\mathcal{S}_{l, \infty, -\infty}}(e \in \gamma) \leq e^{|\sigma|2\pi}$$

(the curve cannot wind around an edge). Summing this over all $4m$ edges in the top and bottom sides we arrive at

$$\sum_{e \in A_{\text{top}} \cup A_{\text{bottom}}} F(e) \leq 4me^{|\sigma|2\pi} \leq \frac{1}{2} C_2 m. \quad (4.5)$$

Combining Equations (4.1), (4.2) and (4.5) for $n = 0$ we deduce that

$$U_0 \leq \frac{2C_1}{m} U_0 + \frac{1}{2} C_2 m \leq \frac{1}{2} U_0 + \frac{1}{2} C_2 m,$$

therefore

$$U_0 \leq C_2 m. \quad (4.6)$$

Estimate (4.6) along with the iterated Estimate (4.4) imply

$$U_n \leq U_0 e^{-c_2 n} \leq C_2 m e^{-c_2 n}. \quad (4.7)$$

Similar reasoning applies to \tilde{F} , yielding the same inequality for V_n . Combining the two inequalities with Estimate (2.3), we obtain the claim:

$$\begin{aligned} \mathbb{P}_{\mathcal{S}_{L, \infty, -\infty}}(\gamma \cap \mathcal{R}_\diamond(m, n) \neq \emptyset) &\leq \sum_{e \in \mathcal{R}_\diamond(m, n)} \mathbb{P}_{\mathcal{S}_{L, \infty, -\infty}}(e \in \gamma) \\ &\leq \frac{1}{2} \sum_{e \in \mathcal{R}_\diamond(m, n)} (F(e) + \tilde{F}(e)) \leq \frac{1}{2} (U_n + V_n) \leq C_2 m e^{-c_2 n}. \end{aligned}$$

□

For a rectangle \mathcal{R} , we define the event $\mathcal{C}_h(\mathcal{R})$ (resp. \mathcal{C}_v) to be the existence of an open path from the left-hand to the right-hand side (resp. from the top to the bottom) of \mathcal{R} . Similarly, we define $\mathcal{C}_v^*(\mathcal{R})$ and $\mathcal{C}_h^*(\mathcal{R})$ in terms of the dual open paths through the rectangle \mathcal{R} shifted by $\frac{1}{2} + \frac{i}{2}$, so that it belongs to the dual lattice. In the next lemma, we bound the probabilities of such events for rectangles of aspect ratio $1/3$ whenever $p < p_{sd}$ (duality provides estimates for $p > p_{sd}$ as well).

Lemma 8 *Let $q > 4$ and $p < p_{sd}$, there exist $0 < c_3, C_3 < \infty$ such that for every $m > 0$,*

$$\mathbb{P}_{p,q}(\mathcal{C}_v(\llbracket 0, 3m \rrbracket \times \llbracket 0, m \rrbracket)) \leq C_3 e^{-c_3 \sqrt{m}} \text{ a.s.}$$

where $\mathbb{P}_{p,q}$ is the unique infinite-volume measure.

This result is not surprising once we consider typical (not formerly proved) sub-critical behavior. Indeed, the probability for two points to be connected by an open path in the sub-critical phase should decay exponentially fast with respect to the distance between them. It implies that the probability for large rectangles to be crossed from bottom to top is extremely low.

Proof: Throughout the proof, we will implicitly round up the side lengths of rectangles involved (for instance, \sqrt{n} will actually mean $\lceil \sqrt{n} \rceil$ in that context).

Fix $p < p_{sd}$ and take m large enough satisfying

$$C_2 m e^{-c_2 \sqrt{m}} < \frac{1}{3}.$$

We will work with $L > 2n$, $n = \sqrt{m}$ and the following events, depicted in Figure 4:

$$A = \mathcal{C}_v(\mathcal{R}^-(m, \sqrt{m})), \quad B = \mathcal{C}_v^*(\mathcal{R}^+(m, \sqrt{m})).$$

Recall that the exploration path is an interface between the open cluster connected to the (wired) bottom side and the dual open cluster connected to the (free) top side. Therefore, if both A and B occur, the exploration path is forced to pass through $\mathcal{R}_\circ(m, \sqrt{m})$, thus Lemma 7 implies the estimate

$$\mathbb{P}_{\mathcal{S}_{L,\infty,-\infty}}(A \cap B) \leq C_2 m e^{-c_2 \sqrt{m}} < \frac{1}{3}. \quad (4.8)$$

Consider the symmetry of the strip exchanging its sides and add $\frac{1+i}{2}$ so that the lattice is mapped to its dual. Note that it preserves Dobrushin boundary conditions, *e.g.* the wired boundary conditions on the bottom part are sent to the dual wired (= free) boundary conditions on the top part. Therefore, by duality, the random-cluster measure $\mathbb{P}_{\mathcal{S}_{L,\infty,-\infty}}$ with parameters $p^*(p, q)$ and q gets mapped to the random-cluster measure on the dual strip with the same boundary conditions and parameters p and q . This symmetry also maps the event A to the event B , so we can write

$$\mathbb{P}_{p,q,\mathcal{S}_{L,\infty,-\infty}}(B) = \mathbb{P}_{p^*,q,\mathcal{S}_{L,\infty,-\infty}}(A) \geq \mathbb{P}_{p,q,\mathcal{S}_{L,\infty,-\infty}}(A),$$

since A is an increasing event and we are assuming that $p^*(p, q) > p$ (since $p < p_{sd}$).

Hence (we return to the fixed parameters p and q), event A has smaller probability than B , and (4.8) implies

$$2\mathbb{P}_{\mathcal{S}_{L,\infty,-\infty}}(A) - 1 \leq \mathbb{P}_{\mathcal{S}_{L,\infty,-\infty}}(A) + \mathbb{P}_{\mathcal{S}_{L,\infty,-\infty}}(B) - 1 \leq \mathbb{P}_{\mathcal{S}_{L,\infty,-\infty}}(A \cap B) < \frac{1}{3},$$

concluding that

$$\mathbb{P}_{\mathcal{S}_{L,\infty,-\infty}}(\mathcal{C}_v(\mathcal{R}^-(m, \sqrt{m}))) = \mathbb{P}_{\mathcal{S}_{L,\infty,-\infty}}(A) < \frac{2}{3}.$$

Letting L go to infinity, the measure $\mathbb{P}_{\mathcal{S}_{L,\infty,-\infty}}$ converges to the random-cluster measure $\mathbb{P}_{p,q}$ in the upper-half plane with wired boundary conditions on \mathbb{Z} . Therefore, for m large enough, the probability of the event $\mathcal{C}_v(\mathcal{R}^-(m, \sqrt{m}))$ given that the bonds of \mathbb{Z} are open is bounded from above by $2/3$.

Since these boundary conditions stochastically dominate all the others and A is an increasing event, we deduce from (1.3) that the probability of A is always smaller than $2/3$, uniformly with

respect to the boundary conditions on \mathbb{Z} – in other words, uniformly on what happens below the rectangle. Consider m large enough, we divide the rectangle $\llbracket 0, 3m \rrbracket \times \llbracket 0, m \rrbracket$ into rectangles R_i (where $i = 1 \cdots \sqrt{m/3}$) with height $\sqrt{3m}$ and width $3m$. Let A_i be the event that R_i is crossed vertically. Notice that for every i , A_i is a translate of the event A . If there is a vertical crossing of $\llbracket 0, 3m \rrbracket \times \llbracket 0, m \rrbracket$, there must exist a vertical crossing for each of these $\sqrt{m/3}$ rectangles so that

$$\mathbb{P}_{p,q}(\mathcal{C}_v(\llbracket 0, 3m \rrbracket \times \llbracket 0, m \rrbracket)) \leq \mathbb{P}_{p,q} \left(\bigcap_{i=1}^{\sqrt{m/3}} A_i \right) = \prod_{i=1}^{\sqrt{m/3}} \mathbb{P}_{p,q}(A_i | A_j, j < i).$$

Estimating the conditional probabilities of events A_i one by one, using the domain Markov property and the uniform bound on boundary conditions, we deduce the claim. \square

We are now in a position to prove our main theorems.

Proof of Theorem 1: First consider the case of $q > 4$. Recall that $\mathbb{P}_{p,q}$ is the unique infinite-volume measure. Even if $p_c \geq p_{sd}$ is already known (see [5]), we give another proof, which provides an explicit estimate for the probability of 0 being connected to a site at distance n .

Consider $p < p_{sd}$ and let $n \geq 1$, define the box $\mathcal{B}_n = \llbracket -n, n \rrbracket^2$ and denote by $\{0 \leftrightarrow \mathcal{B}_n^c\}$ the event that there exists an open connected path from the origin to the boundary of the box \mathcal{B}_n . Define \mathcal{R}_1 to be the rectangle $\llbracket 0, n \rrbracket \times \llbracket -n, n \rrbracket$ and $\mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$ to be its rotations around the origin by the angles $\pi/2, \pi$ and $3\pi/2$ respectively. If 0 is connected to the boundary of the box, one of these four rectangles is crossed by an open cluster joining its two longer sides. Employing Lemma 8, we obtain

$$\mathbb{P}_{p,q}(0 \leftrightarrow \mathcal{B}_n^c) \leq 4C_3 e^{-c_3 \sqrt{n}}. \quad (4.9)$$

Hence, the probability that the origin belongs to an infinite cluster is 0 and p must be sub-critical. We deduce $p_c \geq p_{sd}$ by letting p go to p_{sd} .

For $p > p_{sd}$, consider the sequences of events (E_n) and (F_n) defined as follows:

$$\begin{aligned} E_n &= \mathcal{C}_v(\llbracket 0, 3^n \rrbracket \times \llbracket 0, 3^{n+1} \rrbracket), \\ F_n &= \mathcal{C}_h(\llbracket 0, 3^{n+1} \rrbracket \times \llbracket 0, 3^n \rrbracket). \end{aligned}$$

If all the events E_n and F_n occur simultaneously, out of the corresponding crossings we can build a curve connecting the origin to infinity (an event denoted by $\{0 \leftrightarrow \infty\}$). From the FKG inequality and the estimate of Lemma 8 we obtain

$$\mathbb{P}_{p,q}(0 \leftrightarrow \infty) \geq \prod_{n=0}^{\infty} \mathbb{P}_{p,q}(E_n) \mathbb{P}_{p,q}(F_n) \geq \prod_{n=0}^{\infty} \left(1 - C_3 e^{-c_3 3^{n/2}}\right)^2 > 0.$$

Thus, the origin is connected to infinity with positive probability for every $p > p_{sd}$, which implies $p_{sd} \geq p_c$.

The case $q = 4$ is derived through stochastic domination between random cluster measures (see [5] for details). Indeed, for every $p < p_{sd}$, there exists (p', q) with $q > 4$ and $p' < p_{sd}(q)$ such that the random-cluster measure with parameters (p', q) stochastically dominates the random-cluster measure with parameters $(p, 4)$. Since the former parameters are sub-critical, the latter must also be. Therefore, we obtain $p_c \geq p_{sd}(4) = 2/3$. For every $p > 2/3$, there exists $q > 4$ such that $p > p_c(q)$, a fact which implies the existence of an infinite cluster for the random-cluster measure with parameters p and q . Since $\mathbb{P}_{p,q}$ is stochastically dominated by $\mathbb{P}_{p,4}$, we deduce that there exists an infinite cluster for $q = 4$. Thus, the value p must be super-critical in a way that $p_c(4) \leq p$. The claim follows readily due to the fact that this conclusion holds for every $p > 2/3$. \square

Proof of Theorem 2: The main ingredients in our proof are Lemma 8 and several classical theorems on sharp threshold and exponential decay which can be found in [5]. In brief, we show that our estimate on the rate of decay of the 1-point function is accurate enough to deduce exponential decay.

Denote by C the cluster containing the origin: Inequality (4.9) implies that any finite power of its diameter (hence also of its number of vertices) is integrable; in particular, if we denote by $|C|$ its cardinality, we obtain that

$$\mathbb{E}_{p,q}(|C|^5) < \infty.$$

This is precisely the hypothesis needed in order to apply Theorems (5.64) and (5.66) of [5] and thus obtain exponential decay for every $p' < p$. Since this holds for every $p < p_c$, we obtain the claim of our theorem for every $q > 4$.

The case of $q = 4$ can be derived exactly as in the proof of Theorem 1, using stochastic domination between random-cluster measures. \square

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