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Mécanique statistique et criticalité en dimension deux

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Chapter I

Introduction

This document is a summary of my research up to now, presented for the *habilitation à diriger des recherches (HDR)*. It relies on the publications listed separately at the end (numbered with a label starting with ‘VB’); the works [10], [11], [12], [13] and [14] are mentioned in the main bibliography for completeness.

The unifying theme of this research is statistical mechanics in two dimensions (one-plus-one in the case of particle systems, although even then the main accent is on planar geometrical arguments as often as possible). I tried to gather the contents into mostly independent thematic chapters; in the remainder of this introduction, I briefly introduce the models I studied and their basic properties, deferring the statement of my own results to the next chapters. The aim here is not to be complete nor self-contained, but rather to put the other chapters in context; some shortcuts are silently taken, some theorems are deliberately not stated, and I apologize for that.

1. — LATTICE MODELS

1.1. — PERCOLATION

Perhaps the easiest lattice model to describe, if not to study, is bond percolation [29, 42]. Fix a parameter $p \in (0, 1)$, and in the square lattice \mathbb{Z}^2 declare each bond to be *open* with probability p and *closed* with probability $1 - p$, independently of each other. This defines a random subgraph ω of \mathbb{Z}^2 (a *configuration*) consisting of all vertices but only those edges which are open; denote by P_p the probability distribution of ω . The primary question of interest is that of the connectivity properties of ω .

Denote by $x \longleftrightarrow y$ the existence of a path of open bonds connecting x to y , and by $x \longleftrightarrow \infty$ the event that the connected component of x is infinite; the first natural observable is

$$\theta(p) := P_p[0 \longleftrightarrow \infty].$$

It is easy to show that θ is non-decreasing, thus there exists a *critical value* p_c of the parameter such that:

- If $p < p_c$, then $\theta(p) = 0$ and all connected components of ω are almost surely finite; this is called the *sub-critical regime*. Moreover, there is *exponential decay* of the two-point function [62], *i.e.* there exist $0 < c, C < \infty$ such that, for every $x \in \mathbb{Z}^2$,

$$P_p[0 \longleftrightarrow x] \leq C e^{-c\|x\|}.$$

- If $p > p_c$, then $\theta(p) > 0$ and there is almost surely a unique infinite connected component in ω , with asymptotic density $\theta(p)$; this is called the *super-critical regime*. Here there is exponential decay for finite components, *i.e.*

$$P_p[0 \longleftrightarrow x \mid 0 \not\longleftrightarrow \infty] \leq C e^{-c\|x\|}.$$

Besides, the *phase transition* is non-trivial, in the sense that $p_c \in (0, 1)$. In the case of \mathbb{Z}^2 , it is known that θ is continuous; in particular, there is no infinite component at the critical point, *i.e.* $\theta(p_c) = 0$.

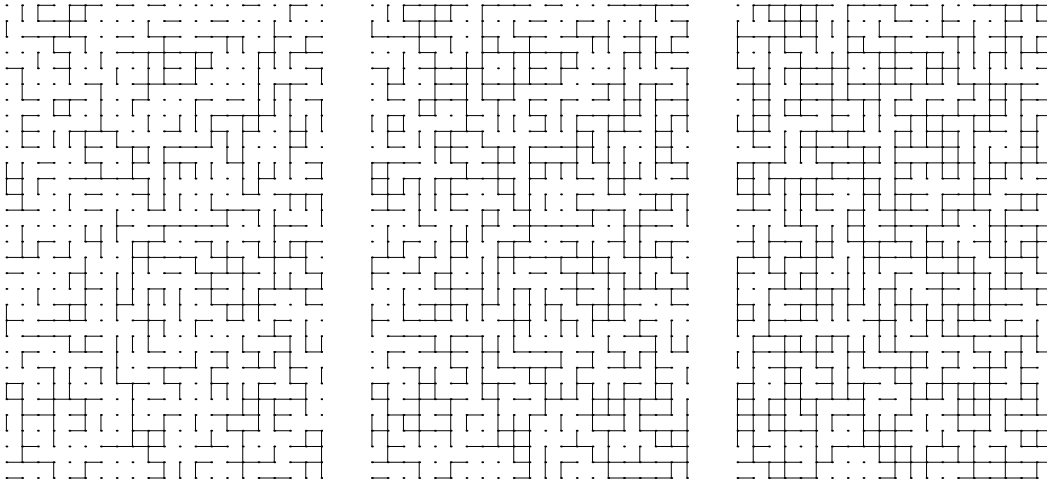


Figure I.1. — Three configurations of percolation in a rectangle (left: $p < p_c$; center: $p = p_c$; right: $p > p_c$).

On the other hand, there are still rather large clusters at criticality, due to what is known as *Russo-Seymour-Welsh theory*, or RSW for short [73, 77]. For positive integers a and b , let $LR(a, b)$ be the event that, within the rectangle $[0, a] \times [0, b]$, there is a path of open edges connecting the opposite vertical sides. One form in which RSW can be used is the following, known as the *box-crossing property* (BCP for short): for every $\lambda > 0$, there exists $\eta(\lambda) \in (0, 1)$ such that, for every n large enough,

$$\eta(\lambda) \leq P_{p_c}[LR(n, \lambda n)] \leq 1 - \eta(\lambda).$$

A consequence of that is the fact that the largest component in a box of size n has diameter of order n , while it is logarithmic in the sub-critical phase.

A celebrated result of Kesten [41] is that, in the case of the square lattice, p_c is equal to $1/2$. The main argument uses *planar duality*. The dual graph of \mathbb{Z}^2 is another copy of \mathbb{Z}^2 ; one can define a percolation model on it by declaring a dual edge open (resp. closed) if the corresponding primal edge is closed (resp. open). Starting from percolation at parameter p , one thus gets percolation at parameter $1 - p$, and the natural guess is that phase transition occurs simultaneously for both models, leading to the relation

$$p_c = 1 - p_c.$$

Going from there to a proof is far from easy; we will come back to the argument in Chapter II.

There is one variant of percolation which is natural to mention here, namely *site percolation* on the triangular lattice \mathbb{T} . Here, instead of putting the randomness on the

edges of the underlying graph, each vertex is declared open with probability p , and one looks for paths and connected components of open vertices. The same results hold, including Kesten's: one still has $p_c = 1/2$, even though the duality argument is a bit different and relies on "color swapping" rather than planar graph duality.

This particular instance of percolation is of special importance because it is the only one (to date) for which the existence of *scaling limits* is known. Here, the main step is Smirnov's proof [78] of the convergence of $P_{p_c}[LR(n, \lambda n)]$, as $n \rightarrow \infty$, to an explicit limit $f(\lambda)$ known as *Cardy's formula* [20], which implies the *conformal invariance* of the limit. More on that and related topics in Chapter III.

1.2. — THE RANDOM-CLUSTER MODEL

Another variant of percolation, introducing dependency between the edges, was defined by Fortuin and Kasteleyn [25] (see also [30] and references therein). Fix two parameters $p \in (0, 1)$ and $q \in [1, \infty)$; the *random cluster model with parameters p and q* can be first defined on a finite graph $G = (V, E)$ as follows: a configuration ω is again an assignment of a state, open or closed, to each edge of the graph, *i.e.* an element of $\{0, 1\}^E$, and its probability is set to be

$$P_{p,q}(\omega) := \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega)}}{Z_{p,q}}, \text{ where}$$

- $o(\omega)$ is the number of open bonds in ω ,
- $c(\omega)$ is the number of closed bonds in ω ,
- $k(\omega)$ is the number of connected components of ω , and
- $Z_{p,q}$ is the partition function, chosen to make $P_{p,q}$ a probability measure.

(The particular case $q = 1$ is exactly the same as percolation.)

Defining the model in the whole lattice \mathbb{Z}^2 can be done by taking a thermodynamical limit, choosing a sequence of finite boxes $\Lambda_n \uparrow \mathbb{Z}^2$, but this requires the choice of appropriate boundary conditions. For now, only two will be useful:

- *Free* boundary conditions are simply looking at Λ_n as a finite graph; the thermodynamical limit in that case will be denoted by $P_{p,q}^0$.
- *Wired* boundary conditions are identifying all the boundary vertices of Λ_n , or equivalently are counting all the connected components of a configuration in Λ_n which touch the boundary as a single contribution to $k(\omega)$; the limit will be denoted by $P_{p,q}^1$.

There is stochastic domination between these: if $p < p'$, then $P_{p,q}^b \prec P_{p',q}^b$ for each $b \in \{0, 1\}$, and $P_{p,q}^0 \prec P_{p,q}^1$.

In fact, it is not difficult to see from convexity arguments that for fixed q these two measures coincide for every $p \in (0, 1)$ except on an at most countable set \mathcal{D}_q . From there, one can define a critical point in the same way as before, *i.e.* such that all connected components are a.s. finite for every $p < p_c$ (and for every boundary condition) while there a.s. exists a unique infinite connected component for $p > p_c$. Moreover, $P_{p,q}^0 = P_{p,q}^1$ as soon as $p \neq p_c$.

The behavior at the critical point is not well understood; on the square lattice \mathbb{Z}^2 , one expects that the nature of the phase transition depends on the value of q :

- If $q \leq 4$, it is expected to be of *second order*, meaning that one would still have $P_{p_c,q}^0 = P_{p_c,q}^1$ and there would be no infinite component at the critical point;
- If $q > 4$, it is expected to be of *first order*, where $P_{p_c,q}^0 \neq P_{p_c,q}^1$, and there would be no infinite component at criticality for the free measure and a unique infinite component for the wired measure.

This is so far only rigorously known to hold in the case of large q (specifically, for $q > 25.72$, see [46, 45]).

Like in the case of percolation, there is a duality relation between random cluster models [44]. Applying the same construction as above starting from the measure $P_{p,q}^b$, for $b \in \{0, 1\}$, one gets the dual measure

$$\left(P_{p,q}^b\right)^* = P_{p^*,q}^{1-b} \quad \text{where} \quad \frac{p^*p}{(1-p^*)(1-p)} = q. \quad (\text{I.1})$$

In other words, q is unchanged, the boundary condition is reversed, and p gets transformed by a non-trivial involution. Still as above, this leads to the definition of the *self-dual point* as the fixed point of the involution:

$$p_{sd} = \frac{\sqrt{q}}{1 + \sqrt{q}}$$

and to the prediction that p_c should be equal to p_{sd} . The proof of that prediction will be the main focus of Section 1 of Chapter II.

1.3. — THE ISING AND POTTS MODELS

The reason for the particular dependency structure of the random cluster model comes from its relation with the Potts model. Let $q \geq 2$ be an integer, and let again $G = (V, E)$ be a finite graph; let $\beta > 0$. The *Potts model with q colors and inverse temperature β* on G is the measure $\mu_{q,\beta}$ on $\{1, \dots, q\}^V$ defined by the Hamiltonian and Gibbs formula

$$H_{q,\beta}(\sigma) = \sum_{x \sim y} \mathbb{1}_{\sigma_x \neq \sigma_y}, \quad \mu_{q,\beta}(\{\sigma\}) = \frac{1}{Z_{q,\beta}} e^{-H_{q,\beta}(\sigma)}.$$

The case $q = 2$ corresponds exactly to the Ising model, which is most likely the best known model of statistical physics.

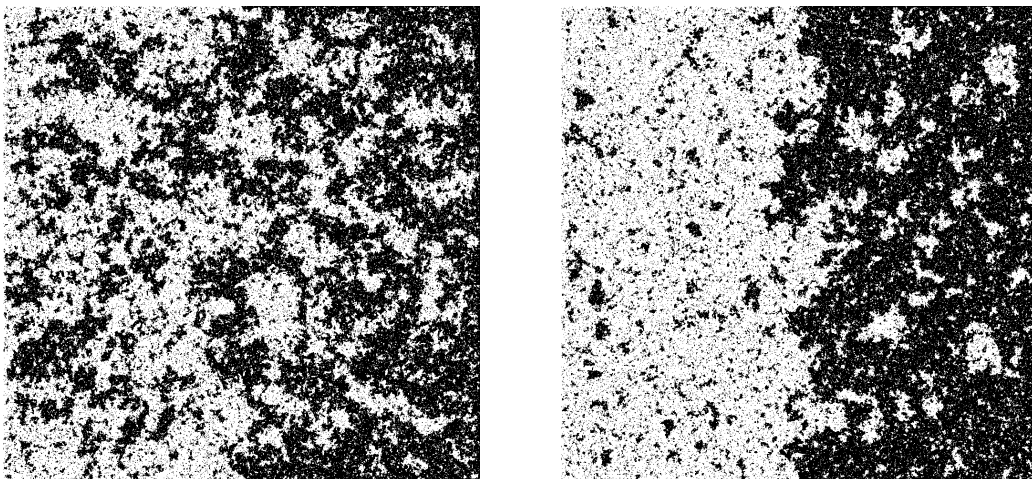


Figure I.2. — Two configurations of the Ising model in a large, square box (left: critical temperature; right: low temperature).

The relation between the Potts model and the random cluster model is given by the Edwards-Sokal coupling. Start with a configuration ω of the random cluster model with

parameters p and q for integer q , on a finite graph. For each connected component of ω , pick an integer uniformly in $\{1, \dots, q\}$, independently, and assign it as the color of all the vertices in the cluster. This leads to a coloring $\sigma \in \{1, \dots, q\}^V$, and it is easy to check that the distribution of σ is exactly given by the Potts model with q colors and inverse temperature

$$\beta = -\log(1 - p).$$

This relates the two-point function of the Potts model to connection probabilities in the random cluster model, namely

$$\mu_{q,\beta}[\sigma_x = \sigma_y] - \frac{1}{q} = \frac{q-1}{q} P_{p,q}[x \longleftrightarrow y].$$

This in turn makes the determination of the critical point for both models equivalent, and the fact that $p_c = p_{sd}$ for the random cluster model has the immediate corollary that

$$\beta_c(q) = \log(1 + \sqrt{q}).$$

1.4. — THE DIVIDE-AND-COLOR MODEL

The third bond model that will appear in this thesis is Häggström's divide-and-color model [34]. It is constructed, say on \mathbb{Z}^2 , as follows. Let $p \in [0, 1]$ and $r \in (0, 1)$ be two parameters. First construct a bond percolation configuration with parameter p . This creates a countable family of clusters; for each of them, color all of its vertices with the same color, black with parameter r and white with parameter $1 - r$, independently of the other clusters (this is in a way similar to the Edwards-Sokal construction, with bond percolation instead of the random cluster model as the underlying bond measure).

This can be seen as a dependent site-percolation model; let $\theta(p, r)$ be the probability that the origin lies in an infinite component of black vertices (using all the bonds of \mathbb{Z}^2). The phase diagram of the model is easy to predict (but not trivial to obtain formally):

- If $p > p_c$, then the underlying configuration a.s. has an infinite cluster, which is black with probability $r > 0$, so $\theta(p, r) > 0$ — and the probability that there is an infinite cluster of black vertices is equal to r ;
- If $p = p_c$, then all the underlying bond clusters are finite, and infinitely many of them contain loops around the origin (this is a consequence of Russo-Seymour-Welsh theory). At least one will be colored white, so $\theta(p, r) = 0$;
- If $p < p_c$, then there is a critical coloring parameter $r_c(p) \in (0, 1)$ such that
 - For $r > r_c(p)$, $\theta(p, r) > 0$ and there a.s. exists an infinite black cluster, while
 - For $r < r_c(p)$, $\theta(p, r) = 0$.

2. — SCHRAMM-LOEWNER EVOLUTION

Schramm-Loewner Evolution, or SLE for short, was introduced by Oded SCHRAMM [75] as a candidate to be the scaling limit of many critical two-dimensional models. A full introduction would go well beyond the scope of this introduction, and there are places where this is done much better than I could [87]. I will still give a summary of basic results and properties, mainly for notational consistency.

Let $\kappa > 0$, and let $(B_t)_{t \geq 0}$ be a standard real-valued Brownian motion. Chordal SLE from 0 to ∞ in the upper-half plane \mathbb{H} is defined as the solution of Loewner's differential equation in \mathbb{H} with driving function $\beta_t := B_{\kappa t}$, namely:

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \beta_t},$$

with initial condition $g_0(z) = z$. For every $t > 0$, this defines a compact subset K_t of $\overline{\mathbb{H}}$ and conformal map $g_t : \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$.



Figure I.3. — An SLE process with parameter $\kappa = 2$.

If Ω is a simply connected strict subset of the complex plane and a and b are two prime ends of Ω , there exists a conformal map $\Phi_{\Omega,a,b} : \Omega \rightarrow \mathbb{H}$ mapping a to 0 and b to ∞ . SLE from a to b in Ω is defined as the inverse image of SLE in \mathbb{H} by the map $\Phi_{\Omega,a,b}$. It is well-defined up to a time reparametrization, due to the existence of a one-parameter family of such conformal maps.

A consequence of that definition is the *conformal invariance* of SLE: given two domains Ω and $\tilde{\Omega}$, and boundary points a and b on $\partial\Omega$, \tilde{a} and \tilde{b} on $\partial\tilde{\Omega}$, there exists a conformal map $\Psi_{\Omega,a,b}^{\tilde{\Omega},\tilde{a},\tilde{b}}$ mapping Ω to $\tilde{\Omega}$, a to \tilde{a} and b to \tilde{b} ; and SLE from \tilde{a} to \tilde{b} in $\tilde{\Omega}$ is the image of SLE from a to b in Ω through the map $\Psi_{\Omega,a,b}^{\tilde{\Omega},\tilde{a},\tilde{b}}$.

The main feature of SLE which we will need is the existence of the *trace*: there a.s. exists a continuous curve $\gamma : \mathbb{R}_+ \rightarrow \overline{\mathbb{H}}$ such that, for every $t > 0$, the unique unbounded connected component of $\overline{\mathbb{H}} \setminus \gamma_{[0,t]}$ is exactly H_t . The trace exhibits two phase transitions:

- If $0 < \kappa \leq 4$, then γ is a.s. a simple curve, and for every $t > 0$, $\gamma_t \in \mathbb{H}$. In particular $H_t = \mathbb{H} \setminus \gamma_{[0,t]}$;
- If $4 < \kappa < 8$, then γ a.s. has double points and a.s. touches the real axis infinitely many times, but it still has measure 0. In that case $H_t \subsetneq \mathbb{H} \setminus \gamma_{[0,t]}$;
- If $8 \leq \kappa$, then γ is a.s. a Peano curve, *i.e.* for every $z \in \overline{\mathbb{H}}$ there exists $t \geq 0$ such that $z = \gamma(t)$. In that case, again $H_t = \mathbb{H} \setminus \gamma_{[0,t]}$.

SLE has been proved to be the scaling limit of several critical discrete models (in a stronger or weaker sense depending on the setup), including

- For $\kappa = 2$: the loop-erased random walk;
- For $\kappa = 3$: the Ising model;

- For $\kappa = 4$: the harmonic explorer and level lines of the Gaussian free field;
- For $\kappa = 6$: percolation;
- For $\kappa = 16/3$: the random cluster model for $q = 2$;
- For $\kappa = 8$: the contour line of the uniform spanning tree.

One of the main goals is then to recover information about the discrete models themselves from features of SLE; for instance, the values of some critical exponents can be computed using SLE tools [80] — though a few exponents seem to remain inaccessible that way, such as the *backbone exponent* of percolation (technically, one can still write a PDE, the explicit solution of which would provide the value of the backbone exponent, so the situation is not that far from the others; but finding such a solution seems to be intractable).

More on the backbone exponent and some others in Chapter II; SLE will be the main focus of Chapter III.

3. — OTHER MODELS

In what remains of the introduction, I quickly introduce two models which are not directly related to 2D lattice statistical physics; results on those are gathered in Chapter IV.

3.1. — INTERACTING PARTICLE SYSTEMS

The Totally Asymmetric Simple Exclusion Process, or TASEP for short, is defined as follows. Let $\Omega := \{0, 1\}^{\mathbb{Z}}$ be the space of configuration; here, the symbol 0 (resp. 1) is meant to represent the absence (resp. presence) of a particle at a given site. Each particle carries a Poisson clock of intensity 1, and whenever its clock rings, it attempts a jump to the right; the jump is performed if the right neighboring site is empty, and suppressed if it is already occupied. In other words, the generator of the process can be written, for $f : \Omega \rightarrow \mathbb{R}$, as

$$[Lf](\eta) = \sum_{x \in \mathbb{Z}} \mathbb{1}_{\eta_x=1} \mathbb{1}_{\eta_{x+1}=0} [f(\eta^{x \rightarrow x+1}) - f(\eta)]$$

(where $\eta^{x \rightarrow x+1}$ denotes the configuration after the jump is performed).

The translation-invariant stationary measures on Ω are classified: they are exactly the products of Bernoulli measures. For $\lambda \in (0, 1)$, let μ_λ be the product of $\mathcal{B}(\lambda)$, and start the dynamics from a configuration distributed according to μ_λ . Let X_t be the location at time t of the particle initially closest to the right of the origin, and let N_t be the number of particles which have jumped from 0 to 1 by time t . Both of these quantities exhibit strong laws of large numbers: as $t \rightarrow \infty$,

$$\frac{X_t}{t} \rightarrow 1 - \lambda \quad \text{and} \quad \frac{N_t}{t} \rightarrow \lambda(1 - \lambda).$$

While TASEP is well understood, even slight modifications of it remain mostly beyond the reach of known methods. In particular, breaking translation invariance also breaks the classification of invariant measures in general. One particular such modification of the dynamics leads to the so-called *slow-bond problem*: simply make the rate of the Poisson clock carried by each particle equal to $1 - \varepsilon$ when the particle sits at the origin (and equal to 1 elsewhere — the perturbation is local). It has been shown that the above two laws of large numbers remain valid for $\varepsilon < 0$, but it is easy to see that the limit of N_t/t , if it exists, cannot be larger than $1 - \varepsilon$, so there is a regime where it is different from its unperturbed value $\lambda(1 - \lambda)$.

In that case, the system builds a zone of higher density $\lambda_+ > \lambda$ (a “traffic jam”) to the left of the origin, and a zone of lower density $\lambda_- < \lambda$ to the right, both of which grow linearly in time. A back-of-the-envelope computation assuming that the measure is still product leads to the equations

$$\lambda_+(1 - \lambda_+) = \lambda_-(1 - \lambda_-) = (1 - \varepsilon)\lambda_+(1 - \lambda_-)$$

which can be solved explicitly; for a given value of ε , the regime behaves as described above, with two moving shocks, when $\lambda \in (\lambda_-, \lambda_+)$.

Of course the assumption that the product structure is preserved fails as soon as $\varepsilon > 0$; but still, it is reasonable to predict that, for each value of λ , there is a critical $\varepsilon_c(\lambda)$ above which such shocks appear, and that the behavior of ε_c as a function of λ matches (at least approximately) the above computations. In particular, the main conjecture is that $\varepsilon_c(\lambda)$ approaches 0 as $\lambda \rightarrow 1/2$ — the physical interpretation being that at $\lambda = 1/2$ the flow of particle is maximized, which makes the system most sensitive to a local slow-down.

3.2. — THE PRUDENT RANDOM WALK

The last model I wish to mention here is called the (*kinetic prudent random walk*); it is a particular self-interacting process for which an exact study turns out to be possible. The definition is quite simple: in the lattice \mathbb{Z}^2 , start a discrete-time process X at the origin at time 0; assuming that its path is known up to time n , its next step is chosen uniformly among the neighbors v of X_n such that the half-line $[X_n, v)$ does not intersect the past $X_{[0, n-1]}$ of the walk. While simple, the process has a non-trivial ballistic behavior, with a random limit, and exhibits aging as $n \rightarrow \infty$.

4. — STRUCTURE OF THE DOCUMENT

Chapter II is concerned with the behavior of the above-mentioned discrete 2D models in statistical physics, mostly at or around their critical point. The most notable result of the chapter is the determination of the critical point of the random-cluster and Potts models on the square lattice \mathbb{Z}^2 ([VB7], joint work with Hugo DUMINIL-COPIN). Other papers presented are [VB4], [VB1], [VB8] and [VB10].

Chapter III is the continuous counterpart of Chapter II, and focuses on Schramm-Loewner evolution (SLE) and more generally on conformal invariance. The main result here is the computation of the Hausdorff dimension of the SLE curve (papers [VB3] and [VB5], generalizing a partial result in my PhD Thesis [10]), although the considerations about natural choices of embeddings ([VB6]) are — I believe — of equal interest, if still at the speculative phase. Also mentioned is [VB2].

Chapter IV gathers works on the slow-bond problem ([VB11] and [VB12], joint with Vlaslas SIDORAVICIUS, Herbert SPOHN and Maria Eulália VARES) and the prudent random walk ([VB9], joint with Sacha FRIEDLI and Yvan VELENIK). It ends with an ongoing project with Vlaslas SIDORAVICIUS about a new approach to the study of once-reinforced random walks.

Percolation and the Random Cluster Model

1. — THE CRITICAL POINT OF THE RANDOM CLUSTER MODEL

I present in this section the main result obtained in [VB7]:

Theorem 1 (B., Duminil-Copin). *For every $q \geq 1$, the critical point $p_c(q)$ of the random cluster model on the square lattice \mathbb{Z}^2 with cluster weight q is equal to its self-dual point $p_{sd}(q)$, namely:*

$$p_c(q) = p_{sd}(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$

Moreover, for every $p < p_c(q)$, there is exponential decay of the two-point function, i.e. there exist $0 < c(p), C(p) < \infty$ such that, for every x and y ,

$$P[x \longleftrightarrow y] \leq C(p)e^{-c(p)\|y-x\|}.$$

Corollary 1. *For every $q \geq 2$, the critical inverse temperature of the Potts model on the square lattice \mathbb{Z}^2 is equal to*

$$\beta_c(q) = \log(1 + \sqrt{q}).$$

A rigorous derivation of the critical point was previously known in three cases:

- For $q = 1$, the model is simply bond percolation, proved by Kesten in 1980 [41] to be critical at $p_c = 1/2$;
- For $q = 2$, the self-dual value corresponds to the critical temperature of the Ising model, as first derived by Onsager in 1944 [65]; for modern proofs in that case, see [2] or the short argument in [VB8], also briefly mentioned below;
- Finally, for sufficiently large q , a proof is known based on the fact that the random-cluster model exhibits a first order phase transition (see [45, 46], the proofs are valid for q larger than 25.72). Physicists derived the critical temperature for the Potts models with (integer) $q \geq 4$ in 1978, using non-geometric arguments based on analytic properties of the Hamiltonian [35].

The overall strategy of our proof is quite close to that of Kesten's, but at every step one needs to take care of the dependencies in the model, especially as far as boundary conditions are concerned (in particular, the box-crossing property is not expected to hold for free boundary conditions if $q > 4$).

1.1. — DUALITY AND THE GENERAL STRATEGY

As mentioned already in the introduction, the core argument of the proof is planar duality: given a configuration in the planar lattice \mathbb{Z}^2 distributed according to the random cluster model with parameters q and p , and either free or wired boundary condition, its dual configuration is distributed according to the random cluster model as well, with the same cluster weight q , parameter p^* as given in (I.1) and the “opposite” boundary condition.

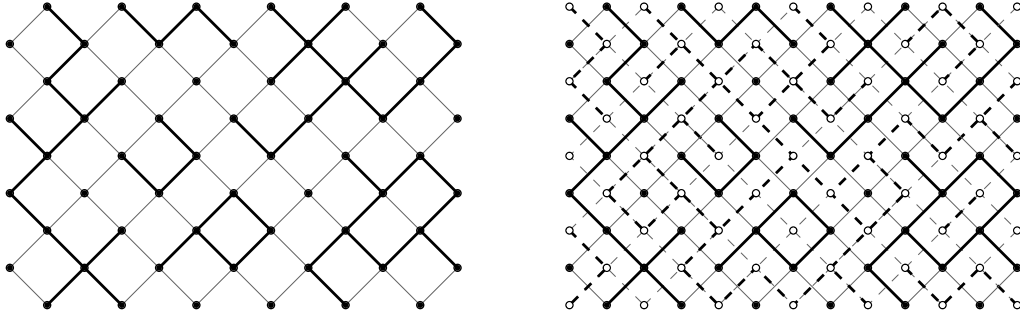


Figure II.1. — Duality for bond models on \mathbb{Z}^2 .

One bound on the critical point is given by what is known as *Zhang’s argument*, based on the impossibility of coexistence between an infinite primal and an infinite dual cluster. The argument goes as follows (see Figure II.2).

1. Show that if an infinite open cluster exists, it is necessarily unique. This is a direct consequence of the fact that the graph \mathbb{Z}^2 is *amenable*, the argument by Burton and Keane in the case of percolation [18] still applies with no modification.
2. Fix $p > p_c(q)$, so that there is indeed an infinite open cluster; for n large enough, the box $[0, n]^2$ is connected to infinity by an open path with probability at least $1 - \varepsilon$. By invariance of the model under rotations of angle $\pi/2$, together with the “square-root trick”, the top side of the box is then itself, with probability at least $1 - \varepsilon^{1/4}$, connected to infinity by an infinite open path which does not enter the box. The application of the square-root trick requires the FKG inequality, which holds for every $q \geq 1$.
3. Assume that p^* is also larger than $p_c(q)$. This implies the existence, with large probability, of dual-open paths connecting the same box to infinity, which in turn shows that the probability of existence of either two infinite clusters or two infinite dual ones is positive as soon as $4\varepsilon^{1/4} < 1$, thus leading to a contradiction.

Hence for every $p > p_c$ we have $p^* \leq p_c$, which shows that $p_c \geq p_{sd}$.

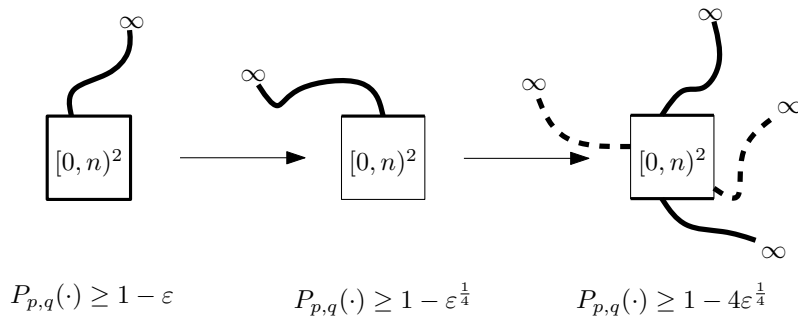


Figure II.2. — Zhang’s argument.

One word of caution is in order here. During the proof below, we will have to consider the random cluster model with periodic boundary conditions, or equivalently the random cluster model on the torus (with no boundary condition since there is no boundary anymore). Here, duality does not hold; however, we were able to introduce a new form of “almost duality” which does hold, in the following sense.

Recall that if ω is a configuration, $o(\omega)$ stands for the number of open bonds in ω , $c(\omega)$ for the number of closed bonds and $k(\omega)$ for the number of connected components of ω ; we will now define an additional parameter $\delta(\omega)$. Call a (maximal) connected component of ω a *net* if it contains two non-contractible simple loops of different homotopy classes, and a *cycle* if it is non-contractible but is not a net. Notice that every configuration ω can be of one of three types:

- One of the clusters of ω is a net. Then no other cluster of ω can be a net or a cycle. In that case, we let $\delta(\omega) = 2$;
- One of the clusters of ω is a cycle. Then no other cluster can be a net, but other clusters can be cycles as well (in which case all the involved, simple loops are in the same homotopy class). We then let $\delta(\omega) = 1$;
- None of the clusters of ω is a net or a cycle. We then let $\delta(\omega) = 0$.

This parameter transforms in a simple way under duality: $\delta(\omega) = 2 - \delta(\omega^*)$. The same proof as that of usual duality, based on Euler’s formula and taking the additional topology into account, then leads to the relation

$$(P_{p,q,n}^p)^*(\{\omega\}) \propto q^{1-\delta(\omega)} P_{p^*,q,n}^p(\{\omega\}). \quad (\text{II.1})$$

This means that even though the dual model of the periodic boundary condition FK model is not exactly an FK model at the dual parameter, it is absolutely continuous with respect to it and the Radon-Nikodym derivative is bounded above and below by constants depending only on q . Another way of stating the same result would be to define a *balanced FK model* with weights

$$\tilde{P}_{p,q,n}^p(\{\omega\}) := \frac{(\sqrt{q})^{1-\delta(\omega)}}{Z} P_{p,q,n}^p(\{\omega\}) :$$

this one is absolutely continuous with respect to the usual FK model and does satisfy exact duality.

1.2. — THE RUSSO-SEYMOUR-WELSH APPROACH

Now we want to prove that for every $p > p_{sd}$, the random cluster model does exhibit infinite clusters; this will provide the converse inequality. The strategy is made of two ingredients: the box-crossing property (BCP) at the self-dual point, and a sharp-threshold argument.

— *The box-crossing property for periodic boundary conditions* —

The main point here is to show that the model at the self-dual point is already “almost super-critical”:

Theorem 2 (BCP for the RCM). *Let $\alpha > 1$ and $q \geq 1$: there exists $c(q, \alpha) > 0$ such that for every $m > \alpha n > 0$,*

$$P_{p_{sd},q,m}^p(LR(\alpha n, n)) \geq c(q, \alpha),$$

where $P_{p_{sd},q,m}^p$ denotes the random-cluster model on the torus $(\mathbb{Z}/m\mathbb{Z})^2$.

(Notice that the same will automatically hold as well for wired boundary conditions.)

As in the case of percolation, one first proves the theorem for $\alpha = 1$ using duality. We would have $c(q, 1) = 1/2$ if we had exact self-duality, because the event $LR(n, n)$ is dual to its complement; otherwise said, we would get a constant $1/2$ if the theorem was stated with the balanced measure rather than the usual one. Because of absolute continuity, here we obtain $c(q, 1) = (1 + q^2)^{-1}$.

The same duality argument extends to symmetric domains with alternating free and wired boundary conditions, as shown in Figure II.3. Again, care needs to be taken concerning boundary conditions: the dual measure of this is the random-cluster measure in the same domain, still alternating boundary conditions, but the wired parts of the boundary wired together rather than separately (which makes the computation of $k(\omega)$ differ by at most 1 and changes the value of the constant from $c(q, 1)$ to $c(q, 1)/(1 + q^2)$).

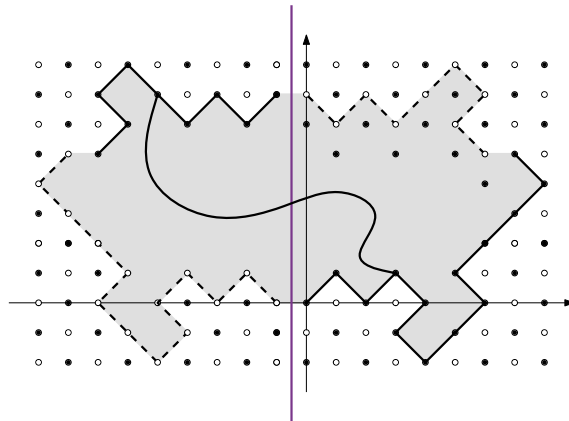


Figure II.3. — The box-crossing property for symmetric domains.

Extending the bound to every α is the “Russo-Seymour-Welsh step”; as usual the difficult step is to show the existence of $c(q, \alpha)$ for *some* $\alpha > 1$, typically $\alpha = 3/2$. The exact construction is rather tedious to describe (see [VB7] for details), but the idea is quite natural: in a $3n/2 \times n$ rectangle, one can fit two extremal, overlapping squares; each one is crossed with positive probability (solid lines in the figure), and given that they are, connecting the corresponding paths “costs” at most as much as crossing a symmetric domain with alternating boundary conditions. Going from $\alpha = 3/2$ to other values can then be done in the same way as for percolation, using the FKG inequality to “glue” crossings of boxes to each other.

Remark. The proof here only works for periodic and “larger” boundary conditions. The question remains open when the result itself extends to general boundaries, or equivalently to free boundary conditions. The general expectation is that it does for $q < 4$ and does not for $q > 4$, due to the different nature of the phase transition. However, a proof is only known in the cases $q = 1$ (percolation) and $q = 2$ (the FK-Ising model — see [22]).

— The sharp-threshold step —

Getting from the box-crossing property to the bound we want on p_c is now a matter of showing that, for every $p > p_{sd}$ (with wired or periodic boundary conditions), large boxes are crossed with probability close to 1. In the case of percolation, this can be done using Russo’s formula,

$$\partial_p P_p(LR(m, n)) = E_p[\mathcal{P}(LR(m, n))]$$

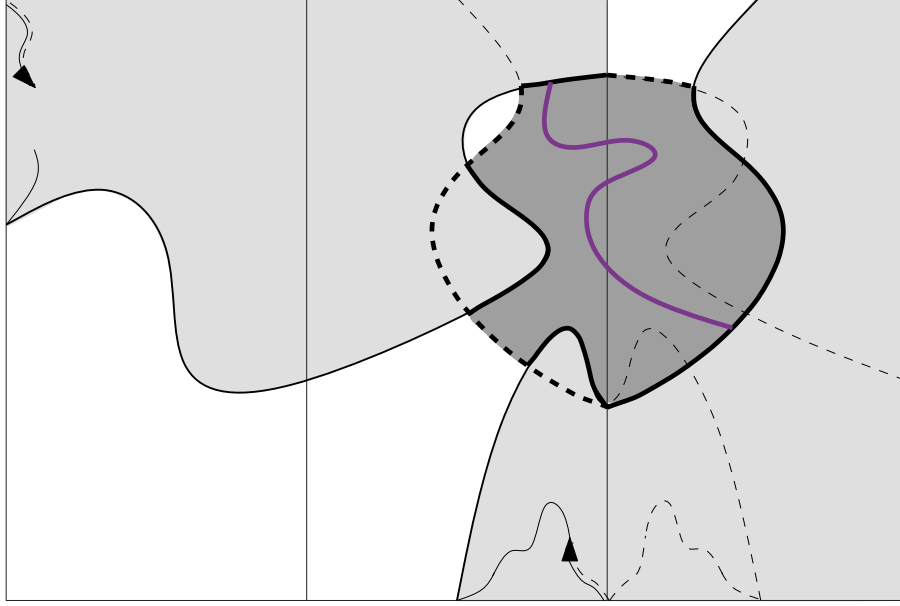


Figure II.4. — The main step in the RSW approach.

where, for an increasing event A , $\mathcal{P}(A)$ denotes the number of *pivotal bonds* for A , i.e. the number of bonds e in a configuration such that changing the state of e changes whether A is realized or not. One can then use the box-crossing property to estimate the number of pivotal bonds, and show that the above derivative is very large (when mn is) whenever $P_p(LR(m, n))$ is close to neither 0 or 1.

In the case of the random-cluster model, one needs additional tools; these come in the shape of recent sharp-threshold results by Graham, Grimmett and Piza. The first replaces Russo's formula with a bound involving *influences*:

Proposition 1 (Grimmett, Piza [33]). *Let $q \geq 1$ and $\varepsilon > 0$; there exists $c = c(q, \varepsilon) > 0$ such that for any random-cluster measure $P_{p,q,G}^\xi$ with $p \in [\varepsilon, 1 - \varepsilon]$ and any increasing event A ,*

$$\partial_p P_{p,q,G}^\xi(A) \geq c \sum_{e \in E} I_A(e),$$

where the influence of the edge e on the event A is defined as

$$I_A(e) := P_{p,q,G}^\xi(A | \omega_e = 1) - P_{p,q,G}^\xi(A | \omega_e = 0).$$

As it turns out, any increasing event of decent probability has at least one edge with an influence which is not too small:

Proposition 2 (Graham, Grimmett [27, 28]). *Let $q \geq 1$ and $\varepsilon > 0$; there exists a constant $c = c(q, \varepsilon) \in (0, \infty)$ such that the following holds. Consider a random-cluster model on a graph $G = (V, E)$. For every $p \in [\varepsilon, 1 - \varepsilon]$ and every increasing event A , there exists $e \in E$ such that*

$$I_A(e) \geq c P_{p,q,G}^\xi(A) (1 - P_{p,q,G}^\xi(A)) \frac{\log |E|}{|E|}.$$

These two results can be combined most efficiently if G is the torus $(\mathbb{Z}/m\mathbb{Z})^2$ and the event A is invariant under all automorphisms of G , because then all influences are

equal implying that the lower bound on $\partial_p P_{p,q,A}^\xi(A)$ is at least of order $\log m$. Now all that remains to be done is to put the pieces together.

Fix $\alpha > 1$; for $n > 0$, let $m = \alpha^2 n$ and A_n be the event that at least one of the $m^2 = \alpha^4 n^2$ translates of the box of size $\alpha^2 n \times n/\alpha$ in the torus $(\mathbb{Z}/m\mathbb{Z})^2$ is crossed horizontally by an open path. The box-crossing property and duality imply the existence of $\eta > 0$, depending only on α and q , such that for all $n > 0$,

$$P_{p_{sd},q,m}^p[A] \in [\eta, 1 - \eta].$$

We can then integrate the above differential inequality to obtain, for every $p > p_{sd}$, a bound of the form

$$P_{p,q,m}^p[A] \geq 1 - Cn^{-c.(p-p_{sd})}.$$

On the other hand, one can find a family of $4\alpha^3$ overlapping translates of the box of size $\alpha n \times n$, set up according to a lattice of spacing $(\alpha n/2, n/2)$, in such a way that the event A implies that one of these is itself crossed horizontally; a variant of the square-root trick then shows that

$$P_{p,q,m}^p(LR(\alpha n, n)) \geq 1 - Cn^{-c.(p-p_{sd})/4\alpha^3}.$$

From there, it is not hard to show that $\theta(p)$ is positive whenever $p > p_{sd}$, in a way similar to the conclusion of the proof of Kesten. The only difference is that boundary conditions need to be taken care of again, so the construction of large clusters needs to be performed “from the outside”, as shown in Figure II.5. The inequality $p_c \leq p_{sd}$ follows.

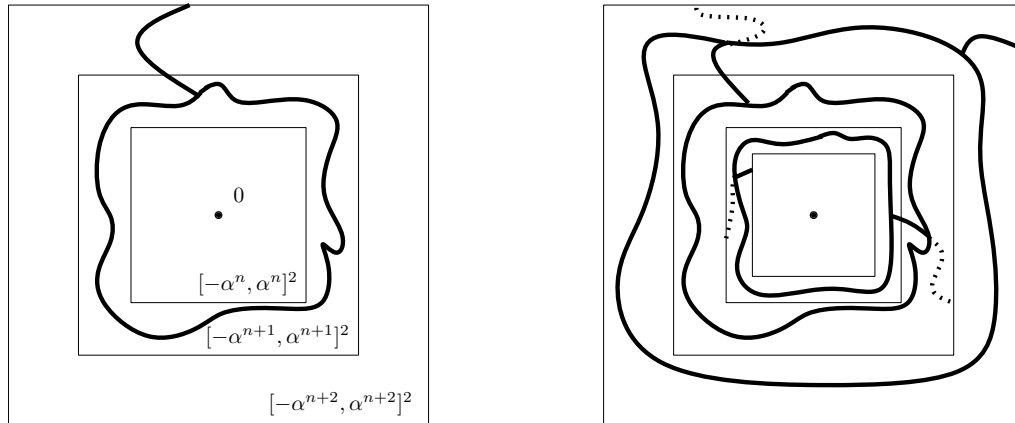


Figure II.5. — The construction of a large open cluster using box crossings.

Showing exponential decay below the critical is then done by proving that the size of the cluster \mathcal{C}_0 of the origin has all finite moments (which comes from the previous estimates and construction); it is well known that the bound $E(|\mathcal{C}_0|^d) < \infty$ is sufficient [30].

1.3. — GENERALIZATIONS

The main possible generalization of the proof described above is to extend it to other lattices. While there is no hope to make anything similar work in a non-planar case (due to constant use of planar duality), in the case of the triangular and hexagonal lattices it can be done. These are dual to each other, but they are also related *via* the star-triangle transformation which relates their critical points through a polynomial relation; this remark and a proof similar to the previous one leads to their computation:

Theorem 3 (B., Duminil-Copin [VB7]). For every $q \geq 1$, the critical point $p_c(q)$ of the random cluster model with cluster weight q satisfies

$$\begin{aligned} y_c^3 + 3y_c^2 - q &= 0 && \text{on the triangular lattice and} \\ y_c^3 - 3qy_c - q^2 &= 0 && \text{on the hexagonal lattice,} \end{aligned}$$

where $y_c := p_c/(1 - p_c)$. Moreover, there is exponential decay in the sub-critical phase.

2. — THE ISING MODEL AWAY FROM CRITICALITY

While we now know that the random cluster model exhibits exponential decay away from criticality, the correlation length is not known explicitly in general. In [VB8] we investigate the case $q = 2$ in a more quantitative and “modern” way, exploiting Smirnov’s fermionic observable. I will first give a construction of the observable on edges to fix notation, and then describe two applications: an alternate proof of Theorem 1 in the $q > 4$ case, and the study of the two-point function of the Ising model in the high-temperature phase.

2.1. — SMIRNOV’S (PARA)FERMIONIC OBSERVABLE

Let G be a discrete, simply connected, finite domain in the square lattice \mathbb{Z}^2 , and let a and b be two boundary points of G ; consider the random cluster model on G with wired boundary condition on one of the two arcs delimited by a and b and free boundary condition on the other one (we will refer to this setup as a *Dobrushin domain*). The dual model is a random cluster model in a shifted copy of G , also with (switched) Dobrushin boundary conditions.

One can encode a configuration in G as a collection of cluster boundaries in the medial graph, as shown in Figure II.6. This collection consists in a single path γ joining a and b , and a family of closed loops; γ is the central object here, and we will refer to it as the *exploration path* of the configuration. For convenience, the edges of the medial graph (in gray in the figure) can be oriented as shown; γ going from a to b then only uses edges in their positive direction.

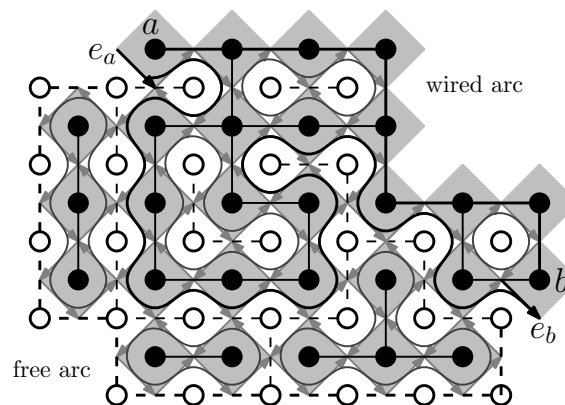


Figure II.6. — The exploration path of a bond model.

One expects the behavior of γ to contain a lot of information on the underlying model. In particular, if $p < p_c$ (resp. $p > p_c$), γ will stay close to the wired (resp. free) boundary arc, so there is a possibility to deduce the value of p_c from features of the exploration path.

One key estimate would be, for a given edge e of the medial graph, the probability that γ goes through it. Unfortunately, this does not seem to be amenable to a direct study;

Smirnov's observable is a modification of this quantity. To define it, one needs to first introduce the *winding* $W_\gamma(e)$ of γ at e : this is simply the total algebraic variation of the argument of the tangent vector of γ from its starting point a to the edge e (assuming that indeed $e \in \gamma$). The observable is then defined as

$$F(e) := E_{p,q} \left(\mathbb{1}_{e \in \gamma} e^{i\sigma W_\gamma(e)} \right),$$

where the *spin* σ is given by the relation

$$\cos \frac{\sigma\pi}{2} = \frac{\sqrt{q}}{2}$$

(σ is chosen to be real for $q < 4$ and pure imaginary for $q > 4$).

Using Euler's formula, the probability of a configuration ω can be re-expressed as a function of $o(\omega)$ and the number $\ell(\omega)$ of loops in its loop representation: namely,

$$P_{p,q}(\{\omega\}) = \frac{x^{o(\omega)} (\sqrt{q})^{\ell(\omega)}}{Z_{p,q}^{\text{loop}}}, \quad \text{where } x = \frac{p}{(1-p)\sqrt{q}}.$$

Notice that the loop picture is invariant under duality, and that this is reflected on the fact that $x = 1$ for $p = p_{sd}(q)$.

The main interest of that particular way of writing the weights of the model is that the parameters involved change in a simple way when the state of a bond is switched. Let e be a bond of the medial graph, and let v be its terminal vertex — v is identified with a bond of the primal lattice. Assume that the primal edge v is closed; assume besides that all the medial edges around it are visited by γ , e being the first one (left-hand picture in Figure II.7). Changing the configuration ω to $s(\omega)$ where v is open adds one open bond and one loop, so the weight is multiplied by $x\sqrt{q}$; but the windings at e and at the last visited edge are not modified. Besides, all the windings involved in that situation can be easily computed given $W_\gamma(e)$.

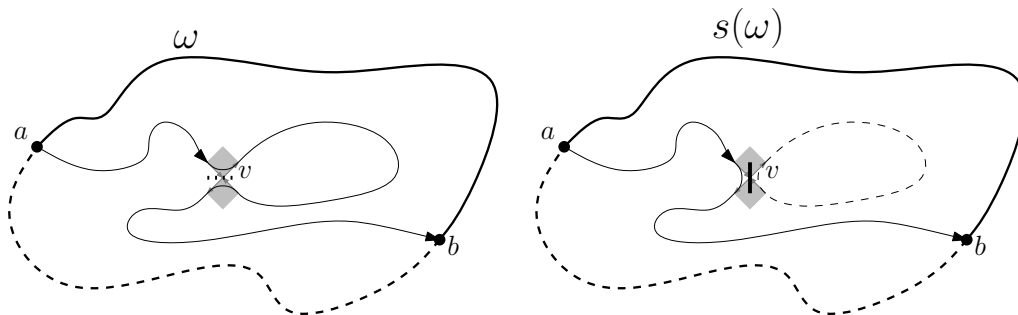


Figure II.7. — The main relation for Smirnov's observable.

From this and the particular choice of σ , one gets the following relation:

$$IN(\omega) + IN(s(\omega)) = \Lambda(x)(OUT(\omega) + OUT(s(\omega))), \quad (\text{II.2})$$

where $IN(\omega)$ (resp. $OUT(\omega)$) is the sum of the two terms of the form $\mathbb{1}_{e \in \gamma} e^{i\sigma W_\gamma(e)}$, e pointing towards (resp. away from) v , and where

$$\Lambda(x) = \frac{e^{i\sigma\pi/2} + x}{e^{i\sigma\pi/2}x + 1}.$$

Taking then the expectation over ω , one obtains the main equation

$$\sum_{\bar{e}=v} F(e) = \Lambda(x) \sum_{\underline{e}=v} F(e) \quad (\text{II.3})$$

(where, for a given medial edge e , \underline{e} is its origin and \bar{e} its target). Notice again the special role of the self-dual point: there, $\Lambda(x) = 1$ and as a consequence F can be seen as a divergence-free flow (and the relation (II.3) as a conservation law).

2.2. — ANOTHER PROOF OF THEOREM 1 ($q > 4$)

The easiest use of Smirnov's observable concerns the case $q > 4$ and $p \neq p_{sd}$. Then, σ is pure imaginary so F is real-valued, and the whole arsenal of real analysis, especially differential inequalities, becomes available. The main guideline here will be the notion of (discrete) *massive harmonic function*: if Ω is a discrete domain in the square lattice, and $m > 0$ is fixed, a function $f : \Omega \rightarrow \mathbb{R}$ is said to be massive harmonic with mass m if, for every $z \in \Omega$, it satisfies

$$[\Delta f](z) = m f(z),$$

where Δ denotes the discrete Laplacian,

$$[\Delta f](z) := \frac{f(z + (1, 0)) + f(z + (-1, 0)) + f(z + (0, 1)) + f(z + (0, -1))}{4} - f(z).$$

Massive harmonic functions automatically exhibit exponential decay in their distance to the boundary. A good way of understanding why is to investigate for which $\theta = (\theta_1, \theta_2)$ the map $z \mapsto \exp(\langle \theta, z \rangle)$ is massive harmonic with mass m , which can be written as

$$\cosh \theta_1 + \cosh \theta_2 = 2m + 2 \quad (\text{II.4})$$

leading to a one-parameter family of such maps in the whole plane; for vanishing m , the above equation becomes asymptotic to $\|\theta\|_2^2 = 4m$ (so we get some kind of rotational invariance in the limit $m \rightarrow 0$ — more on that in the next chapter). If f is massive harmonic in Ω and bounded outside Ω , then there exist $c, C > 0$ such that, for every $z \in \Omega$, we have $|f(z)| \leq C \exp(-c d(z, \partial\Omega))$.

The idea would then be to apply the above to Smirnov's observable and get exponential decay of the two-point function, either in the primal lattice or in its dual, as soon as $\Lambda(x) \neq 1$, i.e. as soon as $p \neq p_{sd}$. This cannot be done directly, because the relation (II.3) is not exactly the same as massive harmonicity. The observable is however massive harmonic "in the average": summing it over a sub-domain of Ω leads to the following

Proposition 3. *Let Ω be a finite Dobrushin domain, let $q \neq 4$ and $p \neq p_{sd}(q)$, and let F be Smirnov's observable in Ω with these parameters: there exists $C = C(p, q) < \infty$ such that for any set of edges $A \subset \Omega$ not containing any edge adjacent to the boundary of Ω , there exists a function $\alpha : \partial A \rightarrow [-C, C]$ such that*

$$\sum_{e \in A} F(e) = \sum_{e \in \partial A} \alpha_e F(e).$$

Let z be a medial edge in Ω , let N denote its distance to the boundary of Ω , and for $n < N$ let S_n be the sum of F over the square Q_n of radius n centered at z . The above proposition and the positivity of F (remember that we are still in the case $q > 4$) lead to

$$S_n \leq C(S_{n+1} - S_n) \quad \text{or equivalently} \quad S_n \leq \frac{C}{1+C} S_{n+1}.$$

Iterating this N times leads to the bound

$$f(z) \leq \left(\frac{C}{1+C}\right)^N S_N \leq C \left(\frac{C}{1+C}\right)^N \sum_{e \in \partial Q_N} F(e).$$

The observable for $q > 4$ is still bounded along the boundary of a convex domain (and in general, it is always bounded along the boundary of any domain, but the bound will depend on the winding of the boundary); so if $\Omega = Q_N$, the above gives an exponential bound on it at the center. In a general domain, ∂Q_N might be far away from $\partial\Omega$, in which case we have no *a priori* bound on it, as the winding term could be quite large; however, the general idea can be made to work without too much trouble.

2.3. — THE CASE $q = 2$

The other case where the direct study of the observable is doable is the FK-Ising model, *i.e.* the case $q = 2$, for which an explicit determination of the asymptotics of the two-point function is possible:

Theorem 4 (B., Duminil-Copin [VB8]). *Let $\beta < \beta_c$ and let $\langle \cdot \rangle_\beta$ denote the (unique) infinite-volume Ising measure at inverse temperature β on the square lattice \mathbb{Z}^2 ; fix $z = (x, y) \in \mathbb{L}$. Then,*

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \langle \sigma(0)\sigma(nz) \rangle_\beta = x \operatorname{argsinh}(sx) + y \operatorname{argsinh}(sy),$$

where $s > 0$ solves the equation

$$\sqrt{1 + (sx)^2} + \sqrt{1 + (sy)^2} = \sinh 2\beta + \frac{1}{\sinh 2\beta}.$$

The main step consists in considering massive harmonic functions in the punctured plane $\mathbb{Z}^2 \setminus \{(0, 0)\}$. The natural way to get such a function is as the green function of a *massive random walk*: for $z \in \mathbb{Z}^2$, let (X_n) denote a simple random walk started at z , and let τ denote its first hitting time of the origin. Give the particle a (deterministic) mass $m_n := (1 + m)^{-n}$, and define

$$G_m(z) := E[m_\tau].$$

It is easy to check that this massive Green function is finite for all $m \in (0, 1)$ and, for all $z \neq 0$, satisfies the equation

$$G_m(z) = \frac{1}{4(1+m)} \sum_{z' \sim z} G_m(z') = \frac{1}{1+m} [\Delta G_m + G_m](z).$$

In other words, G_m is massive harmonic with mass m on $\mathbb{Z}^2 \setminus \{(0, 0)\}$.

The asymptotic behavior of G_m is then given from (II.4) by a standard Wulff duality argument: G_m is asymptotic to $g_m : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$g_m(x, y) := \inf \left\{ e^{ax+by} : (a, b) \in \mathbb{R}^2, \cosh a + \cosh b = 2m + 2 \right\}.$$

The Lagrange multiplier method gives the additional equation $x \sinh b = y \sinh a$ on the minimizing pair (a, b) , or in other words:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log g_m(nx, ny) = ax + by \quad \text{where} \quad \begin{cases} \cosh a + \cosh b = 2m + 2, \\ y \sinh a - x \sinh b = 0. \end{cases} \quad (\text{II.5})$$

The second condition ensures that there exists $s_0 \in \mathbb{R}$ such that $\sinh a = s_0 x$ and $\sinh b = s_0 y$, and the first one then provides the explicit rate of decay,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log g_m(nx, ny) = x \operatorname{argsinh} sx + y \operatorname{argsinh} sy$$

where s is the *negative* solution (so that one actually minimizes e^{ax+by}) of the equation

$$\sqrt{1 + (sx)^2} + \sqrt{1 + (sy)^2} = 2m + 2.$$

The above is not far from a proof of Theorem 4; all that remains would be to show that the two-point function is indeed massive harmonic, and that its mass is given by

$$2m + 2 = \sinh 2\beta + \frac{1}{\sinh 2\beta}.$$

There are a few points to be made here:

- The two-point function itself is *not* massive-harmonic; in fact, no local linear relation is known on it. However, we do have a local linear relation (II.2) on the fermionic observable. Going from the exponential decay of the observable to that of the two-point function itself is a matter of controlling the winding term in the definition of F (which is easy to do along domain boundaries);
- The theorem is stated in terms of the spin-spin correlation, and it can be restated in terms of connectivities in the random-cluster model, while the natural connectivities measured by F are those of the loop representation. They match along domain boundaries;
- The statement of Theorem 4 is about the bulk behavior, and the observable is defined in a most natural way in Dobrushin domains (which have boundaries).

Thus a natural intermediate step is the following. Let $z = (x, y)$ be such that both x and y are positive; define

$$\begin{aligned} L_-(0, 0) &:= \{(u, v) \in \mathbb{Z}^2 : u < 0, v < 0\}, \\ L_+(x, y) &:= \{(u, v) \in \mathbb{Z}^2 : u > x, v > y\} \text{ and} \\ \Omega(x, y) &:= \mathbb{Z}^2 \setminus (L_-(0, 0) \cup L_+(x, y)). \end{aligned}$$

One can define Dobrushin boundary conditions in $\Omega(x, y)$, with marked points at $(-1, 0)$ and $(0, 0)$, which is essentially the same as setting free boundary conditions on $\partial\Omega(x, y)$ (and in fact, up to a little nudge in the definitions, one could also merge the two boundary points at the origin). If F is the observable in $\Omega(x, y)$, then

$$|F(z)| \asymp P_{p, \Omega(x, y)}^{\text{free}}[(0, 0) \longleftrightarrow (x, y)].$$

On the other hand, by convexity arguments, the exponential decay of the last term has to be the same as that of the connection probability in the bulk. It is thus enough to estimate F in the domain $\Omega(x, y)$.

The last obstacle is that the relation (II.2) is not exactly massive harmonicity. Notice though that for $q = 2$, the spin σ is equal to $1/2$; since, by the orientation given to the medial bonds, the winding is deterministic modulo 2π , this means that the argument of F is known (modulo π), or in other words that, up to an explicit and deterministic correction, one can see F as a real-valued observable.

In a more effective way, for general q , the equation (II.2) enables the computation of $F(e)$ at a medial edge e given its value at 3 of the neighboring edges; for $q = 2$, one can

multiply the equation by $e^{i\theta}$ for a well-chosen θ , take the real part, and compute $F(e)$ from the value of F at only 2 of its neighboring bonds.

This manipulation, which is specific to the case $q = 2$, then allows to show that the restriction of F to all the medial bonds pointing to a given direction (which form a lattice isomorphic to \mathbb{Z}^2) is exactly massive harmonic in the bulk, and to conclude the proof.

A natural question here is whether, for other values of q , one could implement the same kind of reasoning. Not much stands in the way, in fact only the very last step is specific to the FK-Ising model. On the other hand, just equation-counting shows that there is a fundamental problem: (II.2) gives one complex-valued equation per medial vertex, while F has one complex value per medial bond, which makes twice as many. Whenever F is real ($q > 4$) or “morally real” ($q = 2$), this can be walked around, but in the general case, there is absolutely no hope to even evaluate F from (II.2) alone.

A credible plan is to try and build a coarse-grained version of F , *i.e.* to find a sublattice \mathbb{L} of \mathbb{Z}^2 and a function $F_{\mathbb{L}} : \mathbb{L} \rightarrow \mathbb{C}$ as a local linear combination of values of F with translation-invariant coefficients, in such a way that $F_{\mathbb{L}}$ gets massive harmonicity from (II.2). In fact, this is exactly what we just did in the case $q = 2$, simply by decimation.

If all that were made to work, a possible outcome would be that, as $p \rightarrow p_c$, the mass term would vanish (at least for $q < 4$) and as a consequence the model would exhibit rotational invariance at the critical point. Since this can be argued to be the missing step towards conformal invariance, this seems worth investigating. However, chances are that the massive harmonicity obtained this way would have to be approximate, at least for generic q , and that the technical complications would be much greater than in the FK-Ising case.

3. — MONOCHROMATIC EXPONENTS FOR PERCOLATION

Most of the contents of this chapter, up to now, dealt with the behavior of percolative systems in the plane away from their critical point, either for their own sake or as a tool to derive the value of that critical point. I would like to switch now to percolation (more precisely site percolation on the triangular lattice) taken at its critical point $p_c = 1/2$.

3.1. — QUASI-MULTIPLICATIVITY AND CRITICAL EXPONENTS

Recall that percolation in this setup satisfies the box-crossing property. For every $n > 0$, let p_n be the probability that the origin is connected to the boundary of the ball of radius n by a path of open vertices (this is called a *one-arm event*). Away from criticality, p_n would either decay exponentially (if $p < p_c$) or converge exponentially quickly to $\theta(p)$ (if $p > p_c$), but at criticality, it is easy to get polynomial bounds from both sides.

Indeed, there exists $\varepsilon > 0$ such that, for every $R > 0$, with probability at least ε , an annulus of radii R and $2R$ contains a circuit of closed vertices surrounding its center, and a $R \times 2R$ rectangle is crossed in the long direction. As soon as one such annulus around the origin contains such a closed circuit, long connections are prevented; if on the other hand a “stairway” of exponentially larger and larger such rectangle are all crossed, this builds a long connection. Since in both cases logarithmically many building blocks are needed to reach scale n , the Harris–FKG inequality leads to the bounds

$$cn^{-1/c} \leq p_n \leq c^{-1}n^{-c} \quad \text{for some } c > 0.$$

In fact a lot more can be said. Let n and n' be both larger than 2. The following remark is key: if there are

- an open path between the origin and the circle of radius n centered at the origin;
- an open path between the circle of radius n and that of radius nn' ;
- a “connector set” composed of an open path between radii $n/2$ and $2n$, an open circuit around the origin between radii $n/2$ and n and another one between radii n and $2n$,

then there is an open path between the origin and the circle of radius nn' . Moreover, the third event above has positive probability, independently of n , by the box-crossing property. On the other hand, obviously if an open path up to radius nn' exists, then the first two of these three events are automatically satisfied. If p_{n_1, n_2} denotes the probability that there is an open path between radii n_1 and n_2 (so that $p_n = p_{1, n}$), we obtain *quasi-multiplicativity*:

$$p_{1, nn'} \asymp p_{1, n} p_{n, nn'} \quad (\text{II.6})$$

(where here and in what follows, the symbol \asymp means that the ratio of the two sides is bounded above and below by positive, absolute constants).

In a continuous framework, one would expect scale invariance at the critical point, meaning in particular that p_{n_1, n_2} would only depend on the ratio n_2/n_1 . This in turn would lead to a new family of inequalities, of the form

$$p_{nn'} \asymp p_n p_{n'};$$

from there and the above *a priori* polynomial bounds, it is standard to derive the existence of an exponent α such that

$$p_n \approx n^{-\alpha}, \quad \text{i.e.} \quad \frac{\log p_n}{\log n} \rightarrow -\alpha.$$

Such an α is called a *critical exponent*, and the computation of critical exponents has been one of the great advances in the topic over the last decade. In this particular case, it is known [54] that

$$\alpha = \frac{5}{48}.$$

In a discrete setup though, no scale invariance makes sense before taking a *scaling limit* which, limited to the present context, means looking at the behavior of $p_{an, bn}$ as $n \rightarrow \infty$. The box-crossing property ensures that this quantity is bounded as a function of n ; proving that it converges as n goes to infinity is enough (besides quasi-multiplicativity) to derive the existence of α , and that convergence is a consequence of Smirnov’s proof of Cardy’s formula. Computing the value of α is a different matter, and it requires the use of SLE.

Let $j > 1$, and fix a sequence σ of j colors, each one white or black; assume that both colors appear somewhere in the sequence. One can define a j -arm event with colors given by σ as the natural generalization of the definition of p_{n_1, n_2} : we will say that the event $A_{j, \sigma}(n_1, n_2)$ is realized if, joining the circles of radii n_1 and n_2 , there exists a family of j disjoint paths, in sequential order around the origin, and each one of the color prescribed by the corresponding entry in σ .

Here, *a priori* polynomial bounds still hold, and so does quasi-multiplicativity but proving it requires quite a bit of additional work because the event $A_{j, \sigma}(n_1, n_2)$ is not increasing. Nevertheless, it can be shown from Smirnov’s result that again critical exponents exist, and their values are known [80]: as $N \rightarrow \infty$ (for fixed n large enough for the existence of the arms to be possible),

$$P[A_{j, \sigma}(n, N)] \approx N^{-\alpha_j} \quad \text{where} \quad \alpha_j = \frac{j^2 - 1}{12}.$$

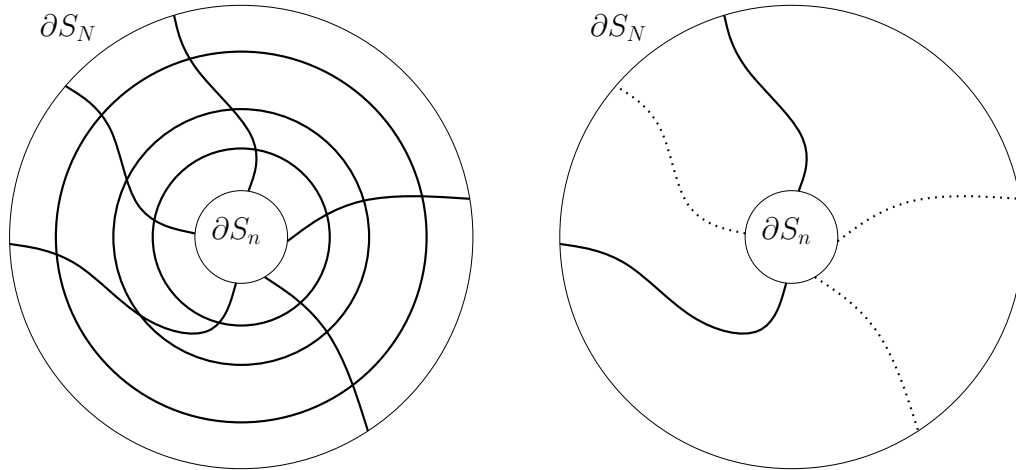


Figure II.8. — Monochromatic vs. polychromatic arm exponents ($j = 5$).

We will refer to α_j as the *polychromatic j -arm exponent*.

The fact that α_j does not depend on the exact sequence σ (as long as it contains both colors) is a consequence of the so-called *color-swapping trick*; the argument does require at least one arm of each color to be applicable, and so does the computation of the α_j using SLE techniques.

3.2. — EXISTENCE OF THE MONOCHROMATIC EXPONENTS

When σ contains only one color (be it white or black, it does not matter because of the symmetry of the model), a new family of exponents appears:

Theorem 5 (B., Nolin [VB10]). *For any $j \geq 2$, there exists an exponent $\alpha'_j > 0$ such that*

$$\pi_j(n, N) := P[A_{j,B\dots B}(n, N)] \approx N^{-\alpha'_j}$$

as $N \rightarrow \infty$ for any fixed $n \geq n_0(j)$. Moreover, for any $j \geq 2$, we have

$$\alpha_j < \alpha'_j < \alpha_{j+1}.$$

The proof of the existence of the exponent is very similar to that of the existence of the polychromatic exponents, using the existence of the scaling limit of percolation. First, fix $\lambda > 1$: convergence of percolation to its full scaling limit [19] shows the existence of $q(\lambda) > 0$ such that

$$\pi_j(n, \lambda n) \xrightarrow[n \rightarrow \infty]{} q(\lambda).$$

From quasi-multiplicativity of π_j , we obtain it for q : there exists $C > 0$ such that

$$C^{-1}q(\lambda)q(\lambda') \leq q(\lambda\lambda') \leq Cq(\lambda)q(\lambda') \text{ for all } \lambda, \lambda' > 1.$$

This in turn implies the existence of α'_j such that

$$q(\lambda) \approx \lambda^{-\alpha'_j}.$$

This is not quite the same as the corresponding claim for π_j , because we have no estimate for the speed of convergence to q above. There is a way around though, which is the same as the one used in [80].

Let $\varepsilon > 0$, and fix $n < N$. There exists $A > 0$ such that, for every $\lambda > A$,

$$\left| \frac{\log q(\lambda)}{\log \lambda} + \alpha'_j \right| < \varepsilon.$$

Let k be the largest exponent satisfying $A^k n < N$. From the quasi-multiplicativity of π_j ,

$$\pi_j(n, N) \asymp \pi_j(n, A^k n) = e^{\mathcal{O}(k)} \prod_{\ell=0}^{k-1} \pi_j(A^\ell n, A^{\ell+1} n)$$

(where the $\mathcal{O}(k)$ is uniform in A). Taking logarithms on both sides:

$$\frac{1}{k} \log \pi_j(n, N) = \mathcal{O}(1) + \frac{1}{k} \sum_{\ell=0}^{k-1} \log \pi_j(A^\ell n, A^{\ell+1} n).$$

As $\ell \rightarrow \infty$, $\pi_j(A^\ell n, A^{\ell+1} n)$ converges to $q(A)$, so by Cesàro, as $N \rightarrow \infty$ for fixed n ,

$$\frac{\log(\pi_j(n, N))}{\log N} \sim \frac{\pi_j(n, N)}{k \log A} = \frac{\log q(A)}{\log A} + \mathcal{O}\left(\frac{1}{\log A}\right).$$

If A is chosen large enough that the error term is bounded by ε , what we obtain is that for N large enough,

$$\left| \frac{\log(\pi_j(n, N))}{\log N} + \alpha'_j \right| \leq 3\varepsilon$$

which is what we wanted.

The strict inequalities $\alpha_j < \alpha'_j < \alpha_{j+1}$ are the actual core of the theorem. The second one is easier: one can see the event $A_{j+1, B \dots B W}(n, N)$ as the intersection of the events $A_{j, B \dots B}(n, N)$ and $A_{1, W}(n, N)$, one of which is increasing while the other is decreasing, so by the Harris–FKG inequality,

$$P[A_{j+1, B \dots B W}(n, N)] \leq P[A_{j, B \dots B}(n, N)] P[A_{1, W}(n, N)] \leq \pi_j(n, N) c^{-1} (n/N)^c$$

for some $c > 0$ (where as above the last inequality derives from the box-crossing property). This and the definition of the exponents lead directly to

$$\alpha'_j \leq \alpha_{j+1} - c < \alpha_{j+1}.$$

The first inequality on the other hand is a bit more interesting. The idea here is the following: given a configuration with j arms involving both colors, there is only one (up to local changes) way to chose these arms in the right order, because the location of the black arms is constrained by the existence of white arms and vice versa; while given a typical configuration with j black arms (and no white arm), there will be many black circuits around the origin and many *macroscopically different* ways to exhibit a family of j arms within it — as shown in Figure II.8.

For every choice of j colors and of $n < N$, one can look at the expected number of ways to choose j arms of the appropriate colors, witnessing for the event $A_{j, \sigma}(n, N)$. This is simply the sum, over all choices of a family of j disjoint paths, of the probability that those j paths have the right color; and written like that, it is apparent that the value of the expectation does not depend on the choice of the colors at all (since white and black have the same probability).

In other words: the expected number of witnesses is the same for all choices of colors, but one expects a given realization of the arm event to contain many more witnesses in the monochromatic case than in the polychromatic one. This would imply

$$P[A_{j,B\dots B}(n, N)] \ll P[A_{j,B\dots BW}(n, N)]$$

which, by an argument similar to the proof of existence of the exponent, is enough to obtain the other inequality $\alpha_j < \alpha'_j$.

Making the proof work along those lines would be nice, but we were not able to do it; the main reason being that polychromatic arm configurations might exhibit more *microscopically different* witnesses than monochromatic ones. This is quite unlikely, but seems very difficult to rule out convincingly — so the actual proof in [VB10] has to rely on estimates for the minimal winding angle of a witness to replace the counting.

The above considerations say nothing about the exact values of the exponents α'_j , and (apart from the special case $j = 1$) they don't seem to be computable using SLE methods. α'_2 in particular would be very interesting to obtain, as it is known as the *backbone exponent* of percolation: a vertex is in the backbone if it is connected to the boundary of a large domain by two disjoint paths, and α'_2 is a measure of how many such points there are within a large box. From numerical simulations, I got the following:

Conjecture 1. *In decreasing amount of confidence:*

$$\alpha'_2 = \frac{17}{48}; \quad \alpha'_3 = \frac{37}{48}; \quad \alpha'_4 = \frac{63}{48}.$$

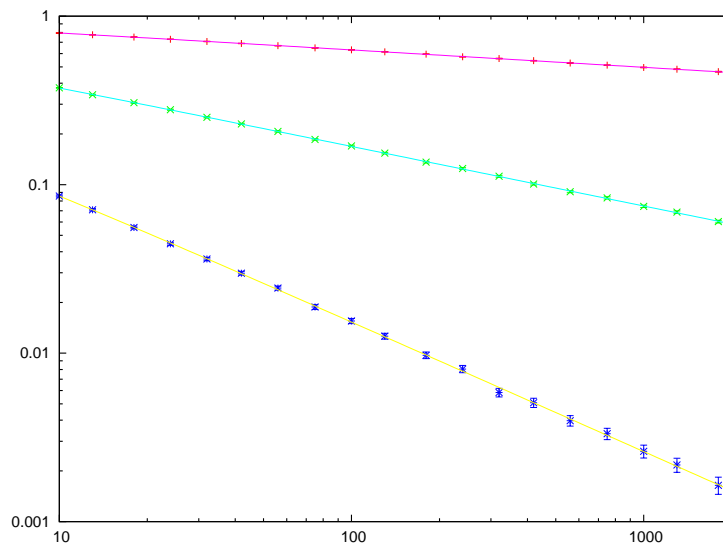


Figure II.9. — Simulation results for the estimation of the exponents α'_j (for $j \in \{1, 2, 3\}$).

The “numerology” here is quite rich: notice for instance that $17 = 5 + 12$ meaning that α'_2 seems to be equal, or at least very close, to $\alpha_2 + \alpha'_1$. From there, one could hope to devise a purely combinatorial proof, coupling for instance a pair of configurations realizing $A_{2,BW} \times A_{1,B}$ to one realizing $A_{2,BB} \times E$ where E is an event with probability of order 1 (e.g., the event that no arm of any color exists). We have not been able to do that yet, unfortunately.

4. — ABOUT THE DIVIDE-AND-COLOR MODEL

I end this chapter on discrete models with a few results and a puzzle (or at least a few puzzling observations) about the divide-and-color model. Apart from Theorem 6, most of the discussion below is extracted from unpublished notes, so the level of detail might be a little higher than in the rest of this memoir — on the other hand, while some of the statements are proved, the general heuristic has yet to lead to a tangible outcome.

4.1. — THE BEHAVIOR OF THE CRITICAL COLORING VALUE

In the construction leading to the divide-and-color model, the cases $p = p_c$ (assuming the box-crossing property) and $p > p_c$ are easy to treat, and the only regime where something non-trivial occurs, at least in terms of $\theta(p, r)$, is for $p < p_c$, which we will assume here. It is known that if the initial lattice is the triangular lattice rather than the square lattice \mathbb{Z}^2 , then $r_c(p)$ is identically equal to $1/2$. On the square lattice, it is not constant but something can still be said:

Theorem 6 (Bálint, B., Tassion [VB1]). *The critical coloring value $r_c^{\mathbb{Z}^2}(p)$ is a continuous function of p on the interval $p \in [0, 1/2)$.*

One promising approach to the model is to see it as site-percolation on a random graph. Let G be the planar square lattice \mathbb{Z}^2 , and let $p \in [0, 1/2)$ be fixed (for now). For every edge e of G , let $\eta_p(e)$ be a Bernoulli variable of parameter p , sampled independently for each edge; let G_p be the random subgraph of G obtained by keeping the edges for which $\eta_p(e) = 1$ and removing the others. This is exactly Bernoulli bond-percolation with parameter $p < p_c$.

For each $x \in G$, let $C(x)$ be the connected component (or *cluster*) of x in G_p ; let $(C_i)_{i \in \mathbb{Z}_+}$ be an enumeration of the clusters of G_p (i.e., C_i and C_j are disjoint for $i \neq j$). For $x \in G$, let $i(x)$ be such that $C(x) = C_{i(x)}$. Finally, let $(\varepsilon_i)_{i \in \mathbb{Z}_+}$ be a sequence of independent Bernoulli variables of parameter $r \in (0, 1)$. This induces a coloring of G , obtained by letting $Z_x = \varepsilon_{i(x)}$, which is one particular way of realizing the divide-and-color model with parameters (p, r) .

— The random graph —

Let \mathcal{C}_p be the graph constructed in the following way: the vertices of \mathcal{C}_p are the connected components of G_p (the clusters of the underlying Bernoulli percolation); there is an edge between C_i and C_j if and only if there is an edge of G between a vertex of C_i and a vertex of C_j . In other words, two clusters are neighbors in \mathcal{C}_p whenever they are at distance 1 from each other in G . We will call \mathcal{C}_p the *cluster graph* of the percolation realization.

The construction of (Z_x) above is exactly that of Bernoulli site-percolation with parameter r on \mathcal{C}_p . Since $p < p_c$, all the C_i are finite, and in particular the percolation event in (Z_x) used to define $\theta(p, r)$ is equivalent to the existence of an infinite chain $(C_{i_n})_{n \in \mathbb{Z}_+}$ of pairwise distinct clusters, such that for every $n \geq 0$, $\varepsilon_{i_n} = 1$ and the clusters C_{i_n} and $C_{i_{n+1}}$ are neighbors in \mathcal{C}_p .

An equivalent way of constructing \mathcal{C}_p is the following: start with the graph G , and contract every edge of G with probability p , independently of the others; then merge multiple edges into single edges and remove loops (i.e. edges from a point to itself).

Recall Wagner's version of Kuratowski's theorem (see e.g. [16, p. 24]): a graph is planar if and only if it does not contain a copy of the complete graph K_5 or of the complete bipartite graph $K_{3,3}$ as a minor. A direct consequence of this is the fact that the graph \mathcal{C}_p is

always planar (for such a minor can obviously not be created by contracting the edges of a planar graph). However, even though planarity holds, the existence for instance of non-simply connected open clusters in G makes embeddings of \mathcal{C}_p in the plane not unique (in the sense that they are not all isomorphic to each other as planar maps), and there is no well-defined notion of faces for \mathcal{C}_p .

Nevertheless, we are interested in the macroscopic features of site-percolation on \mathcal{C}_p , and this allows us to ignore some of its features. First, assume C_i (say) is a non-simply connected open cluster of G . Then its complement in G is not connected, and in turn the subgraph $\mathcal{C}_p \setminus \{C_i\}$ is not connected. This means that C_i is a cut-point of \mathcal{C}_p . But because we are assuming that $p < p_c$, the complement of C_i in G has exactly one unbounded connected component. We call the other components (and their clusters) *surrounded*. Clearly, changing the color of surrounded clusters will not affect the existence of an infinite chain along which ε is 1: letting $\mathcal{C}_p^{(1)}$ be the subgraph of \mathcal{C}_p obtained by keeping only the clusters of G that are not surrounded, there is percolation in \mathcal{C}_p if and only if there is percolation in $\mathcal{C}_p^{(1)}$.

Second, assume that there exist two open clusters of G , say C_i and C_j , which are not surrounded and such that $\mathcal{C}_p^{(1)} \setminus \{C_i, C_j\}$ is not connected. We call the clusters in its finite connected components *2-surrounded*. Let $\mathcal{C}_p^{(2)}$ be the subgraph of $\mathcal{C}_p^{(1)}$ obtained by keeping only the vertices of $\mathcal{C}_p^{(1)}$ that are not 2-connected and the edges between them. Again it is easy to check that (for $p < p_c$) there is an infinite chain of white clusters in \mathcal{C}_p if and only if there is one in $\mathcal{C}_p^{(2)}$: indeed, from any chain in $\mathcal{C}_p^{(1)}$, one can remove all the 2-surrounded vertices, and what remains is still a chain in $\mathcal{C}_p^{(2)}$.

Last, we repeat the argument of the previous paragraph to define 3-surrounded clusters; they correspond to the existence of separating triangles in $\mathcal{C}_p^{(2)}$, and again removing all the 3-surrounded clusters in a chain of vertices of $\mathcal{C}_p^{(2)}$ leaves a chain of non-3-surrounded vertices — because the triangle is a complete graph. Let \tilde{G}_p be the subgraph of $\mathcal{C}_p^{(2)}$ obtained by keeping only the vertices that are not 3-surrounded and the edges between them. We will call \tilde{G}_p the *reduced cluster-graph* of the percolation realization.

\tilde{G}_p is infinite, connected, planar, and has no separating triangle and no cut-vertex (however, it is not necessarily 4-connected or even 3-connected unless G is a triangulation). It has a natural embedding, in the sense that it respects the circular orderings of neighbors that are present in G_p (all the “dangling ends” which prevented such a canonical embedding for G_p are removed in the construction of \tilde{G}_p).

— *In terms of the dual graph* —

Even though it is not true that the graph $\mathcal{C}_p^{(1)}$ is uniquely embeddable in the plane, the previous construction can be followed in terms of the dual graph G^* of G , according to the following “dictionary”:

- Contracting an edge e of G makes it so that the two faces it separated do not share that edge anymore; this corresponds to removing the dual edge e^* from G^* . Equivalently, and not surprisingly, the underlying bond-percolation model G_p is replaced with Bernoulli bond-percolation on G^* with parameter $p^* = 1 - p$;
- A cut-point in G_p determines one infinite component and at least one finite component in its complement; along the boundary of each of these finite components lies a single face of the graph, and the dual vertex corresponding to it in G_p^* is then a cut-point; the operation of removing 1-surrounded clusters is the same on both sides;

- A 2-surrounded cluster corresponds to the existence of two dual edges, the removal of which separates G_p^* . Removing the 2-connected cluster then amounts to replacing these two edge and the finite component of their complement in G_p^* by a single edge (kind of like “flattening a sausage”);
- A 3-surrounded cluster corresponds to the existence of a separating triangle in G_p , which is equivalent to the existence of 3 edges in the dual whose removal disconnects it. Removing the cluster (emptying the separating triangle) is equivalent to contracting the finite component of the complement of the 3 dual edges into a single dual vertex.

It is plainly seen now that performing Bernoulli bond-percolation on G^* and applying all the above simplifications of G_p^* is a well-defined operation if $p < p_c$, and that it leads to the dual graph of \tilde{G}_p .

If $p > p_c$, then G_p^* is not locally finite; if $p = p_c$, a consequence of Russo-Seymour-Welsh theory is that G_p^* essentially looks like a tree, encoding the embedding of the clusters of G_p inside each other — and since any vertex of a tree is a cut-point, \tilde{G}_p^* is empty, as was \tilde{G}_p . The initial cluster graph \mathcal{C}_p on the other hand has a rich structure, in particular in terms of isoperimetric inequalities; it has been considered before by Itai Benjamini, Ori Gurel-Gurevich and Gady Kozma under the name CCCP (for Contracting Clusters of Critical Percolation).

— *The average degree of \tilde{G}_p^** —

Here, we list a few simple properties of the graph \tilde{G}_p . First of all, because \tilde{G}_p is obtained by contracting edges of G , each of its faces in turn corresponds to contracting some of the edges of a face of G , so the maximal size of a face of \tilde{G}_p is not larger than the maximal size of a face of G . In particular, if G is a triangulation of the plane, so is \tilde{G}_p , and if G is the square lattice \mathbb{Z}^2 , every face of \tilde{G}_p has either 3 or 4 sides.

Of central interest to us is the behavior of the system as p approaches $p_c(G)$, *i.e.* as the underlying percolative system is close to being critical. The main intuition one could have is that, because the clusters of the underlying model tend to be very large in that case, \tilde{G}_p would look more and more like a triangulation (and maybe would resemble the Poisson-Delaunay random triangulation of the graph, in some sense).

However, it is not clear how to measure how close a graph is to being a triangulation. The first observable one thinks of is the average number of sides of a face of \tilde{G}_p (or equivalently the average degree of its dual \tilde{G}_p^*): this converging to 3 as $p \rightarrow p_c(G)$ would seem to be natural. This is certainly not the case for G_p if the initial graph G is not a triangulation itself: indeed, every face of G has positive probability to remain a face of G_p (it suffices to ensure that each of its vertices is a one-point cluster of the underlying model, and this has probability of order 1 uniformly in $p < p_c$). This argument does not hold for \tilde{G}_p though, because small clusters are easily surrounded.

Assume that $G = \mathbb{Z}^2$. We define a particular instance of *four dual arms events* as follows. First, let C be a finite, non-empty collection of vertices of G^* . We say that there are 4 dual arms around C if there exist four simple, disjoint, infinite paths in G^* , starting on C and composed of open dual edges (*i.e.* duals of closed edges of G). Let $A_4^*(C)$ be the event that such arms exist. For convenience, let $A_4^*(\emptyset) := A_4^*(f_0)$ where f_0 is a fixed face of G , say the one north-east of the origin.

Notice that, because we are assuming that $G = \mathbb{Z}^2$ and $p < p_c$, all the events $A_4^*(C)$ have positive probability. Recall that C_0 is the cluster of 0 for the underlying model on G ; let ∂C_0 be its boundary in G^* , *i.e.* the set of faces of G which are adjacent to both C_0 and

$G \setminus C_0$. Define

$$\alpha(p) := \frac{E_p[|C_0|^{-1}; A_4^*(\partial C_0)]}{P_p[A_4^*(\emptyset)]} < \infty$$

(α is, for lack of a better name, the mean number of vertices per square face of \tilde{G}_p). From ergodicity and the Euler relation, one can obtain the average degree of a uniformly chosen dual vertex:

$$\delta(\tilde{G}_p^*) = 3 + \frac{1}{2\alpha(p) - 1} > 3.$$

— **Numerology** —

Let L_p be the correlation length of the underlying model at parameter p ; recall that α'_4 is the monochromatic 4-arm exponent at criticality (expected to be close to $63/48$). Then,

$$\alpha(p) \simeq \frac{E_p[|C_0|^{-1}(\text{diam } C_0/L_p)^{\alpha'_4}]}{(1/L_p)^{\alpha_4}} \simeq E_p \left[\frac{(\text{diam } C_0)^{\alpha'_4}}{|C_0|} \right].$$

(Proving this up to constants, replacing $(\text{diam } C_0)^{\alpha_4}$ by the corresponding event and using approximate sub-multiplicativity, should be doable rigorously — however, there is a technicality in the fact that the event $\text{diam } C_0 \geq L_p$ should be treated separately, and it may have a significant probability — though a back-of-the-envelope computation shows that it shouldn't.) Then presumably $|C_0| \simeq (\text{diam } C_0)^{91/48}$ and since $91 > 63$, α_p would be bounded. As a consequence, the average face size of \tilde{G}_p would be bounded below, which is what we expect:

Conjecture 2. *Let \tilde{G}_p be constructed as above from the square lattice $G = \mathbb{Z}^2$, for $p < p_c(G)$. Then, as $p \uparrow p_c(G)$, the average number of edges of the faces of \tilde{G}_p , or equivalently the average degree of its dual graph, remains bounded below by $3 + \eta$ where $\eta > 0$ does not depend on p .*

In other words, as p approaches criticality, the graph \tilde{G}_p seems to *not* look like a random triangulation of the plane, at least in terms of face sizes. Yet simulations seem to agree with the initial intuition, and lead us to still claim the following (with the corollary that average face degree is not the appropriate measure of how close G_p is to being a triangulation):

Conjecture 3. *As $p \rightarrow p_c = 1/2$, the critical coloring parameter $r_c^{\mathbb{Z}^2}(p)$ converges to $1/2$.*

4.2. — MONOTONICITY AND “ALMOST LINEARITY”

The last thing I would like to mention is both an observation of simulation results (obtained by my student Vincent TASSION, cf. Figure II.10) and a natural expectation, but we have not been able to prove it yet:

Conjecture 4. *The function $p \mapsto r_c^{\mathbb{Z}^2}(p)$ is strictly decreasing on the interval $[0, 1/2)$.*

This is reasonable to expect, because as p increases, the reduced cluster graph \tilde{G}_p always evolves *via* transforming squares into triangles (which tends to decrease its critical point) and further reductions of surrounded clusters (which has no effect on its percolative properties); what is in fact missing is an equivalent of the monotone coupling of Bernoulli percolation models, which gives monotonicity of the probabilities of increasing events “for free”.

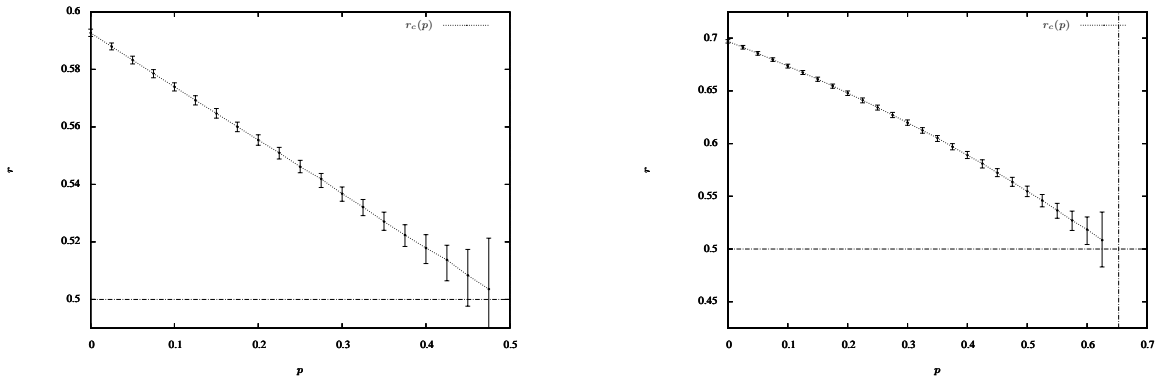


Figure II.10. — “Almost linearity” of the critical coloring parameter (left: square lattice; right: hexagonal lattice, for comparison).

On the other hand, no “easy” proof of monotonicity (in the sense that it relies on a clever coupling without delving into the specifics of the square lattice) can exist:

Proposition 4 (Bálint, B., Tassion [VB1]). *There exists a periodic lattice L for which the critical point p_c^L of bond percolation is strictly between 0 and 1, and such that the map $p \mapsto r_c^L(p)$ is not monotone on the interval $[0, p_c^L)$.*

The square lattice seems to play a special role in another way: looking at the results of the simulations (Figure II.10), it seems that on \mathbb{Z}^2 the map $p \mapsto r_c^{\mathbb{Z}^2}(p)$ is very close to being an affine function (and that is not at all the case on the hexagonal lattice, for which it looks rather concave).

It doesn’t make much sense to conjecture that $r_c^{\mathbb{Z}^2}$ is *exactly* affine on the interval without at least a vague idea of why it would be — which we don’t have. In fact, there even is a credible way of *disproving* linearity: one can follow up on the numerology above and try to get a power-law behavior for $r_c^{\mathbb{Z}^2}(p) - 1/2$ as a function of $p_c - p$, with an explicit exponent written in terms of percolation critical exponents (and hopefully different from 1, contradicting linearity).

Nevertheless, I believe that $r_c^{\mathbb{Z}^2}$ is too close to being linear for it to be a random coincidence, and it has to say something about \mathbb{Z}^2 as opposed to general lattices — possibly a trace of self-duality or something similar. I am convinced that there is something possibly deep to understand here, but so far we were no able to figure out a reasonable explanation.

Chapter III

Conformal invariance and SLE

This chapter gathers my work on conformally invariant continuous models and on scaling limits as such. Some of it was done during my PhD (references [VB2] and [VB3]), and a partial version of Theorem 8, limited to the case $\kappa \neq 4$, appears in my PhD thesis [10]; the general version, valid for all κ , took quite some additional work (as the publication date of [VB5] indicates), even if the basic scheme of the proof is the same as the one in [VB3]. I chose to quickly state the main theorems and their context for completeness, but to omit proofs because they are not the main focus of this memoir. The second part of the chapter is related to reference [VB6].

1. — RESULTS FROM AND ADJACENT TO MY PHD THESIS

1.1. — ON BROWNIAN MOTION

Let $W = (W_t)_{t \geq 0}$ be a standard Brownian motion started at 0 in the complex plane. It is *conformally invariant* in the following sense. Pick two simply connected, bounded, open domains Ω_1 and Ω_2 , both containing 0; for $k \in \{1, 2\}$, let τ_k be the first exit time of Ω_k by W and let $K_k = W_{[0, \tau_k]}$ be its path up to time τ_k . K_1 and K_2 are two random compact sets in the plane. Moreover, let Φ be a conformal map from Ω_1 to Ω_2 fixing the origin (there is a one-parameter family of such maps, indexed by the argument of $\Phi'(0)$, but which one is chosen does not matter here). Then,

$$\Phi(K_1) \stackrel{(d)}{=} K_2 \tag{III.1}$$

which is known as conformal invariance for Brownian motion.

The proof by Lévy of the previous statement uses Itô's formula to obtain a stronger result: the process $(\Phi(W_t))$ is itself a time-change of Brownian motion, where the time change is given explicitly in terms of Φ' along the path of W . In other words, while stated in terms of compact sets, (III.1) is really a local statement about processes.

One hope at the time (which was more than fulfilled by SLE) was to derive geometric information about the whole path from conformal invariance, and a "baby version" of the goal went through the study of families of exceptional points, typically the boundary of K_1 , the set of its cut-points, etc. One of the great victories gotten using SLE has been the computation of *intersection* and *disconnection exponents* for planar Brownian motion, leading to the computation of the Hausdorff dimension of the boundary (4/3, as conjectured by Mandelbrot) and cut-set (3/4) of the Brownian path.

Let $\theta > 0$ be given. A *pivoting point of angle θ* is a cut-point W_s with $s \in (0, 1)$ such that one component of $W_{[0,1]} \setminus \{W_s\}$ can be turned by an angle θ without intersecting the other component (as in Figure III.1); more formally, it has to satisfy the condition

$$\{W_t : 0 \leq t < s\} \cap \{W_t, W_s + e^{i\theta}(W_t - W_s) : s < t \leq 1\} = \emptyset.$$

The endpoints, $s \in \{0, 1\}$, are excluded from the definition because W_0 and W_1 would always be pivoting points.

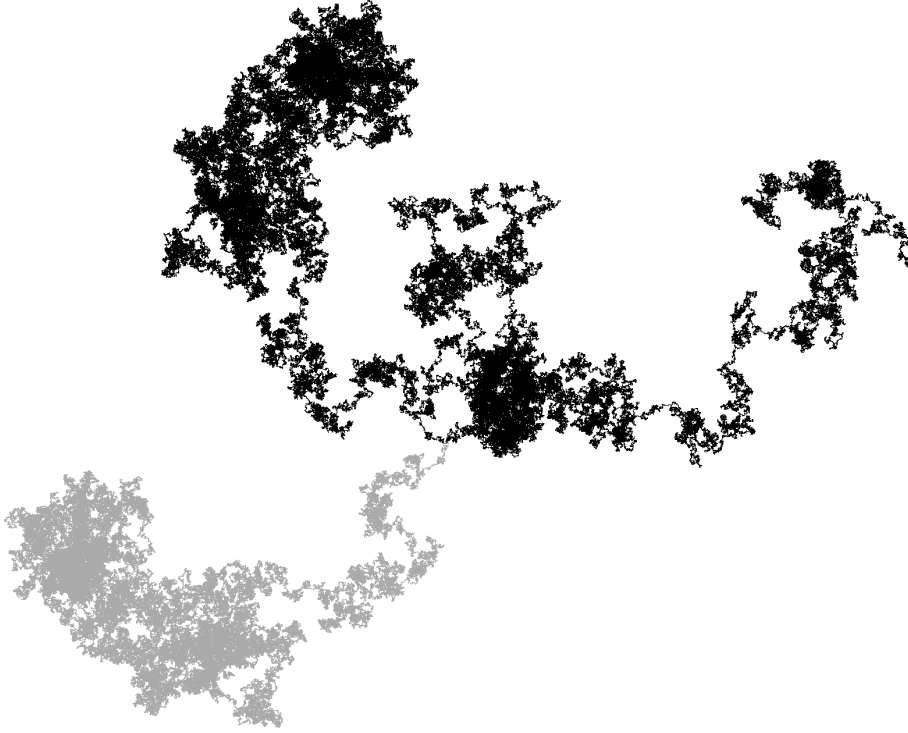


Figure III.1. — A pivoting point of angle $\pi/2$ on a Brownian path.

Theorem 7 (B. [VB2]). *There exists $\theta_0 > 0$ such that, for every $\theta < \theta_0$, the set of pivoting point of angle θ on $W_{[0,1]}$ a.s. has positive Hausdorff dimension (and in particular is a.s. non-empty).*

The proof gives the bound $\theta_0 \geq \log^2 2/2\pi \simeq .076$; simulations suggest that it is close to $3\pi/4$ (Figure III.1 already indicates that it should be at least equal to $\pi/2$), but an exact computation using SLE does not seem to be doable — besides, since the definition is macroscopic, the set of pivoting points of angle θ is not conformally invariant.

One can define *local pivoting points* by making the previous condition microscopic, namely: W_s is a local pivoting point if, for some $\varepsilon > 0$,

$$\{W_t : s - \varepsilon \leq t < s\} \cap \{W_t, W_s + e^{i\theta}(W_t - W_s) : s < t \leq s + \varepsilon\} = \emptyset.$$

Then, the set of local pivoting points is conformally invariant, and it a.s. has positive dimension (and is a.s. everywhere dense on the path $W_{[0,1]}$) for all θ small enough.

1.2. — ON SCHRAMM-LOEWNER EVOLUTION

My main result on SLE processes is the computation of their Hausdorff dimensions (the upper bound was already obtained by Rohde and Schramm [72]):

Theorem 8 (B. [VB3, VB5]). *Let $\kappa \in [0, 8]$, let (K_t) be an SLE_κ in the upper-half plane, let γ be its trace and let $\mathcal{H} := \gamma([0, \infty))$. Then, almost surely,*

$$\dim_H(\mathcal{H}) = 1 + \frac{\kappa}{8}.$$

For $\kappa \geq 8$, SLE is space-filling so in particular it has Hausdorff dimension 2. In the case $4 < \kappa < 8$, for which the trace up to time t is a strict subset, Kenyon had previously conjectured that the dimension of the SLE boundary at a fixed time was given by

$$\dim_H(\partial K_1) = 1 + \frac{2}{\kappa} < \dim_H(\mathcal{H}).$$

This was first proved for $\kappa = 6$ using comparison with Brownian motion and the computation by Lawler, Schramm and Werner [52, 53] of the Brownian disconnection exponents, and then obtained for all $\kappa \in (4, 8)$ as a consequence of Theorem 8 and *SLE duality*, which states that for such values of the parameter, the boundary of SLE_κ is a version of $SLE_{\kappa'}$ for the dual parameter $\kappa' = 16/\kappa$.

2. — UNIVERSALITY AND ISOTROPIC EMBEDDINGS

I would like to focus the larger part of this chapter on a topic which occupied me for quite a while and which I believe is important — although the results I was able to get in this direction are extremely partial. The goal is to understand what is known as *universality* in the context of scaling limits; first, I will recall three cases of known convergence of discrete models to SLE as the lattice mesh goes to 0.

2.1. — SCALING LIMITS OF A FEW DISCRETE MODELS

The three models discussed below exhibit various levels of generality in the way they are proved to be conformally invariant in the limit:

- Critical site-percolation on the equilateral triangular lattice: here the proof is extremely specific and does not seem to apply to any other lattice or model (except for more or less trivial modifications);
- The Ising model on isoradial graphs: here the existence of a (conformally invariant) scaling limit is known for a large family of lattices, but not for all of them, for deep reasons;
- Loop-erased random walks and uniform spanning trees: here the proof has at its core the convergence of simple random walk to planar Brownian motion, and (up to the choice of embedding, as discussed below) this proof is as universal as can be hoped.

— *Site-percolation on the triangular lattice* —

As was mentioned above, the key step in deriving the convergence of percolation in the scaling limit is Smirnov’s proof of Cardy’s formula, *i.e.* of the convergence of crossing probabilities for conformal rectangles. The proof is extremely clever and elegant, but at its core is the fact that all the faces of the underlying lattice are equilateral triangles; it is based on the estimation of a contour integral by a discrete Stokes formula, and rather than proving that the volume integral obtained this way is small through percolation estimates, one remarks that it vanishes exactly due to the geometry of the lattice.

In order to refine Cardy's formula into geometric information on the scaling limit, one has to consider the *exploration process* of percolation. If one views the model as the i.i.d. coloring of the faces of the hexagonal lattice blue or yellow with probability $1/2$, and if on the boundary of a simply connected domain Ω one fixes Dobrushin boundary conditions, the exploration path is simply the connected component of the set of color boundaries which touches $\partial\Omega$ — an example when Ω is a wedge is shown in Figure III.2.

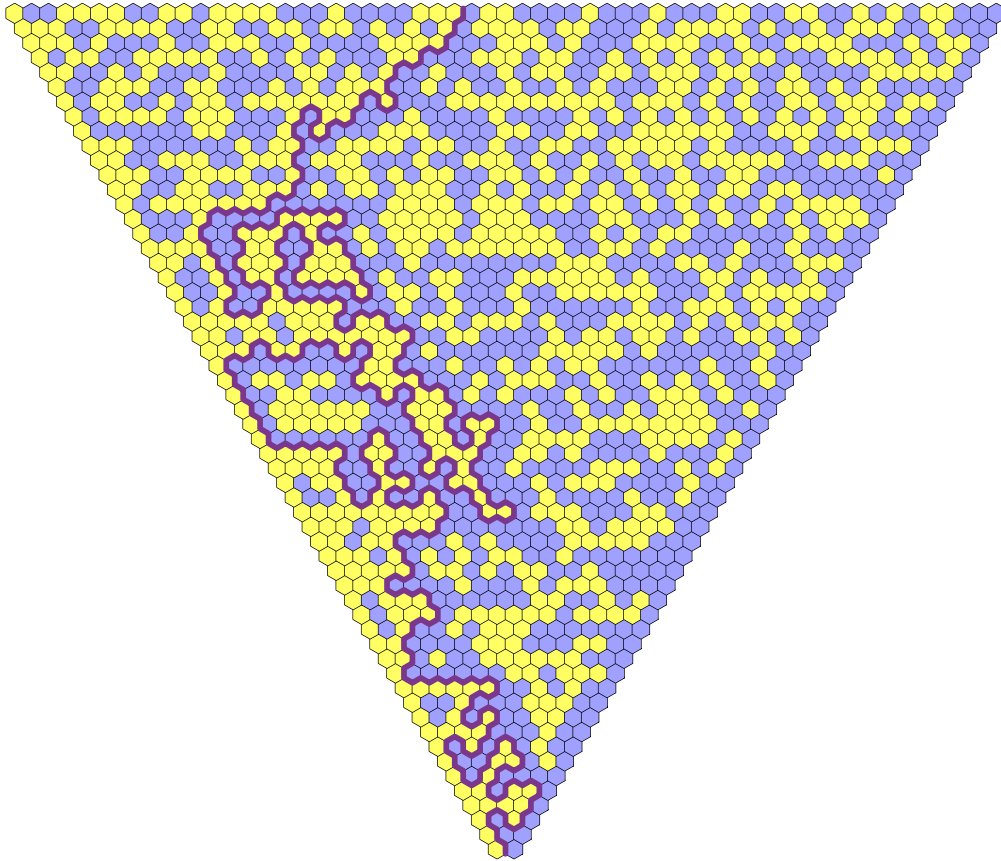


Figure III.2. — The exploration curve of a percolation configuration in a wedge-shaped lattice domain.

The exploration path has a similar role as the path γ in the loop representation of the random-cluster model mentioned earlier. One key difference though is that it is easy to construct it dynamically, by looking at the colors of the hexagons along it one after the other and turning left or right accordingly (the conditional distribution of the configuration away from the currently explored path at a given time is then exactly the same as percolation). As the lattice mesh vanishes, the exploration path converges in distribution to the trace of an SLE_6 process in the domain.

While Smirnov's proof is very specific, universality and conformal invariance have indeed been tested numerically for percolation in various geometries (see *e.g.* [48]), and a partial version of conformal invariance (assuming the existence of the limit) is known in the case of Voronoi percolation (see [15]).

— *The Ising model on isoradial graphs* —

In the case of the Ising model, a partial version of universality is known. Recall that a planar graph $G = (V, E)$ is *isoradial* if all its faces can be inscribed in circles of the same radius — that radius will be the parameter going to 0 in the scaling limit. If a graph

is isoradial, so is its dual graph $G^* = (V^*, E^*)$ if each dual vertex is embedded as the circumcenter of the corresponding face. The *diamond graph* $G^\diamond = (V^\diamond, E^\diamond)$ is the graph having $V^\diamond = V \cup V^*$ as its vertex set, two of its vertices being connected by a bond if and only if one is a dual site and the other is a primal site on the boundary of the corresponding face; the faces of the diamond graph of an isoradial graph are automatically lozenges.

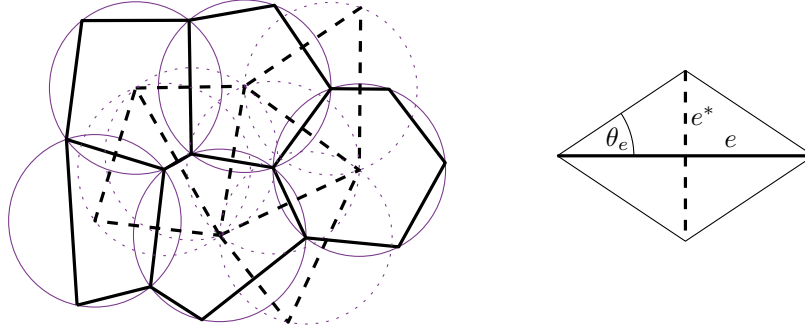


Figure III.3. — A piece of an isoradial graph together with its dual, and the definition of the half-angle θ_e .

Isoradial graphs are the most natural setup for a discrete equivalent of complex analysis, because the diagonals of the faces of the diamond graph are orthogonal and lead to a natural definition of holomorphicity: if $g : V^\diamond \rightarrow \mathbb{C}$ is given, it is *discrete holomorphic* if, whenever (v_1, v_2, v_3, v_4) is a face of G^\diamond , it satisfies

$$\frac{g(v_3) - g(v_1)}{v_3 - v_1} = \frac{g(v_4) - g(v_2)}{v_4 - v_2}$$

(where as usual the plane \mathbb{R}^2 is identified to the complex plane and vertices to the corresponding complex number); in particular, the ratio $(g(v_3) - g(v_1)) : (g(v_4) - g(v_2))$ is pure imaginary.

Fix G to be a doubly periodic planar graph (in the sense that there is a free action of \mathbb{Z}^2 by translations on it); if e is a bond of G , and hence a diagonal of one of the faces of G^\diamond , let $\theta_e \in (0, \pi/2)$ be one half of the angle formed by that face at the ends of e . One can consider the Ising model on G , with interaction strength $J = (J_e)_{e \in E}$ depending on the bond in a periodic way, by defining the formal Hamiltonian

$$H_J(\sigma) := - \sum_{u \sim v} J_e \sigma(u) \sigma(v)$$

and the measure as $\mu_J(\{\sigma\}) \propto \exp(-H_J(\sigma))$.

The *critical surface* is the set of J such that the model is critical, e.g. in the sense that the two-point function does not decay exponentially. It is typically a hypersurface in the space of all parameters. While this can be defined in the most general case, when G is isoradial there is one point on the critical surface which plays a special role:

Proposition 5 (Baxter, Mercat, Boutillier-De Tilière, Chelkak-Smirnov). *The Ising model on an isoradial periodic graph is critical if, for every edge e ,*

$$J_e = \frac{1}{2} \log \frac{1 + \sin \theta_e}{\cos \theta_e}. \quad (\text{III.2})$$

Moreover, the random-cluster observable defined above then has a conformally invariant scaling limit (which does not depend on G) when the lattice mesh goes to 0 for a fixed geometry.

The reason behind this property is one of *exact integrability*: essentially, the proof hinted at earlier in the case of the random-cluster model in \mathbb{Z}^2 still works (up to considerable technical difficulties) when the interaction is chosen in the right way for the given geometry. The Ising model is *universal* in the sense that the scaling limit is the same for all isoradial graphs, provided that it is taken at this particular point.

While this is much more general than in the case of percolation, one could ask the question in a different way: start with a periodic lattice, fix positive interaction parameters on each edge, and introduce the inverse temperature β to define the Ising measure formally as $\mu_J(\{\sigma\}) \propto \exp(-\beta H_J(\sigma))$. This defines a critical point $\beta_c = \beta_c(J)$ — which is equal to 1 if the graph is isoradial and J satisfies (III.2). Does the model, taken at its critical point, still converge to the same scaling limit as the lattice mesh goes to 0? Smirnov and Chelkak’s proof does not seem to extend to this level of generality, the main reason being that there is no natural notion of discrete holomorphic function on such graphs.

— *Uniform spanning trees* —

The last model I would like to speak about here is the uniform spanning tree, which is the first one for which convergence to SLE was proved and one of the main motivations for Schramm to define SLE in the first place.

Let Ω be a simply connected, bounded, discrete domain in \mathbb{Z}^2 , and let a and b be two boundary points; equip Ω with the same Dobrushin boundary condition as above, identifying all the boundary points on one of the two arcs delimited by a and b . A *spanning tree* of Ω is a connected subgraph of Ω (considered with the boundary condition) without loops. The same duality construction as for the random cluster model (Figure III.4) gives rise to a dual tree and, winding between the tree and its dual, to a space-filling *exploration curve* γ . As the lattice mesh vanishes for a fixed domain shape, γ converges in distribution to the trace of an SLE_8 process.

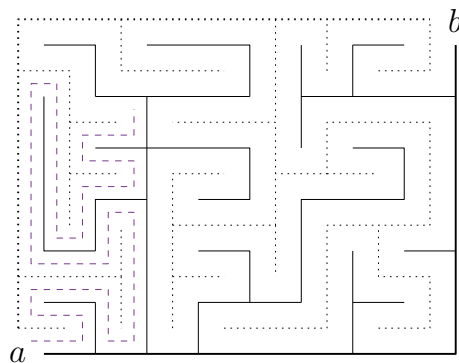


Figure III.4. — A spanning tree (plain lines), its dual (dotted) and the beginning of its exploration path (dashed).

The proof of convergence is rather involved, but as mentioned earlier, the main fact needed for it to go through is the convergence of the simple random walk to planar Brownian motion in the scaling limit — or even the convergence of discrete harmonic measure to its continuous counterpart would be enough. Introducing the same definition in a general periodic planar lattice, as above for the Ising model, still makes sense, and here, universality is known in a much stronger sense: as soon as we have convergence of the SRW to planar Brownian motion, the exploration of the uniform spanning tree still converges to the trace of an SLE_8 .

If the simple random walk does *not* converge to Brownian motion, it still converges to a diffusion with a constant and explicit covariance matrix, and up to a change in

embedding, this can be mapped back to standard Brownian motion, so even though γ does not converge to SLE_8 , it is not far from doing so. The argument is even flexible enough to allow for a periodic family of weights on the edges on the initial lattice (the probability of a given tree being proportional to the product of the weights of its edges), which simply replaces the simple random walk by a walk with different (but explicit) jump rates across different edge types. In other words, spanning trees exhibit universality in a strong sense.

2.2. — UNIVERSALITY AND THE CHOICE OF EMBEDDING

It is a strongly supported conjecture that many discrete models of random media, such as *e.g.* percolation and the Ising model, when taken in dimension 2 at their critical point, exhibit conformal invariance in the scaling limit. Indeed, the *universality* principle implies that the asymptotic behavior of a critical system after rescaling should not depend on the specific details of the underlying lattice, and in particular it should be invariant under rotations (at least under suitable symmetry conditions on the underlying lattice). Since by construction a scaling limit is also invariant under rescaling, it is natural to expect conformal invariance, as the local behavior of a conformal map is the composition of a rotation and a rescaling. However, actual convergence of discrete models to SLE in the scaling limit is known for only a few models.

In fact, it seems that the question of convergence itself has hardly been addressed by physicists, at least in the CFT community — a continuous, conformally invariant object is usually the starting point of their work rather than its outcome. Techniques such as the renormalization group do give reason to expect the existence of a scaling limit and of critical exponents, but they seem to not give much insight into the emergence of rotational invariance.

This is not surprising in itself, for a very trivial reason: take any discrete model for which you know that there is a conformally invariant scaling limit, say a simple random walk on \mathbb{Z}^2 , and deform the underlying lattice, in a linear way, so as to change the aspect ratio of its faces. Then the scaling limit still exists (it is the image of the previous one by the same transformation); but obviously it is not rotationally invariant. Since all the rescaling techniques apply exactly the same way before and after deformation, they cannot be sufficient to derive rotational invariance.

The main question I would like to address here, along the lines of [VB6], is the following: given a discrete model on a doubly periodic planar graph, how to embed this graph into the plane so as to make the scaling limit isotropic? If the graph has additional symmetry (as for instance in the case of the square or triangular lattices), the embedding has to preserve this symmetry, so the question is moot; nevertheless, answering it “the right way” seems to still be a key step towards a proof of conformal invariance.

One surprising thing about the question, besides the fact that it appears to actually be orthogonal to the interests of physicists in that domain, is that for most graphs, the way to choose the right embedding turns out to depend on the model considered. In other words, there is no absolute notion of a “conformal embedding” of a general graph.

In the case of the simple random walk, the answer is quite easy to obtain (though it does not seem to have appeared in the literature in this form); in the case of percolation, which is my main motivation, I could find no reference whatsoever, the closest being the discussion and numerical study of *striated models* in [47] where, instead of looking at a different graph, the parameter p in the model is chosen to depend on the site in \mathbb{Z}^2 in a periodic fashion — which admittedly is a very related question, similar in spirit to what we mentioned earlier about the Ising model on isoradial graphs.

— *Periodic lattices* —

Let T be a finite connected graph of genus 1 (*i.e.*, a graph that is embeddable in the torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ but not in the plane). For ease of notation, assume that T is equipped with a fixed embedding in \mathbb{T}^2 , which we also denote by T . The dual T^* of T (which we also assume to be embedded in the torus once and for all) is then also a graph of genus 1.

Let \hat{T} (resp. \hat{T}^*) be the universal cover of T (resp. T^*): then \hat{T} and \hat{T}^* are mutually dual, infinite, locally finite planar graphs, on which \mathbb{Z}^2 acts by translation. We are interested in natural ways of embedding \hat{T} into the complex plane \mathbb{C} . Let T_i (the meaning of the notation will become clear in a minute) be the embedding obtained by pulling T back using the canonical projection from \mathbb{R}^2 to \mathbb{T}^2 — we will call T_i the *square embedding* of T . For every $\alpha \in \mathbb{C} \setminus \mathbb{R}$, let $\varphi_\alpha : \mathbb{C} \rightarrow \mathbb{C}$ be the \mathbb{R} -linear map defined by $\varphi_\alpha(x + iy) = x + \alpha y$ (*i.e.*, it sends 1 to itself and i to α) and let T_α be the image of T_i by φ_α . We will call T_α the *embedding of modulus α* of T in the complex plane.

Notice that the notation T_α depends on the *a priori* choice of the embedding of T in the flat torus; but, up to rotation and scaling, the set of proper embeddings of \hat{T} obtained starting from two different embeddings of T is the same, so no generality is lost as far as our purpose here is concerned.

We say that an embedding T_α of \hat{T} in the complex plane is *balanced* if each of its vertices is the barycenter (with equal weights) of its neighbors; or, equivalently, if the simple random walk on it is a martingale. The following is almost trivial:

Proposition 6. *Let T be a graph of genus 1: then, for every $\alpha \in \mathbb{H}$, there is a balanced embedding of \hat{T} in the complex plane with modulus α . Moreover, this embedding is unique up to translations of the plane.*

An essential point is that, even though the natural proof uses Euclidean geometry, the fact that the embedding is balanced is a linear condition. In particular, if the embedding T_i is balanced, then so are all the other T_α . The corresponding *a priori* embedding of T itself into the flat torus \mathbb{T}^2 (which is also unique up to translations) will be freely referred to as *the balanced embedding of T into the torus*.

— *Uniqueness of the modulus* —

Given T and a translation-invariant discrete model on \hat{T}^* , it is natural to ask whether it is possible to choose a value for α which provides conformal invariance in the scaling limit (provided that the limit exists). There are two possible strategies: either give an explicit value for which “a miracle occurs” (in physical terms, for which the model is *integrable* — this is what Smirnov did in the case of the triangular lattice), or obtain its existence in a non-constructive way. A reassuring fact is that, whenever such an α exists, it is essentially unique:

Proposition 7. *There are either zero or two values of α such that conformal invariance holds in the scaling limit. In the latter case, the two values are complex conjugates of each other.*

When such a pair of moduli exists, we will denote by $\alpha_T^{(\text{model})}$ the one with positive imaginary part. When an argument does not depend on the specific model (as is the case in the next subsection), we will use the generic notation α_T as a placeholder. Because the value of α_T (when it exists) is uniquely defined by the combinatorics of T and the coefficients of the model, there are cases where additional symmetry specifies its value uniquely, because such symmetries have to be present in the correct embedding. Of course,

identifying α_T in those cases is a long way from a proof of conformal invariance; but it would seem that in the case of percolation for instance, understanding, in the general case, what α_T^{perc} is would be a significant progress in our understanding of the process.

The case of the random walk is easy to treat explicitly. For every edge e and every modulus α , let e_α be the difference between the complex coordinates of the endpoints of e as embedded in T_α ; e_α is an \mathbb{R} -linear function of α . It is a matter of bookkeeping to show the following proposition, which enables an easy and explicit computation of α_T^{SRW} :

Proposition 8. *Assume that the embedding is balanced; then, $n^{-1/2}Z_n$ converges in distribution to a rotationally invariant Gaussian variable if and only if*

$$\sum_{e \in E(T)} (e_\alpha)^2 = 0$$

(which is a second-degree equation in α with real-valued coefficients).

One can also look at a simple random walk on the dual graph T_α^* , and ask for which values of α this dual walk is isotropic in the scaling limit. As it turns out, the modulus one obtains this way is the same as on the initial graph, in other words

$$\alpha_T^{\text{RW}} = \alpha_{T^*}^{\text{RW}}.$$

This is a very weak version of universality, and unfortunately there doesn't seem to be a purely discrete proof of it — say, using a coupling of the two walks.

There is another natural way to obtain the same condition. We are planning on studying convergence of discrete objects to conformally invariant limits, so it is a good idea to look for discrete-harmonic functions on T_α (with respect to the graph Laplacian, which is the same as the generator of the simple random walk on T_α). The condition of balanced embedding is exactly equivalent to saying that the identity map is harmonic on T_α ; it is a linear condition, so it does not depend on the value of α . The main difficulty when looking at discrete holomorphic maps is that the product of two such maps is not holomorphic in general. But we are interested in scaling limits, so maybe imposing that such a product is in fact “almost discrete holomorphic” (in the sense that it satisfies the Cauchy-Riemann equations up to an error term which vanishes in the scaling limit) would be sufficient.

Whether the previous paragraph makes sense or not — let us investigate whether the map $\zeta : z \mapsto z^2$ is discrete-harmonic. For every $z \in T_\alpha$, we can write

$$\Delta\zeta(z) = \frac{1}{3} \sum_{z' \sim z} (z'^2 - z^2) = \frac{1}{3} \sum_{e \in E_z(T)} (z + e_\alpha)^2 - z^2 = \frac{1}{3} \sum_{e \in E_z(T)} e_\alpha^2$$

(the term in $\sum z e_\alpha$ vanishes because the embedding is balanced). So, if ζ is discrete-harmonic, summing the above relation over $z \in T$ gives the very same condition $\sum e_\alpha^2 = 0$ as before; in other words, α_T^{RW} is the embedding for which $z \mapsto z^2$ is *discrete-harmonic on average*. Within the class of 3-regular lattices, this is as far as one can go:

Proposition 9. *The only 3-regular graph on which the map $\zeta : z \mapsto z^2$ is discrete-harmonic is the honeycomb lattice, embedded in such a way that its faces are regular hexagons.*

So, imposing ζ to be harmonic not only determines the embedding, it also restricts T to essentially one graph; but in terms of scaling limits, the condition that ζ is harmonic on the average makes as much sense as the exact condition.

— *Embedding using circle packings* —

There is another way to specify essentially unique embeddings of triangulations, which is very strongly related to conformal geometry, using the theory of circle packings. It is a fascinating subject in itself and a detailed treatment would be outside of the purpose of these notes, so the interested reader is advised to consult the book of Stephenson [81] and the references therein for the proofs of the claims in this subsection and much more. We first give a version of a theorem of K obe, Andreev and Thurston, specialized to our case:

Theorem 9 (Discrete uniformization theorem [81]). *Let T^* be a finite triangulation of the torus, and let \hat{T}^* be its universal cover. There exists a locally finite family $(\mathcal{C}_v)_{v \in V(\hat{T}^*)}$ of disks of positive radii and disjoint interiors, satisfying the following compatibility condition: \mathcal{C}_v and $\mathcal{C}_{v'}$ are tangent if, and only if, v and v' are neighbors in \hat{T}^* .*

Such a family is called a circle packing associated to the graph \hat{T}^ . It is essentially unique, in the following sense: If (\mathcal{C}'_v) is another circle packing associated to \hat{T}^* , then there is a map $\varphi : \mathbb{C} \rightarrow \mathbb{C}$, either of the form $z \mapsto az + b$ or of the form $z \mapsto a\bar{z} + b$, such that for every $v \in V(\hat{T}^*)$, $\mathcal{C}'_v = \varphi(\mathcal{C}_v)$.*

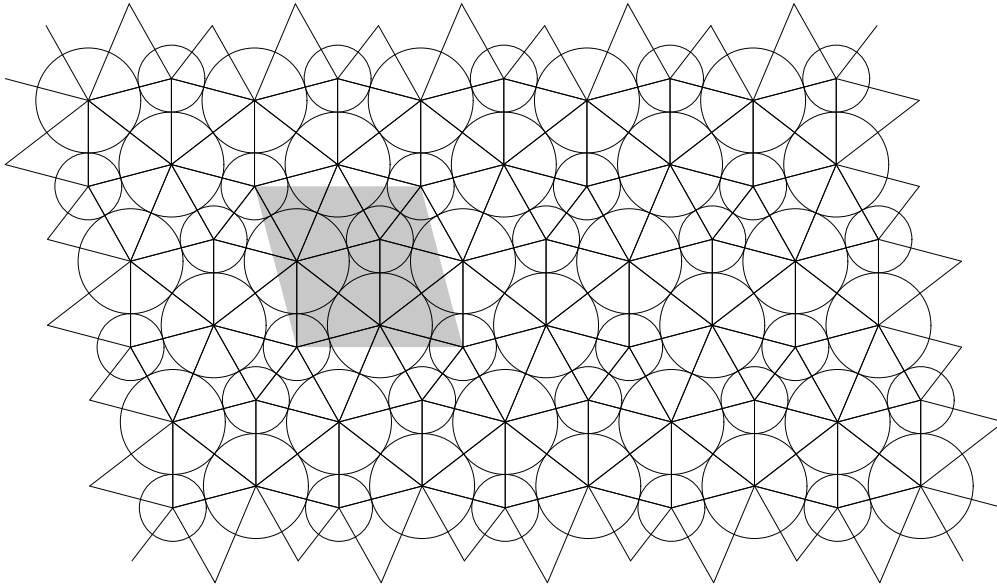


Figure III.5. — Circle-packing of a periodic lattice.

The “existence” part of the above theorem remains true in a much broader class of graphs; essentially all that is necessary is bounded degree and recurrence of the simple random walk on it. (One can see that a packing exists by completing the graph into a triangulation.) The “uniqueness” part however fails in general, as is made clear as soon as one tries to construct a circle packing associated to the square lattice ...

It is easy to check that the circle packing obtained for a periodic triangulation is itself periodic. In other words, as soon as one is given a circle packing associated to a planar graph, it comes with a natural, periodic embedding with modulus α_T^{CP} : every vertex $v \in V(\hat{T}^*)$ is represented by the center of \mathcal{C}_v . One can then specify an embedding of \hat{T} by putting each of its vertex at the center of the disk inscribed in the corresponding triangular face of (the embedding of) \hat{T}^* ; the collection of all those inscribed disks is in fact a circle packing associated with the graph \hat{T} (see Figure III.5).

— *Comparing different methods of embedding* —

Assume for a moment that T is 3-regular. We now have at least two ways of giving a conformal structure to a torus equipped with a triangulation — which is but another way of referring to the choice of α : α_T^{SRW} and α_T^{CP} . A third one is obtained by gluing equilateral conformal triangle according to the combinatorics of T , it leads to a module which we will denote by α_T^R (R here stands either for Riemann surface or for Regge metric). Assuming that critical percolation does have a scaling limit on T , it leads to a fourth natural embedding choice α_T^{perc} of it.

It would seem to be a natural intuition that all these moduli are the same, and correspond to a notion of *conformal embedding* of a triangulation (or a 3-regular graph) in the plane; and they all have a claim to that name. But this is not true in general: we detail the construction of a counterexample. Start with the face-centered square lattice T_s and its dual T_s^* ; and refine one of the “vertical” triangular faces of T_s^* by adding a vertex in the interior of it, connected to its three vertices. In terms of the primal graph, this correspond to replacing one of its vertices by a triangle — see Figure III.6. Let T'_s be the graph obtained that way; we will refer to such a splitting as a *refinement*, and to the added vertex as a *new vertex*.

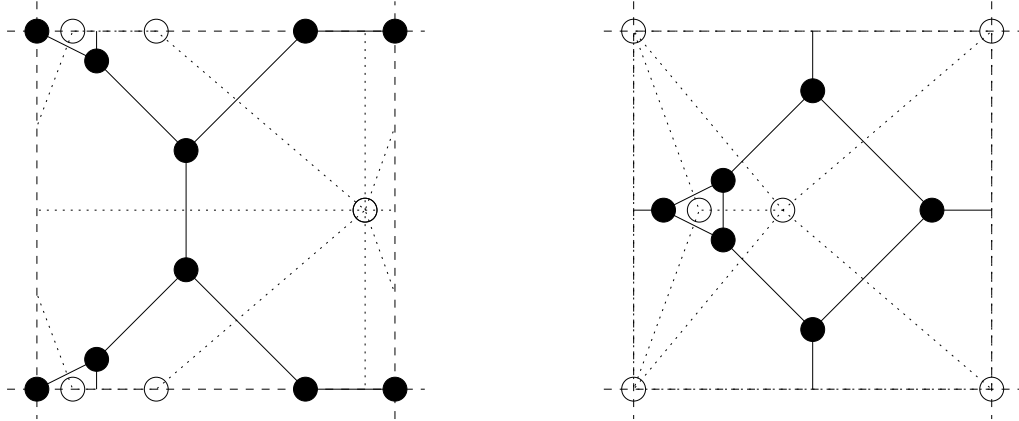


Figure III.6. — Square (but not balanced) embeddings of T'_s (solid) and its dual (dotted); the origin is taken as a point of T'_s on the left, and as a point of the dual on the right.

In terms of circle packings, this changes essentially nothing; the new vertex of $(T'_s)^*$ can be realized as a new disc without modifying the rest of the configuration. In terms of random walks, however, adding edges will modify the covariance matrix in the central limit theorem. The computation can be done easily, as explained above, and one gets the following values:

$$\alpha_{T'_s}^{\text{CP}} = i = \alpha_{T_s}^{\text{CP}} ; \quad \alpha_{T'_s}^{\text{RW}} = i\sqrt{\frac{6}{7}} \neq i = \alpha_{T_s}^{\text{RW}}.$$

So, α^{RW} and α^{CP} are different in general. Is α^{Perc} (provided it exists) one of them? An easy fact to notice is the following: let T^* be a triangulation of the torus and let $(T')^*$ be obtained from it by splitting a triangle into 3 as in the construction of T'_s . Then, consider two realizations of site-percolation at $p_c = 1/2$ on both universal covers, coupled in such a way that the common vertices are in the same state for both models. In other words, start with a realization of percolation on \hat{T}^* and without changing site states, refine a periodic family of triangles of it into 3, choosing the state of each new vertex independently of the others and of the configuration on \hat{T}^* .

If there is a chain of open vertices in \hat{T}^* , this chain is also a chain of open vertices in the refined graph — because all the edges are preserved in the refinement. Conversely, starting from a chain of open vertices in the refinement and removing each occurrence of a new vertex on it, one obtains a chain of open vertices in \hat{T}^* ; the reason for that being that the triangle is a complete graph. Hence, the probability that a large conformal rectangle is crossed is the same in both cases, and if one is conformally invariant in the scaling limit, the other also has to be. In short, if all the terms exist,

$$\alpha_T^{\text{Perc}} = \alpha_{T'}^{\text{Perc}}.$$

Looking at circle packings instead of percolation, we get the same identity (as was mentioned in the particular case of T_s), with a very similar proof: adding a vertex does not change anything to the rest of the picture, and we readily obtain

$$\alpha_T^{\text{CP}} = \alpha_{T'}^{\text{CP}}.$$

This leads us to the following hope, which we state as a conjecture even though it is much closer to being wishful thinking:

Conjecture 5. *Let T^* be a triangulation of the torus. Then, the critical parameter for site-percolation on its universal cover \hat{T}^* is equal to $1/2$, and for every $\alpha \in \mathbb{H}$, critical site-percolation on \hat{T}_α^* has a scaling limit. The value of the modulus α for which the model is conformally invariant in the scaling limit is that obtained from the circle packing associated to \hat{T}^* :*

$$\alpha_T^{\text{Perc}} = \alpha_T^{\text{CP}}.$$

— *Different notions of universality* —

This discussion would not be complete without comparing various meanings given to the term “universality”, or “universality class”, in the literature. Two discrete models can be said to belong to the same universality class if:

- They have exactly the same scaling limit (the same crossing probabilities in the limit, the same SLE behavior for their exploration curve. . .). This is what we have just been considering, and I would like to call it *geometric universality*; typically, the Ising models with appropriate coupling constants on two isoradial graphs are in the same geometric universality class.
- They have almost the same scaling limit, *i.e.* their scaling limits are a linear map away from each other. This is almost as strong as the previous version, and I would like to call it *strong universality*; typically, the simple random walks on two different lattices are in the same strong universality class.
- They have the same critical exponents, but nothing is stated about their scaling limits, not even existence; I would like to call it *scaling universality*. It is implied by strong universality under reasonable assumptions (like the box-crossing property), following [80]; besides, under the (much stronger) assumption that a conformally invariant scaling limit exists, it implies strong universality.

In a very impressive series of papers [31, 32], Grimmett and Manolescu use a version of the star-triangle relation to first obtain the box-crossing property for a wide family of critical bond percolation models, and then to show that, assuming that a critical exponent exists for two such models, it has the same value for both — a statement similar to scaling universality. Unfortunately, their argument seems not to show that crossing probabilities converge in the general case.

Interacting particle systems and self-interacting random walks

This chapter contains results and discussions on topics which do not directly belong to the field of two-dimensional lattice models, but rather to those of interacting particle systems and growth processes, either in dimension 2 (the once-reinforced walk, which we want to interpret as a growth process) or in dimension $1 + 1$ (Ulam's problem and columnar pinning, in relation with the slow-bond problem).

1. — THE SLOW-BOND PROBLEM

The problem was introduced above in the introduction, but before delving into it, I would like to describe a few related questions for which the geometric intuition of our approach is more apparent. This follows references [VB11] and [VB12], and is joint work with Herbert SPOHN, Vladas SIDORAVICIUS and Maria Eulália VARES.

1.1. — A FEW RELATED MODELS AND QUESTIONS

— *Longest increasing subsequences and polymer pinning* —

Let \mathcal{X} be a Poisson process of intensity λ in the plane \mathbb{R}^2 ; let $L > 0$. One basic question on polymer measures is the following: among all 1-Lipschitz functions $f : [0, L] \rightarrow \mathbb{R}$ equal to 0 at both 0 and L (the polymers), which one(s) has a graph passing through the largest number of points in \mathcal{X} , and how many points does it “collect”?

The discrete analog to the same question is known as Ulam's problem: let $n > 0$ and let $\sigma \in \mathfrak{S}_n$ be a permutation of the integers $[n] := \{1, \dots, n\}$ chosen uniformly at random; what is the largest subset of $[n]$ along which σ is increasing?

Let N_L be the largest number of points that a polymer of length L can collect. Looking at the question on the interval $[0, L + L']$ and restricting the polymer to be at 0 at L in addition to 0 and $L + L'$, one gets a super-additivity inequality:

$$N_{L+L'} \geq N_L + \tilde{N}_{L'}$$

where $\tilde{N}_{L'}$ is an independent copy of $N_{L'}$. From this, it is easy to derive the existence of a deterministic *velocity*

$$v(\lambda) := \lim_{L \rightarrow \infty} \frac{N_L}{L}.$$

The scaling property of Poisson measures shows that in addition, for all $\lambda > 0$,

$$v(\lambda) = \lambda^{1/2} v(1).$$

It is well known [5, 38] that for this model, $v(1) = \sqrt{2}$, and the fluctuations are in the Kardar-Parisi-Zhang universality class (in particular the variance of N_L is of order $L^{2/3}$).

Now, let \mathcal{X}' be a Poisson process of intensity μ on the horizontal axis. One can ask the same question as before: how many points of the union $\mathcal{X} \cup \mathcal{X}'$ can a polymer collect on the interval $[0, L]$? By the same super-additivity property, this still scales linearly in L and the velocity $v(\lambda, \mu)$ is well defined; it is obviously non-decreasing in each of its two variables. In particular, one can define a *critical point*:

$$\mu_c(\lambda) := \inf\{\mu : v(\lambda, \mu) > v(\lambda)\}.$$

The bound $v(\lambda, \mu) \geq \mu$ (obtained by considering the x -axis as a polymer) implies that $\mu_c(\lambda)$ is finite; the main question, which is of the same nature as that of the slow-bond problem, is whether $\mu_c(\lambda)$ is positive.

— *Hammersley's broken-line process* —

Given a realization of \mathcal{X} (or $\mathcal{X} \cup \mathcal{X}'$), one natural problem is to find an algorithm to compute N_L explicitly. One such algorithm works as follows. Interpret the plane \mathbb{R}^2 as space-time, the first coordinate being time. From each point of \mathcal{X} in the accessible square

$$Q_L := \{(t, x) \in \mathbb{R}^2 : t \in [0, L], |x| \leq \min(t, L - t)\},$$

launch a particle traveling at speed $+1$ and an anti-particle traveling at speed -1 ; whenever a particle and an anti-particle meet, remove them from the system. The union of the space-time paths of all particles and anti-particles creates a family of broken lines in Q_L ; the number of lines thus created is equal to N_L . (The construction can be performed in other domains than the square, cf. Figure IV.1, but then the interpretation of the number of lines in terms of polymers is not as clear).

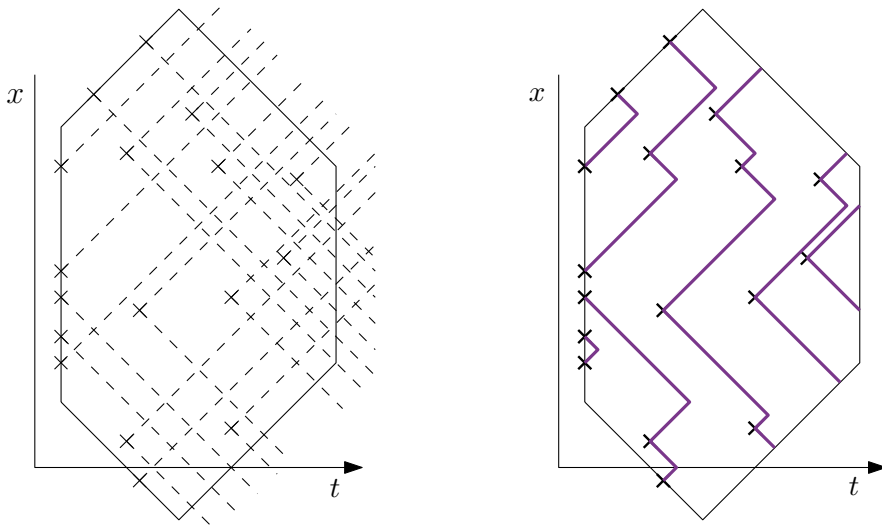


Figure IV.1. — The construction of Hammersley's broken-line process from a Poisson point process in a planar domain.

It is tempting to start from that construction and investigate what happens when adding points along the t -axis, and in fact, this leads to interesting results. First, consider

what happens when one adds one point (denoted by x in Figure IV.2). One can consider the additional pair of particles created at x as existing in the environment provided by the previous process, and see how the modification propagates by constructing the symmetric difference between the two versions step by step: essentially each of the two particles behaves as before in empty space, but it is caught by the lines of the preexisting process when it meets them.

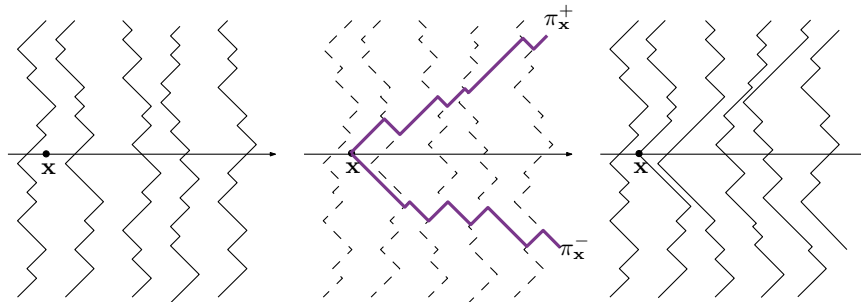


Figure IV.2. — The effect of adding a point on a Hammersley process.

This leads to a *zone of influence* of the point x , which ends when the two added particles annihilate each other. If this occurs while still inside Q_L , the geometry of the lines is modified locally but no large-scale effect is created, in particular the number of lines N_L is not modified; if on the other hand the additional particles exit Q_L , this increases the number of lines by 1. In the latter case, the number of points in a maximizing polymer is increased, and such a polymer in the new point system has to go through x .

Thus, the initial question can now be restated: one adds linearly many points along the t -axis, and asks whether linearly many additional lines are created; if such is the case, v is modified, and moreover we obtain geometric information on the maximizing polymer in $\mathcal{X} \cup \mathcal{X}'$: since it has to collect many points along the axis, it will become localized rather than having fluctuations of order $n^{2/3}$.

1.2. — RESTATEMENT IN TERMS OF A PERCOLATIVE SYSTEM

It is easy though to prove that, far away from the domain boundary, the probability that the influence of a single additional point extends to the boundary is very small; the only way v can be affected is thus through the collaborative effect of many points in \mathcal{X}' . Tracing the influence of just *two* points gives the right pictures, see Figure IV.3; two things can happen if two zones of influence intersect:

- They can meet externally, as for x_3 and x_4 in the figure. Then, two particles of different type touch, annihilate, and the zones of influence merge (and hence their range is extended);
- They be contained in one another, as for x_1, x_2 and x_5 in the figure. Then, as long as the innermost zone of influence survives, so do the outer ones — this is the main mechanism creating linearly many new lines: as soon as one particle pair survives, it is not very difficult to make a second pair survive by “leaning on the first one”.

Of course, the law of the duration of the influence of one point taken in isolation, even if it were known explicitly, is far from enough to obtain the behavior of the whole system: there is strong dependence between what happens to the different particle pairs in play, through the shape of the initial, unperturbed Hammersley process.

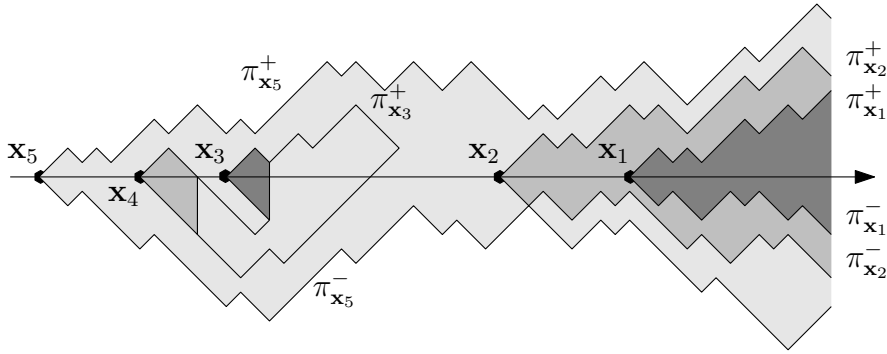


Figure IV.3. — The collaborative effect of the zones of influence created by several points along the axis.

Forgetting dependencies for a second leads to a cute toy-model which we termed *stick percolation*. Let $\alpha > 0$, let (X_n) denote the points of a Poisson process of intensity α in the positive half-line \mathbb{R}_+ , and let (L_n) be a sequence of i.i.d. random variables, also independent of the (X_n) . Define the random subset K of \mathbb{R}_+ as the union of all segments (or “sticks”) $[X_n, X_n + L_n]$. We say that the system *percolates* if K contains a half-line. This is a tail event, so it occurs with probability 0 or 1; in fact the threshold can be obtained explicitly:

Theorem 10. *Let $R : t \mapsto P[L_1 > t]$ denote the tail of the distribution of stick lengths, and let $\varphi(t) := \int_0^t R(u)du$. Then, stick-percolation percolates with probability 1 if and only if*

$$\int_0^{+\infty} e^{-\alpha\varphi(t)} dt < \infty.$$

The most interesting case within the toy-model is when $R(t)$ behaves as β/t when $t \rightarrow \infty$ (typically, if L_n is the absolute value of a Cauchy variable): then, $R(t) \sim \beta \log t$ and the integral converges if and only if $\alpha\beta > 1$, so there is a non-trivial critical point $\alpha_c = 1/\beta$ (which is not so common for a one-dimensional system).

Applying the same kind of ideas to the actual picture above, and studying *influence percolation* as a proxy to the pinning problem, is still ongoing. The promising fact is that there is much positive correlation in the system, so a comparison to the i.i.d. case should be possible — and besides, the tail of the length of one isolated zone of influence is much fatter than $1/x$, so we strongly support the following

Conjecture 6. *For every $\lambda > 0$ (or equivalently for $\lambda = 1$ by a scaling argument), $\mu_c(\lambda) = 0$.*

1.3. — RANDOMIZED PNG

There is one seemingly related model for which we were able to answer the question satisfyingly. Notice that a Hammersley process is a graphical way of representing polynuclear growth (PNG) on the real line; the Poisson points correspond to nucleations, and the number of lines separating (t, x) from the origin is the height of the interface at time t and location x . The additional Poisson process \mathcal{X}' can then be interpreted as an additional source of nucleations above the origin (a “columnar defect”), and the question is for which intensities it changes the growth velocity of the PNG interface.

One can modify PNG into what we called *randomized PNG* in [VB12] by making the paths of the particles involved in the Hammersley construction random rather than

deterministic; giving them a drift changes essentially nothing (and does not make the initial question any easier), so we chose to discretize space and have all particles move as continuous-time random walks jumping at rate 1.

The background particle pairs are created at each edge $\langle x, x + 1 \rangle$ between adjacent sites (one particle at $x + 1$, one anti-particle at x) at rate λ and the additional ones on the edge $\langle 0, 1 \rangle$ at rate μ . Particles and anti-particles annihilate whenever they meet — which can now also occur as soon as one of them is created.

The *height process* h is defined in a similar way as for PNG, but a little care is needed to make things coherent. It is defined on the edges of \mathbb{Z} , and its value on the edge $e = \langle x, x + 1 \rangle$ evolves in the following way:

- It increases by 1 when a nucleation occurs on e , when a particle jumps from x to $x + 1$ and when an anti-particle jumps from $x + 1$ to x ;
- It decreases by 1 when a particle jumps from $x + 1$ to x and when an anti-particle jumps from x to $x + 1$.

Again, one can show the existence of a growth velocity, defined as

$$w(\lambda, \mu) := \lim_{t \rightarrow \infty} \frac{h(t, \langle 0, 1 \rangle)}{t}$$

(there is no exact super-additivity though, so the proof of existence is a little more involved than in the previous cases). The function w is non-decreasing in each variable, and it is positive as soon as λ is, or for $\lambda = 0$, if and only if $\mu > 1$; in fact, it can be computed explicitly, which is our main result in this section:

Theorem 11 (B., Sidoravicius, Vares [VB12]). *For all $\lambda \geq 0$ and $\mu \geq 0$,*

$$w(\lambda, \mu) = \lambda + \max(\mu - 1, 0).$$

In particular, the critical value of randomized PNG is given by

$$\mu_c(\lambda) := \inf\{\mu : w(\lambda, \mu) > w(\lambda, 0)\} \equiv 1.$$

The phenomenological behavior of RPNG is rather different from the one expected from PNG. In the regime $\mu < \mu_c$ (which we believe does not exist at all for PNG) nothing much happens and there are renewal times at which the height function matches that of the system with the same value of λ and $\mu = 0$. When $\mu > 1$, a “needle” appears at the origin, but it only spreads diffusively along the x -axis.

This is in contrast with the PNG height function which, in the super-critical regime, is known to exhibit a linearly growing crystal around the additional nucleation point — which matches actual experiments, such as that produced by M. Myllys et al. ([64], cf. Figure IV.4).

1.4. — BACK TO THE INITIAL PROBLEM

The reasoning above is well adapted to pinning questions for polymers driven by a point process; up to predictable notational changes, it can be mapped to the case of (directed) last-passage percolation on a lattice. To do something similar for TASEP, the idea is starting the same way: one can couple, starting from the same initial configuration, a TASEP with jump rate 1 at every bond and one with a slow bond of rate $1 - \varepsilon$ at the origin, in such a way that any Poisson time of the second process is also a Poisson time for the first one.

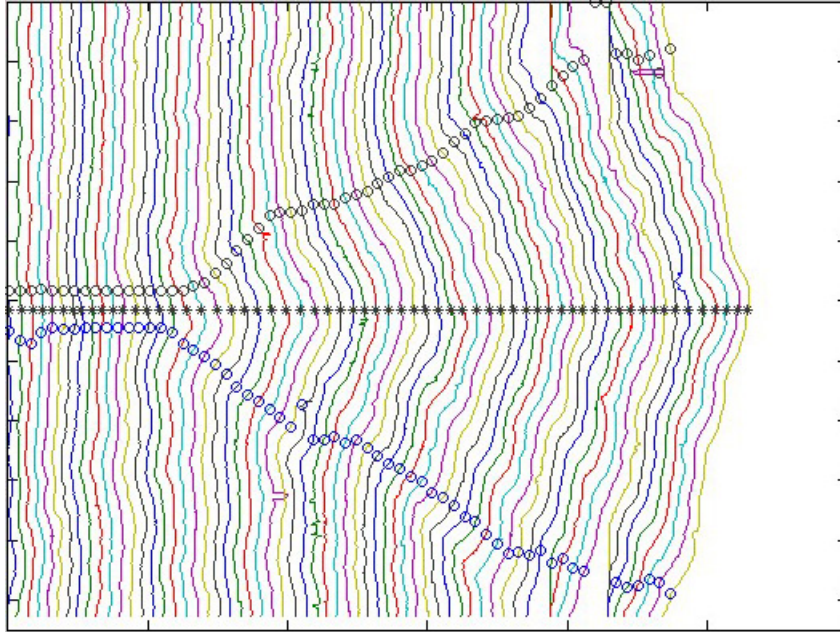


Figure IV.4. — [64] The effect of a local defect on the propagation of a slow-combustion front: the higher concentration of potassium nitrate along a horizontal line changes the shape of otherwise nearly vertical front lines. The higher velocity of propagation of the combustion front along this line affects the macroscopic picture, producing a faceted shape around the defected region.

For a time of order $1/\varepsilon$, the two processes will agree; then a first *decoupling* will occur, when a slow particle does not jump with its fast counterpart. Such a situation is customarily encoded using *second-class (anti-)particles*, in an extended configuration space that can be chosen as $\{0, 1, +, -\}^{\mathbb{Z}}$: at a given site n , 0 and 1 mean that both of the coupled system agree, + signifies that the fast system has a particle but the slow one does not, and – the opposite case.

The dynamics can be transposed to this state space: essentially, all non-0 (*i.e.*, occupied) sites carry a Poisson clock driving jumps to the right, there is an additional source of pairs $(-, +)$ with rate ε on the edge $\langle 0, 1 \rangle$, and particles with label 1 have priority over (are not excluded by) ones labeled + or –. In addition, whenever a – particle attempts a jump onto a + particle, they merge at the target site into a standard particle — meaning that the system *re-couples* locally.

Now, one could copy Figure IV.3 here and re-label it to state that it represents the space-time paths of second-class particles in the coupling; ε here plays the same role as μ there, and the qualitative features of the collaborative behavior is very similar.

Of course, a lot is missing, mostly because the path of a second-class particle is a very dependent process. Nevertheless, the physics behind the models can be compared and the heuristics from stick-percolation applied:

- If the density λ of particles in the initial configuration is not $1/2$, second-class particles have a drift equal to $1 - 2\lambda \neq 0$. A given pair might take a long time to re-couple, but if it does that will occur far away from the origin, and the time for which the span of the pair covers the edge $\langle 0, 1 \rangle$ will be small (integrable of expectation of order $|1 - 2\lambda|^{-1}$). This prevents collaboration if the creation rate of such pairs is small enough, say much smaller than $|1 - 2\lambda|$, and in that case ε_c should be positive.

- If λ is $1/2$, second-class particles have no drift and hence they will remain in the vicinity of the origin for a much longer, non-integrable time (that is closer to the polymer picture of Figure IV.3). Then, collaboration will occur for every positive value of ε , so ε_c should be 0.

Stick-percolation in itself is too simple a model to apply directly here, but it is quite possible that there would be a stronger version of the proof leading to the same behavior for positively correlated stick lengths. What is really missing is a way to exploit collaborative effects within the same framework, possibly having sticks “rebound” if their endpoint is within another stick born later in time. We understand quite well how to do something similar in the case of RPNG (even though the final version of the argument does not make use of it); it is unclear how much detailed information on the second-class particle behavior is needed to make the proof go through.

1.5. — RESTATEMENT AS A UNIVERSALITY QUESTION FOR THE TASEP

Looking at the Hammersley process with additional source at the origin, a natural operation is to split it into two half-space systems. The behavior of one half-space system taken in isolation (with a bulk source of intensity 1 and boundary source of intensity μ , both Poisson) is much better understood: Baik and Rains [7] prove in this case that $\mu_c = v(1)/2$ and they compute the growth velocity $\tilde{v}(\mu) > \mu$ explicitly above criticality. This has a geometric interpretation: without boundary source, all the broken lines exit the system at the boundary in the same direction; at criticality, (asymptotically) half of them touch the boundary at one of the boundary source points.

Now, for the bulk process, one can try what I called the *sewing procedure*, which goes as follows.

- First construct the Hammersley process for the restriction of the bulk process to one half-space. It lies in the range of Baik and Rains’ result, and so has growth velocity $v_0 = \tilde{v}(\mu)$. Out of all the broken lines, intensity μ come from source points and so intensity $\tilde{v}(\mu) - \mu$ are “free” to exit the half-space.
- Now look at the system in the other half-space, and interpret the previous step as building a boundary source process with intensity $\mu_0 = v_0$. This process is **not** Poisson — but let us ignore that for a moment. Performing the Hammersley construction using that source and the initial bulk process creates a growth velocity $v_1 = \tilde{v}(\mu_0)$. Out of all broken lines, intensity μ_0 correspond to source points, and intensity $v_1 - \mu_0$ are “new” lines exiting the system.
- Look at the first half-space again. There is now a compound source on the boundary: intensity μ from the initial source, and intensity $v_1 - \mu_0$ from the previous step of the construction, so the combined source intensity is $\mu_1 = (v_1 - \mu_0) + \mu$. Still ignoring that it is not Poisson, this leads to an updated growth velocity $v_2 = \tilde{v}(\mu_1)$.
- Back in the second half-space, now the intensity of the source is μ from the initial process, plus $v_2 - \mu_1$ from the previous step, which add up to $\mu_2 = (v_2 - \mu_1) + \mu$. Continuing in this vein leads to the recursion

$$\begin{cases} v_{k+1} &= \tilde{v}(\mu_k), \\ \mu_{k+1} &= (v_{k+1} - \mu_k) + \mu. \end{cases}$$

Combining those we get a simple, explicit recursion for (μ_k) :

$$\mu_{k+1} = \mu + \tilde{v}(\mu_k) - \mu_k. \tag{IV.1}$$

The construction converges locally, so one expects the growth velocity of the whole-plane process to be the unique fixed point of (IV.1).

This all fits with the physics heuristics described earlier, and indeed with the conjecture that $\mu_c = 0$. The main obstacle is that, as soon as the second step of the construction, the source process ceases to be Poisson, so Baik and Rains' theorem does not apply (its proof is based on explicit computations and relies heavily on the exact nature of the boundary point process).

It is reasonable to hope though that *any* boundary source satisfying e.g. an ergodic principle would lead to a law of large numbers for the number of Hammersley lines in the half-space. The catch is that, if the source has asymptotic intensity μ , the velocity v has no reason to be equal to $\tilde{v}(\mu)$ in the non-Poisson case, and in fact it is easy to construct examples for which it is not.

There is a natural notion of *effective source intensity* $\tilde{\mu}$ defined by the equation

$$v = \tilde{v}(\tilde{\mu}),$$

but computing it explicitly in the general case seems hard. The universality question in that setup is this: *for which source processes is it the case that $\tilde{\mu} = \mu$?* In other words, what are the point processes which are indistinguishable from Poisson processes as far as velocity growth is concerned?

If it turns out that mild dependency properties are enough (and there are reasons to believe that), then the sewing procedure will be a credible way of answering the initial pinning question.

1.6. — TASEP WITH A COMPLEX BOUNDARY MECHANISM

One related question is a subject I proposed to my student Nicky SONIGO. Consider a TASEP in the half-line \mathbb{Z}_+ , with a Poisson source of particles of intensity $\lambda < 1/2$ at the origin, independent of the dynamics, and the space empty at time 0. Then, the distribution of the configuration converges, as time increases, to a product of Bernoulli variables of intensity λ (which is invariant under the dynamics), and the number N_t of particles present in the system at time t — which is the same as the integrated particle flow at the origin — satisfies a strong law of large numbers with speed $\lambda(1 - \lambda)$.

It is natural to ask what happens if the particle source is not a Poisson process. Keeping it independent of the dynamics leads to artificial systems where the behavior is not stationary in time; rather, Grosskinski proposed to have it locally Poisson, but with an instantaneous intensity depending on the current configuration. Typically, the Poisson times of the source could be taken out of a Poisson process with intensity λ_u where $u \in \{0, 1\}$ is the current state of the site at location 18.

This has the effect that, if $\lambda_0 \neq \lambda_1$, no product measure is invariant anymore; like in the slow-bond case, all the explicit methods break down, and even showing that there is a law of large numbers for the particle flow is nontrivial. Nevertheless, Nicky was able to prove it:

Theorem 12 (Sonigo). *Let $\bar{\lambda} < 1/2$, let $K \in \mathbb{Z}_+$ and fix $\lambda : \{0, 1\}^K \rightarrow [0, \bar{\lambda}]$. One can define a TASEP $(\eta_t)_{t \geq 0}$ with empty initial condition and source intensity at time t given by*

$$\lambda_t = \lambda(\eta_t(0), \dots, \eta_t(K)).$$

If N_t denotes again the number of particles in the system at time t , there exists $v = v(\lambda) \in [0, \bar{\lambda}(1 - \bar{\lambda})]$ such that, almost surely,

$$\frac{N_t}{t} \rightarrow v.$$

If the initial condition is not empty but still dominated by a product of Bernoulli variables of parameter $\tilde{\lambda}$, the same holds and the speed is not affected. Again one can define an effective intensity $\tilde{\lambda}$ for the source by the equation $v = \tilde{\lambda}(1 - \tilde{\lambda})$, but explicit computation in the general case seems impossible.

Interesting things happen in the case of infinite range (*i.e.* $K = +\infty$). The canonical case is asymptotic density:

$$\lambda(\eta) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \eta_t(k)$$

for which *any* product measure with intensity $\lambda < 1/2$ is invariant; in that case, the speed depends not only on the source mechanism but also on the initial configuration. Tamer cases seem to exist and exhibit metastability, though work on this is still in progress.

2. — SELF-INTERACTING RANDOM PROCESSES

This last section contains reports on two works in the field of self-interacting random processes, which are in very different stages of completion: the first one — on the prudent random walk — is essentially over, while the second one — on the once-reinforced random walk — is far from it, even though we were still able to obtain a few positive results in the direction that we hoped.

2.1. — THE PRUDENT RANDOM WALK

— *Statement of the results* —

Let $(X_n)_{n>0}$ be a kinetic prudent random walk in \mathbb{Z}^2 , as defined in the introduction. Our main result, from which the others derive, is that it is a.s. ballistic with deterministic, explicit speed:

Theorem 13 (B., Friedli, Velenik [VB9]). *For every $\varepsilon > 0$,*

$$P \left[\left| \frac{\|X_n\|_1}{n} - \frac{3}{7} \right| \geq \varepsilon \right] = \mathcal{O}(n^{-1/10});$$

otherwise stated, the L^1 -norm of (X_n) satisfies a weak law of large numbers.

In terms of the L^2 -norm by contrast, no law of large numbers is satisfied, and $\|X_n\|_2/n$ remains random in the limit; in fact, the scaling limit of the walk, as the lattice mesh vanishes, is itself nontrivial and can be seen as a jump process with non-integrable jump length. One way to realize the scaling limit uses the excursion decomposition of real Brownian motion, as follows: first, choose a quadrant of \mathbb{Z}^2 at random (say, the north-eastern one); then sample a real Brownian motion $(B_t)_{t>0}$. For each excursion of (B_t) away from 0, toss a Bernoulli(1/2) random variable, independently for different excursions; and according to the outcome, jump either to the north or to the east of a length equal to $3/7$ times the duration of the excursion.

This builds a rectifiable random curve in the plane; the scaling limit of the prudent walk follows this curve with speed $3/7$ (measured in terms of curvilinear coordinate, which here matches the L^1 -norm by construction) — see Figure IV.5 for a large scale picture of a prudent walk.

for the walk to visit two adjacent corners one after the other, it will have to crawl along the corresponding side of the rectangle; and while its location along that side is close to being a symmetric random walk, and in particular is diffusive, the drift in the orthogonal direction is positive the whole time so the other dimension of the rectangle will increase by a very large amount.

If at the beginning of the crossing the length of the side is of order L , the crossing itself will take a time of order L^2 and the other side of the rectangle will have increased by an amount also of order L^2 . On the other hand, while the process remains in the vicinity of one of the corners, it will still alternate between excursions into the two neighboring sides, and this again will make the envelope grow linearly in time.

To conclude, it is enough to see that the distance between the prudent walk and the corner it is in the neighborhood of is always diffusive, while the distance to the nearest corner will, by the previous discussion, increase linearly: there is hence a positive probability that it will only ever visit one of the four corners, and a Borel-Cantelli type argument then implies that only one of the four corners will be visited infinitely many times — which is what we were after.

— *Possible extensions* —

There is another version of the prudent walk, which is closer to polymer measures: one can look at the uniform measure on paths of a given length n satisfying the same condition that, for every $k < n$, the half-line $[X_k, X_{k+1})$ does not intersect the past $X_{[0, k-1]}$. This differs from our “dynamical” version in a similar way as the self-avoiding walk does from the loop-erased random walk. For this version, simulations suggest that the scaling limit is a straight line, and in particular (given the quadrant in which it lives) it seems to be deterministic; our method has no hope of applying there because the law of a walk of length n is not the same in that case as that of the first n steps of a walk of length $n + 1$.

There is another direction in which our approach could still lead to something interesting, namely the three-dimensional analog of the process. This can mean two things, one easy to study and one much more subtle:

- One can forbid the half-line $[X_n, X_{n+1})$ from intersecting the parallelepipedic envelope of $X_{[0, n-1]}$. Then our general method still applies, there is not much geometry to keep track of in the system, and a similar result should hold, but the process is rather artificial;
- One can use the previous definition and forbid the half-line from intersecting the path $X_{[0, n-1]}$ itself (the two definitions match in dimension 2, but obviously not in dimension 3). Here, simulations suggest an interesting intermittent behavior. First, the walk seems to look like a standard random walk, transient but diffusive; this occurs as long as its local density remains small, but once it builds a fat enough droplet, it notices and becomes ballistic for a while, in one of the lattice directions, for as long as it takes to exit the “shadow” of the droplet; then it switches back to the random walk behavior, and so on.

The second version sounds much more interesting than the first, and the basic questions of diffusivity has no clear answer in that case; it will depend on the integrability properties of the lengths of the ballistic parts, which we were not able to estimate.

2.2. — ONCE-REINFORCED RANDOM WALK

This last section describes ongoing joint work with Vladas SIDORAVICIUS about a particular self-interacting process. The model we are interested in is a particular continuous-time version of once-reinforced random walk (or ORRW for short) on the integer lattice \mathbb{Z}^2 . We

choose reinforcement by vertices, although it is very likely that all the results mentioned here would still hold for the corresponding edge-reinforced process. Here is an informal definition of the process. Let $a > 0$ be a positive parameter, which should be considered as large (in a sense to be made precise later).

The discrete-time version of the process starts at the origin; whenever it is at site x , for every neighbor x' of x the process can jump to x' with a probability proportional to 1 if x' has never been visited before, and to $1 + a$ if it has been. In other words, the process is attracted by its previous path.

We chose to rather work with the continuous-time version of the process, defined as follows. $(X_t)_{t \geq 0}$ starts at $X_0 = 0$; whenever it is at site x , for every neighbor x' of x the process can jump to x' with an intensity 1 if x' has been visited before, and $(1 + a)^{-1}$ if it has not. This particular choice of jump rates has the effect that within a large visited domain X behaves like a standard random walk, and that the value of a is only relevant when discovering new sites — and of course the sequence of jumps of X is exactly the same in distribution as for its discrete-time counterpart.

The main question is whether the once-reinforced random walk is recurrent or transient. While recurrence seems like the natural conjecture, no proof is currently known. What I want to present here is work in progress leading to the following:

Conjecture 7. *There exists a positive constant $a_0 > 0$ such that, whenever $a > a_0$, the process (X_t) is recurrent. Moreover, if K_t is the set of points visited by time t , then almost surely*

$$|K_t| \approx t^{2/3}, \quad \text{diam } K_t \approx t^{1/3},$$

and the rescaled set $K_t / \text{diam } K_t$ converges in probability, in the Hausdorff topology, to a deterministic asymptotic shape K_∞ .



Figure IV.7. — Once-reinforced random walks for increasing values of the parameter a (in order: 1, 1.5, 1.75, 2) in a square box, up to their first hitting time of the boundary.

If the conjecture holds, it is very likely that in fact $a_0 = 0$, *i.e.* that arbitrarily small reinforcement is enough to put the process in this sub-diffusive regime. What happens in higher dimension is more dubious: for large a the argument seems to still apply, leading to a diameter growth of order $t^{1/(d+1)}$; on the other hand, it is quite possible that a_0 is positive, because the unperturbed walk is transient.

— **General strategy: ORRW as a growth process** —

To get a feeling of our strategy, we now give a second, equivalent description of the process. Let (Y_t) be a continuous-time random walk in \mathbb{Z}^2 , which jumps with rate 1. One can couple (X_t) and (Y_t) in the following way: whenever (Y_t) jumps, say from y to $y + e$, look at the state of site $X_t + e$. If it is on the previous path of X , let $X_{t+} = X_t + e$ (*i.e.* let X

jump with Y); if it is a site not visited yet by the process, sample a Bernoulli variable Z of parameter $1/(1+a)$, and jump if and only if $Z = 1$. It is easy to check that the process (X_t) obtained in this construction is the same as the one we are interested in.

Consider the case where a is extremely large. This means that, in first approximation, (X_t) should look like a reflected version of (Y_t) . Then one expects the process to behave as follows: given that $K_t = K$ is small enough, for a long time X_t will behave exactly like the reflected random walk in K (*i.e.* the random walk whose jumps outside of K are suppressed). Then, at some random time, depending on the value of a , it will be allowed to discover a new site along the boundary of K . If a is large enough, the typical time between discoveries of new sites should be longer than the averaging time of the reflected walk, meaning that the process (K_t) would be almost Markovian — in fact, for “infinite a ” it would be a simple growth process.

The main heuristic is then that, even on longer time-scales, (X_t) should still exhibit many of the same features as for small times, and that considering it like a growth process rather than random-walk-like object is the right approach to the problem. This way of addressing the question seems to be new; Vervoot’s approach in the one-dimensional case [84, 83] is not far in spirit, but the argument used by Durrett, Kesten and Limic [24] in the case of the tree is much more computational and closer to usual RWRE methods.

— The two-step Eden model —

Let K be a finite connected subset of \mathbb{Z}^2 . Let (Z_t) be a continuous-time random walk on K (or equivalently, a simple random walk on \mathbb{Z}^2 reflected on the outer boundary of K). It is well known that, seen as a Markov process, (Z_t) is irreducible, and that its invariant measure μ_K assigns to each point in K a mass which is proportional to its degree in K (*i.e.* to the number of neighbors it has in K). If $y \in \mathbb{Z}^2 \setminus K$, let

$$\nu_K(y) := \sum_{x \in K, x \sim y} \mu_K(x).$$

According to the heuristic considerations above, one expects that, at least if a is very large with respect to K , and up to a multiplicative factor of order $1/a$, $\nu_K(y)$ is the rate at which site y is “discovered” by the ORRW. This leads us to the following definition:

Definition 1 (two-step Eden models). *We will call discrete-time two-step Eden model the growth process (A_n) defined as follows: $A_0 = \{0\}$, and for all $n \geq 0$, A_{n+1} is obtained by adding to A_n a point of $\partial_e A_n$ chosen according to the probability measure $\frac{1}{\|\nu_{A_n}\|} \nu_{A_n}$.*

We will call continuous-time two-step Eden model (or FPP’ for short, because it is related to first-passage percolation with exponential waiting times) the natural continuous-time counterpart (B_t) of the previous process, where points are added with rate ν ; in other words, it is the process with generator given, for any finite connected subset K of \mathbb{Z}^2 containing the origin, by

$$[\mathcal{L}f](K) := \sum_{y \in \partial_e K} \nu_K(y) [f(K \cup \{y\}) - f(K)].$$

Recall that the usual Eden model is defined by the addition of a new site uniformly on the boundary of the current cluster; in the case of the discrete-time two-step Eden model, one chooses a two-step chain (x', x, y) with x and x' in the current cluster and y on its outer boundary, uniformly among such chains, hence the name of the process. It is easy to apply standard sub-additivity methods to prove a shape theorem for FPP’:

Theorem 14. *Let (B_t) be a continuous-time two-step Eden model in \mathbb{Z}^d . There exists a compact, convex subset \tilde{K} of \mathbb{R}^2 , with all the symmetries of \mathbb{Z}^2 , such that for every $\varepsilon > 0$,*

$$P \left[(1 - \varepsilon)\tilde{K} \subset t^{-\frac{1}{d+1}} B_t \subset (1 + \varepsilon)\tilde{K} \right] \xrightarrow{t \rightarrow \infty} 1.$$

In other words, $t^{-1/(d+1)} B_t$ converges to \tilde{K} in probability for the Hausdorff metric on compact subsets of \mathbb{R}^2 ; convergence actually holds almost surely.

The only point about proving this theorem is to obtain the exponent $\frac{1}{d+1}$; if μ was defined to be the degree measure rather than its normalized version, then the very same argument as for first-passage percolation argument would give a linear growth of the diameter of B_t and of course the same asymptotic shape, so at least we already know something about the geometry of B_t , all that needs to be controlled is the speed of growth. Let R_t be the diameter of B_t . The rate at which the volume $|B_t| \approx R_t^d$ grows is the total mass of ν_{B_t} , which is of order $|\partial B_t|/|B_t|$ i.e. of order R_t^{-1} ; matching exponents leads to the claimed growth rate $R_t \approx t^{1/(d+1)}$. The discrete-time version is actually easier to state, because the volume grows linearly by construction:

Corollary 2. *Let (A_n) be a discrete-time two-step Eden model: then, there exists a positive constant $\lambda > 0$ such that, almost surely,*

$$\frac{1}{\sqrt{n}} A_n \rightarrow \lambda \tilde{K}.$$

— **The scaling Ansatz** —

For a moment, we still work in the general case of a once-reinforced walk in \mathbb{Z}^d for $d \geq 1$, because it makes the argument nicer.

Fix $a > 0$, very large, and let $L > 1$ be a mesoscopic scale: going to infinity with a , but small enough that the mixing time of any connected set of diameter at most L is still much smaller than a — something like $L = a^{1/(2d+1)}$ is a natural choice. As long as K_t is smaller than L , it behaves like a continuous-time two-step Eden model, but the rate at which new sites are visited is $\frac{1}{a} \nu_{K_t}$ rather than ν_{K_t} . In other words, we know (stated informally) that K_t looks like

$$\left(\frac{t}{a} \right)^{\frac{1}{d+1}} \tilde{K}$$

as long as $(t/a)^{1/(d+1)} < L$, i.e. as long as $t < aL^{d+1}$.

Split the lattice \mathbb{Z}^d into hypercubes of width L . A given hypercube can be in one of three states:

- Empty, if the process (X_t) has not visited it yet;
- Partially filled, if it has been visited but not all its vertices have;
- Full, if all its vertices belong to K_t .

Because L is smaller than the scale at which K_t possibly stops looking like a growth process, one expects that most mesoscopic boxes are either empty or full. Besides, as long as a given box \mathcal{B} is partially filled, one can focus at the process within it: it is still at the scale at which we know that the geometry should be reasonably smooth. Let $L_t(\mathcal{B})$ be the local time in the box,

$$L_t(\mathcal{B}) := \int_0^t \mathbb{1}_{X_s \in \mathcal{B}} ds.$$

The growth of the visited portion of \mathcal{B} should still be given by the comparison to the two-step Eden model, in terms of the local time, and in particular, it should become filled when $L_t(\mathcal{B})$ is of order aL^{d+1} . Besides, every time X_t visits \mathcal{B} when it has been explored significantly, it will tend to spend an expected time of order L^2 in it before exiting for the first time, by Brownian scaling. In other words, the typical number of visits to \mathcal{B} needed to make it full should be of order

$$a' := \frac{aL^{d+1}}{L^2} = aL^{d-1}.$$

Let $X'_t := \lfloor X_t/L \rfloor \in \mathbb{Z}^d$ denote the box containing X_t . Within the bulk of full boxes, it looks very much like a time-changed random walk, with Brownian scaling making the new time-variable $t' := t/L^2$ natural. The previous discussion then states that $(X'_{t'})$ should look like a once-reinforced random walk of parameter a' (which is much larger than a).

Admittedly, the above is highly speculative and very far from constituting a proof of anything. Nevertheless, one sign seems to indicate that there is something to it. Let $K'_{t'}$ be the set of full boxes by rescaled time t' . If the above holds, it gives the growth rate

$$\text{diam } K'_{t'} \simeq \left(\frac{t'}{a'} \right)^{\frac{1}{d+1}};$$

writing the rescaled variable in terms of the original ones leads to the same scaling as in the statement of Theorem 14. The coarse-graining scheme is then quite clear: the macroscopic once-reinforced walk matches the initial one over a range of scales, and the growth process over a much larger range; this indicates that the initial walk itself should match the behavior of the growth process for a longer time than aL^{d+1} .

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