Faster Algorithms for List-Decoding Reed-Solomon Codes Using Structured Matrix Computations

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Outline

1. Unique decoding via approximation
   - Encoding and transmission
   - Unique decoding
   - Berlekamp-Welch(-like) algorithm

2. List-decoding Reed-Solomon codes
   - List-decoding
   - The interpolation step (previous work)

3. List-decoding via approximation
   - From interpolation to approximation
   - Solving the approximation problem using structured matrices
   - Extension to the multivariate case (folded Reed-Solomon codes)
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Error-correcting codes

Goal:
Enable **reliable** delivery of data over **unreliable** communication channels

Strategy:
add **redundancy** to the message
add **redundancy** to the message
add **redundancy** to the message

(courtesy of J.S.R. Nielsen)
Encoding: adding redundancy

All intended words

\[(w_0, \ldots, w_k)\]

\[\rightarrow\]

All code words

\[(c_1, \ldots, c_n)\]

\[w = w_0 + w_1 X + \cdots + w_k X^k\]

\[\rightarrow\]

their evaluation at \(x_1, \ldots, x_n\)

\[w(x_1), \ldots, w(x_n)\]
Transmission over an unreliable channel

Assumption: there are \textbf{at most} \( e \) errors during transmission of a code word

\[
c = (c_1, \ldots, c_n) \xrightarrow{\text{noise}} y = (y_1, \ldots, y_n)
\]

with \( \#\{i \mid c_i \neq y_i\} \leq e \) \hspace{1cm} \text{(metric called Hamming distance)}

\( = \) code word

\( \bullet = \) received word
Transmission over an unreliable channel

Assumption: there are at most $e$ errors during transmission of a code word

$$c = (c_1, \ldots, c_n) \xrightarrow{\text{noise}} y = (y_1, \ldots, y_n)$$

with $\#\{i \mid c_i \neq y_i\} \leq e$ (metric called Hamming distance)

Reed-Solomon code:

$$(w(x_1), \ldots, w(x_n)) \xrightarrow{\text{noise}} (y_1, \ldots, y_n)$$

with $\#\{i \mid w(x_i) \neq y_i\} \leq e$

$(y_1, \ldots, y_n)$ is the received word

All possible received words $=$ words in the balls of radius $e$ centered on the code words
Transmission over an unreliable channel

Assumption: there are at most $e$ errors during transmission of a code word

$$c = (w(x_1), \ldots, w(x_n)) \xrightarrow{\text{noise}} y = (y_1, \ldots, y_n)$$

with $\# \{ i \mid w(x_i) \neq y_i \} \leq e$ (metric called Hamming distance)

Possible received words
Unique decoding

Received word \((y_1, \ldots, y_n)\)

Decoding
find a polynomial \(w\) of degree \(\leq k\) such that \(\#\{i \mid w(x_i) \neq y_i\} \leq e\)

Well-defined?
Exactly one such polynomial \(w\) as long as no overlap between the balls of radius \(e\) centered on the codewords
Unique decoding

Received word \((y_1, \ldots, y_n)\)

**Decoding**
find a polynomial \(w\) of degree \(\leq k\)
such that \(#\{i \mid w(x_i) \neq y_i\} \leq e\)

**Well-defined?**
Exactly one such polynomial \(w\) as long
as no overlap between the balls of radius \(e\) centered on the codewords

**Unique decoding**
when
\[2e < d_{\text{min}}\]
Unique decoding

Received word \((y_1, \ldots, y_n)\)

**Decoding**

find a polynomial \(w\) of degree \(\leq k\) such that 

\[
\# \{ i \mid w(x_i) \neq y_i \} \leq e 
\]

**Well-defined?**

Exactly one such polynomial \(w\) as long as no overlap between the balls of radius \(e\) centered on the codewords

**Unique decoding**

when

\[
2e < d_{\text{min}}
\]
Minimum distance

For Reed-Solomon codes:

- for \( w_1 \neq w_2 \) polynomials of degree \( \leq k \) over the base field \( \mathbb{K} \),
  \((w_1(x_1), \ldots, w_1(x_n))\) and \((w_2(x_1), \ldots, w_2(x_n))\) agree at \( \leq k \) positions
  \( \Rightarrow \) distance at least \( n - k \) between two code words

- for \( w_1 = 0 \) and \( w_2 = (X - x_1) \cdots (X - x_k) \), the code words are
  \((0, \ldots, 0)\) and \((0, \ldots, 0, w_2(x_{k+1}), \ldots, w_2(x_n))\)
  \( \Rightarrow \) two code words at distance exactly \( n - k \)

\( \implies \) minimum distance \( d_{\text{min}} = n - k \)

Hence the unique decoding condition: \( e < \frac{n - k}{2} \)
Unique decoding problem

Unique decoding of Reed-Solomon codes

Input:
\[ x_1, \ldots, x_n \text{ the } n \text{ distinct evaluation points in } \mathbb{K}, \]
\[ k \text{ the degree bound, } e \text{ the error-correction radius}, \]
\[ (y_1, \ldots, y_n) \text{ the received word in } \mathbb{K}^n \]

Unique decoding assumption: \( e < \frac{n-k}{2} \)

Output:
The polynomial \( w \) in \( \mathbb{K}[X] \) such that

\[ \deg w \leq k \quad \text{and} \quad \# \{ i \mid w(x_i) \neq y_i \} \leq e. \]
Unique decoding via approximation

Berlekamp-Welch(-like) algorithm

Key equations (unique decoding)

Define the interpolation polynomial

\[ R(X) \text{ such that } R(x_i) = y_i, \]

and the error-locator polynomial

\[ \Lambda(X) = \prod_{i \mid \text{error}} (X - x_i). \]

\( \Lambda(X) \) is an unknown polynomial with \( \deg\Lambda \leq e \)

Key equations

for every \( i \), \[ \Lambda(x_i)R(x_i) = \Lambda(x_i)w(x_i) \]

Quadratic equations in the unknown coefficients of \( w \) and \( \Lambda \)…
Modular key equation (unique decoding)

Recall the interpolation and error-locator polynomials

\[ R(x_i) = y_i, \quad \Lambda(X) = \prod_{i \mid \text{error}} (X - x_i) \]

Key equations

for every \( i \), \( \Lambda(x_i) R(x_i) = \Lambda(x_i) w(x_i) \)

i.e. for every \( i \), \( \Lambda(X) R(X) = \Lambda(X) w(X) \mod (X - x_i) \)
Modular key equation (unique decoding)

Recall the interpolation and error-locator polynomials

\[ R(x_i) = y_i, \quad \Lambda(X) = \prod_{i \mid \text{error}} (X - x_i) \]

Key equations

for every \( i \), \( \Lambda(x_i) R(x_i) = \Lambda(x_i) w(x_i) \)

i.e. for every \( i \), \( \Lambda(X) R(X) = \Lambda(X) w(X) \mod (X - x_i) \)

Define the master polynomial

\[ G(X) = \prod_{1 \leq i \leq n} (X - x_i) \]

Modular key equation

\[ \Lambda(X) R(X) = \Lambda(X) w(X) \mod G(X) \]
Reduction to rational reconstruction

Modular key equation:

\[ \Lambda R \equiv \Lambda w \mod G \]

where \( R(x_i) = y_i \), \( G(X) = \prod_{1 \leq i \leq n}(X - x_i) \), \( \Lambda(X) = \prod_{i \mid \text{error}}(X - x_i) \).

\[ \implies \lambda = \Lambda, \omega = \Lambda w \text{ form a solution of the rational reconstruction problem} \]

\[ \begin{cases} 
\lambda R = \omega \mod G, \\
\deg(\lambda) \leq e, \quad \deg(\omega) < n - e, \quad \lambda \text{ monic.}
\end{cases} \]

(since \( \deg \Lambda w \leq e + k < n - e \) by the unique decoding assumption)

[Modern Computer Algebra, von zur Gathen - Gerhard, 2003]
Berlekamp-Welch(-like) algorithm for unique decoding

\[ \lambda = \Lambda, \omega = \Lambda w \] form a solution of the rational reconstruction problem

\[
\begin{align*}
\lambda R &= \omega \mod G, \\
\deg(\lambda) &\leq e, \quad \deg(\omega) < n - e, \quad \lambda \text{ monic.}
\end{align*}
\]

\[ \Rightarrow \text{unique rational solution } \omega/\lambda, \text{ which has to be } \frac{\Lambda w}{\Lambda} = w ! \]

This solution is computed using the extended Euclidean algorithm in \( O^\sim(n) \) operations in \( K \)

**Conclusion:**
unique decoding in quasi-linear time via an approximation problem
List-decoding Reed-Solomon codes

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Non-unique decoding

How to “decode” when more errors?

transmission with \( \leq e \) errors

where \( e \geq \frac{d_{\min}}{2} \)

\[
\begin{align*}
\text{\( \cdot \)} &= \text{code word} \\
\text{\( \cdot \)} &= \text{received word}
\end{align*}
\]
Non-unique decoding

How to “decode” when more errors?

transmission with $\leq e$ errors where $e \geq \frac{d_{\text{min}}}{2}$

possibly two (or more) code words at the same distance.

the closest code word is not necessarily the one which was sent.
Non-unique decoding

How to “decode” when more errors?

transmission with $\leq e$ errors
where $e \geq d_{\text{min}}/2$

possibly two (or more) code words at the same distance...

the closest code word is not necessarily the one which was sent...

$\Rightarrow$ Return a list of all code words at distance $\leq e$
(called list-decoding)
Problem

For convenience, we use the agreement parameter $t = n - e$

List-decoding Reed-Solomon codes

Input:
- $n$ points $\{(x_i, y_i)\}_{1 \leq i \leq n}$ in $\mathbb{K}^2$, with the $x_i$’s distinct
- $k$ the degree constraint, $t$ the agreement

List-decoding assumption: $t^2 > kn$ [Guruswami - Sudan 1999]

Output:
- all polynomials $w$ in $\mathbb{K}[X]$ such that
  \[
  \text{deg } w \leq k \quad \text{and} \quad \# \{i \mid w(x_i) = y_i\} \geq t.
  \]

Problem also called Polynomial Reconstruction
Polynomial Reconstruction (1/2)

Figure: Polynomial reconstruction (Lagrange interpolation)

\[ R(x) \]

degree \( \leq 4 \)
agreement \( \geq 5 \)
Polynomial Reconstruction (1/2)

Figure: Polynomial reconstruction
Polynomial Reconstruction (1/2)

Figure: Polynomial reconstruction (all solutions)
Consider one solution $w_1$; we still have the modular key equation

$$\Lambda_1 R = \Lambda_1 w_1 \mod G$$

where

$$R(x_i) = y_i, \quad G(X) = \prod_{1 \leq i \leq n} (X - x_i), \quad \Lambda_1(X) = \prod_{i \mid \text{error}_1} (X - x_i).$$

But possibly,

$$\deg(\Lambda_1) + \deg(\Lambda_1 w_1) \geq n = \deg G$$

$$\implies \text{no uniqueness of a rational solution } \omega_1 / \lambda_1 \text{ to the linearized problem } \Lambda_1 R = \omega_1 \mod G \text{ with } \deg \omega_1 \leq e + k$$

(more unknowns than equations)
Why the interpolation step (2/3)

Note that

$$\Lambda_1(R - w_1) = 0 \mod G$$

Now consider two solutions $w_1, w_2$. We have the modular key equation

$$\Lambda(R - w_1)(R - w_2) = 0 \mod G$$

where $\Lambda = \prod_{i \mid \text{error}_{1,2}} (X - x_i) = \gcd(\Lambda_1, \Lambda_2)$.

$\implies w_1, w_2$ are $Y$-roots of the bivariate polynomial

$$Q(X, Y) = \Lambda(Y - w_1)(Y - w_2)$$
Why the interpolation step (3/3)

Consider two solutions \( w_1, w_2 \), then \( \Lambda(R - w_1)(R - w_2) = 0 \pmod{G} \)
and \( w_1, w_2 \) are \( Y \)-roots of

\[
Q(X, Y) = \Lambda(Y - w_1)(Y - w_2)
= \Lambda w_1 w_2 - \Lambda(w_1 + w_2)Y + \Lambda Y^2
\]

Similar remark when considering all \( \ell \) solutions \( w_1, \ldots, w_\ell \)

Properties of \( Q(X, Y) \):

- the unknown degree in \( Y \) of \( Q(X, Y) \) is the number of solutions \( \ell \)
- the unknown coefficients in \( X \) of \( Q(X, Y) \) have small degree
- we have the modular identity \( Q(X, R) = 0 \pmod{G} \)
  or equivalently, for every \( i \), \( Q(x_i, y_i) = 0 \)
Guruswami-Sudan algorithm

It consists of two main steps,

- **Interpolation step**
  
  \[ Q(X, Y) \text{ such that: } w(X) \text{ solution } \Rightarrow Q(X, w(X)) = 0 \]

- **Root-finding step**
  
  \[ \text{find all } Y\text{-roots of } Q(X, Y), \text{ keep those that are solutions} \]

Here we are interested in the **interpolation step**

\[ \Rightarrow \text{ leads to a problem of } \text{Interpolation with Multiplicities}. \]
A problem of Interpolation with multiplicities

Interpolation With Multiplicities

Input:

- $n$ points $\{(x_i, y_i)\}_{1 \leq i \leq n}$ in $\mathbb{K}^2$, with the $x_i$’s distinct
- $k$ the degree constraint, $t$ the agreement
- $\ell$ the list-size, $m$ the multiplicity ($m \leq \ell$)

Output:

- a polynomial $Q$ in $\mathbb{K}[X, Y]$ such that
  
  (i) $Q$ is nonzero,
  (ii) $\deg_Y Q(X, Y) \leq \ell$, \hfill (list-size condition)
  (iii) $\deg_X Q(X, X^k Y) < mt$, \hfill (weighted-degree condition)
  (iv) $\forall i, \ Q(x_i, y_i) = 0$ with multiplicity $m$. \hfill (vanishing condition)
Algorithms based on structured linear systems

[Roth - Ruckenstein, 2000] [Zeh - Gentner - Augot, 2011]

Write

\[ Q(X, Y) = \sum_{0 \leq j \leq \ell} Q_j(X) Y^j \]  
(list-size condition)

where \( \deg Q_j(X) < mt - jk \).  
(weighted-degree condition)

Then, rewrite the vanishing condition so that a solution \( Q(X, Y) \) can be retrieved as a nontrivial solution of a homogeneous mosaic-Hankel linear system (the unknown being the coefficient vector of \( Q(X, Y) \)).

Complexity bound for this method:

\[ O(\ell m^4 n^2) \]

using a modified Feng-Tzeng’s linear system solver [Feng - Tzeng, 1991].
Algorithms based on polynomial lattices

[Alekhnovich, 2002] [Reinhard, 2003] [Beelen - Brander, 2010] [Bernstein, 2011] [Cohn - Heninger, 2011]

Build a polynomial lattice $\mathcal{L}$ such that

$$Q(X, Y) \in \mathcal{L} \iff \text{(list-size condition) + (vanishing condition).}$$

Then, a solution to Interpolation With Multiplicities can be retrieved as a short vector in $\mathcal{L}$ (weighted-degree condition).

Complexity bound for this method:

$$O^\sim(\ell^\omega mn)$$

using an efficient polynomial lattice basis reduction algorithm:

Contributions

[Chowdhury - Jeannerod - Neiger - Schost - Villard, 2014]

1. New approach for the interpolation step
   - Based on an approximation problem
   - Solved using structured linear systems
   - Improved complexity bound
     \[ \mathcal{O}^\sim((\ell^\omega-1)m^2n) \]

2. Extension to the multivariate case (folded Reed-Solomon codes)
   - Based on the same approximation problem
   - Improved complexity bound
     \[ \mathcal{O}^\sim\left(\left(\frac{s + \ell}{s}\right)^{\omega-1}mn\left(\frac{s + m - 1}{s}\right)\right) \]
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   - Based on a approximation problem
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   - Improved complexity bound
     \[ O^\sim(\ell^{\omega-1} m^2 n) \]

2. Extension to the multivariate case (folded Reed-Solomon codes)
   - Based on the same approximation problem
   - Improved complexity bound
     \[ O^\sim \left( \left( \frac{s + \ell}{s} \right)^{\omega-1} mn \left( \frac{s + m - 1}{s} \right) \right) \]
Reduction to an approximation problem (1/2)

Assume that $Q$ satisfies the list-size condition:

$$Q = \sum_{j \leq \ell} Q_j(X) Y^j$$

for some unknown polynomials $Q_0, \ldots, Q_{\ell}$

The vanishing condition can be rewritten as a set of modular equations

$$\forall i < m, \quad \sum_{i \leq j \leq \ell} Q_j(X) \binom{j}{i} R(X)^{j-i} = 0 \mod G(X)^{m-i}$$

where $G(X) = \prod_{1 \leq i \leq n} (X - x_i)$ and $R(X)$ such that $\forall i, R(x_i) = y_i$. 
Reduction to an approximation problem (2/2)

Vanishing condition + list-size condition

$$\forall i < m, \quad \sum_{i < j \leq \ell} Q_j(X) \binom{j}{i} R(X)^{j-i} \equiv 0 \pmod{G(X)^{m-i}}$$

Cost for computing $F_{i,j}$ and $P_i$:

- computing $n(m - i)$ coefficients of $F_{i,j}$ for every $i, j$
  $$\approx \text{computing } nm \text{ coefficients of } R(X)^j \text{ for } 0 \leq j \leq \ell$$
  $$\leadsto \mathcal{O}^\sim(\ell m^2 n) \text{ operations } \in \mathcal{O}(\ell^{\omega-1} m^2 n)$$

- computing $P_i$ for every $i$
  $$\equiv \text{computing the } m \text{ polynomials } G(X), G(X)^2, \ldots, G(X)^m$$
  $$\leadsto \mathcal{O}^\sim(m^2 n) \text{ operations } \in \mathcal{O}(\ell^{\omega-1} m^2 n)$$
Reduction to an approximation problem (2/2)

Vanishing condition + list-size condition + weighted-degree condition

\[ \forall i < m, \sum_{i < j \leq \ell} Q_j(X) \binom{j}{i} R(X)^{j-i} = 0 \pmod{G(X)^{m-i}} \]

with the degree constraints \( \deg Q_j(X) < mt - jk \) for \( j \leq \ell \)

Cost for computing \( F_{i,j} \) and \( P_i \):

- computing \( n(m - i) \) coefficients of \( F_{i,j} \) for every \( i, j \)
  \( \approx \) computing \( nm \) coefficients of \( R(X)^j \) for \( 0 \leq j \leq \ell \)
  \( \leadsto \mathcal{O}^\sim(\ell m^2 n) \) operations \( \in \mathcal{O}(\ell^{\omega-1} m^2 n) \)

- computing \( P_i \) for every \( i \)
  \( = \) computing the \( m \) polynomials \( G(X), G(X)^2, \ldots, G(X)^m \)
  \( \leadsto \mathcal{O}^\sim(m^2 n) \) operations \( \in \mathcal{O}(\ell^{\omega-1} m^2 n) \)
The approximation problem

\[ \forall i < m, \sum_{i \leq j \leq \ell} Q_j(X) \binom{j}{i} R(X)^{j-i} \equiv 0 \pmod{G(X)^{m-i} P_i(X)} \]

with the degree constraints \( \deg Q_j(X) < mt - jk \) for \( j \leq \ell \)

Simultaneous Polynomial Approximations

**Input:**
- **Parameters:** \( \ell \) the list-size, \( m \) the number of equations
- **Moduli:** \( P_i \in \mathbb{K}[X] \) monic of degree \( M_i \), for every \( i < m \)
- **Polynomials:** \( F_{i,j} \in \mathbb{K}[X] \) of degree less than \( M_i \), for \( i < m \) and \( j \leq \ell \)
- **Degree bounds:** \( N_j \) a positive integer, for every \( j \leq \ell \)

**Output:** \( Q_0, \ldots, Q_\ell \in \mathbb{K}[X] \) satisfying

(i') \( Q_j(X) \) are not all zero,
(ii') \( \forall j \leq \ell, \deg Q_j(X) < N_j \),
(iii') \( \forall i < m, \sum_{j \leq \ell} Q_j(X) F_{i,j}(X) = 0 \pmod{P_i(X)} \).

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Simultaneous approximations via a structured system (1/3)

Write \( Q_j(X) = \sum_{r<N_j} Q_j^{(r)} X^r \), then the equations are

\[
\forall i < m, \quad \sum_{i\leq j \leq \ell} \sum_{r<N_j} Q_j^{(r)} X^r F_{i,j}(X) = 0 \pmod{P_i(X)}
\]

Define the companion matrix

\[
C(P_i) = \begin{bmatrix}
0 & 0 & \cdots & 0 & -P_i^{(0)} \\
1 & 0 & \cdots & 0 & -P_i^{(1)} \\
0 & 1 & \cdots & 0 & -P_i^{(2)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -P_i^{(M_i-1)}
\end{bmatrix} \in \mathbb{K}^{M_i \times M_i}
\]

Key property:
multiplication by \( C(P_i) \) on the left is multiplication by \( X \) modulo \( P_i(X) \)
Simultaneous approximations via a structured system (2/3)

Solution $\iff$ nonzero vector in the nullspace of the matrix $A$.

where the block $A_{i,j} \in \mathbb{K}^{M_i \times N_j}$ is defined by its first column

$$c^{(0)} = \begin{bmatrix} F_{i,j}^{(0)} \\ \vdots \\ F_{i,j}^{(M_i-1)} \end{bmatrix}$$

and the subsequent columns $c^{(r+1)} = C(P_i) \cdot c^{(r)}$
Simultaneous approximations via a structured system (3/3)

Let $M = M_0 + \cdots + M_{m-1}$ (number of linear equations), and $N = N_0 + \cdots + N_\ell$ (number of linear unknowns). Define

$$Z_M = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \in \mathbb{K}^{M \times M}$$

Fact: $A - Z_M A Z_N^T$ has rank $\leq m + \ell + 1$

the displacement operator $A \mapsto A - Z_M A Z_N^T$ corresponds to a Toeplitz structure

Conclusion: the matrix of the system is Toeplitz-like with displacement rank $\leq 2\ell$
Complexity bound for this approach

Solving the structured linear system [Bitmead - Anderson, 1980] [Morf, 1980] [Kaltofen, 1994] [Pan, 2001] [Bostan - Jeannerod - Schost, 2007]

Two main operations:

- computing generators
  \[ \approx \text{computing the first and last column of each block} \implies \mathcal{O}^\sim(\ell m^2 n) \]
  \[ + \text{computing the first row of each block} \implies \mathcal{O}^\sim(\ell m^2 n) \]
  \[ \implies \mathcal{O}^\sim(\ell m^2 n) \text{ operations} \]

- solving the system
  at most \( \ell + 1 \) blocks on each row or column,
  the number of equations is \( \sum_i n(m - i) = \mathcal{O}(m^2 n) \)
  \[ \implies \mathcal{O}^\sim(\ell^{\omega-1}m^2 n) \text{ operations} \]

Complexity bound:

\[ \mathcal{O}^\sim(\ell^{\omega-1}m^2 n) \]
Contributions

[Chowdhury - Jeannerod - Neiger - Schost - Villard, 2014]

1. **New approach for the interpolation step**
   - Based on a approximation problem
   - Solved using structured linear systems
   - Improved complexity bound
   
   $\mathcal{O}(\ell^{\omega-1} m^2 n)$

2. **Extension to the multivariate case (folded Reed-Solomon codes)**
   - Based on the same approximation problem
   - Improved complexity bound
   
   $\mathcal{O}(\left(\frac{s + \ell}{s}\right)^{\omega-1} m n \left(\frac{s + m - 1}{s}\right))$
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1. New approach for the interpolation step
   - Based on a approximation problem
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     \[ O^\sim(\ell^{\omega-1} m^2 n) \]

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   - Based on the same approximation problem
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     \[ O^\sim\left(\left(\frac{s + \ell}{s}\right)^{\omega-1} m n \left(\frac{s + m - 1}{s}\right)\right) \]
Multivariate Interpolation with Multiplicities

**Input:**
- $s$ the number of variables
- $n$ points $\{(x_i, y_{i1}, \ldots, y_{is})\}_{1 \leq i \leq n}$ in $\mathbb{K}^{s+1}$, with the $x_i$’s distinct
- $k$ the degree constraint, $t$ the agreement
- $\ell$ the list-size, $m$ the multiplicity

**Output:** a polynomial $Q$ in $\mathbb{K}[X, Y_1, \ldots, Y_s]$ such that

(i) $Q$ is nonzero,
(ii) $\deg_Y Q(X, Y_1, \ldots, Y_s) \leq \ell$, \hspace{1cm} (list-size condition)
(iii) $\deg_X Q(X, X^k Y_1, \ldots, X^k Y_s) < mt$, \hspace{1cm} (weighted-degree condition)
(iv) $\forall i, \ Q(x_i, y_{i1}, \ldots, y_{is}) = 0$ with multiplicity $m$. \hspace{1cm} (vanishing condition)

Application: list-decoding of **folded** Reed-Solomon codes
Reduction to an approximation problem (1/2)

Assume that $Q$ satisfies the list-size condition:

$$Q = \sum_{|j| \leq \ell} Q_j(X) Y^j$$

for some unknown polynomials \{Q_j, \ |j| \leq \ell\}

The vanishing condition can be rewritten as a set of modular equations.

$$\sum_{i \leq j, |j| \leq \ell} Q_j(X) \binom{j_1}{i_1} R_1(X)^{j_1-i_1} \cdots \binom{j_s}{i_s} R_s(X)^{j_s-i_s} = 0 \mod G(X)^{m-|i|}$$

where $G(X) = \prod_{1 \leq i \leq n} (X - x_i)$ and

$R_1(X), \ldots, R_s(X)$ such that $R_1(x_i) = y_{i1}, \ldots, R_s(x_i) = y_{is}$
Reduction to an approximation problem (2/2)

Vanishing condition + list-size condition

\[ \sum_{i \leq j, |j| \leq \ell} Q_j(X) \left( j_1 \atop i_1 \right) R_1(X)^{j_1 - i_1} \ldots \left( j_s \atop i_s \right) R_s(X)^{j_s - i_s} = 0 \mod \frac{G(X)^{m - |i|}}{P_i(X)} \]

for \( i = (i_1, \ldots, i_m) \) such that \( |i| < m \),

Instance of Simultaneous Polynomial Approximations

- list-size \( \binom{s + \ell}{s} \)
- number of linear equations \( mn \binom{s + m - 1}{s} \)
Reduction to an approximation problem (2/2)

Vanishing condition + list-size condition + weighted-degree condition

\[ \sum_{i \leq j, |j| \leq \ell} Q_j(X) \left( \begin{array}{c} j_1 \\ i_1 \end{array} \right) R_1(X)^{j_1-i_1} \cdots \left( \begin{array}{c} j_s \\ i_s \end{array} \right) R_s(X)^{j_s-i_s} = 0 \mod G(X)^{m-|i|} \]

for \( i = (i_1, \ldots, i_m) \) such that \( |i| < m \),

with the degree constraints \( \deg Q_j(X) < mt - |j| k \) for \( |j| \leq \ell \)

Instance of Simultaneous Polynomial Approximations

- list-size \( \binom{s+\ell}{s} \)
- number of linear equations \( mn \binom{s+m-1}{s} \)
Complexity bound in the multivariate case

Improves on [Busse, 2008], [Brander, 2010] and [Nielsen, 2014]

Further extends to
- weight specific to each variable
  \[ \deg_X Q(X, X^{k_1} Y_1, \ldots, X^{k_s} Y_s) < mt \]
- multiplicity specific to each point
  \[ Q(x_i, y_{i_1}, \ldots, y_{i_s}) = 0 \text{ with multiplicity } m_i \]
Contributions

[Chowdhury - Jeannerod - Neiger - Schost - Villard, 2014]

1. **New approach for the interpolation step**
   - Based on a approximation problem
   - Solved using structured linear systems
   - Improved complexity bound
   \[ O^{\sim}(\ell^{\omega-1} m^2 n) \]

2. **Extension to the multivariate case (folded Reed-Solomon codes)**
   - Based on the same approximation problem
   - Improved complexity bound
   \[ O^{\sim} \left( \left( \frac{s + \ell}{s} \right)^{\omega-1} mn \left( \frac{s + m - 1}{s} \right) \right) \]
Contributions

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     \[ O^\sim\left(\left(\frac{s + \ell}{s}\right)^{\omega-1}mn\left(\frac{s + m - 1}{s}\right)\right) \]