The study of 2-dimensional p-adic Galois deformations in the $\ell \neq p$ case

Vincent Pilloni

April 22, 2008

Abstract

We introduce the language and give the classical results from the theory of deformations : deformations and framed deformations, representability, tangent spaces computation via Galois cohomology, formal smoothness...
Then we give a technic, due to Mark Kisin, which allows us to study the general fiber of the deformation spaces.
As an application we describe, after Kisin, the generic fiber of the deformation scheme of a representation $\rho : \mathrm{G} \to \mathrm{GL}_2(\mathbb{F})$, where $\mathbb{F}$ is a finite extension of $\mathbb{F}_p$ and G the absolute Galois group of a local field of residual characteristic $\ell \neq p$.

1 Introduction

These notes are divided in 3 parts.

The first part is a survey of the deformation theory of Galois representations. Our main source is Barry Mazur's original article [5]. Given a continuous Galois representation ρ into a finite $\mathbb{F}$-vector space ($\mathbb{F}$ a topological field) and $\mathcal{O}$ a complete local noetherian ring having residue field $\mathbb{F}$, we define the notions of lifting and deformation of ρ to certain artinian $\mathcal{O}$-algebras. This leads, under mild restrictions on ρ, to the construction of universal $\mathcal{O}$-schemes parameterizing the deformations of ρ. Fundamental geometric properties of these spaces are encoded in the Galois cohomology of the adjoint representation of ρ.

The aim of the second part is to explain a technique, due to Mark Kisin ([2] and [3]), which allows us to study the general fiber of the deformation spaces : let ρ be a Galois representation in $\mathrm{GL}_N(\mathbb{F})$ with $\mathbb{F}$ a finite field and let $\mathcal{O}_E$ be an integer ring of a local field E, whose residue field is finite over $\mathbb{F}$. Let ρ_E be a lifting of ρ to $\mathrm{GL}_N(\mathcal{O}_E)$. One can build the deformation theory for $\rho_E \otimes E$ out of the deformation theory of ρ.
This has the following interesting consequence : let R be the universal deformation ring of ρ. The deformation ρ_E defines an E-point ξ, lying on the generic fiber of R. A corollary of the preceding result is that Spec $R[1/p]$ is locally isomorphic at ξ to the universal deformation space of ρ_E. As a result the local geometry of Spec $R[1/p]$ at ξ can be computed via the Galois cohomology of ρ_E.

The last and main part of these notes is dedicated to the generic fiber of the framed deformation scheme of a representation $\rho : \mathrm{G} \to \mathrm{GL}_2(\mathbb{F})$, where $\mathbb{F}$ is a finite extension of $\mathbb{F}_p$ and G the absolute Galois group of a local field of residual characteristic $\ell \neq p$. We continue the analysis initiated by Mark Kisin in [3] and [4]. We list here the main results and give an idea of proof:

- We show that the generic fiber is a union of smooth components. Since this is a local statement, it is essentially proven by using the results explained in the second part of these notes and reduces to Galois Cohomology computations. We also describe the eventual intersections.
- We classify these components according to the Galois representation family lying over. This is understood by exploiting the rigidity of the situation ($\ell \neq p$) and identifying the parameters that may vary on the components.
- We give multiplicity statements : We discuss the number of occurrence of a component of a given type in the generic fiber and prove that it is almost always less or equal to one. These are the most delicate results since they are of global nature on the generic fiber and since this one does not have an easy modular interpretation. To prove this kind of properties, following an original idea of Kisin, we construct certain resolutions of the deformation space whose geometry is very simple.

2 Deformations

2.1 Basic definitions

Let G be a group. We assume that G arises from one of the following situations :

- *Local non−archimedean situation*: Let ℓ be a prime and let L be a finite extension of $\mathbb{Q}_\ell$, let $\bar{L}$ be an algebraic closure of L, and then $\mathrm{G} = \mathrm{Gal}(\bar{L}/L)$.
- *Local archimedean situation* : $\mathrm{G} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.
- *Global situation*: Let L be a number field, S a finite set of places of L and $L_S \subset \bar{\mathbb{Q}}$ the maximal extension of L unramified outside S. Then $\mathrm{G} = \mathrm{Gal}(L_S/L)$.

Let $\mathbb{F}$ be a topological field. In all applications $\mathbb{F}$ will be either a finite extension of $\mathbb{F}_p$ (with the discrete topology) or a finite extension of $\mathbb{Q}_p$ (with the p-adic topology).
Let $\mathcal{O}$ be a local noetherian complete algebra, whith maximal ideal $\mathfrak{m}_\mathcal{O}$ and with residue field $\mathbb{F}$. We fix an isomorphism $\mathcal{O}/\mathfrak{m}_\mathcal{O} \backsimeq \mathbb{F}$. In all applications $\mathcal{O}$ will be either $\mathbb{F}$ itself (char $\mathbb{F} = 0$) or the ring of integers of a p-adic field (char $\mathbb{F} = p$).
Let $\mathcal{AR}_\mathcal{O}$ be the category of artinian local $\mathcal{O}$-algebras, A with maximal ideal $\mathfrak{m}_A$ such that the structural map $\mathcal{O} \to A$ induces an isomorphism $\mathcal{O}/\mathfrak{m}_\mathcal{O} \backsimeq A/\mathfrak{m}_A$. Maps are local ring homomorphisms compatible with the identification of residue fields with $\mathbb{F}$.
Consider also the category $\widehat{\mathcal{AR}_\mathcal{O}}$ of complete noetherian local $\mathcal{O}$-algebras. One can view $\mathcal{AR}_\mathcal{O}$ as a full subcategory of $\widehat{\mathcal{AR}_\mathcal{O}}$ and any object A in $\widehat{\mathcal{AR}_\mathcal{O}}$ can be written as the projective limit of its quotients $A/\mathfrak{m}_A^n$ which lie in $\mathcal{AR}_\mathcal{O}$.
Any object A in $\widehat{\mathcal{AR}_\mathcal{O}}$ is given the coursest topology which is finer than the $\mathfrak{m}_A$-adic topology and which makes the map $A \to \mathbb{F}$ continuous.

Let $N > 0$ be an integer and let $\rho : \mathrm{G} \to \mathrm{GL}_N(\mathbb{F})$ be a continuous representation of G.
All the representations we consider in the sequel are continuous.

Definition 2.1.1. *Let A be an object in $\widehat{\mathcal{AR}_{\mathcal{O}}}$.*

1. *A representation $\rho_A : \mathrm{G} \to \mathrm{GL}_N(A)$ is a lifting or a framed deformation of the representation ρ if ρ_A composed with the map $\mathrm{GL}_N(A) \to \mathrm{GL}_N(\mathbb{F})$ induced by reduction modulo $\mathfrak{m}_A$ is equal to ρ.*

2. *Two liftings are said to be equivalent if they differ by a conjugation by an element of $\mathrm{GL}_N(A)$ reducing to the identity modulo $\mathfrak{m}_A$.*

3. *A deformation of ρ to A is an equivalence class of liftings of ρ to a representation $\rho_A : \mathrm{G} \to \mathrm{GL}_N(A)$.*

The group $\mathrm{PGL}_{N/\mathcal{O}}$ acts by conjugation on $\mathrm{GL}_{N/\mathcal{O}}$. We set $\widehat{\mathrm{PGL}_N}$ for its completion along the unit section of the special fiber. The group $\widehat{\mathrm{PGL}_N}$ acts on the set of liftings and a deformation is an orbit under this action.

We can define the deformation functor :

$$\begin{aligned} \mathbb{D} : \widehat{\mathcal{AR}_{\mathcal{O}}} &\longrightarrow SET \\ A &\rightsquigarrow \{\text{deformations of } \rho \text{ to } A\} \end{aligned}$$

In the same fashion we define the framed deformation functor :

$$\begin{aligned} \mathbb{D}^{\square} : \widehat{\mathcal{AR}_{\mathcal{O}}} &\longrightarrow SET \\ A &\rightsquigarrow \{\text{framed deformations of } \rho \text{ to } A\} \end{aligned}$$

There is a natural transformation :

$$\Theta : \mathbb{D}^{\square} \longrightarrow \mathbb{D}$$

defined by sending a framed deformation to its deformation class.

We say that the functor $\mathbb{D}$ is representable if there exist a pair (R, ρ_{univ}) with R an object in $\widehat{\mathcal{AR}_{\mathcal{O}}}$ and ρ_{univ} an element of $\mathbb{D}(R)$ such that the canonical map of functors :

$$\rho_{univ} : \mathrm{Hom}(R, \cdot) \longrightarrow \mathbb{D}$$

is an equivalence of category. The pair (R, ρ_{univ}) is unique up to a unique isomorphism. Sometimes we drop the ρ_{univ} and simply say $\mathbb{D}$ is represented by R or Spec R.
We say that the functor $\mathbb{D}^{\square}$ is representable if there exist a pair $(R^{\square}, \rho_{univ})$ satisfying similar properties.

2.2 Determinant condition and change of group

2.2.1 The determinant

Let $\rho : \mathrm{G} \to \mathrm{GL}_N(\mathbb{F})$ a representation of G and $\mathbb{D}$ the deformation functor of ρ. Consider the determinant $\det : \mathrm{GL}_N \to \mathrm{GL}_1$, and let Δ be the deformation functor of the representation $\det\rho$.

If A is in $\widehat{\mathcal{AR}_{\mathcal{O}}}$ and ρ_A is a lifting of ρ then $\det\rho_A$ is a lifting of $\det\rho$.
Hence there is a natural transformation

$$\det : \mathbb{D} \longrightarrow \Delta.$$

If both functors are representable, say respectively by rings R and Λ, then R becomes naturally a Λ-algebra.

2.2.2 Deformations with fixed determinant

Let $\chi : \mathrm{G} \to \mathcal{O}^\times$ be a lifting of $\det\rho$. One can consider a sub-deformation functor $\mathbb{D}^\chi$ of $\mathbb{D}$ by letting $\mathbb{D}^\chi(A) \subset \mathbb{D}(A)$ consist of deformations of ρ to A having determinant equal to

$$\mathrm{G} \xrightarrow{\chi} \mathcal{O}^\times \to A^\times$$

Assume again the representability of $\mathbb{D}$ and Δ by R and Λ. The character χ corresponds to a morphism $\Lambda \to \mathcal{O}$ and the functor $\mathbb{D}^\chi$ is represented by the scheme Spec $R \otimes_\Lambda \mathcal{O}$ which is a closed subscheme of Spec R.

2.2.3 Twisting

Let Δ be the deformation functor for the trivial character. For any A in $\widehat{\mathcal{AR}_{\mathcal{O}}}$ the tensor operation on characters gives $\Delta(A)$ a natural group structure. Hence Δ is a group functor. Let $\rho : \mathrm{G} \to \mathrm{GL}_N(\mathbb{F})$ a representation and $\mathbb{D}$ its deformation functor. Then we have a natural group action $\Delta \times \mathbb{D} \to \mathbb{D}$ sending a pair $(\eta, \rho_A) \in \Delta(A) \times \mathbb{D}(A)$ to $\eta \otimes_A \rho_A$.

2.2.4 Global and local deformations

Suppose now that $\mathrm{G} = \mathrm{Gal}(L_S/L)$, with L a number field, S a finite set of places and $L_S \subset \mathbb{Q}$ the maximal extension of L unramified outside S. Choose, for some prime ideal $\mathfrak{P} \in \mathcal{O}_L$ a decomposition group $D_\mathfrak{P} \subset \mathrm{G}$. Then ρ induces, by restriction, a representation $\rho_\mathfrak{P} : D_\mathfrak{P} \to \mathrm{GL}_N(\mathbb{F})$.
Now let $\mathbb{D}_\mathfrak{P}$ be the deformation functor for $\rho_\mathfrak{P}$. The restriction to $D_\mathfrak{P}$ induces a natural transformation

$$\mathbb{D} \longrightarrow \mathbb{D}_\mathfrak{P}.$$

All this section can be immediately generalized by replacing $\mathbb{D}$ by $\mathbb{D}^\square$.

2.3 Representability

2.3.1 The main theorem

The following proposition follows easily from Schlessinger's representability criterion ([8]):

Proposition 2.3.1. *The framed deformation functor $\mathbb{D}^\square$ is representable.*

The representability of the deformation functor is more delicate :

Theorem 2.3.2. *If the natural map* $\mathbb{F} \to \mathrm{End}_{\mathbb{F}[\mathrm{G}]}(\rho)$ *is an isomorphism the deformation functor is representable.*

Proof. See [6], p 264. □

Corrolary 2.3.3. *If* ρ *is absolutely irreducible the deformation functor is representable.*

2.3.2 The case $N = 1$

Let $\rho : \mathrm{G} \to \mathbb{F}^\times$ be a one dimensional representation. Then by the above theorem the deformation functor is represented by a pair (Λ, ρ_{univ}). One can construct explicitly the pair (following [5], section 1.4):
Let $\mathrm{G}^{ab,p}$ be the abelianized p-completion of G and if g is in G let $\bar{g}$ be its image in $\mathrm{G}^{ab,p}$. Put $\Lambda = \mathcal{O}[[\mathrm{G}^{ab,p}]]$ which is an object of $\widehat{\mathcal{AR}_{\mathcal{O}}}$ (by Class Field theory). Let $\tilde{\rho} : \mathrm{G} \to \mathcal{O}^\times$ be a lifting of ρ and finally let

$$\begin{array}{rcl} \rho_{univ} : \mathrm{G} & \to & \Lambda^\times \\ g & \mapsto & \tilde{\rho}(g) \cdot \bar{g} \end{array}$$

2.3.3 The case $N = 2$

Let $\rho : \mathrm{G} \to \mathrm{GL}_2(\mathbb{F})$ be a 2-dimensional representation. If ρ is absolutely irreducible or if ρ is a non-trivial extension of two distinct characters then the deformation functor $\mathbb{D}$ is representable.

2.4 Tangent space calculations

Let A in $\widehat{\mathcal{AR}_{\mathcal{O}}}$ with maximal ideal $\mathfrak{m}_A$, let $\mathfrak{m}_{\mathcal{O}}$ be the maximal ideal of $\mathcal{O}$ and let $\mathbb{F}[\epsilon]$ be the ring of dual numbers.

Definition 2.4.1. *The Zariski tangent space of* A *is the* $\mathbb{F}$ *vector space* $\mathrm{Hom}(A, \mathbb{F}[\epsilon]) = \mathrm{Hom}_{\mathbb{F}-vect}(\mathfrak{m}_A/\mathfrak{m}_A^2 + \mathfrak{m}_{\mathcal{O}}, \mathbb{F})$*. Its dual we denote* t_A *is the Zariski cotangent space.*

If d is the dimension of t_A and if $\{a_1, ..., a_d\}$ is a collection of elements of $\mathfrak{m}_A$ mapping to a base of t_A, then the map $\mathcal{O}[[X_1, ..., X_d]] \to A$ which sends X_i to a_i is a surjection and induces an isomorphism on the tangent spaces.

Let $\rho : \mathrm{G} \to \mathrm{GL}_N(\mathbb{F})$ be a representation and let $\mathbb{D}$ be the deformation functor of ρ, we define in a similar fashion the tangent space of $\mathbb{D}$ and $\mathbb{D}^\square$:

Definition 2.4.2. *The set* $\mathbb{D}(\mathbb{F}[\epsilon])$ *(respectively* $\mathbb{D}^\square(\mathbb{F}[\epsilon])$*) as a natural* $\mathbb{F}$*-vector space structure and we call it the Zariski cotangent space of* $\mathbb{D}$ *(respectively* $\mathbb{D}^\square$*).*

Let $Ad\rho$ be the adjoint representation of ρ.

Proposition 2.4.3. *There is a natural isomorphism* $\mathbb{D}(\mathbb{F}[\epsilon]) \simeq H^1(\mathrm{G}, Ad\rho)$.

Proof.([5], section 1.2) The construction of the natural isomorphism is as follow : choose a lifting ρ_ϵ of ρ to $\mathbb{F}[\epsilon]$ and write $\rho_\epsilon = \rho(1 + \epsilon C)$ where C is a map from G to $\mathrm{M}_N(\mathbb{F})$. C is a 1-cocycle for the adjoint representation. Changing ρ_ϵ to some other representative of its deformation class changes C by a 1-coboundary. □

Let $\chi : \mathrm{G} \to \mathcal{O}^\times$ be a character lifting detρ and let $\mathbb{D}^\chi$ be the deformation functor with fixed determinant χ. Let $Ad^0\rho$ be the adjoint representation of ρ on trace zero matrices. We have a natural map $H^1(\mathrm{G}, Ad^0\rho) \to H^1(\mathrm{G}, Ad\rho)$. We denote by $H^1(\mathrm{G}, Ad^0\rho)'$ the image of $H^1(\mathrm{G}, Ad^0\rho)$ in $H^1(\mathrm{G}, Ad\rho)$. Note that if N is invertible in $\mathbb{F}$ the above map is injective.
Analogous to the previous proposition we have :

Proposition 2.4.4. *There is a natural isomorphism* $\mathbb{D}^\chi(\mathbb{F}[\epsilon]) \simeq H^1(\mathrm{G}, Ad^0\rho)'$.

Recall that we have a morphism $\Theta : \mathbb{D}^\square \longrightarrow \mathbb{D}$. The fibers of the induced map on tangent spaces $\mathbb{D}^\square(\mathbb{F}[\epsilon]) \to \mathbb{D}(\mathbb{F}[\epsilon])$ are principal homogeneous spaces under $Ad\rho/Ad\rho^{\mathrm{G}}$ and hence:

Proposition 2.4.5. $dim_{\mathbb{F}}\mathbb{D}^\square(\mathbb{F}[\epsilon]) = dim_{\mathbb{F}}\mathbb{D}(\mathbb{F}[\epsilon]) + dim_{\mathbb{F}}Ad\rho - dim_{\mathbb{F}}Ad\rho^{\mathrm{G}}$.

2.5 Smoothness and Dimension

2.5.1 Formal smoothness

Let F and F' be two functors on $\widehat{\mathcal{AR}_{\mathcal{O}}}$ with value in SET, and $\phi : F \to F'$ a natural transformation.

Definition 2.5.1. *The map ϕ is formally smooth if for any surjective map $A \to B$ in $\mathcal{AR}_{\mathcal{O}}$, the morphism*

$$F(A) \to F(B) \times_{F'(B)} F'(A)$$

is surjective.

In general, if we say that a functor is formally smooth it means that it is smooth over the base $\mathcal{O}$. If $A \in \widehat{\mathcal{AR}_{\mathcal{O}}}$ is formally smooth then it is isomorphic to a power series ring over $\mathcal{O}$.
The following proposition follows immediately from the definitions :

Proposition 2.5.2. *The map* $\Theta : \mathbb{D}^\square \longrightarrow \mathbb{D}$ *is formally smooth*

2.5.2 Obstruction

Let $A_1 \to A_0$ be a surjective mapping of artinian rings in $\mathcal{AR}_{\mathcal{O}}$ with kernel I such that $\mathfrak{m}_{A_0} \cdot I = 0$. We may view I as a finite dimensional $\mathbb{F}$-vector space. Let $\rho_0 : \mathrm{G} \to \mathrm{GL}_N(A_0)$, then there is an obstruction class in $H^2(\mathrm{G}, Ad\rho) \otimes I$ depending only on the deformation class of ρ_0 and which vanishes if and only if there exists a lifting $\rho_1 : \mathrm{G} \to \mathrm{GL}_N(A_1)$ of ρ_0. The construction is as follow : Let $\gamma_1 : \mathrm{G} \to \mathrm{GL}_N(A_1)$ be a set theoretic mapping lifting ρ_0, we can form the obstruction 2-cocycle :

$$c(\sigma, \tau) = \gamma_1(\sigma\tau)\gamma_1(\sigma)^{-1}\gamma_1(\tau)^{-1} \in 1 + I \otimes \mathrm{M}_N(\mathbb{F}).$$

Hence we get the following :

Proposition 2.5.3. *The deformation functor $\mathbb{D}$ is formally smooth if* $H^2(\mathrm{G}, Ad\rho) = 0$.

Let d^i be the dimension of $H^i(\mathrm{G}, Ad\rho)$. Assume that $\mathbb{D}$ is representable by a ring R.

Proposition 2.5.4. *The ring R admits a presentation:*

$$\mathcal{O}[[X_1, ..., X_{d_1}]]^{d_2} \to \mathcal{O}[[X_1, ..., X_{d_1}]] \to R \to 0$$

Proof. See [5], 1.6, proposition 2. □

All the above results hold if one replaces $\mathbb{D}$ by $\mathbb{D}^\chi$ and $Ad\rho$ by $Ad^0\rho$.

3 Generic fibres

Let p be a prime, we now assume that $\mathbb{F}$ is a finite field extension of $\mathbb{F}_p$ and that $\mathcal{O}$ is a discrete valuation ring, finite flat over $W(\mathbb{F})$, having $\mathbb{F}$ as residue field. Let $\rho : \mathrm{G} \to \mathrm{GL}_N(\mathbb{F})$ a representation. The aim of this section is to explain a technic, due to Kisin ([2] and [3]), for the study of the generic fibers of the (framed) deformation ring.

3.1 Preliminary

Let E be a totally ramified extension of $\mathcal{O}[1/p]$, with ring of integers $\mathcal{O}_E$. We denote $\mathcal{AR}_E$ the category of artinian E-algebras having residue field E.
Let B in $\mathcal{AR}_E$, we denote by $\mathrm{Int}B$ the category whose objects are finite $\mathcal{O}_E$-subalgebras $A \subset B$ such that $A \otimes_{\mathcal{O}_E} E = B$. Morphisms are given by natural inclusions.
Let $\widehat{\mathcal{AR}}_{\mathcal{O},(\mathcal{O}_E)}$ denote the category whose objects are $\mathcal{O}$-algebras A in $\widehat{\mathcal{AR}_{\mathcal{O}}}$ equipped with a map of $\mathcal{O}$-algebras $A \to \mathcal{O}_E$. Hence $\mathrm{Int}B$ is a subcategory of $\widehat{\mathcal{AR}}_{\mathcal{O},(\mathcal{O}_E)}$.
Let F be a functor on $\widehat{\mathcal{AR}_{\mathcal{O}}}$ with value in SET and let $\xi \in F(\mathcal{O}_E)$. First we define a functor $F_{(\xi)}$ on $\widehat{\mathcal{AR}}_{\mathcal{O},(\mathcal{O}_E)}$. If $(A, A \stackrel{\alpha}{\to} \mathcal{O}_E)$ is in $\widehat{\mathcal{AR}}_{\mathcal{O},(\mathcal{O}_E)}$ we put

$$F_{(\xi)}(A) = \{\eta \in F(A) \text{ such that } F(\alpha)(\eta) = \xi\}.$$

Then we define a functor we call again $F_{(\xi)}$ on $\mathcal{AR}_E$ by setting :

$$F_{(\xi)}(B) = \lim_{\to A\in \mathrm{Int}_B} D_{(\xi)}(A)$$

for B in $\mathcal{AR}_E$.

Proposition 3.1.1. *If F is represented by a complete local $\mathcal{O}$-algebra R, then $F_{(\xi)}$ is (pro)represented by the complete local $\mathcal{O}[1/p]$-algebra $\hat{R}_\xi$ obtained by completing $R \otimes_{\mathcal{O}_E} E$ along the kernel I_ξ of the map $R \otimes_{\mathcal{O}_E} E \to E$ induced by ξ.*

Proof. Let B in $\mathcal{AR}_E$. Any element in $F_{(\xi)}(B)$ comes from a map $R \to A$ for some $A \in \mathrm{Int}B$ which fits in the following commutative triangle of $\mathcal{O}_E$-algebras :

$$\begin{array}{ccc} R \otimes_{\mathcal{O}} \mathcal{O}_E & \to & A \\ \downarrow & \swarrow & \\ \mathcal{O}_E & & \end{array}$$

Thus by extension of scalars to E we get a map $R \otimes_{\mathcal{O}} E \to B$ whose kernel contains a power of I_ξ. On the other hand for any map $R \otimes_{\mathcal{O}} E \to B$ which kills some power of I_ξ we consider A the image of $R \otimes_{\mathcal{O}} \mathcal{O}_E$ in B. Then A is in $\mathrm{Int}B$ and gives rise to an element of $F_{(\xi)}(B)$. □

3.2 Application

Let $\mathbb{D}$ and $\mathbb{D}^\square$ be the deformation and framed deformation functors of ρ. Consider some $\xi \in \mathbb{D}^\square(\mathcal{O}_E)$, whose image $\Theta(\xi)$ in $\mathbb{D}(E)$ we denote again by ξ.
Consider now $\mathbb{D}_\xi$ and $\mathbb{D}^\square_\xi$ the deformation and framed deformation functor of ξ. Both are functors on $\mathcal{AR}_E$. Consider also $\mathbb{D}_{(\xi)}$ and $\mathbb{D}^\square_{(\xi)}$ as defined in the preceding section.

Proposition 3.2.1. *There are natural isomorphisms of functors*

$$\mathbb{D}_{(\xi)} \longrightarrow \mathbb{D}_\xi \text{ and } \mathbb{D}^\square_{(\xi)} \longrightarrow \mathbb{D}^\square_\xi.$$

Proof. Let B in $\mathcal{AR}_E$ and A in Int$B$. An element in $\mathbb{D}_{(\xi)}(A)$ is a $\hat{\mathrm{PGL}}_N(A)$ orbit of some $\rho_A : \mathrm{G} \to \mathrm{GL}_N(A)$ which is equal to ξ after base change to $\mathcal{O}_E$. Hence there is a natural map

$$\mathbb{D}_{(\xi)}(B) = \varinjlim_{A \in \mathrm{Int}B} \mathbb{D}_{(\xi)}(A) \to \mathbb{D}_\xi(B)$$

obtained by sending ρ_A to $\rho_A \otimes_{\mathcal{O}_E} E$.
This map is surjective : Let $\nu \in \mathbb{D}_\xi(B)$ and let A be an object of Int(B). Let $\mathfrak{m}_B$ be the maximal ideal of B kernel of the projection map $b : B \to E$ and define an increasing family of algebras in Int(B) by setting for any positive integer n:

$$A_n = \sum_{j=1}^{\infty} (\mathfrak{m}_B \cap A)^j p^{-nj} + A.$$

We have : $\bigcup_{n \geq 0} A_n = b^{-1}(\mathcal{O}_E)$. As a result $\nu(\mathrm{G}) \subset \mathrm{GL}_N(\bigcup_{n\geq 0} A_n)$ and by compacity of G this map must factor through some A_n.
This map is injective: Let A and A' in Int$B$ and ρ_A and $\rho_{A'}$ be two lifings which are conjugated by an element in $\mathrm{GL}_N(B)$ reducing to the identity modulo $\mathfrak{m}_B$. An adaptation of the preceding argument shows that we can find an A'' in Int$B$ containing A and A' and such that ρ_A and $\rho_{A'}$ become conjugated over A''.
Similarly there is a map

$$\mathbb{D}^\square_{(\xi)}(B) \longrightarrow \mathbb{D}^\square_\xi(B)$$

which is seen to be bijective by the same arguments. □

4 The generic fibre of the deformation ring in the local $\ell \neq p$ case

Let ℓ and p be two distinct primes and let L be a finite extension of $\mathbb{Q}_\ell$ and $\bar{L}$ an algebraic closure of L. In this section G is the Galois group Gal$(\bar{L}/L)$. Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$ and $\rho : \mathrm{G} \to \mathrm{GL}_2(\mathbb{F})$ be a two dimensional Galois representation. Our aim is to analyze the generic fibre of the framed deformation ring of ρ. The important parts 4.6, 4.7 and 4.8 of the following are entirely taken from [3], section 2.6 and [4], section 2.5. They contain the key ideas.

We give here the plan of our study:
In section 4.1 we state the results. Sections 4.2 to 4.10 are dedicated to proving them.
Sections 4.2, 4.3 and 4.4 are preliminary : we recall the needed results in Galois cohomology, local class field theory and some generalities about 2-dimensional representations of G in a p-adic field of characteristic zero.
In section 4.5 we prove a rigidity statement : The representation of inertia is (up to the monodromy logarithm) constant on the connected components of the generic fibers of the deformation space.
Section 4.6 is crucial : we construct a resolution of the deformation space in case ρ is an extension of some character γ by $\gamma(1)$ and are able to exhibit a unique irreducible component of the generic fiber parameterizing such extensions. The kind of arguments of this section will be used extensively in the rest of the work (see also [3]).
In section 4.7 we prove that if $\bar{\rho}$ is unramified, there is a unique irreducible component parameterizing unramified representations.
In section 4.8 we apply the results of sections 4.6 and 4.7 to prove that the deformation space is the union of formally smooth components, and we explain when it happens that two components intersect.

In section 4.9 we study components that parameterize split representations.
Section 4.10 is dedicated to components parameterizing absolutely irreducible representations.
In section 4.11 we explain why, when $p \neq 2$, we also have a determinant condition-free version of the results of 4.1.

4.1 Statement of the theorems

Let $\rho : \mathrm{G} \to \mathrm{GL}_2(\mathbb{F})$ be a representation. Let $R^{\square}$ be the universal framed deformation ring of ρ. Let ψ be a lifting of $\det\rho$ and let $R^{\psi,\square}$ be the universal framed deformation ring with fixed determinant equal to ψ. Let $\rho_{univ} : \mathrm{G} \to \mathrm{GL}_2(R^{\square})$ be the universal representation. Let also (ρ'_{univ}, N_{univ}) be the universal representation of the Weil-Deligne group over Spec $R^{\square}[1/p]$ (see section 4.5). Let $\bar{K}$ be an algebraic closure of $\mathcal{O}[1/p]$. For any geometric point x : Spec $\bar{K} \to$ Spec $R^{\square}$, $\rho_x : \mathrm{G} \to \mathrm{GL}_2(\bar{K})$ is the representation deduced from ρ_{univ}.

4.1.1 Local properties

Theorem 4.1.1. *The generic fiber of* Spec $R^{\psi,\square}[1/p]$ *is* 3*-dimensional, reduced, and the union of formally smooth components.*

4.1.2 Rigidity

Theorem 4.1.2. *Let* x, y : Spec $\bar{K} \to$ Spec $R^{\square}[1/p]$ *be two geometric points lying in the same geometrically connected component. Set* ρ'_x *and* ρ'_y *for the representations* $\mathrm{G} \to \mathrm{GL}_2(\bar{K})$ *deduced from* ρ'_{univ}*. Then the restriction to inertia of* ρ'_x *and* ρ'_y *are isomorphic.*

Remark 4.1.1. The result holds in general, for N-dimensional representations.

4.1.3 Modular description of the irreducible components

Theorem 4.1.3. *Any geometrically irreducible component* $\mathcal{C}$ *of* Spec $R^{\psi,\square}[1/p]$ *admits one of the following descriptions :*

1. *Unipotent monodromy case : There exists a character* $\gamma : \mathrm{G} \to \bar{K}^{\times}$ *such that for any geometric point* x : Spec $\bar{K} \to \mathcal{C}$ *the representation* ρ_x *is an extension of* γ *by* $\gamma(1)$*. We say that* $\mathcal{C}$ *is of unipotent type* γ*.*

2. *The unramified (up to twist by a character) case : There exists a character* $\gamma : \mathrm{G} \to \bar{K}^{\times}$ *such that for any geometric point* x : Spec $\bar{K} \to \mathcal{C}$ *the representation* ρ_x *becomes unramified after a twist by* γ^{-1}*. We say that* $\mathcal{C}$ *is of unramified type* γ*.*

3. *The absolutely irreducible case : There exists an irreducible representation* $\xi : \mathrm{G} \to \mathrm{GL}_2(\bar{K})$ *such that for any geometric point* x : Spec $\bar{K} \to \mathcal{C}$ *the representation* ρ_x *is isomorphic to* ξ*. We say that* $\mathcal{C}$ *is of absolutely irreducible type* ξ*.*

4. *The split ramified case : There are two characters* $\eta, \lambda : \mathrm{G} \to \bar{K}^{\times}$ *such that* $\eta.\lambda^{-1}$ *is ramified and such that for any geometric point* x : Spec $\bar{K} \to \mathcal{C}$ *the representation* ρ_x *is isomorphic to* $\eta.\phi \oplus \lambda.\phi^{-1}$ *where* ϕ *is some unramified lifting of the trivial character. We say that* $\mathcal{C}$ *is of split ramified type* $\{\eta|_{I_{\mathrm{G}}}, \lambda|_{I_{\mathrm{G}}}\}$*.*

If two distinct geometric components intersect then they are of unipotent type γ and unramified type γ. If x is a geometric point lying on there intersection, the representation ρ_x is isomorphic to $\gamma \oplus \gamma(1)$.

4.1.4 Multiplicity

Theorem 4.1.4. *Let $x, y : \mathrm{Spec}\ \bar{K} \to \mathrm{Spec}\ R^{\psi,\square}[1/p]$ be two geometric points. Assume that ρ_x and ρ_y are isomorphic representations. Then x and y lie on the same geometrically irreducible component.*

In the same spirit we also have :

Theorem 4.1.5.

1. *There is at most one component of unipotent type γ, of unramifed type γ or of absolutely irreducible type ξ in $\mathrm{Spec}\ R^{\psi,\square}[1/p]$.*

2. *There is at most one component of split reducible type $\{\eta|_{I_G}, \lambda|_{I_G}\}$ in $\mathrm{Spec}\ R^{\psi,\square}[1/p]$ except when ρ is a split representation isomorphic to $\bar{\eta} \oplus \bar{\lambda}$ with $\bar{\eta}.\bar{\lambda}^{-1}$ a nontrivial unramified character. Then there are exactly two components of split reducible type $\{\eta|_{I_G}, \lambda|_{I_G}\}$ in the generic fiber.*

4.2 Galois cohomology

We recall here some results we need in Galois cohomology.
Let E be a field, either a finite extension of $\mathbb{Q}_p$ or a finite extension of $\mathbb{F}_p$. Let V be a finite dimensional E-vector space equipped with a continuous action of G. Let $V^* = \mathrm{Hom}(V, E(1))$.

Theorem 4.2.1. *The cohomology groups $H^i(\mathrm{G}, V)$ are finite dimensional for $i \geq 0$ and $= 0$ for $i \geq 3$ and we have the following identities :*

$$dim_E H^0(\mathrm{G}, V) = dim_E H^2(\mathrm{G}, V^*)$$

$$dim_E H^0(\mathrm{G}, V) - dim_E H^1(\mathrm{G}, V) + dim_E H^2(\mathrm{G}, V) = 0$$

Proof. For a proof, see [7]. □

Let $\rho : \mathrm{G} \to \mathrm{GL}_2(E)$ be a reducible continuous representation, $\rho \simeq \begin{pmatrix} \eta & b \\ 0 & \lambda \end{pmatrix}$ for two characters η and λ. Then $\lambda^{-1} \cdot b$ gives a class in $H^1(\mathrm{G}, \lambda^{-1} \cdot \eta)$. This class is trivial if and only if the extension splits.

Corrolary 4.2.2. *If E is characteristic 0, and if ρ is non split then $\eta \cdot \lambda^{-1}$ is either trivial or the cyclotomic character.*
If E is characteristic p, and if ρ is non split then $\eta \cdot \lambda^{-1}$ is either trivial or the cyclotomic character modulo p.

Now let $L^{ur} \subset \bar{L}$ be the maximal unramified extension and set $\mathrm{G}^{ur} = \mathrm{Gal}(L^{ur}/L)$.

Proposition 4.2.3. *Let V be a 1-dimensional E vector space on which G^{ur} acts through a character λ. The cohomology groups $H^i(\mathrm{G}^{ur}, V)$ are trivial for all i if $\lambda \neq 1$.*

Proof. See [7]. □

Corrolary 4.2.4. *An unramified 2-dimensional representation is either split or an extension of an unramified character by itself.*

4.3 Characters

Class Field Theory gives an injective map with dense image $L^\times \to \mathrm{G}^{ab}$ which induces an isomorphism $\mathcal{O}_L\times \simeq I_{\mathrm{G}^{ab}}$, where $I_{\mathrm{G}^{ab}}$ is the inertia subgroup of G^{ab}. Hence the pro-p abelianization $G^{ab,p}$ is isomorphic to $\hat{\mathbb{Z}}_p \times H$ with H a finite group. Hence the deformation ring of a character is represented by the algebra $\Lambda \simeq \mathcal{O}[[T]][H]$.

4.4 Representations in characteristic zero

4.4.1 The local monodromy theorem

The group G admits the following two step filtration :

$$1 \to I_{\mathrm{G}} \to \mathrm{G} \to \mathrm{G}^{ur} \to 1$$

$$1 \to P_p \to I_{\mathrm{G}} \to I_{\mathrm{G},p} \to 1.$$

The group I_{G} is the inertia and the group $I_{\mathrm{G},p}$ is its maximal pro-p quotient. It is isomorphic to $\mathbb{Z}_p(1)$ via the tame character $\mathrm{t}_p : I_{\mathrm{G},p} \to \mathbb{Z}_p(1)$.
Let E be a finite field extension of $\mathbb{Q}_p$, V a finite dimensional E-vector space and $\rho : \mathrm{G} \to \mathrm{GL}(V)$ a continuous representation.

Theorem 4.4.1. *There exists a unique $N \in \mathrm{End}(V)(-1)$ nilpotent called the logarithm of monodromy and a finite index subgroup I_1 of I_{G} such that*

$$\forall \sigma \in I_1,\ \rho(\sigma) = exp(\mathrm{t}_p(\sigma)N).$$

Proof. See [9]. □

Corrolary 4.4.2. *We can associate to ρ a pair (ρ', N), where N is the logarithm of monodromy and $\rho' : \mathrm{G} \to \mathrm{GL}(V)$ is defined by the following rule :*

$$\rho'(F^n\sigma) = \rho(F^n\sigma)exp(-\mathrm{t}_p(\sigma)N)$$

where F is a lift of the Frobenius to G *and $\sigma \in I_{\mathrm{G}}$. The group $\rho'(I_{\mathrm{G}})$ is finite.*

The pair (ρ', N) is a representation of the Weil-Deligne group of L (See [1] for more information).

4.4.2 2-dimensional representations

We now assume that V is a 2-dimensional representation and list the possibilities for ρ :

Non trivial monodromy If $N \neq 0$ then ρ is a non split extension of some character $\gamma : \mathrm{G} \to E^\times$ by $\gamma(1)$.

Trivial monodromy, reducible case In this case ρ is an extension of a character γ by a character λ. If $\lambda \neq \gamma$ this extension splits, otherwise it may not split but ρ becomes unramified after a twist by γ^{-1}.

Absolutely irreducible case We now assume ρ is irreducible.

Proposition 4.4.3. *There exists a character γ such that the representation $\rho \otimes \gamma$ has finite image.*

Proof. This statement is valid in any dimension. For F a lift of the Frobenius, $\rho(F)$ acts by conjugation on the finite group $\rho(I_{\mathrm{G}})$. Hence there is an integer $n \geq 1$ such that $\rho(F)^n$ centralizes $\rho(I_{\mathrm{G}})$ and hence $\rho(\mathrm{G})$. By Schur's lemma, $\rho(F)^n$ is a scalar matrix $a\mathbb{I}$. We can take for γ an unramified character which maps F to some n^{th}-root of a^{-1}. □

We choose a basis of V and view ρ as a morphism $\mathrm{G} \to \mathrm{GL}_2(E)$. We also let $\rho^0 : \mathrm{G} \to \mathrm{PGL}_2(E)$ be the projectivization of ρ. It has finite image.
We know that a finite subgroup of $\mathrm{PGL}_2(\bar{\mathbb{Q}}_p)$ is either cyclic, dihedral, isomorphic to A_4 (the alternating group over 4 symbols), to S_4 (the symmetric group over 4 symbols), or to A_5.
The group A_5 is simple of order 60, the group S_4 has order 24, and it admits a filtration $\mathbb{Z}/2 \times \mathbb{Z}/2 \subset \mathrm{A}_4 \subset \mathrm{S}_4$.
Considering the filtration by ramification subgroups in G we see that if ℓ, the residual characteristic of L, is different from 2 then $\rho^0(\mathrm{G})$ is dihedral, whereas in case $\ell = 2$ the image can be either dihedral, isomorphic to A_4 or to S_4. In the dihedral case one gets easily the following proposition :

Proposition 4.4.4. *If $\rho^0(G)$ is dihedral there exists a subgroup $H \subset \mathrm{G}$ of index 2, and a character $\gamma : H \to E^\times$ such that $\rho \simeq \mathrm{Ind}_H^{\mathrm{G}}\gamma$ (remark that the assumption ρ being irreducible forces $\gamma^\sigma \neq \gamma$, for $\sigma \in \mathrm{G}\backslash H$) .*

4.5 Deformations in characteristic zero

Let $\rho : \mathrm{G} \to \mathrm{GL}_N(\mathbb{F})$ a representation. Let $R^\square$ be the universal framed deformation ring. We start with a version of Grothendieck's local monodromy theorem. The proof is the same as in the usual case.

Proposition 4.5.1. *Let $\rho_{univ} : \mathrm{G} \to \mathrm{GL}_N(R^\square[1/p])$ be the universal representation. There exists a unique $N_{univ} \in \mathrm{M}_N(R^\square[1/p])(-1)$ nilpotent and a finite index subgroup I_1 of I_{G} such that*

$$\forall \sigma \in I_1,\ \rho_{univ}(\sigma) = exp(\mathrm{t}_p(\sigma)N_{univ}).$$

We can associate to ρ_{univ} the pair (ρ'_{univ}, N_{univ}) where $\rho'_{univ} : \mathrm{G} \to \mathrm{GL}_N(R^\square[1/p])$ is defined by the following rule :

$$\rho'_{univ}(F^n\sigma) = \rho_{univ}(F^n\sigma)exp(-\mathrm{t}_p(\sigma)N_{univ})$$

where F is a lift of the Frobenius to G and $\sigma \in I_{\mathrm{G}}$. The group $\rho'(I_{\mathrm{G}})$ is finite. The pair (ρ'_{univ}, N_{univ}) is a representation of the Weil-Deligne group.

Let $x : \mathrm{Spec}\ \bar{K} \to \mathrm{Spec}\ R^\square[1/p]$ be a geometric point. We denote by ρ_x the representation $\mathrm{G} \to \mathrm{GL}_N(\bar{K})$ deduced from x and also by (ρ'_x, N_x) the induced representation of the Weil-Deligne group.

Proposition 4.5.2. *Let $x, y : \mathrm{Spec}\ \bar{K} \to \mathrm{Spec}\ R^\square[1/p]$ be two geometric points lying in the same geometrically connected component. Then the restrictions of ρ'_x and ρ'_y to the inertia I_{G} are isomorphic (i.e conjugate).*

Proof. Let $K_1 \subset \bar{K}$ be a finite extension of $\mathcal{O}[1/p]$ over which the isomorphism classes of irreducible representations of I_{G}/I_1 are defined. Set $R^{\square}_{K_1} = R^{\square} \otimes_{\mathcal{O}} K_1$, and let $\mathcal{C} = \mathrm{Spec}\ C$ be a closed, irreducible and reduced subscheme of $\mathrm{Spec} R^{\square}_{K_1}[1/p]$.
The representation ρ'_{univ} induces a representation $\rho'_C : \mathrm{G} \to \mathrm{GL}_2(C)$. For any $\sigma \in I_{\mathrm{G}}$ we let $P(\rho'_C, \sigma) \in C[T]$ be the characteristic polynomial of $\rho'_C(\sigma)$.

Lemma 4.5.3. *The polynomial $P(\rho'_C, \sigma)$ has its coefficients in K_1.*

Let m be the number of isomorphism classes of N-dimensional representations of I_{G}/I_1 in K_1, let $\{\eta_1, ..., \eta_m\}$ be representatives of these isomorphism classes and let $P(\eta_j, \sigma) \in K_1[T]$ be the characteristic polynomial of $\eta_j(\sigma)$. For any $j = 1, ..., m$ we define the set $Max_j \subset \mathrm{Spec}\ C$ of all maximal ideals $\mathfrak{m}$ such that $P(\rho_C, \sigma) - P(\eta_j, \sigma) = 0$ modulo $\mathfrak{m}$. We claim that there exist j_0 such that Max_{j_0} is dense in Spec C. Indeed C is a Jacobson ring, so closed points are dense and on the other hand Spec C is not the union of finitely many strict closed subschemes because C is a domain. As a result $P(\rho'_C, \sigma) = P(\eta_{j_0}, \sigma)$.

We go back to the proof of the proposition. If $x, y : C \to \bar{K}$ are as in the proposition, the lemma shows that

$$x(P(\rho'_C, \sigma)) = y(P(\rho'_C, \sigma))$$

for any $\sigma \in I_{\mathrm{G}}$. As a result $\rho'_x|_{I_{\mathrm{G}}}$ and $\rho'_y|_{I_{\mathrm{G}}}$ have the same character and are isomorphic. Since this is true for any subdomain $\mathcal{C}$ we get the proposition. □

This proposition implies theorem 4.1.2.

4.6 Unipotent monodromy

We let $\chi : \mathrm{G} \to \mathbb{Z}_p^{\times}$ be the cyclotomic character and we assume that ρ is an extension of 1 by χ mod p, that is $\rho \simeq \begin{pmatrix} \chi \bmod p & \star \\ 0 & 1 \end{pmatrix}$. Let us consider $\mathbb{D}^{\square}$ the framed deformation functor and $\mathbb{D}^{\chi,\square}$ the framed deformation functor with fixed determinant χ. These functors are respectively represented by $\mathcal{O}$-algebras $R^{\square}$ and $R^{\chi,\square}$. We now define a new functor :

$$\mathbb{L}^{\chi,\square} : \widehat{\mathcal{AR}_{\mathcal{O}}} \longrightarrow SET$$

by setting $\mathbb{L}^{\chi,\square}(A) = \{\rho_A,\ L_A\}$ with $\rho_A : \mathrm{G} \to \mathrm{GL}_2(A)$ a framed deformation of ρ and L_A a rank 1 direct factor A-submodule of ρ_A acted on by χ.
We have a natural morphism $\mathbb{L}^{\chi,\square} \to \mathbb{D}^{\chi,\square}$ defined on A-points by sending $\{\rho_A,\ L_A\}$ to ρ_A.

Proposition 4.6.1. *The morphism $\mathbb{L}^{\chi,\square} \to \mathbb{D}^{\chi,\square}$ is represented by a projective morphism $\Pi : \mathcal{L}^{\chi,\square} \to \mathrm{Spec} R^{\chi,\square}$.*

Proof. Over Spec $R^{\chi,\square}$ is the universal rank 2 free $R^{\chi,\square}$-module. Let $\mathbb{P}$ be the projectivization of this bundle and $\hat{\mathbb{P}}$ its completion along the closed point of Spec $R^{\chi,\square}$. Then $\mathbb{L}^{\chi,\square}$ is represented by the closed subscheme of $\hat{\mathbb{P}}$ with equation "$\sigma.[v] - \chi(\sigma).[v]$" where $[v]$ is the universal line over $\hat{\mathbb{P}}$. By formal GAGA this subspace comes from a unique projective $R^{\chi,\square}$-scheme $\mathcal{L}^{\chi,\square}$. □

Proposition 4.6.2. *The scheme $\mathcal{L}^{\chi,\square}$ is formally smooth over* Spec $\mathcal{O}$. *Its generic fibre is connected.*

Proof. Let $A' \to A$ be a surjective map in $\mathcal{AR}_{\mathcal{O}}$. An element η in $\mathcal{L}^{\chi,\square}(A)$ gives a class $c(\eta) \in \mathrm{Ext}^1_{A[\mathrm{G}]}(A, A(1))$ and to lift η to an element η' in $\mathcal{L}^{\chi,\square}(A')$ it is enough to lift the class $c(\eta)$

to a class in $\mathrm{Ext}^1_{A'[\mathrm{G}]}(A', A'(1))$. Hence we are done if we can prove that for any p-order, finite $\mathbb{Z}_p$-module M the canonical map :

$$H^1(\mathrm{G}, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} M \to H^1(\mathrm{G}, M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1))$$

is an isomorphism. By dévissage it is enough to prove it when $M = \mathbb{Z}/p^n$, in this case this map is seen to be injective with cokernel equal to $H^2(\mathrm{G}, \mathbb{Z}_p(1))[p^n]$. This last group is trivial because by local duality $H^2(\mathrm{G}, \mathbb{Z}_p(1))$ is isomorphic to $\mathbb{Z}_p$.

Let us prove that $\mathcal{L}^{\chi,\square}[1/p]$ is connected. Set $\Pi_\star \mathcal{O}_{\mathcal{L}^{\chi,\square}} = \mathcal{L}$. This is a finite $R^{\chi,\square}$-algebra and hence is equal to a finite product of local rings because $R^{\chi,\square}$ is henselian. First notice that $\mathcal{L}$ is already local. Indeed the fiber of Π above the closed point of Spec $R^{\chi,\square}$ is either a single point or a $\mathbb{P}^1$ in case $\mathbb{F} \simeq \mathbb{F}(1)$ and ρ is split - that is in case ρ is trivial. In both cases this fiber is connected. Now let $e \in \mathcal{L}[1/p]$ be an idempotent, and let π be a uniformizing element in $\mathcal{O}$. Then choose the smallest $n \in \mathbb{Z}$ such that $\pi^n e$ is in $\mathcal{L}$. Then we get $\pi^{2n} e^2 = \pi^{2n} e$. But recall that $\mathcal{L}^{\chi,\square} \to \mathrm{Spec}\ \mathcal{O}$ is formally smooth and as a result the special fiber is reduced. The only possibility is $n = 0$ and e is a global idempotent and is equal to 1. □

Let E be a finite extension of $\mathcal{O}[1/p]$ and $\xi \in \mathcal{L}^{\chi,\square}(E)$ whose image $\Pi(\xi)$ in $R^{\chi,\square}$ is a representation we denote again by $\xi : \mathrm{G} \to \mathrm{GL}_2(E)$.

Proposition 4.6.3. *The map* $\Pi : \mathbb{L}^{\chi,\square}_{(\xi)} \to \mathbb{D}^{\chi,\square}_{(\xi)}$ *makes* $\mathbb{L}^{\chi,\square}_{(\xi)}$ *a subfunctor of* $\mathbb{D}^{\chi,\square}_{(\xi)}$. *In case* ξ *is indecomposable,* Π *is an isomorphism.*

Proof.. Let $B \in \mathcal{AR}_E$ and let ξ_B be a framed deformation of ξ and let L_B be a χ-variant line in ξ_B. We have to show that L_B is unique. Indeed $\mathrm{Hom}(B(1), \xi_B/L_B) = 0$ because $\det\xi_B = \chi$ and ξ_B/L_B is a trivial, free of rank 1 B-module.
Suppose now ξ undecomposable. In this case $\mathbb{D}^{\chi}_{(\xi)}$ is representable by E itself because

$$dim_E H^1(\mathrm{G}, Ad\rho^0) = dim_E H^0(\mathrm{G}, Ad\rho^0) + dim_E H^2(\mathrm{G}, Ad\rho^0(1)) = 0.$$

As a result any deformation is trivial, meaning that ξ_B is obtained from ξ by extension of scalars (up to some conjugation). □

We can now compute that the relative dimension of $\mathcal{L}^{\chi,\square}[1/p]$ over $\mathcal{O}$ is 3, indeed we do the calculation at a point ξ undecomposable as above and we have :

$$dim_E \mathbb{D}^{\chi,\square}_{(\xi)}(E[\epsilon]) = dim_E \mathbb{D}^{\chi}_{(\xi)}(E[\epsilon]) + dim_E Ad\xi - dim_E H^0(\mathrm{G}, Ad\xi) = 3$$

Let Spec $R^{\chi,1,\square}$ denote the scheme-theoretic image of $\mathcal{L}^{\chi,\square}$ in Spec $R^{\chi,\square}$ via the morphism Π.

Proposition 4.6.4. *The closed subscheme* Spec $R^{\chi,1,\square}$ *is a domain, its generic fibre is smooth and has dimension* 3.
Let E be a finite extension of $\mathcal{O}[1/p]$, then a morphism $\xi : R^{\chi,\square} \to E$ factors through $R^{\chi,1,\square}$ if and only if the corresponding 2*-dimensional representation is an extension of E by $E(1)$.*

Proof. Put as before $\mathcal{L} = \Pi_* \mathcal{O}_{\mathcal{L}^{\chi,\square}}$. Proposition 4.6.2 shows that $\mathcal{L}$ is a domain, since $R^{\chi,1,\square}$ is a subring of $\mathcal{L}$ it is also a domain.
Since $\mathcal{L}^{\chi,\square}$ is projective, the induced map $\Pi : \mathcal{L}^{\chi,\square} \to \mathrm{Spec} R^{\chi,1,\square}$ is surjective. By the preceding

proposition it is an isomorphism at the level of the generic fibers. As a result Spec $R^{\chi,1,\square}[1/p]$ is formally smooth over $\mathcal{O}[1/p]$ of dimension 3 and the last assertion follows easily. $\square$

By twisting by a character we can amplify the last proposition. Let $\gamma : \mathrm{G} \to \mathcal{O}^\times$ a continuous character, whose reduction modulo $\mathfrak{m}_\mathcal{O}$ we denote $\bar{\gamma}$. Suppose now that ρ is an extension of $\bar{\gamma}$ by $\bar{\gamma}(1)$.

Proposition 4.6.5. *There exists a closed subscheme* Spec $R^{\chi\gamma,\gamma,\square}$ *of* Spec $R^\square$*. It is a domain, its generic fibre is smooth and it has dimension* 3*.*
Let E be a finite extension of $\mathcal{O}[1/p]$, then a morphism $\xi : R^\square \to E$ factors through $R^{\chi,1,\square}$ if and only if the corresponding 2*-dimensional representation is an extension of γ by $\gamma(1)$.*

This proposition implies the parts of theorems 4.1.3, 4.1.4, 4.1.5 concerning unipotent type representations.

4.7 Unramified liftings

Assume that $\rho : \mathrm{G} \to \mathrm{GL}_2(\mathbb{F})$ is unramified, and let $\psi : \mathrm{G} \to \mathcal{O}^\times$ an unramified lifting of detρ.

Proposition 4.7.1. *There is a closed subscheme* Spec $R^{ur,\psi,\square}$ *of* Spec $R^\square$ *which corresponds to unramified framed deformations of ρ with determinant ψ. It is formally smooth over $\mathcal{O}$ of relative dimension* 3*.*

Proof. Let L^{ur} be the maximal unramified extension of L. Set $\mathrm{G}^{ur} = \mathrm{Gal}(L^{ur}/L)$, and recall the Galois cohomology : $dimH^2(\mathrm{G}^{ur}, Ad\rho^0) = 0$, $dimH^1(\mathrm{G}^{ur}, Ad\rho^0) = 1$ if Frobenius has distinct eigenvalues and $dimH^1(\mathrm{G}^{ur}, Ad\rho^0) = 3$ if Frobenius has only one eigenvalue. $\square$

Of course one can amplify the result by twisting by a character and in this way get the parts of theorems 4.1.3, 4.1.4, 4.1.5 concerning unramified (up to a twist) representations.

4.8 Local property

Let $\rho : \mathrm{G} \to \mathrm{GL}_2(\mathbb{F})$ be a representation. Let ψ be a lifting of detρ and let $R^{\psi,\square}$ be the framed deformation universal ring with fixed determinant equal to ψ.

Proposition 4.8.1. *The generic fiber of* Spec $R^{\psi,\square}[1/p]$ *is* 3*-dimensional, reduced and the union of formally smooth components.*
If two distinct geometric components intersect, there is a character γ such that one component is parameterizing representations becoming unramifed after a twist by γ^{-1} and the other is parameterizing extensions of γ by $\gamma(1)$.
For any geometric point x : Spec
$barK \to$ Spec $R^{\psi,\square}$ *lying on the intersection, $\rho_x \simeq \gamma \oplus \gamma(1)$.*

Proof. Let E be a finite field extension of $\mathcal{O}[1/p]$ and $\xi : R^{\psi,\square} \to E$ an E-point. First of all Spec $R^{\psi,\square}[1/p]$ is formally smooth at ξ provided $H^2(\mathrm{G}, Ad\xi^0) = 0$ and in this case the tangent space dimension is

$$dim_E H^1(\mathrm{G}, Ad\xi^0) + dim_E Ad\xi - dim_E H^0(\mathrm{G}, Ad\xi) = 3.$$

Suppose now that $H^2(\mathrm{G}, Ad\xi^0) \neq 0$ and recall that by local duality

$$dim_E H^2(\mathrm{G}, Ad\xi^0) = dim_E H^0(\mathrm{G}, Ad\xi^0(1)).$$

The calculation shows that this last group is not trivial if and only if ξ is isomorphic to a sum of characters $\gamma \oplus \gamma(1)$ (here we use the fact that $H^1(\mathrm{G}, E(-1)) = 0$). Moreover in this case $dim_E H^2(\mathrm{G}, Ad\xi^0) = 1$ and the tangent space has dimension 4.
Let us suppose that ξ is isomorphic to $\gamma \oplus \gamma(1)$, because otherwise there is nothing to do. After an extension of the scalars we can assume that γ is rational (actually γ is already rational if $p \neq 2$), and after a twist by γ^{-1} we can assume that γ is trivial.
Consider the schemes Spec $R^{ur,\psi,\square}[1/p]$ (which parameterizes unramified representations) and Spec $R^{\chi,1,\square}[1/p]$ (which parameterizes the extensions of 1 by χ). We claim that these are two distinct irreducible, 3-dimensional, formally smooth, subschemes of Spec $R^{\psi,\square}[1/p]$ passing through ξ. To prove that they are distinct we prove that there tangent spaces at ξ are distinct. We construct two infinitesimal deformations : Let $c \in H^1(\mathrm{G}, E(1))$ a non zero class which has to be ramified and let also $\mu : \mathrm{G} \to E[\epsilon]^\times$ be a non trivial unramified deformation of the trivial character. Consider over $E[\epsilon]$ the framed deformations $\begin{pmatrix} \chi & c \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} \chi \cdot \mu & 0 \\ 0 & 1 \cdot \mu^{-1} \end{pmatrix}$.

The complete local ring of Spec $R^{\psi,\square}[1/p]$ is isomorphic to the quotient of $\mathcal{O}[1/p][[X_1, ..., X_4]]$ by a monogenous ideal J. If the ideal J were zero then Spec $R^{\psi,\square}[1/p]$ would be formally smooth everywhere. In this case we get a contradiction if we take any closed point of Spec $R^{\psi,\square}[1/p]$ lying on Spec $R^{\chi,1,\square}[1/p]$ inducing a non-split representation (it exists !). The dimension at this point is only 3, but it is on the same irreducible component as ξ so it should have dimension 4. As a result J is not zero and Spec $R^{\psi,\square}[1/p]$ has dimension 3.
Let $\mathcal{C}$ be a geometric irreducible component passing by ξ. Let x be a geometric point lying on this component. The representation ρ'_x is unramified by theorem 4.1.2. Either $N_x = 0$ and x lies in Spec$R^{ur,\psi,\square}[1/p]$ or $N_x \neq 0$ and x lies in Spec $R^{\chi,1,\square}[1/p]$. As a result the union of Spec $R^{\chi,1,\square}[1/p]$ and Spec $R^{ur,\psi,\square}[1/p]$ is dense and closed in $\mathcal{C}$, so $\mathcal{C}$ must be equal to one of these two. □

This proposition completes the proof of theorem 4.1.1 and of the last part of theorem 4.1.3.

4.9 The split ramified case

Let $\eta : \mathrm{G} \to \mathcal{O}^\star$ be a ramified character. We denote by $\bar{\eta}$ its reduction modulo $\mathfrak{m}_\mathcal{O}$. Let $\rho : \mathrm{G} \to \mathrm{GL}_2(\mathbb{F})$ be an extension of 1 by $\bar{\eta}$, that is $\rho \simeq \begin{pmatrix} \bar{\eta} & \star \\ 0 & 1 \end{pmatrix}$. We let as usual $R^{\eta,\square}$ be the universal ring representing the functor $\mathbb{D}^{\eta,\square}$.

4.9.1 Preliminary

We define a functor:

$$\mathbb{L}^{\eta,\square} : \widehat{\mathcal{AR}_\mathcal{O}} \longrightarrow SET$$

by setting $\mathbb{L}^{\eta,\square}(A) = \{\rho_A,\ L_A\}$ where ρ_A is a framed deformation of ρ to A with determinant η and L_A is a line in the space of ρ_A on which G acts via $\eta.\psi(L_A)$ where $\psi(L_A)$ is some unramified character depending on L_A.
We have a natural morphism $\mathbb{L}^{\eta,\square} \to \mathbb{D}^{\eta,\square}$ defined by forgetting the line.

Proposition 4.9.1. *The morphism $\mathbb{L}^{\eta,\square} \to \mathbb{D}^{\eta,\square}$ is represented by a projective morphism of schemes $\Theta : \mathcal{L}^{\eta,\square} \to \mathrm{Spec}R^{\eta,\square}$.*

Proposition 4.9.2. *The scheme $\mathcal{L}^{\eta,\square}$ is formally smooth.*

Proof. Let $A' \to A$ be a surjective morphism in $\mathcal{AR}_\mathcal{O}$. Let $\{\rho_A,\ L_A\}$ be a point in $\mathcal{L}^{\eta,\square}(A)$. We want to lift it to an A'-point. We first lift the character $\psi(L_A)$ to a character $\psi' : \mathrm{G} \to A'^\times$. Then

we need only to show the surjectivity of the map : $H^1(\mathrm{G}, A'(\eta.\psi'^2)) \to H^1(\mathrm{G}, A(\eta.\psi(L_A)^2))$.
We identify $\mathcal{O}[[T]]$ with the universal unramified deformation ring of the trivial character. Hence any topological $\mathcal{O}[[T]]$-module M is naturally a G-module, G acting through G^{ur} and the Frobenius acting by multiplication by $1+T$. Moreover we let $M(\eta)$ be the module M with twisted Galois action by η.
The above surjectivity will follow from the lemma :

Lemma 4.9.3. *Let M be any $\mathcal{O}[[T]]$-module of finite length. Then the map*

$$H^1(\mathrm{G}, \mathcal{O}[[T]](\eta)) \otimes_{\mathcal{O}[[T]]} M \to H^1(\mathrm{G}, M(\eta))$$

is surjective.

By Nakayama's lemma we can assume that M is of π-torsion (π is the uniformizer of $\mathcal{O}$). Then we reduce to the case where $M = \mathbb{F}[T]/T^n$. We take the following resolution of $M(\eta)$

$$0 \to \mathcal{O}[[T]](\eta) \xrightarrow{\begin{pmatrix} \pi \\ T^n \end{pmatrix}} \mathcal{O}[[T]](\eta) \oplus \mathcal{O}[[T]](\eta) \xrightarrow{\begin{pmatrix} \pi & T^n \end{pmatrix}} \mathcal{O}[[T]](\eta) \to 0.$$

Taking the associated spectral sequence we see that the obstruction to the surjectivity is in the H^1 of the following complex concentrated in degree $0, 1$ and 2

$$H^2(\mathrm{G}, \mathcal{O}[[T]](\eta)) \xrightarrow{\begin{pmatrix} \pi \\ T^n \end{pmatrix}} H^2(\mathrm{G}, \mathcal{O}[[T]](\eta)) \oplus H^2(\mathrm{G}, \mathcal{O}[[T]](\eta)) \xrightarrow{\begin{pmatrix} \pi & T^n \end{pmatrix}} H^2(\mathrm{G}, \mathcal{O}[[T]](\eta))$$

But we can check that $H^2(\mathrm{G}, \mathcal{O}[[T]](\eta)) = 0$ because we have the classical isomorphism

$$H^2(\mathrm{G}, \mathcal{O}[[T]](\eta)) \simeq \varprojlim_{cor} H^2(\mathrm{G}_n, \mathcal{O}(\eta))$$

where G_n is the Galois group of the unramified extension of L of degree p^n and because all the $H^2(\mathrm{G}_n, \mathcal{O}(\eta))$ are zero by local duality. □

Proposition 4.9.4. *The generic fiber of $\mathcal{L}^{\eta,\square}$ is connected except when ρ is split and $\bar{\eta}$ is non trivial and unramified in which case there are two connected components.*

Proof. The connected components of the generic fiber are in bijection with those of the fiber of $\mathcal{L}^{\eta,\square}$ over the closed point of Spec $R^{\eta,\square}$ (see proposition 4.6.2). □

Proposition 4.9.5. *Let E be a finite extension of K and let ξ :* Spec $E \to \mathcal{L}^{\eta,\square}$ *be a point. We call again ξ its image in* Spec $R^{\eta,\square}$. *Then the following maps of functors from $\mathcal{AR}_E$ to SET is an isomorphism :*

$$\mathbb{L}^{\eta,\square}_{(\xi)} \to \mathbb{D}^{\eta,\square}_{(\xi)}$$

Proof.. The point ξ induces a diagonal E-representation which is isomorphic to $\eta.\psi_\xi \oplus \psi_\xi^{-1}$. We define a subfunctor

$$\mathbb{D}^{diag,\square}_{\xi} : \mathcal{AR}_E \to SET$$

by setting $\mathbb{D}^{diag,\square}_x(B) = \{$reducible and split liftings of x to $B\}$. The proof of the following lemma is left to the reader.

Lemma 4.9.6. *The functor $\mathbb{D}^{diag,\square}_x$ is representable by a formally smooth scheme of dimension* 3.

We deduce that we have a natural isomorphism $\mathbb{D}^{diag,\square}_{\xi} \to \mathbb{D}^{\eta,\square}_{(\xi)}$. Let B in $\mathcal{AR}_E$ and ξ_B be a framed deformation of ξ. The above result implies that ξ_B is isomorphic to $\eta.\psi_B \oplus \psi_B^{-1}$ for some unramified character ψ_B. Furthermore there is a unique stable line in the space of ξ_B on which the inertia acts via $\eta|_{I_G}$ and the proposition follows. □

Corrolary 4.9.7. *The map* $\mathcal{L}^{\eta,\square} \to \mathrm{Spec}\ R^{\eta,\square}$ *induces an open and closed immersion at the level of the generic fiber.*

We define a subfunctor $\mathbb{L}^{\eta,1,\square}$ of $\mathbb{L}^{\eta,\square}$ by setting $\mathbb{L}^{\eta,\square}(A) = \{\rho_A,\ L_A\}$ where ρ_A is a framed deformation of ρ to A with determinant η and L_A is a line in the space of ρ_A on which G acts via η. This functor is represented by a closed subscheme $\mathcal{L}^{\eta,1,\square}$ of $\mathcal{L}^{\eta,\square}$.

Proposition 4.9.8. *The scheme* $\mathcal{L}^{\eta,1,\square}$ *is formally smooth.*

Proof. As in proposition 4.6.2. it is enough to prove that for any $\mathbb{Z}_p$-module M, the map:

$$H^1(\mathrm{G}, \mathbb{Z}_p(\eta)) \otimes M \to H^1(\mathrm{G}, M(\eta))$$

is an isomorphism. By dévissage it is enough to prove it for $M = \mathbb{Z}/p^n$, and it follows from $H^2(\mathrm{G}, \mathbb{Z}_p(\eta)) = 0$. □

Proposition 4.9.9. *The generic fiber of the scheme* $\mathcal{L}^{\eta,1,\square}$ *is connected.*

Proof. The fiber of $\mathcal{L}^{\eta,1,\square}$ over the closed point of Spec $R^{\eta,\square}$ consists either of one point or of a $\mathbb{P}^1$ if ρ is trivial. □

Corrolary 4.9.10. *Assume that* $x, y : \mathrm{Spec}\ \bar{K} \to \mathrm{Spec}\ R^{\eta,\square}[1/p]$ *are two geometric points, whose associated representations are both isomorphic to* $\eta \oplus 1$. *Then* x *and* y *lie on the same geometrically irreducible component.*

4.9.2 Generalization

We go back to a general situation : let $\rho : \mathrm{G} \to \mathrm{GL}_2(\mathbb{F})$ be a representation, $\psi : \mathrm{G} \to \mathcal{O}^\times$ a lifting of $\det\rho$ and $R^{\psi,\square}$ be the universal framed deformation ring with determinant ψ.

Proposition 4.9.11. *Let* $x : \mathrm{Spec}\ \bar{K} \to \mathrm{Spec}\ R^{\psi,\square}$ *be a geometric point. Suppose that* $\rho_x : \mathrm{G} \to \mathrm{GL}_2(\bar{K})$ *is isomorphic to a diagonal representation* $\eta \oplus \lambda$ *and that the character* $\eta.\lambda^{-1}$ *is ramified. Let* $\mathcal{C}$ *be the geometrically connected component in* Spec $R^{\psi,\square}[1/p]$ *which contains the image of* x. *Then for any geometric point* $y : \mathrm{Spec}\ \bar{K} \to \mathcal{C}$ *the representation* ρ_y *is isomorphic to a diagonal representation* $\eta.\phi \oplus \lambda.\phi^{-1}$ *for some unramified character* ϕ.

Proof. After twisting we can assume that $\lambda = 1$, ρ is an extension of 1 by η. We need only show that x factorizes through $\mathcal{L}^{\eta,\square}$. The representation ρ_x is defined on a finite extension $\mathcal{O}'$ of $\mathcal{O}$ and we remark that there is a stable $\mathcal{O}'$-line in the space of ρ_x on which G acts via η and the result follows from corollary 4.9.7. □

Definition 4.9.12. *We say that a component* $\mathcal{C}$ *as above is of split ramified type* $\{\eta|_{I_\mathrm{G}}, \lambda|_{I_\mathrm{G}}\}$.

Proposition 4.9.13.

1. *There is at most one component of split ramified type* $\{\eta|_{I_\mathrm{G}}, \lambda|_{I_\mathrm{G}}\}$ *in* Spec $R^{\eta,\square}$, *except if* ρ *is split diagonal, isomorphic to* $\bar{\eta} \oplus \bar{\lambda}$ *with* $\bar{\eta}.\bar{\lambda}^{-1}$ *non trivial and unramified, in which case there are exactly two components of split ramified type* $\{\eta|_{I_\mathrm{G}}, \lambda|_{I_\mathrm{G}}\}$.
2. *Assume that* $x, y : \mathrm{Spec}\ \bar{K} \to R^{\eta,\square}[1/p]$ *are two geometric points, whose associated representations are both isomorphic to* $\eta \oplus \lambda$, *with* $\eta.\lambda^{-1}$ *ramified. Then* x *and* y *lie on the same geometrically irreducible component.*

Proof.. This is corollary 4.9.10 and proposition 4.9.4. □

These two propositions implies the parts of theorems 4.1.3, 4.1.4, 4.1.5 concerning split ramified representations.

4.10 The absolutely irreducible case

Let $\rho : \mathrm{G} \to \mathrm{GL}_2(\mathbb{F})$ be a representation, let $\psi : \mathrm{G} \to \mathcal{O}^\times$ be a lift of the determinant and let $R^{\psi,\square}$ be the universal framed deformation ring of ρ.
Set $X[1/p]$ equal the generic fiber of the space Spec $R^{\psi,\square}$. We make the assumption that there is a geometric point $x : \mathrm{Spec}\ \bar{K} \to X[1/p]$, such that the representation $\rho_x : \mathrm{G} \to \mathrm{GL}_2(\bar{K})$ deduced by pull back is absolutely irreducible. Note that, in general, this does not imply that ρ itself is irreducible.

4.10.1 Exceptional case

We assume that the residual characteristic ℓ of L is 2, and that the image $\rho_x^0(\mathrm{G})$ in $\mathrm{PGL}_2(\bar{K})$ is either isomorphic to A_4 or S_4.

Lemma 4.10.1. *The residual representation ρ is irreducible.*

Proof.. We claim that there is a subgroup $H \subset \mathrm{G}$ of finite index such that $\rho^0(H)$ is a dihedral group of order 4. Indeed recall that we have a filtration $\mathbb{Z}/2 \times \mathbb{Z}/2 \subset \mathrm{A}_4 \subset \mathrm{S}_4$.
In particular it means that $\rho_x|_H \simeq \mathrm{Ind}_{H'}^{H}\gamma$, where $H' \subset H$ has index 2 and γ is a character of H'.
Moreover if $\sigma \in H' \backslash H$, then $\gamma^\sigma.\gamma^{-1}$ is a character of order 2.
This proves that $\rho|_H$ is residually irreducible, because $\bar{\gamma}^\sigma.\bar{\gamma}^{-1}$ is again non trivial since $p \neq \ell$. □

Proposition 4.10.2. *The deformation functor of ρ with determinant ψ is isomorphic to $\mathcal{O}$ and $R^{\psi,\square}$ is a formal power series ring in 3 variables.*
The generic fiber $X[1/p]$ is irreducible and for any geometric point $y : \mathrm{Spec}\ \bar{K} \to X[1/p]$ the representation ρ_y is isomorphic to ρ_x.

Proof. The proof follows from the computation of the cohomology groups $H^i(\mathrm{G}, Ad^0\rho)$. □

4.10.2 Rigidity

Proposition 4.10.3. *Let $y : \mathrm{Spec}\ \bar{K} \to X[1/p]$ be another geometric point lying in the same geometrically irreducible component as x. Then ρ_y is isomorphic to ρ_x.*

Proof. We already gave a proof in the exceptional case and can restrict our attention to the case where ρ_x is induced from a character. We start with a lemma :

Lemma 4.10.4. *There is a closed subscheme $X[1/p]^{ab} \hookrightarrow X[1/p]$, such that a geometric point $z : \mathrm{Spec}\ \bar{K} \to X[1/p]$ factorizes through $X[1/p]^{ab}$ if and only if ρ_z has an abelian image.*

Over $X[1/p]$ we have the Weil-Deligne representation (ρ'_{univ}, N_{univ}). Let F be a lift of the Frobenius in G, and $\sigma_1, ..., \sigma_t$ be element of I_G generating the finite group $\rho'_{univ}(I_\mathrm{G})$. We define $X[1/p]^{ab}$ by the relations

$$N_{univ} = 0,$$
$$\rho'_{univ}(\sigma_i)\rho'_{univ}(\sigma_j) = \rho'_{univ}(\sigma_j)\rho'_{univ}(\sigma_i) \text{ for } 1 \leq i, j \leq t,$$
$$\rho'_{univ}(\sigma_i)\rho'_{univ}(F) = \rho'_{univ}(F)\rho'_{univ}(\sigma_i) \text{ for } i = 1...t.$$

Now we base change $X[1/p]$ to a finite extension $K_1 \subset \bar{K}$ of $\mathcal{O}[1/p]$, such that all irreducible components of $X[1/p]_{K_1}$ are geometrically irreducible and such that the isomorphism classes of irreducible representations of G in $\mathrm{GL}_2(\bar{K})$ with determinant ψ are defined over K_1.
We need a second lemma :

Lemma 4.10.5. *Let $\mathcal{C}$ be the geometrically irreducible component where x lies. The subset of closed points in $\mathcal{C}$ with irreducible associated representation is dense in $\mathcal{C}$.*

We can assume that $\rho_x \simeq \mathrm{Ind}_H^G \gamma$ for a character γ, and a subgroup H of index 2 in G. Let $\sigma \in \mathrm{G}\backslash H$, we recall that $\gamma^\sigma \neq \gamma$.
First we consider the case where H corresponds to the unramified extension of L. Then σ is a lift of the frobenius. We easily see that

$$\rho_y|_{I_\mathrm{G}} \simeq \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^\sigma \end{pmatrix}$$

where $\gamma^\sigma|_{I_\mathrm{G}} \neq \gamma|_{I_\mathrm{G}}$. The relation $\rho_y(\sigma)\rho_y(\tau)\rho_y(\sigma)^{-1} = \rho_y(\sigma\tau\sigma^{-1})$ for any $\tau \in I_\mathrm{G}$ forces $\rho_y(\sigma) = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ and ρ_y to be absolutely irreducible. Now we consider the case where H corresponds to a ramified extension, and we let I_H be the inertia subgroup of H, which is of index 2 in I_G. We have $\rho_x|_{I_\mathrm{G}} \simeq \mathrm{Ind}_{I_H}^{I_\mathrm{G}} \gamma$. Then there are two possibilities, either $\gamma^\sigma|_{I_H} \neq \gamma|_{I_H}$ and $\rho_x|_{I_\mathrm{G}}$ is irreducible or $\gamma^\sigma|_{I_H} = \gamma|_{I_H}$ and

$$\rho_y|_{I_\mathrm{G}} \simeq \begin{pmatrix} \gamma & 0 \\ 0 & \gamma.\chi_H \end{pmatrix}$$

where χ_H is the order 2 character $\mathrm{G} \to \mathrm{G}/H$. But there are no reducible and non-split 2-dimensional representation with such a restriction to inertia, this proves that ρ_y is either abelian or absolutely irreducible.

We conclude the proof of the proposition with a third lemma whose proof, left to the reader, is similar to that of lemma 4.5.3. We set $\mathcal{C} = \mathrm{Spec}\ C$ and we let $\rho_\mathcal{C} : \mathrm{G} \to \mathrm{GL}_2(C)$ be the universal representation above $\mathcal{C}$.

Lemma 4.10.6. *For any $g \in \mathrm{G}$, the characteristic polynomial $P(\rho_\mathcal{C}, g) \in C[T]$ lies in $K_1[T]$.*

We deduce the equality of the characteristic polynomials $P(\rho_y, g) = P(\rho_x, g)$ for any $g \in \mathrm{G}$ and conclude that ρ_x and ρ_y are isomorphic. □

This proposition completes the proof of theorem 4.1.3.

4.10.3 Number of components

We say that a geometrically irreducible component $\mathcal{C}$ is of absolutely irreducible type if there is a geometric point $\xi : \mathrm{Spec}\bar{K} \to \mathcal{C}$ such that the associated representation still denoted by ξ is absolutely irreducible. The last proposition shows that the isomorphism class of ξ is an invariant of the component.

Definition 4.10.7. *We say that a geometrically irreducible component $\mathcal{C}$ as above is of absolutely irreducible type ξ.*

We use the notation of the last part : X is the universal deformation space of ρ and $x : \mathrm{Spec}\ \bar{K} \to X[1/p]$ is a geometric point whose associated representation ρ_x is absolutely irreducible. We can suppose that x maps to a rational point.

Proposition 4.10.8. *There is exactly one geometrically irreducible component in $X[1/p]$ of irreducible type ρ_x.*

Proof. We already proved the proposition in the exceptional case and we restrict our attention to the case where $\rho_x \simeq \mathrm{Ind}_H^\mathrm{G} \gamma$, for a subgroup $H \subset \mathrm{G}$ of index 2 and a character $\gamma : H \to \mathcal{O}_{\bar{K}}^\times$. We choose $\sigma \in \mathrm{G}\backslash H$, and we recall that $\gamma^\sigma \neq \gamma$. We set $\bar{\gamma}$ for the residual character of γ

If we assume that $\bar{\gamma}^\sigma \neq \bar{\gamma}$ then ρ is residually irreducible and $R^{\psi,\square}$ is a formal power series ring. The proposition follows.

Assume now that $\bar{\gamma}^\sigma = \bar{\gamma}$. We shall construct a scheme $\tilde{X} \to X$, whose generic fiber is connected and maps onto the irreducible components of type ρ_x of $X[1/p]$.

First we remark that $\rho|_H$ is a reducible representation, as a result we can assume $\rho|_H = \begin{pmatrix} \bar{\gamma} & \alpha \\ 0 & \bar{\gamma} \end{pmatrix}$. We consider the following functor :

$$\mathbb{D}^H : \widehat{\mathcal{AR}_{\mathcal{O}}} \longrightarrow SET$$

by setting $\mathbb{D}^H(A) = \{\rho_A, L_A\}$ where $\rho_A : H \to \mathrm{GL}_2(A)$ is a framed deformations of $\rho|_H$ with determinant ψ and L_A is a stable line acted on via γ. We claim that this functor is representable by a formally smooth scheme Y. We have a universal representation over $\mathcal{O}_Y$, $\rho_H = \begin{pmatrix} \gamma & \alpha_{univ} \\ 0 & \gamma^\sigma \end{pmatrix}$. To any point $\xi = (\rho_A, L_A) \in Y(A)$, we attach the set $\mathcal{M}(\xi)$ of matrices $M_\sigma \in \mathrm{GL}_2(A)$ such that

- $M_\sigma^2 = \rho_A(\sigma^2)$.
- $\mathrm{det} M_\sigma = \psi(\sigma)$.
- $M_\sigma^{-1}\rho_A M_\sigma = \rho_A(\sigma^{-1}.\sigma)$.
- $M_\sigma = \rho(\sigma)$ modulo $\mathfrak{m}_A$.

We define a functor $\tilde{\mathbb{D}} : \widehat{\mathcal{AR}_{\mathcal{O}}} \longrightarrow SET$ by setting $\tilde{\mathbb{D}}(A) = \{\xi \in Y(A), M_\sigma \in \mathcal{M}(\xi)\}$. This functor is represented by a scheme $\tilde{X}$, affine over Y.

Lemma 4.10.9. *The scheme $\tilde{X}[1/p]$ is connected.*

We describe $\tilde{X}[1/p]$. Let $\rho(\sigma) = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$. Let $\widehat{\mathbb{A}}^4/_{\mathcal{O}}$ be the completion of the affine space (of 2×2 matrices) $\mathbb{A}^4$ at the point $(\bar{a}, \bar{b}, \bar{c}, \bar{d})$ of the special fiber. Let $M_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the universal matrix over $\tilde{X}$. We embed $\tilde{X}$ in $\widehat{\mathbb{A}}^4/_Y$ and compute the equations defining $\tilde{X}$ at the level of the generic fiber. We remark that, on the generic fiber, α_{univ} is proportional to $\gamma - \gamma^\sigma$ (because the extension class vanishes), and hence that conjugating by M_σ sends α_{univ} to $-\alpha_{univ}$. We easily deduce that $\tilde{X}[1/p]$ is defined by the equations :

- $a = -d$.
- $a^2 + bc = \gamma(\sigma^2)$.
- $a^2.(\gamma^\sigma - \gamma) = \alpha_{univ}ac$.
- $ac.(\gamma^\sigma - \gamma) = \alpha_{univ}c^2$.
- $ab.(\gamma^\sigma - \gamma) = \alpha_{univ}bc$.

Since $Y[1/p]$ is a domain c needs to be invertible and hence the two coordinates (a, c) are parameters of $\tilde{X}[1/p]$ over $Y[1/p]$. We define a structure of $\widehat{\mathrm{GL}_1}$-torsor on $\tilde{X}[1/p]$ over $Y[1/p]$ by setting $\lambda.(a, c) = (\lambda a, \lambda c)$ for $\lambda \in \widehat{\mathrm{GL}_1}$ which proves the lemma.

We have a map $\Pi : \tilde{X} \to X$ defined as follow : to $(\xi = (\rho_A, L_A), M_\sigma) \in \tilde{X}(A)$ we associate the framed deformation ρ'_A defined by $\rho'_A|_H = \rho_A$ and $\rho'_A(\sigma) = M_\sigma$.

Lemma 4.10.10. *Let $y : \mathrm{Spec}\ \bar{K} \to X$ a geometric point such that $\rho_y \simeq \rho_x$. Then y factorizes through Π.*

Since $\rho_y \simeq \mathrm{Ind}_H^{\mathrm{G}}\gamma$, $\rho_y|_H$ comes from some $\bar{K}$-point y^0 of Y. And the pair $(y^0, M_\sigma = \rho_y(\sigma))$ defines a point on $\tilde{X}$.

To conclude the proof of the proposition it is enough to prove that all geometric points inducing representations isomorphic to ρ_x are in the same connected component as x and this follows from the two lemmas. □

This proposition completes the proof of theorem 4.1.4 and 4.1.5.

4.11 The determinant

As usual let $\rho : \mathrm{G} \to \mathrm{GL}_2(\mathbb{F})$ be any representation and $R^\square$ its universal framed deformation ring.

Most of the analysis of the preceding sections has been done for the closed subscheme Spec $R^{\psi,\square}$ of Spec $R^\square$ of deformations with determinant ψ. Nevertheless, in the case $p \neq 2$, it is easy to extend all the results to Spec $R^\square$. We shall explain this.

Let Y be the universal deformation space for the trivial character and let $Y' = Y$ which we see as the universal deformation space of the character $\det\rho$.
There is an action $Y \times \mathrm{Spec}\ R^\square \to \mathrm{Spec}\ R^\square$: If $\rho_A : \mathrm{G} \to \mathrm{GL}_2(A)$ is a lifting of ρ to A and if $\eta : \mathrm{G} \to A^\times$ is a lifting of the trivial character, we can construct the representation $\rho_A \otimes_A \eta$ which is still a lifting of ρ to A.
Similarly there is an action $Y \times Y' \to Y'$ defined by sending a pair (η, μ) to $\eta^2 \cdot \mu$ where η and μ are characters $\mathrm{G} \to A^\times$ lifting respectively the trivial character and $\det\rho$.
These two actions are compatible with the determinant map Spec $R^\square \to Y'$.

Lemma 4.11.1. *Assume that $p = char\,\mathbb{F}$ is odd. Then the action $Y \times Y' \to Y'$ is transitive and free.*

Proof. A p-group is uniquely 2-divisible if $p \neq 2$. □

Theorem 4.11.2. *We make the assumption that $p \neq 2$.*
Let ψ be a lifting of $\det\rho$ *and let $R^{\psi,\square}$ be the universal framed deformation ring with determinant ψ.*
The map $Y \times$ Spec $R^{\psi,\square} \to$ Spec $R^\square$ *is an isomorphism.*
In particular the generic fiber Spec $R^\square[1/p]$ *is a union of 4-dimensional formally smooth irreducible components.*

References

[1] P. Deligne *Les constantes des équations fonctionnelles des fonctions L*, proc. Antwerpen conference, vol. 2, Lecture Notes in Math. 349, Springer-Verlag, 1973.

[2] M. Kisin *Overconvergent modular forms and the Fontaine-Mazur conjecture*, Invent. Math. 153(2) (2003), 373-454.

[3] M. Kisin *Moduli of finite flat group schemes, and modularity*, Annals of Math. to appear.

[4] M. Kisin *Modularity of 2-adic Barsotti-Tate representations*, Preprint (2005).

[5] B. Mazur, *Deforming Galois representations*, Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ. 16, Springer, New York-Berlin, pp. 395-437, 1989.

[6] B. Mazur, *An introduction to the deformation theory of Galois representations*, Modular forms and Fermat's last theorem (Boston, MA, 1995), Springer, New York, pp. 243-311, 1997.

[7] J.S. Milne *Arithmetic duality theorems*, Perspective in Mathematics 1, Academic Press, Boston, 1986.

[8] M. Schlessinger, *Functors on Artin rings*, Trans A.M.S 130 (1968), pp. 208-222.

[9] J.P Serre et J. Tate *Good reduction of abelian varieties*, Ann. of Math., 88 3 (1968).