PROC. INT. CONG. OF MATH. – 2018
Rio de Janeiro, Vol. 1 (291–318)

p-ADIC VARIATION OF AUTOMORPHIC SHEAVES

FABRIZIO ANDREATTA, ADRIAN IOVITA AND VINCENT PILLONI

Abstract

We review the construction of analytic families of Siegel modular cuspforms based on the notion of overconvergent modular forms of p-adic weight. We then present recent developments on the following subjects: the halo conjecture, the construction of p-adic L-functions, and the modularity of irregular motives.

1 Introduction

We start by fixing a number field F, a prime integer $p > 0$ and an integer $n \geq 1$ and by denoting G_F and respectively $\mathbb{A}_F$ the absolute Galois group and the ring of adeles of F. We fix an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$. One of the most mysterious conjectures in number theory, known as the *Langlands, Clozel, Fontaine-Mazur Conjecture* is the statement of the existence of a bijection respecting L-functions between the following families of isomorphic classes of representations:

$\mathrm{Rep}^{\mathrm{geom}}_{p,F} :=$ {Irreducible representations $\rho\colon G_F \to \mathrm{GL}_n(\bar{\mathbb{Q}}_p)$ which are continuous, ramified only at a finite number of places and de Rham at the places dividing p}

and

$\mathrm{Aut}^{\mathrm{alg}}_F :=$ {Cuspidal algebraic automorphic representations π of the group $\mathrm{GL}_n(\mathbb{A}_F)$}.

The algebraicity condition of an automorphic form π is the condition that the infinitesimal character (also called the weight) of the π_v is algebraic for all infinite places v of F (Clozel [1990], Buzzard and Gee [2011]). It is known that there are only countably many isomorphism classes of algebraic automorphic representations (Harish-Chandra [1968], Thm. 1). On the other hand the objects in $\mathrm{Rep}^{\mathrm{geom}}_{p,F}$ should arise from the cohomology of proper smooth varieties over F (Fontaine and Mazur [1995], Conj. 1). The bijection we are seeking should therefore be a bijection of (conjecturally) countable sets.

Having written this let us remark that in fact each of these countable sets can be embedded in certain analytic varieties over $\mathbb{Q}_p$. To be more precise, for $\mathrm{Rep}^{\mathrm{geom}}_{p,F}$, one can relax the condition of the representations being de Rham at places dividing p and obtain a moduli space of more general, non-geometric p-adic Galois representations enjoying reasonable finiteness properties (Mazur [1989]).

MSC2010: primary 11G18; secondary 11G40.

Concerning $\mathrm{Aut}_F^{\mathrm{alg}}$, one would also like to see them as a subset of a larger set of p-adic automorphic forms. For the moment there are several good definitions of a p-adic automorphic form depending on the way one endows the space $\mathrm{Aut}_F^{\mathrm{alg}}$ with a p-adic topology.

The general method to study elements of $\mathrm{Aut}_F^{\mathrm{alg}}$ is to realize them (when this is possible!) in the Betti cohomology of a locally symmetric space or in the coherent cohomology of a Shimura variety (Harris [1990]). These cohomology groups naturally carry structures of finite dimensional $\mathbb{Q}$-vector spaces and these structures can be used to equip the automorphic forms with a p-adic topology. Once a cohomological realization has been chosen, one can start to vary the levels and the weights of the algebraic automorphic representations. Miraculously, we find in many cases that the countable set of systems of eigenvalues associated to the cuspidal algebraic automorphic forms is not isolated in the space of p-adic automorphic forms and that its closure acquires the structure of an analytic space.

The point of view of p-adically deforming the elements of $\mathrm{Rep}_{p,F}^{\mathrm{geom}}$ and $\mathrm{Aut}_F^{\mathrm{alg}}$ proves to be very fruitful as it is sometimes easier to study and work with these analytic varieties than to work with the geometric Galois representations and algebraic automorphic forms individually.

In this note we begin by explaining one possible approach for the construction of the p-adic analytic spaces (also called eigenvarieties) attached to p-adic automorphic forms based on the coherent cohomology realization in a Shimura variety of PEL type[1]. We shall actually limit ourselves to the Siegel moduli spaces of polarized abelian varieties with level structure. We explain how to vary p-adically the weight of the automorphic forms. One has as a guiding principle that in order to be able to deform automorphic forms one needs to allow them, seen as global sections of certain automorphic sheaves, to have essential singularities at non-ordinary points. Restricting the automorphic vector bundles to the complement of these non-ordinary points has the advantage that they (the automorphic vector bundles) acquire extra structures arising from the universal p-divisible group via the Hodge-Tate period map. Our main task is to define overconvergent modular forms of any p-adic weight. This is a refinement of the definition of p-adic modular forms of Serre, Katz [1973] and Hida [1986] and an interpolation of the notion of overconvergent modular forms of integral weight considered by Dwork, Coleman and Mazur [1998]... This material has already appeared in print (for example in Andreatta, Iovita, and Pilloni [2015]). Next we present three recent developments in this area: the halo conjecture, the construction of triple product p-adic L functions in the finite slope case, and the modularity of certain irregular motives.

[1]This choice is made to the expense of ignoring interesting automorphic forms because the condition of admitting a cohomological realization in the coherent cohomology of a Shimura variety is restrictive. There are other approaches based on the Betti realization but for the sake of brevity we will not discuss them here.

2 Vector bundles with marked sections

In this section we review some constructions which applied to various contexts provide examples of interpolation of automorphic sheaves in the subsequent sections. We have tried to isolate the key representation theoretic ideas outlining a general method to get interpolations that might be useful in other situations.

2.1 Some classical representation theory. We start by recalling the construction of the irreducible representations of the group GL_g. Let $\mathrm{B} \subset \mathrm{GL}_g$ be the upper triangular Borel and let T be the usual diagonal torus. Let $X(\mathrm{T})$ be the character group of T. This group is isomorphic to $\mathbb{Z}^g$ via the map sending $(k_1, \cdots, k_g)$ to the character $\mathrm{diag}(t_1, \cdots, t_g) \mapsto \prod_{i=1}^{g} t_i^{k_i}$. We let $X(\mathrm{T})^+$ be the cone of dominant weights given by the condition $k_1 \geq k_2 \cdots \geq k_g$.

If $k \in X(\mathrm{T})^+$, we define the algebraic induction $V^k = \{f : \mathrm{GL}_g \to \mathbb{A}^1,\ f(gb) = k(b)f(g),\ \forall (g,b) \in \mathrm{GL}_g \times \mathrm{B}\}$ where k has been extended to a character of B via the projection $\mathrm{B} \to \mathrm{T}$. The group GL_g acts on this space via $f \mapsto f(g \cdot -)$.

2.2 p-Adic representation theoretic variations. Next we explain how to interpolate the weights k and the spaces V^k, for $k \in X(\mathrm{T})^+$. We let $\Lambda = \mathbb{Z}_p[\![T(\mathbb{Z}_p)]\!]$ be the Iwasawa algebra of the torus. The universal continuous character of $\mathrm{T}(\mathbb{Z}_p)$ is the tautological character:

$$k^{\mathrm{un}} \colon \mathrm{T}(\mathbb{Z}_p) \longrightarrow \Lambda^\times.$$

We can consider the formal spectrum $\mathfrak{W} = \mathrm{Spf}\,\Lambda$ and denote by $\mathcal{W}$ its rigid analytic fiber over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. This is a finite union of open unit polydiscs of dimension g. Given a complete Huber pair (B, B^+) over $(\mathbb{Q}_p, \mathbb{Z}_p)$, the morphisms $\mathrm{Spa}(B, B^+) \to \mathcal{W}$ correspond to $\mathrm{Hom}_{\mathrm{cont}}(\mathrm{T}(\mathbb{Z}_p), B^\times)$. In particular $\mathcal{W}(\mathbb{Q}_p)$ contains the algebraic weights $X(\mathrm{T})$. Observe that $X(\mathrm{T})$ is totally disconnected in $\mathcal{W}$ but Zariski dense and that $\mathcal{W}$ has only a finite number of connected components.

The character k^{un} interpolates the algebraic characters $X(\mathrm{T})$. We now explain how to interpolate the representations $\{V^k\}_{k \in X(\mathrm{T})^+}$ over $\mathcal{W}$ (Stevens [2000]). We switch to the analytic setting and let now $\mathrm{GL}_g, \mathrm{T}, \mathrm{B}$ denote the analytifications over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ of the respective group schemes. Let $\mathrm{Iw} \subset \mathrm{GL}_g(\mathbb{Z}_p)$ be the Iwahori subgroup of matrices which are upper triangular modulo p. For any number $w \in \mathbb{Q}_{>0} \cup \{\infty\}$, we denote by $\mathrm{Iw}_w \subset \mathrm{GL}_g$ the adic analytic subgroup of GL_g of integral matrices which are congruent modulo p^w to an element of Iw and we let $\mathrm{B}_w = \mathrm{B} \cap \mathrm{Iw}_w$ and $\mathrm{T}_w = \mathrm{T} \cap \mathrm{Iw}_w$. Let $\mathcal{U} \hookrightarrow \mathcal{W}$ be an open subspace. Let w be such that the universal character extends to a pairing $k^{\mathrm{un}}_{\mathcal{U}} \colon \mathrm{T}_w \times \mathcal{U} \to \mathbb{G}_m$. In this case we say that k^{un} is w-analytic over $\mathcal{U}$. Remark that if $\mathcal{U}$ is quasi-compact, then $k^{\mathrm{un}}_{\mathcal{U}}$ is always w-analytic for some $w \in \mathbb{Q}_{>0}$. We may define a representation of the group Iw_w as the analytic induction

$$V_w^{k^{\mathrm{un}}_{\mathcal{U}}} = \{f : \mathrm{Iw}_w \times \mathcal{U} \to \mathbb{A}^1 \times \mathcal{U},\ \forall (i,b) \in \mathrm{Iw}_w \times \mathrm{B}_w \ \ f(ib) = k^{\mathrm{un}}_{\mathcal{U}}(b) f(i)\}.$$

If $\mathcal{U} = \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ and $k^{\mathrm{un}}_{\mathcal{U}} = k \in X(\mathrm{T})^+$ (k is algebraic and therefore w-analytic for all w) we have an inclusion $V^k \hookrightarrow V_w^{k^{\mathrm{un}}_{\mathcal{U}}}$. Observe that unless $g = 1$, the space $V_w^{k^{\mathrm{un}}_{\mathcal{U}}}$

is an infinite dimensional Banach space and the inclusion is not an isomorphism. This should not be a surprise as for $g \geq 2$ the dimensions of the spaces $\{V^k\}_{k\in X(\mathrm{T})^+}$ vary and the only possibility to interpolate them is to embed them in larger spaces (infinite dimensional) which can then be interpolated. It is moreover possible to characterize V^k inside $V_w^{k^{\mathrm{un}}_{\mathfrak{U}}}$ by using some differential operators (analytic BGG resolution, Jones [2011]).

2.3 Relative constructions. The classical case. We use the notation of Section 2.1. Let X be a scheme, let $\mathcal{E}$ be a locally free sheaf of rank g over X and denote by $\mathcal{E}^\vee = \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{O}_X)$ the dual sheaf. We associate to any dominant weight $k \in X(\mathrm{T})^+$ a locally free sheaf $\mathcal{E}^k$ over X as follows. Consider the GL_g-torsor $f : \mathcal{T}(\mathcal{E}) \to X$ associated to $\mathcal{E}$, namely $\mathcal{T}(\mathcal{E}) := \mathrm{Isom}(\mathcal{O}_X^g, \mathcal{E}^\vee)$. Define $\mathcal{E}^k = f_*\mathcal{O}_{\mathcal{T}(\mathcal{E})}[k]$, the functions on $\mathcal{T}(\mathcal{E})$ transforming via k under the action of B. One gets a finite, locally free $\mathcal{O}_X$-module which locally on X is isomorphic to the space V^k introduced in Section 2.1.

2.4 Relative constructions. *p*-Adic variations. We now assume that $\mathfrak{X}$ is an analytic adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Let $\mathcal{E}$ be a locally free sheaf of rank g over $\mathfrak{X}$ and let $\mathcal{E}^+$ be an integral structure, namely a subsheaf of finite and locally free $\mathcal{O}_{\mathfrak{X}}^+$-modules of rank g such that $\mathcal{E} = \mathcal{E}^+ \otimes_{\mathcal{O}_{\mathfrak{X}}^+} \mathcal{O}_{\mathfrak{X}}$. Let $w \in \mathbb{Q}_{>0} \cup \{\infty\}$. We now provide a formalism which leads to the construction of families of sheaves interpolating the sheaves $\{\mathcal{E}^k\}_{k\in X(\mathrm{T})^+}$ on $\mathfrak{X}$ and which, locally on $\mathfrak{X}$, are isomorphic to the spaces V_w^k of Section 2.2. The new essential ingrediants are the "marked sections" $s_1, \ldots, s_g \in \mathrm{H}^0(\mathfrak{X}, \mathcal{E}^+/p^w\mathcal{E}^+)$ with the property that the induced map $(\mathcal{O}_{\mathfrak{X}}^+/p^w\mathcal{O}_{\mathfrak{X}}^+)^g \to \mathcal{E}^+/p^w\mathcal{E}^+$ is bijective.

Define $\mathcal{T}_w(\mathcal{E}^+, \{s_1, \ldots, s_g\})$ as the functor that associates to any adic space $t : \mathcal{Z} \to \mathfrak{X}$ the set of sections $(\rho_1, \cdots, \rho_g) \in \mathrm{H}^0\big(\mathcal{Z}, t^*(\mathcal{E}^+)^\vee\big)$ such that $(\langle t^*(s_i), \rho_j\rangle)_{1\leq i,j\leq g} \in \mathrm{Iw} \mod p^w$. Here $t^\star(\mathcal{E}^+)$ is the sheaf $t^{-1}(\mathcal{E}^+) \otimes_{t^{-1}\mathcal{O}_{\mathfrak{X}}^+} \mathcal{O}_{\mathcal{Z}}^+$ and $t^\star(\mathcal{E}^+)^\vee$ is its $\mathcal{O}_{\mathcal{Z}}^+$-dual. One proves easily that $\mathcal{T}_w(\mathcal{E}^+, \{s_1, \ldots, s_g\})$ is representable by an adic space and is a Iw_w-torsor. We now assume that there is a map $\mathfrak{X} \to \mathfrak{W}$ and that the character $k_{\mathfrak{X}}$ pulled back from the universal character on $\mathfrak{W}$ is w-analytic. Under this assumption we define the sheaf

$$\mathcal{E}_w^k := f_*\mathcal{O}_{\mathcal{T}_w(\mathcal{E}^+,\{s_1,\ldots,s_g\})}[k_{\mathfrak{X}}].$$

This sheaf is a relative version of the construction of $V_w^{k_{\mathfrak{X}}}$ given in Section 2.2.

We now describe a slight variant of this construction where we only assume that we have a partial set of sections. In this situation it is still possible to realize a partial interpolation. Let $1 \leq r \leq g$. We define the subgroup $\mathrm{Iw}_{w,r}$ of GL_g of integral matrices of the form

$$\begin{pmatrix} A & D \\ B & C \end{pmatrix},$$

where $A \in \mathrm{GL}_r$ and $A \mod p^w$ is upper triangular with entries in $\mathbb{Z}_p/p^w$, $D \in \mathrm{M}_{r,g-r}$, $C \in \mathrm{GL}_{g-r}$, $B \in \mathrm{M}_{g-r,r}$ and $B = 0 \mod p^w$. We denote by $\mathrm{T}_{w,r} =$

$\mathrm{T} \cap \mathrm{Iw}_{w,r}$, $\mathrm{B}_{w,r} = \mathrm{B} \cap \mathrm{Iw}_{w,r}$ and $\mathrm{T}_{w,r} = \mathrm{T} \cap \mathrm{Iw}_{w,r}$. We assume that we have sections $s_1, \cdots, s_r \in \mathrm{H}^0(\mathfrak{X}, \mathcal{E}^+/p^w\mathcal{E}^+)$ such that the induced map $(\mathcal{O}_{\mathfrak{X}}^+/p^w\mathcal{O}_{\mathfrak{X}}^+)^r \to \mathcal{E}^+/p^w\mathcal{E}^+$ is injective with locally free cokernel of rank $g-r$. We define $\mathcal{T}_w(\mathcal{E}^+, \{s_1, \ldots, s_r\})$ as the functor that associates to any adic space $t : \mathcal{Z} \to \mathfrak{X}$ the set of basis $(\rho_1, \cdots, \rho_g) \in \mathrm{H}^0\big(\mathcal{Z}, t^\star(\mathcal{E}^+)^\vee\big)$ such that:

- $(\langle t^\star(s_i), \rho_j \rangle)_{1 \le i,j \le r} \in \mathrm{GL}_r(\mathbb{Z}_p) \mod p^w$ and is upper triangular modulo p^w,
- $\langle t^\star(s_i), \rho_j \rangle = 0 \mod p^w$ for all $1 \le i \le r$ and $g - r + 1 \le j \le g$.

It is clear that $\mathcal{T}_w(\mathcal{E}^+, \{s_1, \ldots, s_r\})$ is an $\mathrm{Iw}_{w,r}$-torsor. We now assume that the character $k_{\mathfrak{X}}$ extends to a character of $\mathrm{T}_{w,r}$ and we denote by $\mathcal{E}_w^k := f_*\mathcal{O}_{\mathcal{T}_w(\mathcal{E}^+,\{s_1,\ldots,s_r\})}[k_{\mathfrak{X}}]$.

We remark that the relative constructions in Sections 2.2 and 2.4 could have been made exactly in the same way by working with an invertible ideal $I \subset \mathcal{O}_{\mathfrak{X}}^+$ such that $I \cap \mathbb{Z}_p = p^w\mathbb{Z}_p$, instead of with $p^w\mathcal{O}_{\mathfrak{X}}^+$.

3 Variations in the Siegel case.

Let GSp_{2g} be the group of similitudes of $(\mathbb{Z}^{2g}, \langle , \rangle)$ where $\langle , \rangle$ is the alternating form given by $\langle e_i, e_{2g-i+1} \rangle = 1$ if $1 \le i \le g$ and $\langle e_i, e_j \rangle = 0$ if $i + j \ne 2g + 1$. Let $K \subset \mathrm{GSp}_{2g}(\mathbb{A}_f)$ be a neat compact open subgroup, where $\mathbb{A}_f$ denotes the ring of finite adels of the rationals. Let $Y_K \to \mathrm{Spec}\ \mathbb{Q}$ be the Siegel moduli space of polarized abelian varieties A of dimension g and level structure K. Its complex analytification $(Y_K \times \mathrm{Spec}\ \mathbb{C})^{an}$ is the locally symmetric space $\mathrm{GSp}_{2g}(\mathbb{Q})\backslash\big(\mathcal{H}_g \times \mathrm{GSp}_{2g}(\mathbb{A}_f)/K\big)$ where $\mathcal{H}_g = \{M \in \mathrm{M}_g(\mathbb{C}), M^t = M,\ \mathrm{Im}(M) \text{ is definite positive or negative}\}$ is the Siegel space i.e. the union of the Siegel upper and lower half-spaces.

3.1 The classical construction. To any g-uple $k = (k_1, \ldots, k_g) \in \mathbb{Z}^g$ satisfying $k_1 \ge k_2 \ge \ldots \ge k_g$, one attaches an automorphic locally free sheaf ω^k on Y_K using the construction of Section 2.3 for the sheaf ω_A of invariant differentials of the universal abelian scheme over Y_K. Let X_K be a toroidal compactification of Y_K (Faltings and Chai [1990]). The sheaf ω^k extends canonically to X_K. The global sections form the space of classical holomorphic Siegel modular forms of weight k and level K. This is a finite dimensional $\mathbb{Q}$-vector space. It carries an action of the Hecke algebra $\mathcal{C}_c^\infty(\mathrm{GSp}_{2g}(\mathbb{A}_f)//K, \mathbb{Z})$ of locally constant and compactly supported functions which are left and right K invariant on $\mathrm{GSp}_{2g}(\mathbb{A}_f)$.

After tensoring with $\mathbb{C}$, these Siegel modular forms can be described as holomorphic vector valued functions on $\mathcal{H}_g$ satisfying a transformation property with respect to a congruence subgroup of $\mathrm{GSp}_{2g}(\mathbb{Q})$. The cuspidal forms (those vanishing on $D = X_K \setminus Y_K$) define (via a usual lifting process) special vectors in the space of algebraic automorphic forms for the group $\mathrm{GSp}_{2g}(\mathbb{A})$. Here and elsewhere $\mathbb{A}$ denotes the ring of adels of $\mathbb{Q}$.

3.2 Interpolation. Let $p > 0$ be a prime integer. We now assume that $K = K^p K_p$ where $K^p \subset \mathrm{GSp}_{2g}(\mathbb{A}_f^{(p)})$ and $K_p = \mathrm{GSp}_{2g}(\mathbb{Z}_p)$. In this setting, Y_K and X_K admit canonical models over Spec $\mathbb{Z}_{(p)}$ and we denote by $\mathcal{Y}$ and respectively $\mathcal{X}$ the associated analytic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Over $\mathcal{Y}$ there is a universal p-divisible group $A[p^\infty]$, which comes with a quasi-polarisation: $A[p^\infty] \simeq (A[p^\infty])^D$.

We review the method of Andreatta, Iovita, and Pilloni [2015] to construct a sheaf interpolating the classical automorphic sheaves ω^k. We shall work over $\mathcal{Y}$ for simplicity, but everything extends to $\mathcal{X}$. This construction relies on the Hodge-Tate period map

$$\mathrm{HT} : T_p(A) \to \omega_A$$

where $T_p(A)$ is the Tate-module of the p-divisible group $A[p^\infty]$, a pro-étale sheaf locally isomorphic to $\mathbb{Z}_p^{2g}$. Over the ordinary locus $\mathcal{Y}^{\mathrm{ord}}$ we have an étale-multiplicative extension $0 \to T_p(A)^m \to T_p(A) \to T_p(A)^{et} \to 0$ and the Hodge-Tate map factors through a map $T_p(A)^{et} \to \omega_A$ which induces an isomorphism of pro-étale sheaves $T_p(A)^{et} \otimes \mathcal{O}_{\mathcal{Y}^{\mathrm{ord}}} \to \omega_A|_{\mathcal{Y}^{\mathrm{ord}}}$. Thus the GL_g-torsor ω_A arises from a $\mathrm{GL}_g(\mathbb{Z}_p)$-torsor over the ordinary locus and this allows the interpolation of the sheaves ω^k over $\mathcal{Y}^{\mathrm{ord}}$. It is nevertheless important in order to have compact operators and for the construction of eigenvarieties to go beyond the ordinary locus.

Given an integer $r \geq 0$ we let $\mathcal{Y}_r \subset \mathcal{Y}$ be the open defined by the valuations x satisfying the inequality $|\widetilde{\mathrm{Ha}}^{p^{r+1}}|_x \geq |p|_x$ where $\widetilde{\mathrm{Ha}}$ is locally defined as a (any) lift of the Hasse invariant on the special fiber of Y_K. Each $\mathcal{Y}_r$ should be thought of as a tubular neighborhood of the ordinary locus $\mathcal{Y}^{\mathrm{ord}}$ in $\mathcal{Y}$, where $\mathcal{Y}^{\mathrm{ord}}$ is defined by the condition that $|\widetilde{\mathrm{Ha}}|_x \geq 1$. It follows from the theory of the canonical subgroup that the p^r-torsion of A over $\mathcal{Y}_r$ contains a canonical subgroup $H_r \subset A[p^r]$ (see Fargues [2011]). Over $\mathcal{Y}^{\mathrm{ord}}$ it coincides with the multiplicative part of $A[p^r]$.

In order to apply the general machinery of Section 2.4 we need to exhibit a vector bundle with marked sections. Consider the finite étale cover of adic spaces $\mathcal{IG}_r \to \mathcal{Y}_r$ classifying trivializations $\psi : (\mathbb{Z}/p^r\mathbb{Z})^g \cong H_r^D$. Then $\mathcal{IG}_r$ carries several sheaves:

1) we have the sheaf H_r and its Cartier dual H_r^D;

2) we have a sheaf ω_A^+, resp. ω_A of $\mathcal{O}^+_{\mathcal{IG}_r}$-modules, resp. of $\mathcal{O}_{\mathcal{IG}_r}$-modules, which are locally free and finite of rank g. Over affinoids $\mathrm{Spa}(B, B^+) \subset \mathcal{IG}_r$ such that the pull-back of A extends to an abelian scheme $\widetilde{A}$ over B^+, the value of ω_A^+ and of ω_A are the module of invariant differentials of $\widetilde{A}$, resp. of A;

3) we have a sheaf $\omega^+_{H_r}$ of $\mathcal{O}^+_{\mathcal{IG}_r}$-modules and a morphism $\mathrm{HT}\colon H_r^D \to \omega^+_{H_r}$. Over affinoids $\mathrm{Spa}(B, B^+) \subset \mathcal{IG}_r$ such that the pull-back of H_r extends to a finite and flat group scheme $\widetilde{H}_r \hookrightarrow \widetilde{A}$ over B^+, the value of $\omega^+_{H_r}$ is the module of invariant differentials $\omega_{\widetilde{H}_r}$ and the map HT is the Hodge-Tate map.

Notice that we have a natural morphism $\omega_A^+ \to \omega^+_{H_r}$. With this we define a *modification* $\omega_A^\sharp \subset \omega_A^+$ as the inverse image in ω_A^+ of $\mathrm{HT}\big(H_r^D \otimes \mathcal{O}^+_{\mathcal{IG}_r}\big)$. One proves that this is a finite and locally free sheaf of $\mathcal{O}^+_{\mathcal{IG}_r}$-modules over $\mathcal{IG}_r$ of rank g and that for every rational number $0 < w \leq r - \frac{1}{p(p-1)}$ the morphism HT defines an isomorphism of $\mathcal{O}^+_{\mathcal{IG}_r}/p^w$-modules

$$\mathrm{HT}_w : H_r^D \otimes \mathcal{O}^+_{\mathfrak{IG}_r}/p^w \cong \omega_A^\sharp/p^w\omega_A^\sharp.$$

which is a good substitute of the comparison map we had over the ordinary locus.

Consider $\mathcal{E} := \omega_A$, $\mathcal{E}^+ := \omega_A^\sharp$ and the sections $s_1, \ldots, s_g$ of $\omega_A^\sharp/p^w\omega_A^\sharp$ provided by the images of the canonical basis of $(\mathbb{Z}/p^r\mathbb{Z})^g$ via $\mathrm{HT}_w \circ \psi$. Let $\mathfrak{U} \subset \mathfrak{W}$ be an open subset where the character $k_{\mathfrak{U}}^{un}$ is w-analytic. Applying the construction explained in Section 2.4 we get the sheaves we are looking for

$$\omega_{r,\mathfrak{U}}^{k^{\mathrm{un}}} := \pi_\star\big(\mathcal{O}_{\mathcal{T}_w(\omega_A^\sharp,\{s_1,\ldots,s_g\})\times\mathfrak{U}}\big)[k^{\mathrm{un}}],$$

where $\pi : \mathcal{T}_w(\omega_A^\sharp, \{s_1, \ldots, s_g\}) \to \mathfrak{IG}_r$ is the torsor of trivializations of $\omega_A^\sharp$ with marked sections $s_1, \ldots, s_g$.

Actually, denote by $\mathcal{Y}_{\mathrm{Iw},r} \to \mathcal{Y}_r$ be the covering parametrizing full flags of H_1^D. Then the sheaf $\omega_{r,\mathfrak{U}}^{k^{\mathrm{un}}}$ descends canonically along the natural map $\mathfrak{IG}_r \to \mathcal{Y}_{\mathrm{Iw},r}$. Moreover it extends without much difficulties to the toroidal compactification $\mathcal{X}_{\mathrm{Iw},r}$ of $\mathcal{Y}_{\mathrm{Iw},r}$.

3.2.1 A perfectoid digression. We'd like to explain the construction of the previous section when $g = 1$ in more elementary terms. Let N be an integer, $N \geq 3$ and $(N, p) = 1$. Recall (see Katz [1973]) that a modular form f of weight $k \in \mathbb{Z}$ with level $\Gamma_1(N)$ over $\mathbb{Z}[1/N]$ can be viewed as a functorial rule mapping a triple $(E/R, P, \omega)$ (consisting of an elliptic curve $E \to \operatorname{Spec} R$ for a $\mathbb{Z}[1/N]$-algebra R, a point $P \in E[N]$ of order exactly N, and a nowhere vanishing differential form ω) to $f(E/R, P, \omega) \in R$ satisfying the additional transformation property: $f(E, P, \lambda.\omega) = \lambda^{-k} f(E, P, \omega)$ for any $\lambda \in R^\times$ and some growth condition at infinity.

Over $\mathbb{C}$, we can pull back f to a function on the Poincaré upper half plane by setting $F(\tau) = f(\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}), \frac{1}{N}, \mathrm{d}z)$ for the coordinate z on $\mathbb{C}$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$, multiplication by $(c\tau + d)^{-1}$ on $\mathbb{C}$ identifies the triples $(\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \frac{1}{N}, \mathrm{d}z)$ and $(\mathbb{C}/(\mathbb{Z} + \gamma.\tau\mathbb{Z}), \frac{1}{N}, (c\tau + d)\mathrm{d}z)$ and therefore F satisfies a descent condition with respect to the action of $\Gamma_1(N)$, namely $F(\gamma.\tau) = (c\tau + d)^k F(\tau)$.

We now express our definition of overconvergent modular forms of some $\mathbb{C}_p$-valued character $k : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ in similar terms. Assume that k is w-analytic and choose a positive integer r such that $r - 1 < w \leq r - \frac{1}{(p-1)p}$ (this can be achieved at the expense of increasing w). Then an r-overconvergent modular form f of weight k is a rule associating to every quadruple (E, P, ψ, ω) an element $f(E, P, \psi, \omega) \in \mathbb{C}_p$, where E is an elliptic curve over $\mathbb{C}_p$ such that $|\widetilde{\mathrm{Ha}}^{p^{r+1}}(E)| \geq |p|$, P is a point of order N, ψ is point of order p^r of the dual canonical subgroup H_r^D, ω is an integral differential form on E such that $\omega \mod p^w = \mathrm{HT}_w(\Psi)$. Moreover we demand that f is "analytic", extends to the cusps, and satisfies the functional equation $f(E, P, \lambda\psi, \lambda\omega) = k^{-1}(\lambda) f(E, P, \psi, \omega)$ for all $\lambda \in \mathbb{Z}_p^\times(1 + p^w\mathbb{C}_p)$.[2]

[2] In order to make sense of the functional equation it is necessary to restrict to differential forms which "arise" from the dual canonical subgroup

Following Chojecki, Hansen, and Johansson [2017], one can describe an analogue of the passage from f to F in the p-adic world. Let $\mathfrak{X}(\infty) \to \mathfrak{X}$ be the prefectoid modular curve of level $\Gamma(p^\infty) \cap \Gamma_1(N)$ constructed by Scholze [2015] and $\mathfrak{Y}(\infty)$ the complement of the boundary. Over $\mathfrak{Y}(\infty)$ we have a universal trivialization $\psi_\infty\colon \mathbb{Z}_p^2 \cong T_p(E)$ and the Hodge-Tate map $\mathrm{HT}\colon T_p(E) \to \omega_E$ induces a period map:

$$\pi_{\mathrm{HT}}\colon \mathfrak{X}(\infty) \longrightarrow \mathbb{P}^1$$

which, over $\mathfrak{Y}(\infty)$, is characterized by the fact that $\pi_{\mathrm{HT}}^\star\big(\mathcal{O}_{\mathbb{P}^1}(1)\big) = \omega_E$ and the pullback of the two canonical sections s_0 and s_1 of $\mathcal{O}_{\mathbb{P}^1}(1)$ are the images via $\mathrm{HT} \circ \psi_\infty$ of the canonical basis e_0, e_1 of $\mathbb{Z}_p \oplus \mathbb{Z}_p$. For any $v \in \mathbb{Q}_{>0}$, let $\mathbb{P}^1_v$ be the open of $\mathbb{P}^1$ defined by the condition $|s_1| \le |p^v s_0|$.

Let $\mathfrak{X}(\infty)_v = \pi_{\mathrm{HT}}^{-1}(\mathbb{P}^1_v)$ and $\mathfrak{Y}(\infty)_v = \mathfrak{Y}(\infty) \cap \mathfrak{X}(\infty)_v$. For v large enough, $(E, P, \psi_\infty) \in \mathfrak{Y}(\infty)_v$ has a canonical subgroup of level r which is generated by the image of $\psi_\infty(e_1)$ in $E[p^r]$, and $\psi_\infty(e_0)$ maps to a generator $\overline{\psi_\infty(e_0)}$ of H_r^D. We can therefore pullback f to a function on $\mathfrak{Y}(\infty)_v$ by setting $F(E, P, \psi_\infty) = f(E, P, \overline{\psi_\infty(e_0)}, s_0 = \mathrm{HT}(\psi_\infty(e_0)))$.

This identifies the space of overconvergent modular forms of weight k with a space of functions on the open $\mathfrak{X}(\infty)_v$ of $\mathfrak{X}(\infty)$. These functions satisfy a descent condition which reminds us of the descent condition on the upper half plane. Namely, let n be the smallest integer greater than v. We consider the subgroup $K_0(p^n) \subset \mathrm{GL}_2(\mathbb{Z}_p)$ of elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $c \in p^n\mathbb{Z}_p$. For any $\gamma \in K_0(p^n)$ as above, we find that $F(E, P, \psi_\infty \circ \gamma) = k^{-1}(a + b\frac{s_1}{s_0})F(E, P, \psi_\infty)$. [3]

3.3 Eigenvarieties. The sheaves $\omega^{k^{\mathrm{un}}}_{r,\mathfrak{U}}$ produce variations of Hecke eigensystems as follows. The global sections of $\omega^{k^{\mathrm{un}}}_{r,\mathfrak{U}}$ over $\mathfrak{X}_{\mathrm{Iw},r}$, vanishing at the boundary, form the Banach module of r-overconvergent, w-analytic cuspidal Siegel modular forms of weight parametrized by $\mathfrak{U}$. Passing to the limit over r and w we obtain the space of overconvergent, locally analytic cuspidal Siegel modular forms of weight parametrized by $\mathfrak{U}$. Let N be the product of primes different from p for which $K_\ell \neq \mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$. This space carries an action of the commutative spherical Hecke algebra $\mathbb{T}^{Np} := \mathcal{C}_c^\infty(\mathrm{GSp}_{2g}(\mathbb{A}_f^{(Np)})//K^{Np}, \mathbb{Z})$. Let $\mathrm{Iw}_p \subset \mathrm{GSp}_{2g}(\mathbb{Z}_p)$ be the Iwahori parahoric of upper triangular matrices modulo p. At p, there is an action of the dilating Hecke algebra $\mathbb{U}_p := \mathbb{Z}[U_{p,1}, \cdots, U_{p,g}]$ [4] $\subset \mathcal{C}_c^\infty(\mathrm{GSp}_{2g}(\mathbb{Q}_p)//\mathrm{Iw}_p, \mathbb{Z})$, and the operator $U = \prod_i U_{p,i}$ is compact.

Let f be a classical cuspidal eigenform of weight k and level $K^p\mathrm{Iw}_p$. We denote by $\Theta_f\colon \mathbb{T}^{Np} \otimes \mathbb{U}_p \to \bar{\mathbb{Q}}$ the associated character.[5] We have the following:

[3] $F(E, P, \psi_\infty \circ \gamma) = f(E, P, \overline{\psi_\infty(ae_0)}, \mathrm{HT} \circ \psi_\infty(ae_0 + be_1)) = f(E, P, \overline{\psi_\infty(ae_0)}, (a + b\frac{s_1}{s_0})\mathrm{HT} \circ \psi_\infty(e_0)) = k^{-1}(a + b\frac{s_1}{s_0})F(E, P, \psi_\infty)$.

[4] If $i \in \{1, \cdots, g-1\}$, $U_{p,i}$ is the characteristic function of the double class $\mathrm{Iw}_p\mathrm{diag}(p^2 Id_i, pId_{2g-2i}, Id_i)\mathrm{Iw}_p$, $U_{p,g}$ is the characteristic function of the double class $\mathrm{Iw}_p\mathrm{diag}(pId_g, Id_g)\mathrm{Iw}_p$

[5] Since f has Iwahori level at p, then $\Theta_f(U_{p,i}) \neq 0$ and f is of finite slope.

Theorem 3.1. *There is a rigid analytic space $\mathcal{E}$, called the eigenvariety of tame level K^p, equipped with a weight map $w : \mathcal{E} \to \mathcal{W}$ which is locally on the source and the target finite and torsion free and there is a universal Hecke character $\Theta : \mathbb{T}^{Np} \otimes \mathbb{U}_p \to \mathcal{O}_{\mathcal{E}}$ with dense image such that:*

- *Any classical cuspidal eigenform f of weight k and level $K^p\mathrm{Iw}_p$ provides a unique point x_f on $\mathcal{E}$ such that $\Theta|_{x_f} = \Theta_f$ and $w(x_f) = k$,*

- *Conversely, any point $x \in \mathcal{E}$ satisfying $w(x) = (k_1, \ldots, k_g) \in X(\mathrm{T})^+$ satisfying $v(\Theta|_x(U_{p,i})) < k_{g-i} - k_{g-i+1} + 1$ for $1 \le i \le g-1$ and $v(\Theta|_x(U_{p,g})) < k_g - \frac{g(g+1)}{2}$ arises from a cuspidal eigenform f of weight k and level $K^p\mathrm{Iw}_p$.*

Remark 3.3.1. 1) The case $g = 1$ of Theorem 3.1 was first proved by Coleman and Mazur in Coleman and Mazur [1998]. They used a different construction of the p-adic families of modular forms in which the Eisenstein family plays a crucial role and which could not be generalized for $g > 1$.

2) The cuspidality condition is crucial for the theorem for $g \ge 2$. We prove an acyclicity result for the cuspidal sheaves $\omega^{k^{\mathrm{un}}}_{r,\mathcal{U}}(-D)$ using that $\mathcal{X}_{\mathrm{Iw,r}}$ has affine image in the minimal compactification and showing that the relative cohomology of cuspidal sheaves between the toroidal and the minimal compactifications vanishes in degrees greater than 1. In particular the acyclicity allows us to prove that the degree zero cohomology of $\omega^{k^{\mathrm{un}}}_{r,\mathcal{U}}(-D)$ commutes with specializations in the weight space $\mathcal{U} \subset \mathcal{W}$.

3) We outlined the construction for Siegel modular varieties but the same method applies more generally for PEL type Shimura varieties having dense ordinary locus, see Andreatta, Iovita, and Pilloni [2016a] for the Hilbert case and Brasca [2016] for the general case.

4) Even for Shimura varieties with empty ordinary locus, one can proceed in a similar way. The ordinary locus is replaced by the so called μ-ordinary locus, introduced by T. Wedhorn, and the Hasse invariant is replaced by the μ-Hasse invariant, defined at various levels of generality by G. Boxer, W. Goldring-M.H. Nicole, V. Hernandez, J.S. Koskivirta-T. Wedhorn. The last ingredient one needs is a replacement for the canonical subgroup and the Hodge-Tate map. We refer to Kassaei [2013] and Brasca [2013] for the case of Shimura curves and to Hernandez [2016] for the more general case of PEL type Shimura varieties and for a thorough account of the problem.

5) The last point of the Theorem 3.1 is proven in Bijakowski, Pilloni, and Stroh [2016] (and already by Coleman and Kassaei for $g = 1$). It is a *classicity criterion* which roughly asserts that small slope overconvergent modular forms are classical. It is a crucial result in order to study eigenvarieties as it provides a dense set of points, the classical ones. In a certain sense, these classical points characterize uniquely the eigenvariety. In particular, it often happens that a given property known at the classical points can be inferred by continuity for the whole eigenvariety.

4 Variations at infinity.

We now restrict to the case $g = 1$. In this case we have an eigencurve $\mathcal{E} \to \mathcal{Z} \to \mathcal{W}$ where $\mathcal{Z} \hookrightarrow \mathcal{W} \times \mathbb{G}_m$, the so called *spectral curve*, is the zero locus of the characteristic series $\mathcal{P}(X)$ of the U-operator acting on the space of overconvergent modular forms. The map $\mathcal{E} \to \mathcal{Z}$ is finite and both $\mathcal{E}$ and $\mathcal{Z}$ are equidimensional of dimension 1. Therefore the geometry of $\mathcal{E}$ can be understood, to some extent, by studying the apparently simpler space $\mathcal{Z}$.

4.1 The spectral halo. Recall that the weight space $\mathcal{W}$ is the rigid analytic fiber of $\operatorname{Spf} \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$. One can consider a slightly bigger space $\mathcal{W}^{\mathrm{an}}$ defined by the analytic points of the adic space $\operatorname{Spa}(\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!], \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!])$. Recall that $\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] \cong \mathbb{Z}_p\big[(\mathbb{Z}/p\mathbb{Z})^\times\big][\![T]\!]$, where T is defined by imposing that the grouplike element $\exp(p)$ is equal to $T + 1$. The complement of $\mathcal{W}$ in $\mathcal{W}^{\mathrm{an}}$ consists of finitely many points in characteristic p, corresponding to the T-adic valuations on $\mathbb{F}_p[(\mathbb{Z}/p\mathbb{Z})^\times](\!(T)\!)$ and $\mathcal{W}^{\mathrm{an}}$ is a compactification of $\mathcal{W}$, obtained by adding a point at the boundary of each rigid analytic open unit disc.

Coleman observed that the characteristic series $\mathcal{P}(X)$ of the U-operator on the eigencurve has coefficients in Λ and, hence one can consider the extended spectral curve $\mathcal{Z}^{an} \hookrightarrow \mathcal{W}^{an} \times \mathbb{G}_m = V(\mathcal{P})$. The fiber of $\mathcal{Z}^{an}$ over a boundary point $k : \operatorname{Spa}\big(\mathbb{F}_p(\!(T)\!), \mathbb{F}_p[\![T]\!]\big) \to \mathcal{W}^{an}$ is the zero set of the specialization $\mathcal{P}_k(X)$ at k, over the non-archimedean field $\mathbb{F}_p(\!(T)\!)$.

In Andreatta, Iovita, and Pilloni [2018] we prove a conjecture of Coleman in which he stated the existence of a Banach space over $\mathbb{F}_p(\!(T)\!)$ and of a compact operator whose characteristic series is $\mathcal{P}_k(X)$. More precisely, we prove the following result. Let $\mathcal{X}_{\mathcal{W}^{\mathrm{an}}}$ be the analytic adic space defined by the pull back of the modular curve to $\mathcal{W}^{\mathrm{an}}$. Given $v \in \mathbb{Q}_{\geq 0}^\times$ we define $\mathcal{X}_{\mathcal{W}^{\mathrm{an}}}(v)$ to be the open consisting of the points x satisfying the condition $|\widetilde{\mathrm{Ha}}|_x \geq \sup\big\{|T^v|_x, |p^v|_x\big\}$.

Theorem 4.1. *For $v > 0$ small enough we have an invertible sheaf $\omega^{k^{\mathrm{un}}}$ over $\mathcal{X}_{\mathcal{W}^{\mathrm{an}}}(v)$, endowed with an action of the Hecke operators, that coincides with the construction in Section 3.2 over $\operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$.*

Moreover, given a boundary point k of $\mathcal{W}^{\mathrm{an}}$, the sections of the fiber of $\omega^{k^{\mathrm{un}}}$ at k form a Banach module over $\mathbb{F}_p(\!(T)\!)$ such that the characteristic series of the U-operator is Coleman's series $\mathcal{P}_k(X)$.

The sections of the characteristic p fiber of $\omega^{k^{\mathrm{un}}}$ are called *T-adic overconvergent modular forms* (of radius of convergence v). They are actually functions on certain overconvergent Igusa tower in characteristic p. Using the sheaf $\omega^{k^{\mathrm{un}}}$ over $\mathcal{X}_{\mathcal{W}^{\mathrm{an}}}(v)$ one manages to extend the Coleman-Mazur eigencurve to an eigencurve $\mathcal{E}^{\mathrm{an}}$ over the whole $\mathcal{Z}^{\mathrm{an}}$.

Each finite slope eigenform f in characteristic p defines a point on $\mathcal{E}^{\mathrm{an}}$ and we can associate to it a semi-simple two dimensional Galois representation

$$\rho_f : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2\big(\overline{\mathbb{F}_p[\![T]\!]}\big)$$

unramified at the primes different form p and not dividing the tame level. Here $\overline{\mathbb{F}_p[\![T]\!]}$ denotes an algebraic closure of $\mathbb{F}_p[\![T]\!]$. If f is ordinary, ρ_f has already been constructed by Hida. For finite slopes we get new, mysterious objects in the realm of Galois representations that deserve further study and understanding. Here are some questions that we find interesting.

Given ρ_f as above, one can construct an equicharacteristic p, étale (φ, Γ)-module $\mathfrak{D}(\rho)$ over the Robba ring for the discretely valued field $\bar{\mathbb{F}}_p(\!(t)\!)$ (as in Berger and Colmez [2008]). Is $\mathfrak{D}(\rho)$ trianguline (i.e., extension of one dimensional (φ, Γ)-modules)?

Does this characterize the two dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with values in finite extensions of $\bar{\mathbb{F}}_p(\!(T)\!)$ which arise from T-adic overconvergent modular eigenforms of finite slope?

4.2 The halo conjecture. We would now like to discuss the halo conjecture and some questions related to the global geometry of $\mathcal{E}^{an}$. Let $x\colon \mathrm{Spa}(K, \mathcal{O}_K) \to \mathcal{W}^{an}$ be a rank one point. The choice of a pseudo uniformizer ϖ allows to normalize the associated valuation $v\colon K \to \mathbb{R} \cup \{\infty\}$ by $v(\varpi) = 1$. In that case we write v_ϖ for v. There are in general two natural choices of pseudo-uniformizer in $\mathcal{O}_K$, namely p and T, except at the boundary when $p = 0$ and at the very center $T = 0$. One can attach to the characteristic series $\mathcal{P}_k(X) = \sum_{n \geq 0} a_n X^n$ and a choice of pseudo uniformizer ϖ a Newton polygon $NP_\varpi(\mathcal{P}_k)$ which is the convex enveloppe of the points $(n, v_\varpi(a_n)) \subset \mathbb{R}^2$. This is the graph of a piecewise linear function and the sequence of slopes of $NP_\varpi(\mathcal{P}_k)$ is giving the sequence of ϖ-adic valuations of finite slope eigenvalues of U.

Conjecture 1 (Coleman-Mazur-Buzzard-Kilford)**.** *Let $k \in \mathcal{W}^{an}$ be a boundary point. Then there exists a positive rational number r such that for all rank one points*

$$k'\colon \mathrm{Spa}\big(K, \mathcal{O}_K\big) \to \mathcal{W}^{an}$$

in a neighbourhood $\mathcal{U} = \{x,\ |p^r|_x \leq |T|_x\}$ of k we have $NP_T(\mathcal{P}_{k'}) = NP_T(\mathcal{P}_k)$. Moreover, the slopes in $NP_T(\mathcal{P}_k)$ form a finite union of arithmetic progressions.

Before discussing what is known about this conjecture, let us describe some of the consequences. The first implication is that $\mathcal{Z}^{an}|_{\mathcal{U}} = \coprod_s \mathcal{Z}^{an}(s)$ splits as a disjoint union of components according to the slopes s occurring in $NP_T(\mathcal{P}_k)$. Each component $\mathcal{Z}^{an}(s)$ is finite flat over $\mathcal{U}$. In certain numerical examples it actually maps isomorphically onto $\mathcal{U}$. In this case the complement of the points at infinity can be visualized as halos, explaining the name of the conjecture.

A second implication is that the p-adic slopes tend to zero as one approaches the boundary. In particular, T-adic overconvergent modular eigenforms of finite slope (for the U-operator) are limits of classical modular forms of arbitrary fixed weight $k \geq 2$ (of course of increasing level at p) by Coleman's classicity theorem. It is known that each irreducible component of $\mathcal{Z}^{an}$ has image in the weight space equal to the complement of a finite number of points. Therefore, any irreducible component of $\mathcal{Z}^{an}$ contains at least one irreducible component of some $\mathcal{Z}^{an}(s)$. Thus each irreducible component of $\mathcal{Z}^{an}$ contains infinitely many classical points of a given weight $k \geq 2$.

Another consequence of the fact that the p-adic slopes tend to zero approaching infinity is that an irreducible component of $\mathcal{Z}^{an}$ is finite over the weight space if and only if it is ordinary (i.e., the slope is 0) (Liu, Wan, and Xiao [2017], prop. 3.24).

The conjecture has not yet been proved for the whole eigencurve $\mathcal{E}^{an}$ but it is known for all the irreducible components that arise from the p-adic Jaquet-Langlands correspondence thanks to Liu, Wan, and Xiao [ibid.]. Independently, in Bergdall and Pollack [2016] it is proved that the constancy of the Newton Polygon implies the second part of the halo conjecture, namely that the slopes form a finite union of arithmetic progressions.

Motivated by the boundary behavior provided by the conjecture and by a conjecture of Buzzard's for classical weights, Bergdall and Pollack have elaborated a unifying conjecture in Bergdall and Pollack [2017], called *ghost conjecture*, predicting (under some extra assumptions) the slopes of overconvergent cuspforms over the whole weight space.

Finally let us remark that, even though we discussed only the elliptic case, eigenvarieties might be defined over the whole analytic adic weight space for more general Shimura varieties. We refer to Andreatta, Iovita, and Pilloni [2016b] for the Hilbert case and to Johansson and Newton [2016] for a Betti cohomology approach. In contrast with the elliptic case, where at infinity the weight space consists of a finite set of points, in the Hilbert case, for a totally real field of degree g, at infinity the weight space has components of dimension $g-1$.

5 p-Adic variation of de Rham automorphic sheaves.

In this section we use the notations and results of Section 2 and Section 3 for $G = \mathrm{GL}_2/\mathbb{Q}$, i.e., for $g = 1$. Here we briefly present the constructions and results of Andreatta and Iovita [2017], using adic analytic spaces instead of formal schemes. The interested reader should consult Andreatta and Iovita [ibid.] for more details.

Before getting into technicalities let us briefly explain the problem we are faced with and explain how we chose to solve it. Let $p > 2$ be a prime integer, $N \geq 3$ an integer relatively prime to p, $\mathfrak{X}$ the adic analytic projective modular curve of level $\Gamma_1(N)$ over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ and $\alpha\colon E \longrightarrow \mathfrak{X}$ the generalized, universal elliptic curve. We denote by $\big(\mathrm{H}_E, \mathrm{Fil}_\bullet, \nabla\big)$ the data consisting of:

i) the relative de Rham cohomology sheaf of E over $\mathfrak{X}$, i.e.

$$\mathrm{H}_E := \mathbb{R}^1\alpha_*\big(\Omega^\bullet_{E/\mathfrak{X}}(\log(\alpha^{-1}(\mathrm{cusps}))\big),$$

ii) the Hodge filtration $\mathrm{Fil}_\bullet$ of H_E, i.e., $\mathrm{Fil}_0 := \omega_E = \alpha_*\big(\Omega^1_{E/\mathfrak{X}}(\log(\alpha^{-1}(\mathrm{cusps}))\big)$, $\mathrm{Fil}_i = 0$ for $i < 0$ and $\mathrm{Fil}_i = \mathrm{H}_E$ for $i \geq 1$.

iii) $\nabla\colon \mathrm{H}_E \longrightarrow \mathrm{H}_E \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathbb{Q}_p}\big(\log(\mathrm{cusps})\big)$, the Gauss-Manin connection, an integrable connection satisfying Griffith's transversality property.

We now consider the following family of data indexed by the integers:

$$(*) \qquad \Big(\big(\mathrm{Sym}^n(\mathrm{H}_E), \mathrm{Fil}_{n,\bullet}, \nabla_n\big)\Big)_{n\in\mathbb{Z}},$$

where $\mathrm{Fil}_{n,\bullet}$ and ∇_n are the natural increasing filtrations and connections on the n-th symmetric powers of H_E.

Over the complex numbers one can use the Hodge decomposition of H_E in order to describe the global sections of $\omega_E^{k-r} \otimes \mathrm{Sym}^r(\mathrm{H}_E)$ as suitable C^∞-functions on the upper half plane, called *nearly holomorphic modular forms* of weight k and order $\le r-1$. Using this interpretation, the Gauss-Manin connection takes the form of the so called *Maass-Shimura differential operator* $\delta_k(f) = \frac{1}{2\pi i}\left(\frac{\partial f}{\partial \tau} + \frac{k}{2iy} f\right)$ where τ is the standard coordinate on the upper half plane and $y = \mathrm{Im}(\tau)$. For $k > 2r$ one also has a *holomorphic projection* $\mathrm{H}^{\mathrm{hol}}$ to weight k modular forms and, hence, a q-expansion of nearly holomorphic forms. See Urban [2014, §2] for details. This is used, for example, to study special values of triple product L-functions as follows.

Let f, g, h be a triple of normalized primitive cuspidal classical eigenforms of weights k, ℓ, m, characters χ_f, χ_g, χ_h and tame levels N_f, N_g, N_h respectively. We write $f \in S_k(N_f, \chi_f)$, $g \in S_\ell(N_g, \chi_g)$, $h \in S_m(N_h, \chi_h)$. We assume that (k, ℓ, m) is unbalanced, i.e., there is an integer $t \ge 0$ such that $k - \ell - m = 2t$. We set $N := \ell.\mathrm{c.m.}(N_f, N_g, N_h)$ and $\mathbb{Q}_{f,g,h} := \mathbb{Q}_f \cdot \mathbb{Q}_g \cdot \mathbb{Q}_h$ the number field generated over $\mathbb{Q}$ by the Hecke eigenvalues of f, g, h. We assume that $\chi_f \cdot \chi_g \cdot \chi_h = 1$.

A result of Harris and Kudla [1991], previously conjectured by H. Jacquet and recently refined by Ichino [2008] and Watson [2002] implies that there are choices of Hecke-equivariant embeddings of $S_k(N_f, \chi_f)$, $S_\ell(N_g, \chi_g)$, $(S_m(N_h, \chi_h)$ into $S_k(N)$, $S_\ell(N)$, $S_m(N)$ respectively such that the images f^o, g^o, h^o of f, g, h respectively satisfy Ichino's formula, i.e.,

$$L\left(f, g, h, \frac{k+\ell+m-2}{2}\right) = \text{(non-zero algebraic constant)} \times |I(f^o, g^o, h^o)|^2,$$

where

$$I(f^o, g^o, h^o) := \frac{\langle (f^o)^*, \mathrm{H}^{\mathrm{hol}}\big(\delta^t (g^o)^{[p]} \times h^o\big)\rangle}{\langle (f^o)^*, (f^o)^*\rangle}.$$

Here $L(f, g, h, s)$ is the complex Garrett-Rankin triple product L-function attached to f, g, h. We have denoted by $\langle\ ,\ \rangle$ the Peterson inner product on the space of weight k-modular forms, $(f^o)^* = f^o \otimes \chi_f^{-1}$, $(g^o)^{[p]}$ is defined on q-expansions by: $(g^o)^{[p]}(q) := \sum_{n=1,(n,p)=1}^\infty a_n q^n$ if $g^o(q) = \sum_{n=1}^\infty a_n q^n$ and finally δ, $\mathrm{H}^{\mathrm{hol}}$ are the operators on nearly holomorphic forms introduced above.

5.1 p-Adic variations of $(\mathrm{H}_E, \mathrm{Fil}_\bullet)$. The task before us is to p-adically interpolate the constructions over the complex numbers previously described. We fix $I := [0, b]$ a closed interval, with $b \in \mathbb{Z}_{>0}$ and let $\mathcal{W}_I$ be the open adic subspace of $\mathcal{W}$ defined by

$$\mathcal{W}_I := \{x \in \mathcal{W} \quad | \quad |T^b|_x \leq |p|_x \neq 0\}.$$

Let $r \geq 0$ be an integer and denote by $\mathfrak{X}_{r,I}$ the open adic subspace of $\mathfrak{X}\times\mathcal{W}_I$ defined as in Section 3 by the valuations x such that $|\widetilde{\mathrm{Ha}}^{p^{r+1}}|_x \geq |p|_x$. If $E_{r,I}$ is the inverse image of the universal generalized elliptic curve over $\mathfrak{X}$. We remark that the universal character k^{un} of $\mathcal{W}_I$ is r-analytic and there is a canonical subgroup $H_r \subset E_{r,I}[p^r]$ of order p^r over $\mathfrak{X}_{r,I}$. Let H_r^D denote the Cartier dual of H_r.

We denote by $\mathfrak{IG}_{r,I} := \underline{\mathrm{Isom}}_{\mathfrak{X}_{r,I}}\big(\mathbb{Z}/p^r\mathbb{Z}, H_r^D\big)$ the adic space over $\mathfrak{X}_{r,I}$ of trivializations of H_r^D. Then $\mathfrak{IG}_{r,I} \longrightarrow \mathfrak{X}_{r,I}$ is a finite, étale and Galois cover with Galois group $\big(\mathbb{Z}/p^r\mathbb{Z}\big)^\times$. We introduce the ideals:

i) Hdg, the ideal of $\mathcal{O}^+_{\mathfrak{X}_{r,I}}$ locally generated by any lift of the Hasse invariant Ha modulo p.

ii) $\underline{\beta}_r$, the ideal of $\mathcal{O}^+_{\mathfrak{X}_{r,I}}$ locally generated by $\dfrac{p}{\mathrm{Hdg}^{\frac{p^r-1}{p-1}}}$.

iii) $\underline{\delta}$, the ideal of $\mathcal{O}^+_{\mathfrak{IG}_{r,I}}$ locally generated by a precisely defined $(p-1)$-st root of Hdg. For $p \geq 5$ one considers the overconvergent modular form D of weight 1 which is a certain precisely defined $(p-1)$-st root of the Eisenstein series E_{p-1}. Then D locally generates $\underline{\delta}$.

In Section 3 we have exhibited the pair of sheaves (ω_E, ω_E^+) over $\mathfrak{IG}_{r,I}$ which are invertible $\mathcal{O}_{\mathfrak{IG}_{r,I}}$ and respectively $\mathcal{O}^+_{\mathfrak{IG}_{r,I}}$-modules and the modification $\omega_E^\#$ of ω_E^+, an $\mathcal{O}^+_{\mathfrak{IG}_{r,I}}$ submodule of ω_E^+ which is itself invertible. In fact in the $g=1$ case, the situation is very simple and we happen to have $\omega_E^\# = \underline{\delta}\cdot\omega_E^+$, which implies that over $\mathfrak{IG}_{r,I}$ for $p \geq 5$ we have $\omega_E^\# = D\cdot\mathcal{O}^+_{\mathfrak{IG}_{r,I}}$, i.e., it is globally free.

Moreover if $\psi\colon \mathbb{Z}/p^r\mathbb{Z} \cong H_r^D$ denotes the universal trivialization of H_r^D over $\mathfrak{IG}_{r,I}$ then $P := \psi(1)$ is a universal generator of H_r^D over $\mathfrak{IG}_{r,I}$ and $s := \mathrm{HT}(P)$ is a $\mathcal{O}^+_{\mathfrak{IG}_{r,I}}/\underline{\beta}_r$-basis of $\omega_E^\#/\underline{\beta}_n\omega_E^\#$. In other words the pair $(\omega_E^\#, s)$ is a locally free sheaf with a marked section as in Section 2.

Let us now denote by H_E^+ the locally free $\mathcal{O}^+_{\mathfrak{X}_{r,I}}$-module of rank 2 characterized by the following property. For $U = \mathrm{Spa}(B, B^+) \subset \mathfrak{X}_{r,I}$ an open such that the generalized elliptic curve E/B is in fact defined over B^+ we let $\pi : E \longrightarrow \mathrm{Spf}(B^+)$ be the structural morphism. Then

$$\mathrm{H}_E^+|_U = \mathbb{R}^1\pi_*\big(\Omega^\bullet_{E/B^+}(\log(\pi^{-1}(\mathrm{cusps})))\big).$$

We have $\mathrm{H}_E^+\otimes_{\mathcal{O}^+_{\mathfrak{X}_{r,I}}}\mathcal{O}_{\mathfrak{X}_{r,I}} = \mathrm{H}_E$ and H_E^+ has a natural Hodge filtration $\mathrm{Fil}_\bullet^+$, expressed by the exact sequence:

$$0 \longrightarrow \omega_E^+ \longrightarrow \mathrm{H_E}^+ \longrightarrow (\omega_E^+)^{-1} \longrightarrow 0.$$

We also have a connection on H_E^+ but we will discuss it later.

It is natural to consider: $\mathrm{H}_E^{\#} := \underline{\delta} \cdot \mathrm{H}_E^{+}$ and $\mathrm{Fil}_{\bullet}^{\#} := \underline{\delta} \cdot \mathrm{Fil}_{\bullet}^{+}$ as $\big(\mathrm{H}_E^{\#}, s = \mathrm{HT}(P)\big)$ is a pair consisting of a locally free sheaf of rank 2 with a marked section and $(\omega_E^{\#}, s) \subset (\mathrm{H}_E^{\#}, s)$ is a compatible filtration. Let us then consider the sequence of adic spaces and morphisms

$$\mathfrak{T}_{\underline{\beta}_n}(\mathrm{H}_E^{\#}, s) \xrightarrow{u} \mathfrak{IG}_{r,I} \xrightarrow{v} \mathfrak{X}_{r,I}$$

and denote by $\rho := v \circ u$. Here $\mathfrak{T}_{\underline{\beta}_r}(\mathrm{H}_E^{\#}, s)$ denotes the VBMS of Section 2 associated to the pair $(\mathrm{H}_E^{\#}, s)$ and ideal sheaf $\underline{\beta}_n$. This VBMS was denoted $\mathbb{V}_0(\mathrm{H}_E^{\#}, s)$ in Andreatta and Iovita [2017]. Then we have a natural action of $\mathbb{Z}_p^*$ on the sheaf $\mathbb{W}^+ := \rho_*\big(\mathcal{O}^+_{\mathbb{V}_0(\mathrm{H}_E^{\#}, s)}\big)$.

Definition 5.1.1. *We denote by $k : \mathbb{Z}_p^\times \longrightarrow \Lambda_I^\times$ a weight in $\mathfrak{W}_I$ (it could be the universal weight or not), denote by $\mathbb{W}_k^+$ the $\mathcal{O}^+_{\mathfrak{X}_{r,I}}$-module $\mathbb{W}^+[k]$, i.e. $\mathbb{W}_k^+$ is the sub-$\mathcal{O}^+_{\mathfrak{X}_{r,I}}$ module of sections of $\mathbb{W}^+$ on which $\mathbb{Z}_p^\times$ acts by multiplication with the values of k. The formalism of vector bundles with marked sections implies that $\mathbb{W}_k^+$ has a filtration by locally free, coherent $\mathcal{O}^+_{\mathfrak{X}_{r,I}}$-submodules $\mathrm{Fil}^+_{k,\bullet}$.*

We let $\mathbb{W}_k := \mathbb{W}_k^+ \otimes_{\mathcal{O}^+_{\mathfrak{X}_{r,I}}} \mathcal{O}_{\mathfrak{X}_{r,I}}$. It is a sheaf of Banach modules on $\mathfrak{X}_{r,I}$ with a filtration $\mathrm{Fil}_{k,\bullet}$ and $\mathrm{Fil}_{k,0}$ coincides with the sheaf $\omega_{r,I}^k$ of Section 3.2.

5.2 p-Adic variations of the connection. In order to obtain a connection on $\mathbb{W}_k$ we need to first choose a formal model of the morphism $\pi : E \longrightarrow \mathfrak{X}_{r,I}$, say $\tau : \mathcal{E} \longrightarrow \mathfrak{X}$. Our favorite such formal model is obtained by taking for $\mathfrak{X}$ the partial blow-up of the base change of the formal completion of the modular curve $X_1(N)$ over $\mathbb{Z}_p$ to the formal weight space $\mathrm{Spf}(\Lambda_I)$, with respect to the ideal $(p, \mathrm{Hdg}^{p^{r+1}})$ and taking $\mathcal{E}$ to be the inverse image of the generalized elliptic curve over $X_1(N)$. We also obtain a natural formal model $\mathfrak{IG}$ of $\mathfrak{IG}_{r,I}$ given by the normalization of $\mathfrak{X}$ in $\mathfrak{IG}_{r,I}$. Having fixed these formal models we obtain: a canonical $\mathcal{O}^+_{\mathfrak{IG}_{r,I}}$-submodule $\Omega^{1,+}_{\mathfrak{IG}_{r,I}/\mathfrak{W}_I}(\log)$ of $\Omega^1_{\mathfrak{IG}_{r,I}/\mathfrak{W}_I}(\log)$ and a natural connection

$$\nabla^+ : \mathrm{H}_E^+ \longrightarrow \mathrm{H}_E^+ \otimes_{\mathcal{O}^+_{\mathfrak{IG}_{r,I}}} \Omega^{1,+}_{\mathfrak{IG}_{r,I}/\mathfrak{W}_I}(\log),$$

whose generic fiber is the connection ∇ described at the beginning of this section.

The connection ∇^+, the weight $k : \mathbb{Z}_p^\times \longrightarrow \Lambda_I^\times$ and the formalism of VBMS produce a connection

$$\nabla_k : \mathbb{W}_k \longrightarrow \mathbb{W}_k \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\mathfrak{W}_I}(\log(\mathrm{cusps}))$$

whose properties are described in the next theorem.

Theorem 5.2.1. *a) The connection ∇_k satisfies Griffith's transversality property with respect to the filtration i.e. $\nabla_k(\mathrm{Fil}_{k,i}) \subset \mathrm{Fil}_{k,i+1} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\mathfrak{W}_I}(\log)$, for all $i \geq 0$.*

b) If $\alpha \in \mathbb{Z}_{>0} \cap \mathfrak{W}_I(\mathbb{Q}_p, \mathbb{Z}_p)$ then the specialization at α of $\big(\mathbb{W}_k, \mathrm{Fil}_{k,\bullet}, \nabla_k\big)$, which we denote by $\big(\mathbb{W}_\alpha, \mathrm{Fil}_{\alpha,\bullet}, \nabla_\alpha\big)$, has $\big(\mathrm{Sym}^\alpha(\mathrm{H}_E), \mathrm{Fil}_{\alpha,\bullet}, \nabla_\alpha\big)$ as canonical submodule with filtration and connection. Moreover their global sections with slopes $h < \alpha - 1$ are equal (classicity).

For every Λ_I-valued weight k of $\mathcal{W}_I$ the elements $\mathrm{H}^0(\mathfrak{X}_{r,I}, \mathbb{W}_k)$ have natural *q-expansions* (for details see Andreatta and Iovita [2017].)

Another very interesting occurrence is the fact that given a Λ_I-valued weight k of $\mathcal{W}_I$ satisfying certain conditions (see below) the integral powers $(\nabla_k)^n$ of the connection ∇_k, for all $n \in \mathbb{Z}_{>0}$ (when we write $(\nabla_k)^n$ we really mean $\nabla_{k+2(n-1)} \circ \nabla_{k+2(n-2)} \circ \ldots \circ \nabla_k$) can be interpolated p-adically on $\mathrm{H}^0(\mathfrak{X}_{r,I}, \mathbb{W}_k)^{U_p=0}$ to the expense of possibly increasing r. More precisely we have (see Andreatta and Iovita [ibid.] for more details).

Theorem 5.2.2. *For every pair of weights γ, k in $\mathcal{W}_I$ satisfying the assumptions Andreatta and Iovita [ibid.] Assumption 4.1 there is $b \geq r$ such that for every $w \in \mathrm{H}^0\big(\mathfrak{X}_{r,I}, \mathbb{W}_k\big)^{U_p=0}$ we have a unique section $\nabla_k^\gamma(w) \in \mathrm{H}^0(\mathfrak{X}_{b,I}, \mathbb{W}_{k+2\gamma})$ satisfying the property: if the q-expansion of w is $w(q) := \sum_{n=0}^\infty a_n q^n$ then the q-expansion of $\nabla_k^\gamma(w)$ is $\nabla_k^\gamma(w)(q) := \sum_{n=1,(p,n)=1}^\infty \gamma(n) a_n q^n$.*

5.3 The overconvergent projection. Finally, in view of the applications to the triple product p-adic L-functions which we have in mind, we define the "overconvergent projection" which is seen as a p-adic analogue of Shimura's "holomorphic projection".

Let us fix a Λ_I-valued weight k of $\mathcal{W}_I$ and denote by $\mathbb{W}_k^\bullet$ the complex of sheaves $\mathbb{W}_k \xrightarrow{\nabla_k} \mathbb{W}_k \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\mathcal{W}_I}$ on $\mathfrak{X}_{r,I}$. We denote by $\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}_{r,I}, \mathbb{W}_k^\bullet)$ for $i \geq 0$, the i-th hypercohomology group with values in the complex $\mathbb{W}_k^\bullet$.

We have natural actions of all the Hecke operators on these cohomology groups and remark that if $h \geq 0$ is a finite slope, we have natural slope decompositions for the action of the operator U_p of the groups $\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}_{r,I}, \mathbb{W}_k^\bullet)$ and we denote by $\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}_{r,I}, \mathbb{W}_k^\bullet)^{\leq h}$ the subgroup of slope less then or equal to h classes for the action of U_p (see Andreatta and Iovita [ibid.] section §3.8). If we denote by $\mathfrak{K}$ the total ring of fractions of Λ_I, we can describe the base change of $\mathrm{H}^1_{\mathrm{dR}}(\mathfrak{X}_{r,I}, \mathbb{W}_k^\bullet)^{\leq h}$ to $\mathfrak{K}$ as follows:

$$\mathrm{H}^1_{\mathrm{dR}}(\mathfrak{X}_{r,I}, \mathbb{W}_k^\bullet)^{\leq h} \otimes_{\Lambda_I[1/p]} \mathfrak{K} \cong \mathrm{H}^0(\mathfrak{X}_{r,I}, \omega_{r,I}^{k+2})^{\leq h} \otimes_{\Lambda_I[1/p]} \mathfrak{K}.$$

Therefore the "overconvergent projection" denoted $H^\dagger$ is the natural map obtained as the composition:

$$\mathrm{H}^0(\mathfrak{X}_{r,I}, \mathbb{W}_k)^{\leq h} \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathfrak{X}_{r,I}, \mathbb{W}_k^\bullet)^{\leq h} \otimes_{\Lambda_I[1/p]} \mathfrak{K} \cong \mathrm{H}^0(\mathfrak{X}_{r,I}, \omega_{r,I}^{k+2})^{\leq h} \otimes_{\Lambda[1/p]} \mathfrak{K}.$$

5.4 Application: the triple product p-adic L-function in the finite slope case. Let f, g, h be a triple of normalized primitive cuspidal classical eigenforms of weights k, ℓ, m, characters χ_f, χ_g, χ_h and tame levels N_f, N_g, N_h respectively. Write f^o, g^o, h^o for their images in $S_k(N), S_\ell(N), S_m(N)$ respectively as explained at the beginning of this section. We assume that f has finite slope a and that (k, ℓ, m) is unbalanced, i.e., there is an integer $t \geq 0$ such that $k - \ell - m = 2t$. We denote by K a finite extension of $\mathbb{Q}_p$ which contains all the values of χ_f, χ_g, χ_h. Let $\alpha_f, \alpha_g, \alpha_h$ denote overconvergent families of modular forms interpolating f^o, g^o, h^o in weights k, ℓ, m respectively. More precisely there are: a non-negative integer r, closed intervals I_f, I_g and I_h such that the weights of these families, denoted respectively $k_f : \mathbb{Z}_p^\times \to \Lambda_{I_f,K}^\times$, $k_g : \mathbb{Z}_p^\times \to \Lambda_{I_g,K}^\times$, $k_h : \mathbb{Z}_p^\times \to \Lambda_{I_h,K}^\times$ are all adapted to a certain integer $n \geq 0$. This data gives an adic space $\mathfrak{X}_{r,I} \longrightarrow \mathfrak{X}$, where I is a closed interval containing $I_f \times I_g \times I_h$.

We denote by $\omega^{k_f}, \omega^{k_g}, \omega^{k_h}$ the respective modular sheaves (over $\mathfrak{X}_{r,I}$), then $\alpha_f \in \mathrm{H}^0(\mathfrak{X}_{r,I_f}, \omega^{k_f})$, $\alpha_g \in \mathrm{H}^0(\mathfrak{X}_{r,I_g}, \omega^{k_g})$, $\alpha_h \in \mathrm{H}^0(\mathfrak{X}_{r,I_h}, \omega^{k_h})$. We make the following assumption on the weights of $\alpha_f, \alpha_g, \alpha_h$:

1) Suppose that the weights k_f, k_g, k_h are such that $k_f - k_g - k_h$ is even, i.e., there is a weight $u : \mathbb{Z}_p^\times \longrightarrow (\Lambda_{I,K})^\times$ with $2u = k_f - k_g - k_h$.

2) the weights k_g, u are each of the form: a finite order character multiplied a strongly analytic weight (see Andreatta and Iovita [ibid.]).

We see $\alpha_f, \alpha_g, \alpha_h$ as global sections of $\mathrm{Fil}_0(\mathbb{W}_{k_f}^{\mathrm{an}})$, $\mathrm{Fil}_0(\mathbb{W}_{k_g}^{\mathrm{an}})$ and $\mathrm{Fil}_0(\mathbb{W}_{k_h}^{\mathrm{an}})$ respectively. In particular we have that $(\nabla_{k_g})^u(\alpha_g^{[p]})$ makes sense and

$$(\nabla_{k_g})^u(\alpha_g^{[p]}) \in \mathrm{H}^0(\mathfrak{X}_{r',I}, \mathbb{W}_{k_g+2u}^{\mathrm{an}}),$$

for some positive integer $r' \geq r$. Therefore we have a section

$$(\nabla_{k_g})^u\big(\alpha_g^{[p]}\big) \times \alpha_h \in \mathrm{H}^0(\mathfrak{X}_{r',I_u}, \mathbb{W}_{k_f}^{\mathrm{an}}).$$

Consider its class in $\mathrm{H}^1\big(\mathfrak{X}_{r',I_u}, \omega^{k_f-2}\big) \otimes \mathfrak{K}_f$ via the natural projection and its overconvergent projection

$$H^\dagger\Big((\nabla_{k_g})^u\big(\alpha_g^{[p]}\big) \times \alpha_h\Big) \in \mathrm{H}^0\big(\mathfrak{X}_{r',I_u}, \omega^{k_f}\big) \otimes_{\Lambda_{I_f}} \mathfrak{K}_f,$$

to which we can apply the slope smaller or equal to a projector, $e^{\leq a}$:

$$e^{\leq a}\Big(H^\dagger\Big((\nabla_{k_g})^u\big(\alpha_g^{[p]}\big) \times \alpha_h\Big)\Big) \in \mathrm{H}^0\big(\mathfrak{X}_{r',I_u}, \omega^{k_f}\big)^{\leq a} \otimes_{\Lambda_{I_f}} \mathfrak{K}_f.$$

We are finally able to define the Garrett-Rankin triple product p-adic L-function attached to the triple $(\alpha_f, \alpha_g, \alpha_h)$ of p-adic families of modular forms, of which α_f has finite slope $\leq a$, to be:

$$\mathfrak{L}_p^f(\alpha_f, \alpha_g, \alpha_h) := \frac{\Big\langle \alpha_f^*, e^{\leq a}\Big(H^\dagger\Big((\nabla_{k_g})^u\big(\alpha_g^{[p]}\big) \times \alpha_h\Big)\Big)\Big\rangle}{\langle \alpha_f^*, \alpha_f^* \rangle} \in \mathfrak{K}_f \widehat{\otimes} \Lambda_{k_g,K} \widehat{\otimes} \Lambda_{k_h,K}.$$

By the definition of the overconvergent projection the p-adic L-function $\mathfrak{L}_p^f(\alpha_f, \alpha_g, \alpha_h)$ has only finitely many poles, i.e., it is meromorphic.

Remark 5.4.1. The triple product p-adic L-function attached to a triple of **ordinary** families of modular forms has been defined in Darmon and Rotger [2014], using q-expansions.

Let now $x \in \mathcal{W}_{I_f}$, $y \in \mathcal{W}_{I_g}$, $z \in \mathcal{W}_{I_g}$ be a triple of unbalanced classical weights, i.e., such that $x, y, z \in \mathbb{Z}_{\geq 2}$ and such that there is $t \in \mathbb{Z}_{\geq 0}$ with $x - y - z = 2t$. Let us denote by f_x, g_y, h_z the specializations of $\alpha_f, \alpha_g, \alpha_h$ at x, y, z respectively, seen as sections over $\mathfrak{X}_{r',I_u}$ of $\omega^x \subset \mathrm{Fil}_{x-2}(\mathbb{W}^{\mathrm{an}}_{x-2}) = \mathrm{Symm}^{x-2}(\mathrm{H}_E)$, $\omega^y \subset \mathrm{Fil}_{y-2}(\mathbb{W}^{\mathrm{an}}_{y-2}) = \mathrm{Symm}^{y-2}(\mathrm{H}_E)$, $\omega^z \subset \mathrm{Fil}_{z-2}(\mathbb{W}^{\mathrm{an}}_{z-2}) = \mathrm{Symm}^{z-2}(\mathrm{H}_E)$ respectively.

If we fix embeddings of $\overline{\mathbb{Q}}$ in $\mathbb{C}$ and $\mathbb{C}_p$ respectively, using the identifications between the p-adic overconvergent projection and the complex holomorphic one and between the Gauss-Manin connection and the Shimura-Maass differential operator on the one hand and the classical expressions of the special values of the complex triple product L-functions on the other, we obtain:

$$|\mathcal{L}_p^f(\alpha_f, \alpha_g, \alpha_h)(x, y, z)|_p = (\text{explicit constant}) \times \Big(\mathrm{L}^{\mathrm{alg}}(f_x, g_y, h_z, \frac{x+y+z-2}{2})\Big)^{\frac{1}{2}}.$$

In particular for $x = k$, $y = \ell$, $z = m$ we have $\mathcal{L}_p^f(\alpha_f, \alpha_g, \alpha_h)(k, \ell, m) \neq 0$ which implies that $\mathcal{L}_p^f(\alpha_f, \alpha_g, \alpha_h) \neq 0$.

6 Higher coherent cohomology.

The purpose of this last section is to explain how the higher coherent cohomology of automorphic bundles enters the picture and how this is related to irregular motives. Let $K \subset \mathrm{GSp}_{2g}(\mathbb{A}_f)$ be a compact open subgroup. Let $X_K \to \mathrm{Spec}\,\mathbb{C}$ be a toroidal compactification of the Siegel variety of genus g and level K. For any classical weight $k = (k_1, \cdots, k_g)$, we can consider the cuspidal cohomology $\mathrm{H}^i(X_K, \omega^k(-D))$, as well as the usual cohomology $\mathrm{H}^i(X_K, \omega^k)$. They don't depend on the choice of the toroidal compactification. Let us define the interior coherent cohomology $\overline{\mathrm{H}}^i(X_K, \omega^k) = \mathrm{Im}(\mathrm{H}^i(X_K, \omega^k(-D)) \to \mathrm{H}^i(X_K, \omega^k))$. Recall that $\lim_K \overline{\mathrm{H}}^i(X_K, \omega^k)$ is an admissible $\mathrm{GSp}_{2g}(\mathbb{A}_f)$-representation. We first recall the following result, saying that a generic weight has only cohomology in one degree.

Theorem 6.0.1 (Harris [1990], Li and Schwermer [2004], Lan [2016]). *There is an (explicit) constant C[6] which only depends on g such that if $(k = (k_1, \cdots, k_g))$ and:*

1. $|k_i - k_{i+1}| \geq C$ *for all* $1 \leq i \leq g-1$,
2. $|k_i + k_j| \geq C$ *for all* $1 \leq i \leq j \leq g$

then $\overline{\mathrm{H}}^\star(X_K, \omega^k)$ *is concentrated in one degree.*

Let us explain how one should think about this theorem. Identify $\mathbb{Z}^g$ with the space of characters of the maximal diagonal torus of the group Sp_{2g}. We make a choice of

[6] One can be more precise. For example, if $g = 1$, $\overline{\mathrm{H}}^\star(X_K, \omega^k)$ is concentrated in degree 0 if $k \geq 2$ and in degree 1 if $k \leq 0$. If $g = 2$, $\overline{\mathrm{H}}^i(X_K, \omega^k)$ is concentrated in degree 0 if $k_2 \geq 4$; degree 1 if $k_2 \leq 0$ and $k_1 + k_2 \geq 5$; degree 2 if $k_2 + k_1 \leq 1$ and $k_1 \geq 3$; degree 3 if $k_1 \leq -1$.

positive roots R^+ to be the union of the compact positive roots $R_c^+ = \{e_i - e_j\}_{1\leq i<j\leq g}$ and non-compact positive roots $R_{nc}^+ = \{e_i + e_j\}_{1\leq i\leq j\leq g}$.

We can associate to k the g-uple $\lambda = (\lambda_1, \cdots, \lambda_g) = (k_1 - 1, \cdots, k_g - g) = k + \rho_c - \rho_{nc} \in \mathbb{Z}^g$ where ρ_c is half the sum of the positive compact roots, and ρ_{nc} is half the sum of the positive non-compact roots. We see that k is dominant if and only if λ is R_c^+ regular: $\langle \lambda, \alpha\rangle > 0$ for all $\alpha \in R_c^+$.

The theorem above says that if k is such that λ is far enough from all the walls perpendicular to all the roots, then $\overline{\mathrm{H}}^\star(X, \omega^k)$ is concentrated in one single degree which can be determined as follows: let $C \subset \mathbb{Z}^g$ be the chamber defined by $\lambda_1 > \ldots > \lambda_g \geq 0$; the cohomological degree is the minimum of the length of the elements of the Weyl group $W_{Sp_{2g}} = (\mathbb{Z}/2\mathbb{Z})^g \rtimes S_g$ that take λ to an element of C. Let λ be far enough from the walls and let w be an element of the Weyl group of minimal length such that $w.\lambda \in C$. Although the Hecke modules $\overline{\mathrm{H}}^{\ell(w)}(X_K, \omega^{\lambda-\rho_c+\rho_{nc}})$ and $\overline{\mathrm{H}}^0(X_{\mathbb{C}}, \omega^{w.\lambda-\rho_c+\rho_{nc}})$ are rarely isomorphic (except for $g = 1$), they are closely related [7].

So from that perspective, a generic weight has cohomology in one single degree, and moreover, one can often reduce to degree 0 cohomology. In that sense, Theorem 3.1 is optimal as long as we want to work over the total weight space.

We'd now like to consider "singular" weights λ that lie on the walls $\langle\lambda, \alpha\rangle = 0$ for $\alpha \in R_{nc}^+$. The main reason is that the corresponding cohomology groups of weight $\lambda - \rho_c + \rho_n$ are conjecturally related to irregular motives. Moreover, they don't admit a Betti cohomology realization and can only be seen in the coherent cohomology. In one direction, one knows how to attach compatible systems of Galois representations to automorphic forms realized in the coherent cohomology [8] (Deligne and Serre [1974], Taylor [1991], Goldring [2014], Pilloni and Stroh [2016], Boxer [2015], Goldring and Koskivirta [2017]). The method is to establish congruences with automorphic forms which are holomorphic discrete series at infinity and whose Galois representations can (often) be constructed in the étale cohomology of a Shimura variety.

Example 1 (Limits of discrete series). Let π be an automorphic representation for the group $\mathrm{GSp}_{2g}/\mathbb{Q}$ for which π_∞ is a limit of discrete series with infinitesimal character λ lying on such non-compact wall. Then π_f is realized in $\lim_K \overline{\mathrm{H}}^i(X_K, \omega^{\lambda-\rho_c+\rho_{nc}})$. Moreover, it will often be realized (for instance if the associated parameter has trivial centralizer) in several consecutive degrees (the number of consecutive degrees is the number of non-compact roots $\alpha \in R_{nc}^+$ such that $\langle\lambda, \alpha\rangle = 0$). For the standard $2g + 1$ dimensional representation of the L-group $\mathrm{GSp}_{2g}^L \to \mathrm{GL}_{2g+1}$, the associated compatible system has (conjectural) Hodge-Tate weights $(\lambda_1, -\lambda_1, \cdots, \lambda_g, -\lambda_g, 0)$. The simplest situation is $g = 1$, $\lambda = 0$, $k = 1$. There is an isomorphism of Hecke modules

[7] If λ is far enough from the walls, all the cohomology is represented by automorphic forms $\pi_\infty \otimes \pi_f$ which are discrete series at infinity. The L-packet corresponding to π_∞ contains a holomorphic discrete series π_∞^h. It the global Langlands parameter associated to π has trivial centralizer, then $\pi_\infty^h \otimes \pi_f$ is still automorphic and realized in the degree 0 cohomology

[8] For automorphic forms which are not holomorphic limits of discrete series, they are very "weak" compatible systems since they are not known to be de Rham.

$\overline{\mathrm{H}}^0(X_K, \omega^1) = \overline{\mathrm{H}}^1(X_K, \omega^1)$ and $\lim_K \overline{\mathrm{H}}^i(X_K, \omega^1) = \oplus \pi_f$ where π_f runs over all admissible $\mathrm{GL}_2(\mathbb{A}_f)$-modules for which $\pi_\infty \otimes \pi_f$ is cuspidal automorphic for π_∞ the unique limit of discrete series of $\mathrm{GL}_2(\mathbb{R})$ [9]. For the tautological 2-dimensional representation of the L-group, the associated compatible system arises from an Artin motive (Deligne and Serre [1974]).

When $g = 1$ we have the following theorem (a particular case of Artin's conjecture):

Theorem 6.0.2 (Langlands [1980], Tunnell [1981], Buzzard, Dickinson, Shepherd-Barron, and Taylor [2001], Khare and Wintenberger [2009], Kisin [2009], Kassaei [2013], Kassaei, Sasaki, and Tian [2014], Pilloni and Stroh [2016], Calegari and Geraghty [2018])**.** *There is a bijective correspondence between isomorphism classes of continuous irreducible odd Galois representations $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ and cuspidal automorphic forms $\pi = \pi_\infty \otimes \pi_f$ on $\mathrm{GL}_2/\mathbb{Q}$ such that π_∞ is a limit of discrete series. This bijection satisfies $L(\rho, s) = L(\pi \otimes |\,|^{-\frac{1}{2}}, s)$.*

Remark 6.0.3. The theorem holds also for totally odd irreducible two dimensional representations of the Galois group G_F of a totally real finite field extension F of $\mathbb{Q}$. Under mild technical hypothesis one can also prove that a representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ which is totally odd, irreducible and geometric with Hodge-Tate weights all equal to 0 is an Artin representation. See Pilloni and Stroh [2016].

The case $g = 2$ is also particularly interesting. Let $A \to \operatorname{Spec} \mathbb{Q}$ be a simple abelian surface and denote by $\mathrm{H}^1(A)$ the associated motive. For every prime p, we can define a Weil-Deligne representation $WD_p(\mathrm{H}^1(A)) : \mathrm{WD}_{\mathbb{Q}_p} \to \mathrm{GSp}_4(\mathbb{C})$. By Gan and Takeda [2011] there is a local L-packet $\Pi_p(A)$ whose Langlands parameter is $WD_p(\mathrm{H}^1(A)) \otimes |\,|^{\frac{3}{2}}$. This local L-packet contains at most two elements and exactly one generic element π_p^g. At the infinite place there is a local L-packet $\Pi_\infty(A)$ which consists of the two limits of discrete series (respectively holomorphic and generic) $\{\pi_\infty^h, \pi_\infty^g\}$ with infinitesimal character $\lambda = (1, 0)$[10]. We let $\Pi(A) = \prod_p \Pi_p(A) \times \Pi_\infty(A)$. The following is a particular example of Langlands's conjectures:

Conjecture 2. *The global L-packet $\Pi(A)$ contains a cuspidal automorphic form. As a consequence the complex L-function $L(\mathrm{H}^1(A), s)$ extends to an entire function over $\mathbb{C}$ and satisfies a functional equation as predicted in Serre [1970].*

Remark 6.0.4. 1) If $\mathrm{End}(A) \neq \mathbb{Z}$ (the GL_2-type case), the conjecture is known thanks to the works Khare and Wintenberger [2009], Kisin [2009] and Yoshida [1984].

2) $\pi_f = \otimes_p \pi_p \in \prod_p \Pi_p(A)$ is realized (with multiplicity one) in $\lim_K \overline{\mathrm{H}}^0(X_K, \omega^{(2,2)})$ provided $\pi_\infty^h \otimes \pi_f$ is automorphic, and in $\lim_K \overline{\mathrm{H}}^1(X_K, \omega^{(2,2)})$ provided $\pi_\infty^g \otimes \pi_f$ is automorphic.

3) The character formula of Labesse and Langlands [1979] describes which elements of $\Pi(A)$ should be cuspidal automorphic. If $\mathrm{End}(A) = \mathbb{Z}$, all elements of $\Pi(A)$ should

[9] π is normalized by imposing that the central character of is $|.| \times \chi$ where χ is a finite character

[10] we normalize here $\{\pi^h, \pi^g\}$ by asking that the central character is $|\,|^2$ on the connected component of the center

be cuspidal automorphic. If $\mathrm{End}(A) \neq \mathbb{Z}$ an element $\pi_\infty \otimes \pi_p$ should be automorphic if and only if the number of non-generic representations occurring in the product is even. If $\mathrm{End}(A) = \mathbb{Z}$ we can choose the particular element of $\Pi(A)$ which is generic at all finite places and π_∞^h at infinity. This representation has a unique line which is generated by a lowest weight vector at ∞ and is invariant under the paramodular group of a certain level $N(A)$ at all finite places. Therefore there should be a well determined (up to scalar) holomorphic weight $(2,2)$ cuspform of paramodular level $N(A)$ with rational Hecke eigenvalues attached to A. This is the paramodular conjecture of Brumer and Kramer [2014].

Theorem 6.0.2 and Conjecture 2 share high similarities: the Hodge-Tate weights of the motives considered have multiplicity two and the relevant automorphic forms contribute to two coherent cohomology degrees of the same automorphic sheaf. We will now explain how these singular weights behave in p-adic families: this appears to be a crucial tool in the proof of Theorem 6.0.2 and in the approaches to Conjecture 2.

6.1 Modular curves and weight one forms. We slightly change notations. Let p be a prime integer and $N \geq 3$ an integer prime to p. Let X be the modular curve of level $\Gamma_1(N)$ over Spec $\mathbb{Z}_p$ and X_{Iw} the modular curve of level $\Gamma_1(N) \cap \Gamma_0(p)$. We now examine the behaviour of p-adic families at weight one. We restrict ourselves to ordinary families because weight one modular forms of finite slope at p are necessarily ordinary.

Theorem 6.1.1 (Hida [1986]). *There is a finite projective $\Lambda = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$-module M* [11] *such that for all $k \in \mathbb{Z}_{\geq 2}$,*

$$M \otimes_{\Lambda,k} \mathbb{Z}_p = \mathrm{ordH}^0(X_{\mathrm{Iw}}, \omega^k(-D)).$$

Here $\mathrm{ord} = \lim_n U_p^{n!}$ is the ordinary projector for U_p. There is a control theorem in weight 1, but it is more complicated to state. By construction of M, there is an injective map $\mathrm{H}^0(X_{Iw}, \omega(-D)) \to M \otimes_{\Lambda,1} \mathbb{Z}_p$. In order to state the classicity theorem in weight 1, we need to look at the Galois representation picture. For any $k \in \mathbb{Z}$, and any eigenform f in $M \otimes_{\Lambda,k} \overline{\mathbb{Z}}_p$, there is an associated two dimensional Galois representation ρ_f whose restriction to inertia at p is nearly ordinary (χ_p is the p-adic cyclotomic character):

$$\rho_f|_{I_{\mathbb{Q}_p}} \simeq \begin{pmatrix} 1 & \star \\ 0 & \chi_p^{1-k} \end{pmatrix}$$

If $k \geq 2$ the representation ρ_f is automatically de Rham which is consistent with the control theorem. If $k = 1$, the representation is de Rham if and only if it is unramified at p and the classicity theorem in weight 1 states:

Theorem 6.1.2 (Buzzard and Taylor [1999], Pilloni and Stroh [2016]). *An eigenclass $f \in M \otimes_{\Lambda,1} \overline{\mathbb{Z}}_p$ is classical if and only if the associated Galois representation is unramified at p.*

[11] Moreover, M carries an action of the Hecke algebra and the control isomorphism is Hecke equivariant.

This classicity theorem is one of the key steps towards the proof of Theorem 6.0.2 via the strategy envisionned in Buzzard and Taylor [1999][12]: establish a congruence between an icoashedral artin representation and a modular elliptic curve modulo 5, and prove a modular lifting theorem. There is a difficulty to prove a modular lifting theorem with weight one forms because congruences are obstructed (since $\mathrm{H}^1(X,\omega) \neq 0$) and the usual Taylor-Wiles method doesn't apply. The strategy is to prove the modular lifting theorem with the module of ordinary p-adic modular forms of weight one instead (that is the module $M \otimes_{\Lambda,1} \mathbb{Z}_p$, for which the usual Taylor-Wiles method applies) and then argue via this classicity theorem. In order to obtain a full proof of Theorem 6.0.2, it is necessary to combine this strategy with solvable base change, and therefore one needs an extension of theorem Theorem 6.1.2 over totally real fields (Pilloni and Stroh [2016]). Observe that Theorem 6.0.2 was first proved in full generality as a consequence of Serre's modularity conjecture. Calegari and Geraghty [2018] found a way to modify the Taylor-Wiles method in order to apply it directly to weight one forms, therefore eliminating the use of Theorem 6.1.2. This method is very promising but as its application depends on certain conjectural inputs it has not yet given a complete new proof of Theorem 6.0.2.

6.2 The group GSp_4 and potentially modular abelian surfaces. We now let $X \to \operatorname{Spec} \mathbb{Z}_p$ be the Siegel threefold of hyperspecial level at p (and some fixed level away from p). We let $X_{\mathrm{Iw}} \to X_{\mathrm{Kli}} \to X$ be the Siegel threefolds of Iwahori and Klingen level at p. As we have seen, for all weights $k = (k_1,k_2)$ with $k_1 \geq k_2$, we have an automorphic vector bundle ω^k. The tempered part (at infinity) of the cohomology $\overline{\mathrm{H}}^\star(X_{\mathbb{C}}, \omega^{(k_1,k_2)})$ is concentrated in degree 0 if $k_2 \geq 3$ while there is cohomology in degree 0 and 1 if $k_2 = 2$. The situation resembles that of modular curves and weight one forms, except that there are now infinitely many singular weights: all those of the form $(k,2)$ for $k \geq 2$.

Let $\Lambda_1 = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$ and $\Lambda_2 = \mathbb{Z}_p[\![(\mathbb{Z}_p^\times)^2]\!]$. We first state the main theorem of classical Hida theory:

Theorem 6.2.1 (Hida [1986], Pilloni [2011])**.** *There exists a finite projective Λ_2-module M such that for (k_1,k_2) with $k_1 \geq k_2 \geq 4$, $M \otimes_{\Lambda_2,(k_1,k_2)} \mathbb{Z}_p = \mathrm{ord}\mathrm{H}^0(X_{\mathrm{Iw}}, \omega^{(k_1,k_2)}(-D))$.*

Here ord is the ordinary projector for the operator $U_{p,1}U_{p,2}$. The bound $k_2 \geq 4$ is an accident and the expected optimal bound is $k_2 \geq 3$. It is instructive to look at the Galois representation picture. For all eigenclasses $f \in M \otimes_{\Lambda_2,(k_1,k_2)} \overline{\mathbb{Z}}_p$ there is an associated nearly ordinary Galois representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$ such that:

$$\rho_f|_{I_{\mathbb{Q}_p}} \simeq \begin{pmatrix} 1 & \star & \star & \star \\ 0 & \chi_p^{-k_2+2} & \star & \star \\ 0 & 0 & \chi_p^{-k_1+1} & \star \\ 0 & 0 & 0 & \chi_p^{-k_1-k_2+3} \end{pmatrix}$$

[12] Granting the fact that it is known in the solvable image case by automorphic methods

Such a representation is automatically geometric if $k_1 \geq k_2 \geq 3$. It is tempting to believe that an eigenclass of weight $k_1 \geq k_2 = 2$ is classical if and only if

$$\rho_f|_{I_{\mathbb{Q}_p}} \simeq \begin{pmatrix} 1 & 0 & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & \chi_p^{-k_1+1} & 0 \\ 0 & 0 & 0 & \chi_p^{-k_1+1} \end{pmatrix}$$

(the analogue of Theorem 6.1.2). The techniques of Theorem 6.1.2 which crucially depend on the explicit relation between q-expansion and Hecke eigenvalues don't appear to generalize to this case. In particular, it seems impossible to generalize the strategy of Buzzard and Taylor [1999] to Conjecture 2.

We now state the main theorem of Higher Hida theory which deals with the singular weights:

Theorem 6.2.2 (Pilloni [2017]). *There exists a perfect complex $M^\bullet$*[13] *of Λ_1-modules of amplitude $[0,1]$ such that for all $k \in \mathbb{Z}_{\geq 2}$:*

$$M^\bullet \otimes^L_{(k,2)} \mathbb{Q}_p = \mathrm{ord}'\mathrm{R}\Gamma\big(X_{\mathrm{Kli}}, \omega^{(k,2)}(-D) \otimes \mathbb{Q}_p\big).$$

Here ord' is the ordinary projector for the operator $U_{p,1}$. The control theorem is sharp. Let us explain briefly the construction of $M^\bullet$. Let $X \to \operatorname{Spec} \mathbb{Z}_p$ be the Siegel threefold of level prime to p. We let $\mathfrak{X}$ be the p-adic completion of X and denote by $\mathfrak{X}^{\geq i}$ the p-rank stratification on $\mathfrak{X}$. As we have seen in Section 3, over the ordinary locus $\mathfrak{X}^{\geq 2}$ we have a multiplicative subgroup of rank 2 of $G[p^\infty]$ which provides the extra structure on ω_G allowing for the interpolation property. Over $\mathfrak{X}^{\geq 1}$ there is still an extra structure as we can choose a multiplicative Barsotti-Tate group of height 1, $H_\infty \hookrightarrow G[p^\infty]$ and for such a choice we have an exact sequence

$$0 \to \omega_{G[p^\infty]/H_\infty} \to \omega_{G[p^\infty]} \to \omega_{H^\infty} \to 0$$

and the Hodge-Tate map realizes an isomorphism $T_p(H_\infty^D) \otimes \mathcal{O}_{\mathfrak{X}^{\geq 1}(p^\infty)} \to \omega_{H_\infty}$. Thus, we end up with half the extra structure we had over the ordinary locus and, this allows the interpolation of the automorphic sheaves in one direction. It is quite important to work over this larger base since $\mathfrak{X}^{\geq 1}$ is morally of cohomological dimension 1, while $\mathfrak{X}^{\geq 2}$ is of cohomological dimension 0[14].

The projective module M is obtained by considering the (degree 0) ordinary cohomology of an interpolation sheaf over $\mathfrak{X}^{\geq 2}$ as explained in Section 3, while the complex $M^\bullet$ is obtained by considering the ordinary cohomology of an other interpolation sheaf, whose weight is parametrized by Λ_1 over $\mathfrak{X}^{\geq 1}$.

The cohomology $M^\bullet \otimes^L_{\Lambda_1,2} \mathbb{Z}_p$ is an integral modification of $\mathrm{ord}'\mathrm{R}\Gamma\big(X_{\mathrm{Kli}}, \omega^{(2,2)}(-D)\big)$. One important property of $M^\bullet$ is that it is concentrated in two degrees, while this is not known to hold for $\mathrm{R}\Gamma\big(X, \omega^{(2,2)}(-D)\big)$. In Boxer, Calegari, Gee, and Pilloni [2018]

[13] $M^\bullet$ carries an action of the Hecke algebra and the control theorem is Hecke equivariant

[14] Precisely, for all k, $\mathrm{H}^i(\mathfrak{X}^{\geq 1}, \omega^k(-D)) = 0$ if $i > 1$ and $\mathrm{H}^i(\mathfrak{X}^{\geq 2}, \omega^k(-D)) = 0$ if $i \geq 1$

we manage to study the Galois representation supported by $M^\bullet$ and prove under some technical assumptions that it is ordinary. As a corollary, $M^\bullet \otimes^L_{\Lambda_1,2} \mathbb{Z}_p$ can be used to construct modified Taylor-Wiles systems in the sense of Calegari and Geraghty [2018]. It is an important ingredient in the proof of the following theorem:

Theorem 6.2.3 (Boxer, Calegari, Gee, and Pilloni [2018])**.** *Let $A/\mathbb{Q}$ be an abelian surface. Then there is a finite field extension F of $\mathbb{Q}$ such that $\mathrm{H}^1(A|_F)$ is automorphic. In particular $L(\mathrm{H}^1(A), s)$ has a meromorphic continuation to $\mathbb{C}$.*

The theorem holds also when $\mathbb{Q}$ is replaced by a totally real field.

References

Fabrizio Andreatta and Adrian Iovita (2017). "Triple product p-adic L-functions associated to finite slope p-adic families of modular forms, with an appendix by Eric Urban". arXiv: 1708.02785 (cit. on pp. 302, 305–307).

Fabrizio Andreatta, Adrian Iovita, and Vincent Pilloni (2015). "p-adic families of Siegel modular cuspforms". *Ann. of Math. (2)* 181.2, pp. 623–697. MR: 3275848 (cit. on pp. 292, 296).

– (2016a). "On overconvergent Hilbert modular cusp forms". *Astérisque* 382, pp. 163–193. MR: 3581177 (cit. on p. 299).

– (2016b). "The adic, cuspidal, Hilbert eigenvarieties". *Res. Math. Sci.* 3, Paper No. 34, 36. MR: 3544971 (cit. on p. 302).

– (2018). "Le Halo Spectral". To appear in the Ann. Sci. ENS (cit. on p. 300).

John Bergdall and Robert J. Pollack (2016). "Arithmetic properties of Fredholm series for p-adic modular forms". *Proc. Lond. Math. Soc. (3)* 113.4, pp. 419–444. MR: 3556487 (cit. on p. 302).

– (2017). "Slopes of modular forms and the ghost conjecture". In: (cit. on p. 302).

Laurent Berger and Pierre Colmez (2008). "Familles de représentations de de Rham et monodromie p-adique". *Astérisque* 319. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules, pp. 303–337. MR: 2493221 (cit. on p. 301).

Stéphane Bijakowski, Vincent Pilloni, and Benoît Stroh (2016). "Classicité de formes modulaires surconvergentes". *Ann. of Math. (2)* 183.3, pp. 975–1014. MR: 3488741 (cit. on p. 299).

George A. Boxer (2015). "Torsion in the Coherent Cohomology of Shimura Varieties and Galois Representations". PhD thesis. Harvard University, p. 219. arXiv: 1507.05922. MR: 3450451 (cit. on p. 309).

George A. Boxer, Frank Calegari, Toby Gee, and Vincent Pilloni (2018). "Abelian surfaces over totally real fields are potentially modular" (cit. on p. 314).

Riccardo Brasca (2013). "p-adic modular forms of non-integral weight over Shimura curves". *Compos. Math.* 149.1, pp. 32–62. MR: 3011877 (cit. on p. 299).

– (2016). "Eigenvarieties for cuspforms over PEL type Shimura varieties with dense ordinary locus". *Canad. J. Math.* 68.6, pp. 1227–1256. MR: 3563721 (cit. on p. 299).

Armand Brumer and Kenneth Kramer (2014). "Paramodular abelian varieties of odd conductor". *Trans. Amer. Math. Soc.* 366.5, pp. 2463–2516. MR: 3165645 (cit. on p. 311).

Kevin Buzzard, Mark Dickinson, Nick Shepherd-Barron, and Richard Taylor (2001). "On icosahedral Artin representations". *Duke Math. J.* 109.2, pp. 283–318. MR: 1845181 (cit. on p. 310).

Kevin Buzzard and Toby Gee (2011). "The conjectural connection between automorphic representations and Galois representations". In: Proceedings of the LMS Durham Symposium 2011 (cit. on p. 291).

Kevin Buzzard and Richard Taylor (1999). "Companion forms and weight one forms". *Ann. of Math. (2)* 149.3, pp. 905–919. MR: 1709306 (cit. on pp. 312, 313).

Frank Calegari and David Geraghty (2018). "Modularity lifting beyond the Taylor–Wiles method". *Invent. Math.* 211.1, pp. 297–433. MR: 3742760 (cit. on pp. 310, 312, 314).

Przemysław Chojecki, D. Hansen, and C. Johansson (2017). "Overconvergent modular forms and perfectoid Shimura curves". *Doc. Math.* 22, pp. 191–262. MR: 3609197 (cit. on p. 298).

Laurent Clozel (1990). "Motifs et formes automorphes: applications du principe de fonctorialité". In: *Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988)*. Vol. 10. Perspect. Math. Academic Press, Boston, MA, pp. 77–159. MR: 1044819 (cit. on p. 291).

Robert Coleman and Barry Mazur (1998). "The eigencurve". In: *Galois representations in arithmetic algebraic geometry (Durham, 1996)*. Vol. 254. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, pp. 1–113. MR: 1696469 (cit. on pp. 292, 299).

Henri Darmon and Victor Rotger (2014). "Diagonal cycles and Euler systems I: A p-adic Gross-Zagier formula". *Ann. Sci. Éc. Norm. Supér. (4)* 47.4, pp. 779–832. MR: 3250064 (cit. on p. 307).

Pierre Deligne and Jean-Pierre Serre (1974). "Formes modulaires de poids 1". *Ann. Sci. École Norm. Sup. (4)* 7, 507–530 (1975). MR: 0379379 (cit. on pp. 309, 310).

Gerd Faltings and Ching-Li Chai (1990). *Degeneration of abelian varieties*. Vol. 22. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. With an appendix by David Mumford. Springer-Verlag, Berlin, pp. xii+316. MR: 1083353 (cit. on p. 295).

Laurent Fargues (2011). "La filtration canonique des points de torsion des groupes p-divisibles". *Ann. Sci. Éc. Norm. Supér. (4)* 44.6. With collaboration of Yichao Tian, pp. 905–961. MR: 2919687 (cit. on p. 296).

Jean-Marc Fontaine and Barry Mazur (1995). "Geometric Galois representations". In: *Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993)*. Ser. Number Theory, I. Int. Press, Cambridge, MA, pp. 41–78. MR: 1363495 (cit. on p. 291).

Wee Teck Gan and Shuichiro Takeda (2011). "The local Langlands conjecture for $\mathrm{GSp}(4)$". *Ann. of Math. (2)* 173.3, pp. 1841–1882. MR: 2800725 (cit. on p. 310).

Wushi Goldring (2014). "Galois representations associated to holomorphic limits of discrete series". *Compos. Math.* 150.2, pp. 191–228. MR: 3177267 (cit. on p. 309).

Wushi Goldring and Jean-Stefan Koskivirta (2017). "Strata Hasse invariants, Hecke algebras and Galois representations". Preprint (cit. on p. 309).

Harish-Chandra (1968). *Automorphic forms on semisimple Lie groups*. Notes by J. G. M. Mars. Lecture Notes in Mathematics, No. 62. Springer-Verlag, Berlin-New York, pp. x+138. MR: 0232893 (cit. on p. 291).

M. Harris (1990). "Automorphic forms and the cohomology of vector bundles on Shimura varieties". In: *Automorphic forms, Shimura varieties, and L-functions. Vol. II, Proceedings of the conference held at the University of Michigan, Ann Arbor, Michigan, July 6–16, 1988*. Ed. by Laurent Clozel and James S. Milne. Vol. 10. Perspectives in Mathematics. Academic Press, Inc., Boston, MA, pp. xii+437 (cit. on pp. 292, 308).

Michael Harris and Stephen S. Kudla (1991). "The central critical value of a triple product L-function". *Ann. of Math. (2)* 133.3, pp. 605–672. MR: 1109355 (cit. on p. 303).

Valentin Hernandez (2016). "La filtration canonique des O-modules p-divisibles". arXiv: 1611.07396 (cit. on p. 299).

Haruzo Hida (1986). "Iwasawa modules attached to congruences of cusp forms". *Ann. Sci. École Norm. Sup. (4)* 19.2, pp. 231–273. MR: 868300 (cit. on pp. 292, 311, 312).

– (2002). "Control theorems of coherent sheaves on Shimura varieties of PEL type". *J. Inst. Math. Jussieu* 1.1, pp. 1–76. MR: 1954939.

Atsushi Ichino (2008). "Trilinear forms and the central values of triple product L-functions". *Duke Math. J.* 145.2, pp. 281–307. MR: 2449948 (cit. on p. 303).

Christian Johansson and James Newton (Apr. 2016). "Extended eigenvarieties for overconvergent cohomology". arXiv: 1604.07739 (cit. on p. 302).

Owen T. R. Jones (2011). "An analogue of the BGG resolution for locally analytic principal series". *J. Number Theory* 131.9, pp. 1616–1640. MR: 2802138 (cit. on p. 294).

Payman L. Kassaei (2013). "Modularity lifting in parallel weight one". *J. Amer. Math. Soc.* 26.1, pp. 199–225. MR: 2983010 (cit. on pp. 299, 310).

Payman L. Kassaei, Shu Sasaki, and Yichao Tian (2014). "Modularity lifting results in parallel weight one and applications to the Artin conjecture: the tamely ramified case". *Forum Math. Sigma* 2, e18, 58. MR: 3294624 (cit. on p. 310).

Nicholas M. Katz (1973). "p-adic properties of modular schemes and modular forms", 69–190. Lecture Notes in Mathematics, Vol. 350. MR: 0447119 (cit. on pp. 292, 297).

Chandrashekhar Khare and Jean-Pierre Wintenberger (2009). "Serre's modularity conjecture. I". *Invent. Math.* 178.3, pp. 485–504. MR: 2551763 (cit. on p. 310).

Mark Kisin (2009). "Modularity of 2-adic Barsotti-Tate representations". *Invent. Math.* 178.3, pp. 587–634. MR: 2551765 (cit. on p. 310).

J.-P. Labesse and R. P. Langlands (1979). "L-indistinguishability for SL(2)". *Canad. J. Math.* 31.4, pp. 726–785. MR: 540902 (cit. on p. 311).

Kai-Wen Lan (2016). "Vanishing theorems for coherent automorphic cohomology". *Res. Math. Sci.* 3, Paper No. 39, 43. MR: 3571345 (cit. on p. 308).

Robert P. Langlands (1980). *Base change for* GL(2). Vol. 96. Annals of Mathematics Studies. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, pp. vii+237. MR: 574808 (cit. on p. 310).

Jian-Shu Li and Joachim Schwermer (2004). "On the Eisenstein cohomology of arithmetic groups". *Duke Math. J.* 123.1, pp. 141–169. MR: 2060025 (cit. on p. 308).

Ruochuan Liu, Daqing Wan, and Liang Xiao (2017). "The eigencurve over the boundary of weight space". *Duke Math. J.* 166.9, pp. 1739–1787. MR: 3662443 (cit. on p. 302).

Barry Mazur (1989). "Deforming Galois representations". In: *Galois groups over* **Q** *(Berkeley, CA, 1987)*. Vol. 16. Math. Sci. Res. Inst. Publ. Springer, New York, pp. 385–437. MR: 1012172 (cit. on p. 291).

Vincent Pilloni (2011). "Prolongement analytique sur les variétés de Siegel". *Duke Math. J.* 157.1, pp. 167–222. MR: 2783930 (cit. on p. 312).

– (2017). "Higher coherent cohomology and p-adic modular forms of singular weight" (cit. on p. 313).

Vincent Pilloni and Benoît Stroh (2016). "Surconvergence, ramification et modularité". *Astérisque* 382, pp. 195–266. MR: 3581178 (cit. on pp. 309, 310, 312).

Peter Scholze (2015). "On torsion in the cohomology of locally symmetric varieties". *Ann. of Math. (2)* 182.3, pp. 945–1066. MR: 3418533 (cit. on p. 298).

Jean-Pierre Serre (1970). "Facteurs locaux des fonctions zêta des varietés algébriques (définitions et conjectures)". In: *Séminaire Delange-Pisot-Poitou. 11e année: 1969/70. Théorie des nombres. Fasc. 1: Exposés 1 à 15; Fasc. 2: Exposés 16 à 24*. Secrétariat Math., Paris, p. 15. MR: 3618526 (cit. on p. 310).

Glenn Stevens (2000). *Families of modular forms with positive slopes, cours à l'Institut Henri Poincaré* (cit. on p. 293).

Richard Taylor (1991). "Galois representations associated to Siegel modular forms of low weight". *Duke Math. J.* 63.2, pp. 281–332. MR: 1115109 (cit. on p. 309).

Jerrold Tunnell (1981). "Artin's conjecture for representations of octahedral type". *Bull. Amer. Math. Soc. (N.S.)* 5.2, pp. 173–175. MR: 621884 (cit. on p. 310).

Eric Urban (2014). "Nearly overconvergent modular forms". In: *Iwasawa theory 2012*. Vol. 7. Contrib. Math. Comput. Sci. Springer, Heidelberg, pp. 401–441. MR: 3586822 (cit. on p. 303).

Thomas Crawford Watson (2002). *Rankin triple products and quantum chaos*. Thesis (Ph.D.)–Princeton University. ProQuest LLC, Ann Arbor, MI, p. 81. MR: 2703041 (cit. on p. 303).

Hiroyuki Yoshida (1984). "On Siegel modular forms obtained from theta series". *J. Reine Angew. Math.* 352, pp. 184–219. MR: 758701 (cit. on p. 310).

Received 2017-12-10.

Fabrizio Andreatta
fabrizio.andreatta@unimi.it

Adrian Iovita
adrian.iovita@concordia.ca

Vincent Pilloni
vincent.pilloni@ens-lyon.fr