On Hyperbolic Geometry Structure of Complex Networks

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Abstract
Various real world phenomena can be modeled by a notion called complex network. Much effort has been devoted into understanding and manipulating this notion. Recent research hints that complex networks have an underlying hyperbolic geometry that gives them navigability, a highly desirable property observed in many complex networks. In this internship, a parameter called $\delta$-hyperbolicity, which is related to the underlying hyperbolic geometry of a graph, is studied in contrast of navigability. Graph-theoretical and computational properties of $\delta$-hyperbolicity are studied and used to investigate $\delta$-hyperbolicity of several network models. A seemingly gap between $\delta$-hyperbolicity and navigability is found.

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1 Introduction

Social network, electrical grids and the Internet, complex networks are everywhere. They are not mathematically well-defined, but rather they share a set of unusual properties, including the small-world phenomenon, navigability, etc. Due to their importance, complex networks attract much research effort, especially in understanding the source of these peculiar properties. Many models are proposed to emulate these unusual properties. They are sometimes regarded as adequate substitutes of real-world networks in research.

Recently, there has been studies hinting that complex networks have an underlying geometry that might explain their navigability, one of their interesting properties. In this internship, we pursue this trait of research, try to understand the underlying geometry of complex networks and their effects. More precisely, a graph parameter called $\delta$-hyperbolicity, which is related to hyperbolic geometry, is studied in the context of complex networks.

This report is structured as follows. Further explanation of complex networks and $\delta$-hyperbolicity will be given in the rest of this section. In the next section, we will discuss some graph-theoretical properties of $\delta$-hyperbolicity, also its computation. In Section 3, $\delta$-hyperbolicity of several complex network models and real-world networks will be investigated in contrast of network navigability. Finally, we conclude by some further discussion about the role of $\delta$-hyperbolicity in complex network.

1.1 Complex network

Many important structures can be modeled as networks, e.g. social networks and biological metabolic networks. Studies on their underlying graphs have succeeded in capturing many of their behaviors.

While studying networks arisen from different contexts, researchers found that many networks share some common peculiar properties. Here a few are named.

1. Small-world phenomenon: diameter is extremely small comparing to size of the network, usually in polylogarithmic to the size.

2. Power law degree sequence: degree sequence of the network follows a power law. In this case, the network is also called scale-free, and it has a few “hubs” of very high degree, but most of nodes are of very low degree.

3. Navigability: using only local information, we can navigate from a node to any destination taking a relatively short path, comparing to diameter of the graph.

4. High clustering coefficient: nodes sharing a same neighbor are more likely to be connected than in random setting.

Each property plays a different role in complex networks. To replicate and study these properties, many graph models are proposed, including Barabási’s preferential attachment [1], Fan Chung’s power law graph model [8], Kleinberg’s lattice-based model [14], etc. More models and researches can be found in a survey by Boccaletti et al. [2].
In complex networks, like social networks and the Internet, where transportation or communication is a concern, navigability is a desired property. Small diameter does not automatically result in good navigability, expanders make a good counter-example. Finding a reasonably short path is more crucial than simply having a short path in this case.

The six-degree separation experiment by Milgram [20] is often considered as a pioneer work on complex network. Not only did it demonstrate that social network has small diameter, it also showed that people are capable of finding a short path in the social network using only local information and very restricted information about destination.

There are several models devoted to replicate this property. Kleinberg proposed in [14] a lattice-based directed graph model that has a good navigation property. It is suggested by Franceschetti and Meester [10] that navigability in Kleinberg’s model might result from scale invariance of connections.

In a more general context, it is suggested that an underlying hyperbolic geometry hints good navigability [3]. Kleinberg proposed an embedding of graphs into hyperbolic plane in [16] such that greedy routing always succeeds. A network model based on the Poincaré disk in [18] also recover some properties of complex network. In this internship, we would like to investigate this possible connection between navigability and the possible underlying hyperbolic geometry.

1.2 δ-hyperbolicity

Hyperbolic geometry rests in negative curvature, which is not well-defined on discrete structures like graphs. However, it has a generalization in the context of metric space.

A metric space \((X, d)\) is a set \(X\) of points with a distance function \(d : X^2 \rightarrow \mathbb{R}_+\) satisfying the following conditions for any points \(u, v, w \in X\):

1. (symmetry) \(d(u, v) = d(v, u)\),
2. \(d(u, v) = 0\) if and only if \(u = v\),
3. (triangle inequality) \(d(u, v) \leq d(u, w) + d(w, v)\).

In a metric space \((X, d)\), we define a geodesic segment \([u, v]\) between two points \(u, v\) to be the image of a function \(\rho : [0, d(u, v)] \rightarrow [u, v]\) satisfying \(\rho(0) = u\), \(\rho(d(u, v)) = v\), \(d(\rho(s), \rho(t)) = |s - t|\) for any \(s, t \in [0, d(u, v)]\). We say that a metric space is geodesic if every pair of its points has a geodesic segment, not necessarily unique.

In a metric space, it is sometimes convenient to consider distances between point sets in the following way. We say that a set \(S\) is within distance \(k\) to another set \(T\) if \(S\) is contained in the ball \(B(T, k)\) of all points within distance \(k\) to some certain point in \(T\). We say that \(S, T\) are within distance \(k\) to each other if \(S\) is within distance \(k\) to \(T\) and vice versa.

Simple undirected graph \(G = (V, E)\) is naturally a discrete metric space \((V, d)\), with distance \(d\) defined by distance on the graph. \((V, d)\) can be extended into a geodesic metric space with every edge interpreted as a segment of length 1. We will consider a graph \(G\) also as its induced metric space via notation abusing hereinafter. When only distance is concerned, we will consider its discrete metric.
space. In the case when geodesics are also involved, we will use the geodesic extension.

In [13], Gromov defined a notion of hyperbolic metric space. He then defined hyperbolic groups to be finitely generated groups with a Cayley graph that is hyperbolic. There are several equivalent definitions (up to a multiplicative constant) of Gromov’s hyperbolic metric space [4]. We would use the following in this report.

**Definition 1.1** (Four-point condition). In a metric space \((X, d)\), given \(u, v, w, x\) with \(d(u, v) + d(w, x) \geq d(u, x) + d(v, w) \geq d(u, w) + d(v, x)\) in \(X\), we note \(\delta(u, v, w, x) = d(u, v) + d(w, x) - d(u, x) - d(w, v)\). \((X, d)\) is called \(\delta\)-hyperbolic if for any four points \(u, v, w, x \in X\), \(\delta(u, v, w, x) \leq 2\delta\). We note \(\delta(X, d)\) the smallest possible value of such \(\delta\), if ever exists.

A similar notion of hyperbolicity is also proposed by Rips on geodesic metric space.

**Definition 1.2** (Rips condition). In a geodesic metric space \((X, d)\), given \(u, v, w\) in \(X\), we note \(\Delta(u, v, w) = [u, v] \cup [v, w] \cup [w, u]\) a geodesic triangle. \([u, v], [v, w], [w, u]\) are called sides of \(\Delta(u, v, w)\).

A geodesic triangle is called \(\delta\)-slim if any point on a side is within distance \(\delta\) to the union of other two sides. \((X, d)\) is called Rips \(\delta\)-hyperbolic if every geodesic triangle is \(\delta\)-slim. We note \(\delta_{\text{Rips}}(X, d)\) the smallest possible value of such \(\delta\), if ever exists.

According to [12], \(\delta(X, d)\) and \(\delta_{\text{Rips}}(X, d)\) differ only within a multiplicative constant. Since we are only concerned by asymptotic growth of \(\delta(X, d)\) rather than exact value, we shall sometimes use Rips condition for Gromov’s \(\delta\)-hyperbolicity.

On continuous metric spaces, \(\delta\)-hyperbolicity is a measure of negative curvature. Euclidean plane is not hyperbolic, but Poincaré disk is constantly hyperbolic. On graphs, \(\delta\)-hyperbolicity is somewhat a “tree-like” measure. Trees and block graphs are \(0\)-hyperbolic. In contrast, cycles \(C_n\) are \(O(n)\)-hyperbolic. \(k\)-chordal graphs are at least \(\lfloor k/2 \rfloor/2\)-hyperbolic [23].

Although originated from geometric group theory, \(\delta\)-hyperbolicity is proved to be useful in algorithmic. When \(\delta(G)\) is bounded, some problems can be solved more efficiently. Examples include linear time diameter approximation, exact center computing and construction of distance approximating tree in a graph[6], distance labeling [11], approximate ball packing and covering [7], PTAS for TSP [17], etc.

Negatively curved spaces also are hinted to be behind complex networks. In [21] it is showed that empirically the Internet embeds better into a hyperbolic space than into an Euclidean space. We have also mentioned the possible link between navigability and hyperbolic geometry of some complex networks at the end of the previous subsection. Since \(\delta\)-hyperbolicity reflects hyperbolic geometry of graphs, we would like to study its effects on behavior of complex networks, especially navigability.

## 2 Properties and computation of \(\delta\)-hyperbolicity

Before investigating \(\delta\)-hyperbolicity and its effects on both complex network models and real-world networks, we want to first study \(\delta\)-hyperbolicity itself.
Lemma 2.1. For any graph studied in a graph-theoretical aspect.

Though rooted in the theory of metric space, δ-hyperbolicity can also be considered as a graph property in the context of graph theory. Therefore, it can be studied in a graph-theoretical aspect.

The diameter \( \text{diam}(G) \) of a graph \( G \) gives an upper bound of \( \delta(G) \).

Lemma 2.1. For any graph \( G \), \( \delta(G) \leq \text{diam}(G)/2 \).

Proof. There exists \( u, v, w, x \in X \) with \( d(u, v) + d(w, x) \geq d(u, x) + d(w, v) \geq d(u, w) + d(v, x) \) such that \( 2\delta(G) = d(u, v) + d(w, x) - d(u, x) - d(w, v) \).

Moreover, \( 2\delta(G) \leq d(u, v) + d(w, x) - (d(u, x) + d(x, v) + d(u, w) + d(w, x))/2 \leq 2 \min(d(u, v), d(w, x)) \leq 2\text{diam}(G) \) and we conclude.

Given this lemma, we should be more interested in graphs with \( \delta(G) \) significantly smaller than \( \text{diam}(G)/2 \). In the case of graph families, we are more interested in families such that \( \delta(G) = o(\text{diam}(G)) \) when the size of graph grows to infinity. We say that the graph family is of good hyperbolicity in this case. This concern leads me to find out the following construction of δ-hyperbolic graphs.

Definition 2.1 (Glue sum of graphs). Let \( G_1 = (V_1, E_1) \), \( G_2 = (V_2, E_2) \), \ldots, \( G_k = (V_k, E_k) \) be graphs such that, there is a metric space \( (X, d) \) isometric to some \( V_i \subseteq V_i \) for all \( G_i \). We note the glue sum \( (G_1 \oplus \ldots \oplus G_k)|_X \) the graph formed by identifying all \( V_i \) with \( X \) according to the isometries. Each \( G_i \) is called a component of \( G \).

It is easy to see that distance functions are preserved in each component of a glue sum graph. I proposed the following lemma about hyperbolicity of glue sum graphs.

Lemma 2.2. We define \( \delta_{\text{Rips}}^X(G) \) be the maximum of \( \delta_{\text{Rips}}(G \cup \{u, v\}) \) for any \( u, v \in X \) where \( \{u, v\} \) is a segment of length \( d(u, v) \) attached between \( u, v \).

Let \( G = (G_1 \oplus \ldots \oplus G_k)|_X \). We have the following:

\[
\delta_{\text{Rips}}^X(G) \leq \max_{1 \leq i \leq k} (\delta_{\text{Rips}}^X(G_i)) + \text{diam}(X).
\]

Proof. Firstly, \( \delta_{\text{Rips}}^X(G) \leq \delta_{\text{Rips}}^X(G_i) \) because the attached segment does not affect distances between other points.

Let \( u, v, w \) be points in \( G \). We want to calculate how slim \( \Delta(u, v, w) \) is.

If \( u, v, w \) are all in a certain component \( G_i \), \( \Delta(u, v, w) \) is \( \delta_{\text{Rips}}^X(G_i) \)-slim.

If \( u, v, w \) sit in three different components, then there exists \( x \in [u, v] \cap X, y \in [u, w] \cap X, z \in [u, w] \cap X \). \( x, u, x, z \) can be considered in the same component \( G_i \), therefore \( [u, x] \subseteq B([u, z] \cup [z, x], \delta_{\text{Rips}}^X(G_i)) \subseteq B([u, z], \delta_{\text{Rips}}^X(G_i) + \text{diam}(X)) \) because \( x, z \in X \). Idem \( [v, x] \subseteq B([v, y], \delta_{\text{Rips}}^X(G_j) + \text{diam}(X)) \) for a certain \( G_j \). By cyclic rotation, we conclude that \( \Delta(u, v, w) = \max_{1 \leq i \leq k} (\delta_{\text{Rips}}^X(G_i)) + \text{diam}(X) \)-slim.
If $u, v$ sit in a component $G_i$ and $w$ in another $G_j$, there exists $x \in [u, w] \cap X$, $y \in [v, w] \cap X$. We know from the previous case $[w, x]$ and $[w, y]$ are within distance $\tilde{d}^X_{\text{Rips}}(G_i) + \text{diam}(X)$ to each other. Let $z \in \langle x, y \rangle$ be a point such that $d(w, x) = d(z, x) = d(z, y)$. This is always possible due to triangle inequality. In this case, $[z, x] \cup [x, u]$ and $[z, y] \cup [y, v]$ are geodesics between $z, u$ and $z, v$ respectively. As $\Delta(z, u, v) = \delta_{\text{Rips}}^X(G_i)$-slim, we conclude that $\Delta(u, v, w)$ is $\max_{1 \leq i \leq k}(\delta_{\text{Rips}}^X(G_i)) + \text{diam}(X)$-slim.

We can conclude that $\delta_{\text{Rips}}(G) \leq \max_{1 \leq i \leq k}(\delta_{\text{Rips}}^X(G_i)) + \text{diam}(X)$. 

The case where $X$ is a singleton is particularly useful.

**Corollary 2.1.** If $G = (G_1 \oplus \ldots \oplus G_k)|_X$ with $X$ a singleton, we have:

$$\delta_{\text{Rips}}(G) \leq \max_{1 \leq i \leq k}(\delta_{\text{Rips}}^X(G_i)).$$

**Proof.** We apply directly Lemma 2.2 by noticing that, when $X$ is a singleton, $\text{diam}(X) = 0$, and $\delta_{\text{Rips}}^X(G_i) = \delta_{\text{Rips}}(G_i)$. 

### 2.2 Computational aspect of $\delta$-hyperbolicity

To study $\delta$-hyperbolicity of real-world networks, we need to do fast computation of $\delta$-hyperbolicity on these graphs. Given a general graph $G$, we want to compute $\delta(G)$.

Direct application of four-point condition leads to a naïve algorithm of time complexity $\Theta(2^n)$, where $n$ is the number of nodes. This is impractical for real-world networks containing millions of nodes. However, it seems that no faster algorithm is known. The following lemma suggests that there may not be algorithm computing $\delta(G)$ with near-linear runtime.

**Lemma 2.3.** In the setting of weighted graph where weight is interpreted as distance, computation of diameter can be reduced in linear time to computation of $\delta$-hyperbolicity in RAM model, with a growth of problem size at most linear.

**Proof.** As connectivity can be checked in linear time in RAM model, we suppose that graphs are connected.

Let $G$ be the graph whose diameter is to be computed, and let $n$ be its number of nodes. We construct a graph $G'$ by adding to $G$ two new nodes $u, v$ and edges of weight (therefore distance) $n$ from all nodes in $G$ to $u$ and $v$. As $\text{diam}(G') \leq n$, this does not affect distances on $G$.

Consider $\delta(G')$. We check all possibilities in four-point condition. Given $a, b, c, d$ in $G'$, if none of them is $u$ or $v$, $\delta(a, b, c, d) \leq 2\delta(G) \leq \text{diam}(G)$. If only one of them is $u$ or $v$, $\delta(a, b, c, u) \leq \max(d(a, b), d(b, c), d(c, a)) \leq \text{diam}(G)$. The last possibility is that $u, v$ are both among these four points.

For any two points $a, b$ in $G$, $\delta(a, b, u, v) = d(u, v) + d(a, b) - d(u, a) - d(v, b) = d(a, b)$. Therefore, $\delta(G') = \text{diam}(G')/2$. This reduction can be done in linear time, and the growth of problem size is at most linear.

Therefore, computing $\delta(G)$ seems to be more difficult than computing $\text{diam}(G)$, at least in the setting of weighted graph, but we can also expect the same situation in unweighted graphs. The best result in computing $\text{diam}(G)$ for unweighted graphs is in $O(n^{2.376})$ using fast matrix multiplication for $m =$
We should probably not expect to compute $\delta(G)$ of unweighted graphs faster than that, though we cannot prove this to be theoretically impossible.

Given this speculation, we would rather look for heuristics to simplify and accelerate the computation.

We can reduce the size of $G$ to check by the following lemma.

**Lemma 2.4.** Given $G = (V,E)$ and $u \in V$ of degree 1, $\delta(G) = \delta(G - \{u\})$.

**Proof.** Let $v$ be the only neighbor of $u$. The lemma results directly from the fact that, for any $w \neq u$, $d(w,u) = d(w,v) + 1$.

By repeatedly applying this lemma, we can reduce size of $G$ while keeping $\delta(G)$ invariant. We can also apply Corollary 2.1 on articulation points to split $G$ into smaller pieces to reduce computation time.

We can also accelerate the computation of $\delta(G)$ by checking fewer 4-tuples for four-point condition. The following lemma states that, for four-point condition, not all 4-tuples are necessary to be checked.

**Lemma 2.5.** For any $u,v,w,x$ in a graph $G$ with $d(u,v) + d(w,x) \geq d(u,x) + d(w,v)$, if there exists $u'$ neighbor of $u$ such that $d(u',v) = d(u,v) + 1$, we have $\delta(u,v,w,x) \leq \delta(u',v,w,x)$.

**Proof.** This lemma results from the fact that, for $u'$ neighbor of $u$, $d(u,x) \leq d(u,v) + 1$.

We say that a pair of nodes $u,v$ is **locally far apart** if no neighbor of $u$ is farther to $v$ than $u$, and vice versa. According to the lemma, when computing $\delta(G)$ using the four-point condition, we only need to check $u,v,w,x$ such that the condition of lemma do not apply. That is, $u,v,w,x$ should consist of two locally far apart pairs. In some cases this can reduce computation time empirically by an order of magnitude.

3 $\delta$-hyperbolicity and navigability of different networks

In large complex networks, there is always a small core formed by relatively few high degree nodes called hubs. The core itself is very connected, and with all the hubs, nodes outside the core can reach the core within relatively few steps. Intuitively, a shortest path between any two nodes should have a good probability of passing by the core, which is similar to the case of a hyperbolic metric space. In the case of navigation, it seems favorable to first find a hub, then go to the destination via the hub, which also looks similar to the case of a hyperbolic metric space.

Therefore, it will be interesting to find out whether these similarities are fundamental. We would like to use $\delta$-hyperbolicity for “hyperbolic-like” measure of networks.

We will now investigate $\delta$-hyperbolicity of various network models and try to find its relation with navigability.
3.1 Kleinberg’s model

Kleinberg’s model [14] has been cited in introduction. This model reflects geographic limit and navigability of social networks. In this model, \( n \) nodes form a square lattice on a plane, neighboring nodes are connected. Moreover, each node is permitted with one long jump onto a random node chosen with probability proportional to \( r^{-\alpha} \), where \( r \) is the distance between these two nodes on the lattice and \( \alpha \) a parameter. This model can be extended to square lattice of arbitrary dimension.

Kleinberg proved that, when based on a two-dimensional lattice, with a greedy routing scheme using only local information, we can reach from one node to any other passing by \( O(\log^2 n) \) edges only when \( \alpha = 2 \). More generally, when based on a \( d \)-dimensional lattice, it is shown in [19] that the above result holds only for \( \alpha = d \). Hence, this model has good navigability on a sweet spot \( \alpha = d \).

Guangda Hu in our team proved the following theorem about \( \delta \)-hyperbolicity of Kleinberg’s model. We note \( K(n, d, \alpha) \) the graph of Kleinberg’s model based on \( d \)-dimensional lattice with parameter \( \alpha \).

**Theorem 3.1.** For \( d \) fixed, with high probability with respect to \( n \),

\[
\forall \varepsilon > 0, \delta(K(n, d, \alpha)) \geq (\log n)^{1/((1.5(d+1))+\varepsilon)}.
\]

\[
\delta(K(n, d, \alpha)) = \Omega(\log n) \text{ when } 0 \leq \alpha < d.
\]

This is done by calculating the probability of short-cuts (path following links not on lattice) appearing inside a small square. For a square with suitable size, the probability of having short-cuts other than paths on the lattice between four corners is vanishing. Using the four-point condition, we obtain this lower bound.

However, the diameter of Kleinberg’s model in these cases is also small. In [19], it is shown that \( \text{diam}(K(n, d, d)) = \Theta(\log n) \). We cannot conclude that \( \delta \)-hyperbolicity plays no role in navigability of this model, but it hints that we do not need extremely good hyperbolicity to achieve good navigability.

3.2 Fan Chung’s power-law graph

Many complex networks have a power-law degree sequence, meaning that the fraction of nodes having a certain degree \( d \) is proportional to \( d^{-\beta} \), where \( \beta \) is called the exponent. Fan Chung’s model is a random graph model that emulates this property.

In Fan Chung’s model [8], given number of nodes \( n \), expected average degree \( d \), expected maximum degree \( m \) and exponent \( \beta \), we number \( n \) nodes from \( i_0 \) to \( n + i_0 - 1 \), where \( i_0 = n(d^\beta - 2)/m(\beta - 1) \). A node numbered \( i \) is given an expected degree \( w_i = c_i^{1/(\beta - 1)} \), where \( c = 2^{-2/\beta}dn^{1/(\beta - 1)} \). Two nodes numbered \( j \) and \( k \) are connected by an edge with probability \( w_jw_k/\sum w_i \). When \( n \to \infty \), this model gives random graphs with a power-law degree sequence of the expected exponent \( \beta \), with all other expected characteristics. We note it \( F(n, \beta, m, d) \).

Fan Chung’s model shows a very interesting graph structure. In [8], it is proved that, inside the giant connected component, with high probability, there is a core of size about \( n^{1/\log \log n} \) of diameter \( O(\log \log n) \), almost all nodes
are within distance $O(\log \log n)$ to the core, and the diameter of the graph is
$O(\log n)$, with conditions $d > 1$, $\log m \gg \log n / \log \log n$, $2 < \beta < 3$.

This is very similar to some behaviors of complex networks. The core corre-
sponds to the set of hubs, most nodes can reach a hub in a few hops, and the
diameter is in logarithm of the size. Therefore, we consider it to be an adequate
model for complex networks.

It seems to be difficult to deduce mathematically the $\delta$-hyperbolicity of this
model, due to the instability of metric with respect to adding or deleting an
edge. Therefore, we try to compute $\delta(G)$ for Fan Chung’s model to see its
variation related to size of graph.

In Figure 1 is the result. I sampled $G = F(n, 2.5, 0.1n, 3)$, with $n$
varying from 1000 to 10000 step 1000. Five graphs are sampled for each $n$. Heuristics
in Section 2.2 are used to accelerate the computation. On average, Lemma
2.4 trims half of the nodes. However, Corollary 2.1 does not provide good
acceleration as there are few articulation points. Though accelerated, 10000 is
about the limit of size such that $\delta(G)$ can be computed in reasonable time, and
such a graph would take about 3 days.

From Figure 1, we observe that $\delta(G)$ grows very slowly with $n$, which is
expected as $\text{diam}(G) = \Theta(\log n)$ gives an upper bound by Lemma 2.1. However,
we cannot confirm whether Fan Chung’s model has good $\delta$-hyperbolicity, i.e.
whether $\delta(G) = o(\log n)$. On the plus side, we can observe that there is a
significant gap between diameter of graph, even when trimmed with Lemma 2.4,
and $2\delta(G)$, thus we can at least say that $\delta(G)$ is clearly smaller than $\text{diam}(G)/2$,
and $\delta$-hyperbolicity is not bad in this sense.

### 3.3 Ringed tree

To emulate the hyperbolic geometry behavior of the Poincaré disk, we proposed
a new graph model called ringed tree. It can be seen as a discretization of the
Poincaré disk.
3.3.1 A deterministic model

Starting with a complete binary tree with \( k \) layers, we layout the tree on a plane and connect nodes on the same layer with a single ring. Rings are continuous in the sense that neighbors on the ring have the same parent or neighboring parents on the upper ring. We call this structure ringed tree, and noted \( \text{Rt}(k) \). Figure 2 provides an example.

A fast estimation of circumstance of a circle centered at 0 gives the fact that circumstance of a circle grows roughly exponentially with respect to radius, which means we can embed a ringed tree into the Poincaré disk with low distortion. This justifies the fact that ringed tree is a discretization of the Poincaré disk.

Let \( n \) be the number of nodes in \( \text{Rt}(k) \), it is clear that \( \text{diam}(\text{Rt}(k)) = \Theta(\log n) \). We can also see that \( \text{Rt}(k) \) is navigable by its tree structure.

Before analysing hyperbolicity of ringed tree, we shall first study properties of its geodesics. We define the ring distance \( \text{rd}(u,v) \) of \( u,v \) on the same layer to be their distance on the ring. Ringed trees have the following fundamental property.

**Lemma 3.1.** For \( u,v \) on the same layer and \( u', v' \) be their parents respectively, we have \( \text{rd}(u',v') \leq (\text{rd}(u,v) + 1)/2 \).

**Proof.** On the ring, there is \( \text{rd}(u,v)+1 \) nodes on segment between \( u,v \), belonging to at most \( (\text{rd}(u,v) + 1)/2 + 1 \) parents and correspond to at most \( (\text{rd}(u,v) + 1)/2 + 1 \) nodes on ring segment between \( u',v' \), which concludes the proof.

We number layers from root to leaves by 0, \ldots, \( k - 1 \). For a geodesic, we call its layer sequence the sequence of layers it passes by. Lemma 3.1 indicates that the layer sequence of any geodesic must be reversed unimodal: first decreasing, then increasing.

The following lemma further characterizes geodesics in ringed-trees.
Lemma 3.2. Let $u, v$ be two nodes, $u$ on layer $\ell$. Suppose $[u, v]$ intersects layer $\ell - 1$, and let $t$ be the intersection closest to $u$. Then $d(u, t) \leq 2$, the segment $[u, t]$ of $[u, v]$ and $\{u, u'\}$ are within distance 2 to each other, where $u'$ is the parent of $u$.

Proof. Let $t'$ be the node just before $t$ to $u$ on $[t, u]$. $d(t, u) \leq 1 + rd(t, u') \leq 1 + (rd(t', u) + 1)/2$ by Lemma 3.1. As $t$ is the closest node on layer $\ell - 1$ on the geodesic to $u$, $d(t, u) \geq 1 + rd(t', u)$. We get $d(t, u) \leq 2$ by combining these two inequalities. The segment $[u, t]$ of $[u, v]$ and $\{u, u'\}$ are within distance 2 to each other since $d(t, u) \leq 2$.

Therefore, for a geodesic that stays on the same layer for a few moves before going up towards the root, we can flip this portion to make it goes up first. In the notation of Lemma 3.2, we can replace $[u, t]$ by $\{u, u'\} \cup [u', t]_r$ in $[u, v]$, where $[u', t]_r$ is the shortest path on the ring. This flipping construction leads to the following lemma.

For two nodes $u, v$, we note $\langle u, v \rangle$ the canonical geodesic from $u$ to $v$, which first goes up, then moves 2 or 3 steps on the ring, then goes down, or which has distance 1. We will show in the next lemma that any geodesic on a ringed tree are not far from the canonical geodesic.

Lemma 3.3. For any geodesic $[u, v]$, $[u, v]$ and $\langle u, v \rangle$ are within distance 2 to each other.

Proof. By flipping, we can rectify the geodesic to the shape of straight up, a few moves on the ring, then straight down. Each portion of $[u, v]$ on different layers is within distance 2 to the rectified one by Lemma 3.2. We can see that a geodesic can only make at most 5 moves on the same layer, with two endpoints in $\langle u, v \rangle$. By case analysis we conclude the proof.

Now we consider geodesic triangles formed by canonical geodesics in the following lemma.

Lemma 3.4. A geodesic triangle $\Delta(u, v, w)$ formed by canonical geodesics is $3$-thin on any $Rt(k)$.

Proof. We proceed by induction on the number of layers. A ringed tree of 1 layer is reduced to a point. It satisfies the induction hypothesis. Suppose that, for any $k < k_0$, the ringed tree of $k$ layers satisfies the induction hypothesis. We now consider the ringed tree with $k_0$ layers.

For any three points $u, v, w$, with loss of symmetry, we can assume that $u$ is in layer $k_0 - 1$. Or else we can conclude by induction hypothesis. If $\langle u, v \rangle, \langle u, w \rangle$ all have their first step going to parent of $u$, then by induction hypothesis we conclude by replacing $u$ by its parent.

Now we suppose that $\langle u, v \rangle$ leaves $u$ by a move on the ring. As $u$ is the outermost layer, by the definition of canonical geodesics, $v$ is on the same layer with $u$, and they are of distance at most 3. Distance between ancestors of $u, v$ on the same layer is at most 1. Therefore, in any cases, this geodesic triangle is 3-thin, and we conclude by recurrence.

This leads to the following theorem on $\delta$-hyperbolicity of ringed trees.

Theorem 3.2. $Rt(k)$ is constantly hyperbolic for any $k$. 

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Proof. We use Rips condition here. For any \( u, v, w \), \( \langle u, v \rangle \subseteq B(\langle u, w \rangle \cup \langle v, w \rangle, 3) \) by Lemma 3.4. \( [u, v] \subseteq B([u, w] \cup [v, w], 2) \) ⊆ \( B([u, w] \cup [v, w], 5) \). From Lemma 3.3 we also have \( \langle u, w \rangle \cup \langle v, w \rangle \subseteq B([u, w] \cup [v, w], 2) \). We then conclude by \( [u, v] \subseteq B([u, w] \cup [v, w], 7) \), and \( \delta_{\text{Rips}}(Rt(k)) = 7 \) is constant.

We see that ringed trees are of good \( \delta \)-hyperbolicity. This result was proved collectively. This simplified version is due to myself. We can see that this proof is valid even when we allow more general trees, for example trees with all internal nodes having bounded but at least 2 children and with all leaves of the same depth. The whole analysis still applies.

Using the same technique, with little modification, we can also prove that regular tessellations on the Poincaré Disk (see Figure 2) is constantly hyperbolic. This result is to be expected, as tessellation is a kind of discretization, and the Poincaré disk is itself constantly hyperbolic as a metric space.

We should also note that, unlike ringed tree, tessellation graphs are homogeneous. There is no distinguished point. This suggests that concentration like hubs and cores is not a necessary ingredient in good \( \delta \)-hyperbolicity.

3.3.2 A variant with limited jumps

We want to incorporate randomness into ringed trees, since general complex networks often contain some randomness. I proposed and analysed the following variant.

With a ringed tree of \( n \) nodes, we allow extra edges to be formed between nodes on the same layer, but only when the ring distance of these two nodes are lower than a certain bound. More precisely, for \( u, v \) on the same layer, we allow an extra edge (called a jump) only when \( \text{rd}(u, v) \leq f(n) \), where \( f \) is a certain function. We note \( Rt(k, f(n)) \) the class of graphs satisfying this condition.

We notice that a geodesic cannot pass too many jumps on the same layer.

Lemma 3.5. For a graph in the class \( Rt(k, f(n)) \), for \( u, v \) on the same layer, if \( [u, v] \) never go inside, then \( d(u, v) \leq \max(16, 4 \log_2 f(n)) \).

Proof. \( \text{rd}(u, v)/f(n) \leq d(u, v) \leq 2 \log_2 \text{rd}(u, v) \). Let \( k = d(u, v) \), we have \( k \leq 2 \log_2 kf(n) \) by going along the tree, \( f(n) \geq 2^{k/2}/k \geq 2^{k/4} \) for \( k \geq 16 \), thus \( k = d(u, v) \leq \max(16, 4 \log_2 f(n)) \).

This results in a similar result as Lemma 3.3.

Corollary 3.1. For a graph in the class \( Rt(k, f(n)) \), for two nodes \( u, v \), let \( (u, v) \) be a path linking them on the tree. Any geodesic \( [u, v] \) and \( (u, v) \) are within distance \( 2 \max(16, 4 \log_2 f(n)) \) to each other.

Proof. This can be done with Lemma 3.1 by induction on \( \max(d(u, t), d(v, t)) \), where \( t \) is the lowest common ancestor of \( u, v \).

Therefore we have the following result on the \( \delta \)-hyperbolicity of \( Rt(k, f(n)) \). The proof is nearly the same as that of Theorem 3.2.

Theorem 3.3. \( \delta(Rt(k, f(n))) = O(\log f(n)) \) for any positive monotone \( f \).
We can see that this theorem applies whenever jumps on rings are limited by a certain function $f$. We can set up a random graph family that has a hard limit on jumps, which makes it to be surely in some class $Rt(k, f(n))$, or a distribution that makes it to have jumps violating a certain limit with vanishing probability, which makes it to be in some class $Rt(k, f(n))$ with high probability. In both cases, we obtain a random graph family that is of $O(\log f(n))$-hyperbolic, with certainty or with high probability.

With this theorem, we can construct random graphs with good hyperbolicity. For example, $Rt(k, \log n)$ is $O(\log \log n)$-hyperbolic. We also note that $\text{diam}(Rt(k, f(n))) = k = \Theta(\log n)$. By Lemma 2.1, for any $\epsilon > 0$, any $f(n) = \Omega(n^{\epsilon})$, $\delta(Rt(k, f(n))) = \Theta(\text{diam}(Rt(k, f(n))))$. We cannot prove good $\delta$-hyperbolicity in this case. But for any positive monotone $f$, $Rt(k, f(n))$ is always navigable, due to its underlying tree structure. This suggests a gap between $\delta$-hyperbolicity and navigability.

### 3.3.3 A variant inspired by Kleinberg’s model

We have mentioned Kleinberg’s model [14] and its good navigability. Inspired by this model, I proposed and analysed the following variant of ringed tree, which is a ringed tree enhanced with random jumps according to the distribution of Kleinberg’s model.

Starting from a ringed tree with $k$ layers, we consider the set $V$ of $(n+1)/2 = 2^k - 1$ leaves. For each node $u \in V$, we pick another node $v \in V$ and link them with an edge with probability $rd(u,v)^{-\alpha} \rho^{-1}$, where $\rho = \sum_{v \in V-u} rd(u,v)^{-\alpha}$. We pick an edge from each node. This totally counts for $(n+1)/2$ edges. We will note such a random graph $Rt(k, \alpha)$.

Before analysing $\delta(Rt(k, \alpha))$, we first look at a lemma about this wiring.

**Lemma 3.6.** If we add an edge between $u, v$ on the outermost ring of a ringed tree with $rd(u,v) \geq cn^\gamma$, then the resulted graph $G$ (possibly with other edges on the outermost ring) is at best $\Omega(\log n)$-hyperbolic.

**Proof.** Let $w$ be the lowest common ancestor of $u, v$. By Lemma 3.1, $d(w,u) = d(w,v) \geq (\gamma \log_2 n + \log_2 c)/\log_2 3$. We consider the midpoint $x$ of $[u, w]$. For any point $y$ in $[v, w]$, by considering the path $(x, y)$ in the context of Lemma 3.3, we know that $d(x, y) \geq (\gamma \log_2 n + \log_2 c)/2 \log_2 3 - 3$. Therefore $\Delta(x, v, w)$ is at most $(\gamma \log_2 n + \log_2 c)/2 (\log_2 3 - 3)$-thin, $\delta_{Rips}(G) = \Omega(\log n)$. □

A probabilistic version comes naturally as the following corollary.

**Corollary 3.2.** For a random graph $G$ formed by linking edges on leaves of a ringed tree and $0 < \gamma < 1$, if with high probability there exists an edge linking some $u, v$ with $rd(u,v) = \Theta(n^\gamma)$, then with high probability $\delta(G) = \Theta(\log n)$.

**Proof.** Diameter of ringed tree gives $O(\log n)$ upper bound. Lemma 3.6 gives $\Omega(\log n)$ lower bound. □

We can now estimate the hyperbolicity of $Rt(k, \alpha)$.

**Theorem 3.4.** $\delta(Rt(k, \alpha)) = \Theta(\log n)$ for any $k, \alpha > 0$. 

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Proof. For some fixed node \( u \), the probability that the \( v \) we picked has \( rd(u, v) \leq (n/2)^\gamma \) (we say that it is good) is \( P = 2\rho^{-1} \sum_{d=1}^{(n/2)^\gamma} d^{-\alpha} \).

For \( 0 < \alpha < 1 \), \( P = O(n^{(1-\alpha)/2}) = o(1) \). For \( \alpha = 1 \), \( P = \gamma + o(1) \). In these two cases, all edges are good with probability at most \((\gamma + o(1))^{(n/2)^\gamma} = o(1)\).

For \( \alpha > 1, \rho = o(1) \). We take \( Q = 1 - P \). By simple calculation, \( Q = O(n^{(1-\alpha)}) \). By picking \( \gamma = 1/2(\alpha - 1) \), we have \( Q = O(n^{-1/2}) \). All edges are good with probability \((1 - Q)^{n/2} = O(e^{-\gamma}) = o(1)\).

In all three cases, by Corollary 3.2, with high probability \( \delta(Rt(k,\alpha)) = \Theta(\log n) \).

Therefore, this random graph model is not of good hyperbolicity. This phenomenon occurs largely because the distribution we use has heavy tail. If we use some fast-decreasing weight function, such as \( e^{-rd(u,v)} \), we may have good \( \delta \)-hyperbolicity in finite parameter using Theorem 3.3. This is showed in the following theorem.

**Theorem 3.5.** Consider \( Rt'(k,\alpha) \) be the graph model \( Rt(k,\alpha) \) with probability of linking edges be \( e^{-rd(u,v)\alpha}/\rho \), where \( \rho = \sum_{v \in V\setminus u} e^{-rd(u,v)\alpha} \).

We have \( \delta(Rt'(k,\alpha)) = O(\log\log n) \), for any \( \alpha > 0 \).

*Proof.* \( e^\alpha \leq \rho = \sum_{v \in V\setminus u} e^{-rd(u,v)\alpha} \leq 2 \sum_{i=1}^{\infty} e^{i\alpha} \). Therefore, \( \rho = \Theta(1) \). A node \( u \) has a jump of distance greater than \( k \) with probability \( \Theta(e^{-k\alpha})/\rho \). Let \( k = \frac{2}{\alpha} \log n \), we know that a node has a jump of distance greater than \( \frac{2}{\alpha} \log n \) with probability \( \Theta(1/n^k) = o(1/n) \).

Therefore, with high probability, jumps never exceed ring distance \( \frac{2}{\alpha} \log n \). From Theorem 3.3 follows \( \delta(Rt'(k,\alpha)) = O(\log\log n) \), for any \( \alpha > 0 \). 

Another Kleinberg’s model [15], which is also navigable, provides another type of probability of choosing edges \( u,v \). Instead of being proportional to \( rd(u,v)^{-\alpha} \), we can choose it to be proportional to \( 2^{-h(u,v)} \), where \( h(u,v) \) is the distance from \( u,v \) to their lowest common ancestor. In this case, however, we also have bad hyperbolicity. It is enough to observe that, if \( u,v \) have their lowest common ancestor within distance \( \gamma \log n \), then with high probability \( rd(u,v) > n^{\gamma/2} \), and in this case we can transfer the previous calculation in Theorem 3.4.

Despite bad \( \delta \)-hyperbolicity of this variant, it is still of good navigability, due to the underlying tree. We can see that there is a discrepancy between good hyperbolicity and good navigation property.

### 3.4 Real-world networks

Actual real-world networks are too large and out of the scope of our computational power. Here we cite a result of de Montgolfier, Soto and Viennot [9]. They computed the \( \delta \)-hyperbolicity and related distribution of Internet AS network, and showed that they are of very good hyperbolicity \((\delta = 2)\). The Internet is of course navigable.

### 4 Conclusion

Recent results show that complex networks have an underlying hyperbolic geometry structure, measurable by \( \delta \)-hyperbolicity. It is also hinted that this
underlying hyperbolic geometry is responsible for good navigation properties of some complex networks. Motivated by these results, in this internship, we tried to analyse $\delta$-hyperbolicity of different complex network models and real-world networks.

We first investigated some properties of $\delta$-hyperbolicity in the context of graph theory, both for understanding the structure of $\delta$-hyperbolic graphs and for computation of $\delta(G)$. Though elementary, such studies are rarely reported in literature.

With these tools, we tried to analyse $\delta$-hyperbolicity of different complex network models and real-world networks, and its relation with navigability of complex networks. Size of real-world networks is well beyond the scope of our computational method, and many random network models resist precise analysis. The difficulty of analysing $\delta$-hyperbolicity for many random graph models is that $\delta$ is not stable facing addition or deletion of edges, but in random graphs we cannot effectively distinguish cases whether a certain edge exists.

However, we did obtain some results concerning $\delta$-hyperbolicity and navigability of some complex network models. We showed mathematically an lower bound of $\delta(G)$ on Kleinberg’s model, which is a model with good navigability. We showed empirically that Fan Chung’s power-law graph model, a model that exhibits many interesting phenomena in real-world complex networks, has a gap between $\delta(G)$ and $\text{diam}(G)/2$, given the bound of Lemma 2.1, though we cannot determine yet whether Fan Chung’s model is of good hyperbolicity. We know nothing about its navigability neither.

We then proposed a graph model called ringed tree, and proved it to be constantly hyperbolic. Two variants of this ringed tree model were then proposed and analysed by myself, one signaling that we can keep good $\delta$-hyperbolicity while introducing limited randomness, another signaling that with certain background structure, bad $\delta$-hyperbolicity can still lead to good navigability and there is a discrepancy between these two notions.

We found the notion of $\delta$-hyperbolicity too strict for navigation in complex networks. It is related to the metric, which demands all paths to be the shortest. But in the case of navigation, we can relax on this restriction and be satisfied by a path not the shortest, but short enough comparing to the shortest. This requires us to find the underlying structure making complex networks to be navigable. It is hinted in [15] and [22] that this may due to a hidden hierarchical structure. These ideas can be further investigated.

The instability of $\delta$-hyperbolicity is also very troubling, because addition and deletion of edges are very common in a real-world network, and often this do not affect their navigability or underlying geometry by much. Therefore, to extract useful properties, we may need a more stable notion of hyperbolicity to describe the geometric aspect of complex networks. This aspect also worths further investigation.

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