# SEMISTABLE CONJECTURE VIA K-THEORY

WIESŁAWA NIZIOŁ

Abstract. We show that the Semistable conjecture of Fontaine-Jannsen is true for proper vertical fine and saturated log-smooth families with reduction of Cartier type (for example proper schemes with simple semistable reduction). We derive it from Suslin's comparison theorem between motivic cohomology and étale cohomology. This gives a new proof of the Semistable conjecture showing motivic character of p-adic period maps.

## 1. Introduction

The purpose of this paper is to give a motivic proof of the Semistable Conjecture of Fontaine-Jannsen [8] for proper vertical fine and saturated log-smooth families with reduction of Cartier type. Recall the formulation of this conjecture. Let $K$ be a complete discrete valuation field of mixed characteristic $(0,p)$ with ring of integers $V$ and a perfect residue field $k$. Let $X^{\times}$ be a fine and saturated log-smooth proper vertical $V^{\times}$-scheme, where $V$ is equipped with the log-structure associated to the closed point, such that the special fiber $X_0^{\times}$ is of Cartier type. A scheme $X$ over $V$ with a simple semistable reduction would be a standard example.

**Conjecture 1.1. (Semistable Conjecture)** *There exists a natural $B_{\mathrm{st}}$-linear period isomorphism*

$$\alpha_{st}: H^*(X_{\overline{K}},\mathbf{Q}_p)\otimes_{\mathbf{Q}_p} B_{\mathrm{st}} \simeq H^*_{\mathrm{cr}}(X_0^{\times}/W(k)^0)\otimes_{W(k)} B_{\mathrm{st}}$$

*preserving Galois action, monodromy, filtration and Frobenius.*

Here $\overline{K}$ is an algebraic closure of $K$, $W(k)^0$ is the ring of Witt vectors $W(k)$ equipped with the log-structure associated to $(\mathbf{N}\to W(k), 1\mapsto 0)$, and $B_{\mathrm{st}}$ is a certain ring of periods introduced by Fontaine [8]. The ring $B_{\mathrm{st}}$ is equipped with Galois action, Frobenius and monodromy operators. The log-crystalline cohomology groups $H^*_{\mathrm{cr}}(X^{\times}/W(k)^0)[1/p]$ are also equipped with Frobenius and monodromy operators. Moreover, the ring $B_{\mathrm{st}}$ maps into another ring of periods $B_{dR}$, which is equipped with a decreasing filtration. There is also a canonical isomorphism of $K\otimes_{W(k)} H^*_{\mathrm{cr}}(X_0^{\times}/W(k)^0)$ with the de Rham cohomology groups $H^*_{dR}(X_K/K)$ which are equipped with the Hodge filtration. The base change of the period isomorphism to $B_{dR}$ does yield an isomorphism on the filtrations. As a corollary, one gets that the étale cohomology as a Galois representation can be recovered from the log-crystalline cohomology:

$$H^*(X_{\overline{K}},\mathbf{Q}_p)\simeq (H^*_{\mathrm{cr}}(X_0^{\times}/W(k)^0)\otimes_{W(k)} B_{\mathrm{st}})^{N=0,\phi=1}\cap F^0(B_{dR}\otimes_K H^*_{dR}(X_K/K)).$$

The Semistable Conjecture is now a theorem. It was first proved by Kato [21] and Tsuji [33] for proper, vertical, log-schemes with semistable reduction. Their method is basically local and relies on a comparison theorem between sheaves of $p$-adic vanishing cycles and log-syntomic sheaves (which relate in a known way to étale and log-crystalline cohomology, respectively). This local comparison theorem is proved by explicite computations of certain gradings of both sheaves via symbols. Later the conjecture was proved by Faltings [6]. His method is based on the theory of almost étale extensions he has developed applied to certain sheaves of coherent vanishing cycles.

These are two of the three currently existing methods of proving $p$-adic comparison theorems. The third one is due to the author and was used before to prove the Crystalline Conjecture [25]. It relies on the postulate that the étale cohomology of $X_{\overline{K}}$ for high enough Tate twists realizes motivic cohomology (understood as Bloch's higher Chow groups or $\gamma$-graded pieces of $K$-theory). This is the Beilinson-Lichtenbaum Conjecture. It follows from the Bloch-Kato Conjecture whose proof was recently announced

This research was supported in part by an NSF grant and the Alfred Sloan Foundation Fellowship.

by Voevodsky. Similarly, one could expect that the log-syntomic cohomology of $X^{\times}_{\overline{V}}$ realizes yet-to-be-defined log-motivic cohomology of $X^{\times}_{\overline{V}}$. The period morphism would then simply be a $p$-adic incarnation of the localization map $j^*$ in motivic cohomology for the open immersion $j : X_{\overline{K}} \hookrightarrow X^{\times}_{\overline{V}}$. The Semistable Conjecture would then be equivalent to the map $j^*$ being an isomorphism. That in turn should follow from the fact that mod $p$ $K$-theory of varieties in characteristic $p$ — that controls the kernel and cokernel of $j^*$ — is "stably" trivial (for an appropriate definition of "stably"). This line of argument should in particular imply a uniqueness statement for $p$-adic period morphisms: a $p$-adic period morphism is uniquely determined by its values on higher cycle classes. We refer the reader to [27] for a survey of results on the relationship between $p$-adic motivic cohomology and other cohomologies of arithmetic interest (in the stable range).

Since the Beilinson-Lichtenbaum Conjecture is not yet fully proved and log-motivic cohomology is not yet constructed we are forced to work here with some substitutes. They are though good enough to carry out the above ideas to prove the Semistable Conjecture. They also give a uniqueness statement for $p$-adic period morphisms that imply that the morphisms of Tsuji, Faltings and the author are equal [28].

We will now sketch our proof of the conjecture. Recall that, by a standard argument (see [9], [5]), it suffices to construct a map

$$\alpha : H^*(X_{\overline{K}}, \mathbf{Q}_p) \to H^*_{\mathrm{cr}}(X^{\times}_0/W(k)^0) \otimes_{W(k)} B_{\mathrm{st}} \tag{1.1.1}$$

compatible with all the structures, and, in addition, with Poincaré duality and the trace map. Since there is no monodromy action on étale cohomology, the image of $\alpha$ lies in the kernel of the monodromy [21]:

$$(H^*_{\mathrm{cr}}(X^{\times}_0/W(k)^0) \otimes_{W(k)} B^+_{\mathrm{st}})^{N=0} \simeq \mathbf{Q} \otimes \operatorname{proj}\lim_n H^*_{\mathrm{cr}}(X^{\times}_{\overline{V},n}/W_n(k)),$$

where $\overline{V}$ is the integral closure of $V$ in $\overline{K}$ and $B^+_{\mathrm{st}}$ is a subring of $B_{\mathrm{st}}$. Taking into account filtration and Frobenius we see that we need to construct (at least for large enough $i$) a well-behaved family of maps into log-syntomic cohomology

$$\alpha^n_{a,i} : H^a(X_{\overline{K}}, \mathbf{Z}/p^n(i)) \to H^a(X^{\times}_{\overline{V},n}, s'_n(i)).$$

The motivic proof can be summarized by the following diagram

$$\begin{array}{ccc} \operatorname{gr}^i_{\gamma} K_j(X_{\overline{V}}; \mathbf{Z}/p^n) & \xrightarrow[j^*]{\sim} & \operatorname{gr}^i_{\gamma} K_j(X_{\overline{K}}; \mathbf{Z}/p^n) \\ \downarrow \overline{c}^{\mathrm{cr}}_{ij} & & \wr\downarrow \overline{c}^{\text{ét}}_{ij} \\ H^{2i-j}(X^{\times}_{\overline{V},n}, s'_n(i)) & \xleftarrow{\alpha^n_{2i-j,i}} & H^{2i-j}(X_{\overline{K}}, \mathbf{Z}/p^n(i)), \end{array}$$

where $K_j(\cdot; \mathbf{Z}/p^n)$ is the K-theory with coefficients and $\operatorname{gr}^i_{\gamma} K_j(\cdot; \mathbf{Z}/p^n)$ is the $\gamma$-grading playing the role of motivic cohomology. The term in the left upper corner stands for the limit of K-theory groups of (global) regular resolutions of the log-schemes $X^{\times}_{V'}$, for $V'$ a finite extension of $V$, and is a substitute for not-yet-defined log-motivic cohomology. The maps $\overline{c}^{\text{ét}}_{ij}$ and $\overline{c}^{\mathrm{cr}}_{ij}$ are the étale and the log-syntomic Chern class maps, respectively. A priori, because of the nature of the term $\operatorname{gr}^i_{\gamma} K_j(X_{\overline{V}}; \mathbf{Z}/p^n)$ the log-syntomic Chern class lands in the log-syntomic cohomology groups of the regular resolutions of the log-schemes $X^{\times}_{V'}$. However, the map from the resolution to the original log-scheme $X^{\times}_{V'}$ is a log-blow-up and it follows (Proposition 2.3) that the cohomology of these resolutions is isomorphic to the cohomology of the original log-schemes $X^{\times}_{V'}$.

We define the period map $\alpha^n_{2i-j,i}$ to make the above diagram commute. To do that, first we prove (in Lemma 3.5) that for $j > d + 1$ the localization map $j^*$ is an isomorphism modulo some constants depending only on the dimension $d$ of $X_K$ and $i, j$. This follows from the fact that the $K'$-theory with mod-$p$ coefficients of the special fiber vanishes for $j > d$ (a theorem of Geisser-Levine [13]). The reader will notice that this vanishing is entirely a $p$-type phenomena: it is not true for $K'$-theory mod $l$, $l \neq p$. Next, we show (Proposition 3.2) that, for $j \geq 2d$, the étale Chern class map $\overline{c}^{\text{ét}}_{ij}$ is an isomorphism modulo a constant depending only on $d$ and $i, j$. This should be thought of as a $K$-theory incarnation of the theorem of Suslin comparing higher Chow groups with étale cohomology [31]. Modulo some constants, we set $\alpha^n_{2i-j,i} = \overline{c}^{\mathrm{cr}}_{ij}(j^*)^{-1}(\overline{c}^{\mathrm{\acute{e}t}}_{ij})^{-1}$, take the projective limit over $n$, and after tensoring with $\mathbf{Q}$ we get our map $\alpha_{2i-j,i}$. Our construction of the map $\alpha_{2i-j,i}$ makes it now very easy to check its compatibility with Poincaré duality and trace maps.

Notice that the above period map goes from étale cohomology to log-syntomic cohomology. That is in the opposite direction than the period morphisms of Tsuji and Faltings. This has the nice consequence that to prove that the map is an isomorphism we can simply use Poincaré duality and avoid all the delicate local computations of the other two methods. On the other hand we do not get any information about the sheaves of $p$-adic vanishing cycles like Tsuji does.

**Acknowledgments.** This paper was written during my stay at the Max-Planck Institut in Bonn and at the Institut Henri Poincaré in Paris. I would like to thank these institutions for their support and hospitality. I would also like to thank T. Geisser for many helpful and inspiring conversations. Special thanks go to the referees for a very careful reading of the manuscript.

## 2. Preliminaries

Throughout the paper, let $p$ be a fixed prime, let $\overline{K}$ denote a chosen algebraic closure of a field $K$, and, for a scheme $X$, let $X_n = X \otimes \mathbf{Z}/p^n$.

Let $V$ be a complete discrete valuation ring with fraction field $K$ of characteristic 0 and with perfect residue field $k$ of characteristic $p$. Let $W(k)$ be the ring of Witt vectors with coefficients in $k$ with fraction field $K_0$. Set $G_K = \mathrm{Gal}(\overline{K}/K)$, $\mathbf{C}_p = \widehat{\overline{K}}$, and let $\sigma$ be the absolute Frobenius on $W(\overline{k})$. For a $V$-scheme $X$, let $X_0$ denote the special fiber of $X$. We will denote by $V$, $V^\times$, and $V^0$ the scheme $\mathrm{Spec}(V)$ with the trivial, canonical (i.e., associated to the closed point), and $(\mathbf{N} \to V, 1 \mapsto 0)$ log-structure respectively, and, for a log-scheme $X^\times$, we will denote by $X$ the underlying scheme. Unless otherwise stated, we work in this paper in the category of fine log-schemes.

2.1. **The rings of periods.** Let's recall the definitions of the rings $B_{\mathrm{cr}}$, $B_{dR}$, $B_{\mathrm{st}}$ of Fontaine [7], [8], [9]. We have

$$B^+_{\mathrm{cr},n} = H^0_{\mathrm{cr}}(\mathrm{Spec}(\overline{V}_n)/W_n(k)), \quad B^+_{\mathrm{cr}} = \operatorname{proj}\lim_n B^+_{\mathrm{cr},n}[1/p], \quad B_{\mathrm{cr}} = B^+_{\mathrm{cr}}[t^{-1}],$$

where $\overline{V}$ is the integral closure of $V$ in $\overline{K}$ and $t$ is a certain element of $B^+_{\mathrm{cr}}$ (see [7] for a precise definition of $t$). The ring $B^+_{\mathrm{cr}}$ is a topological $K_0$-module equipped with a Frobenius $\phi$ coming from the crystalline cohomology and a natural $G_K$-action. We have that $\phi(t) = pt$ and that $G_K$ acts on $t$ via the cyclotomic character.

The canonical morphism $B^+_{\mathrm{cr},n} \to \overline{V}/p^n$ is surjective. Let $J_{\mathrm{cr},n}$ denote its kernel. Let

$$B^+_{dR} = \operatorname{proj}\lim_r (\mathbf{Q} \otimes \operatorname{proj}\lim_n B^+_{\mathrm{cr},n}/J^{[r]}_{\mathrm{cr},n}), \quad B_{dR} = B^+_{dR}[t^{-1}].$$

The ring $B^+_{dR}$ has a discrete valuation given by the powers of $t$. Its quotient field is $B_{dR}$. We set $F^n B_{dR} = t^n B_{dR}$. This defines a descending filtration on $B_{dR}$.

Let $\pi$ be a uniformizer of $V$. Choose a sequence of elements $s = (s_n)$ of $\overline{V}$ such that $s_0 = \pi$ and $s^p_{n+1} = s_n$. Fontaine associates to it an element $u_s$ of $B^+_{dR}$. Let $B^+_{\mathrm{st}}$ denote the subring of $B_{dR}$ generated by $B^+_{\mathrm{cr}}$ and $u_s$. Fontaine shows that $u_s$ is transcendental over $B^+_{\mathrm{cr}}$. Hence $B^+_{\mathrm{st}}$ is a polynomial algebra in one variable over $B^+_{\mathrm{cr}}$. The ring $B^+_{\mathrm{st}}$ *does* depend on the choice of $\pi$ but not of $s$. The action of $G_K$ on $B^+_{dR}$ restricts well to $B^+_{\mathrm{st}}$. The Frobenius $\phi$ extends to $B^+_{\mathrm{st}}$ by $\phi(u_s) = pu_s$ and one defines the monodromy operator $N : B^+_{\mathrm{st}} \to B^+_{\mathrm{st}}$ as the unique $B^+_{\mathrm{cr}}$-derivation such that $Nu_s = -1$. We have $N\phi = p\phi N$. Let $B_{\mathrm{st}} = B_{\mathrm{cr}}[u_s]$. Different choices of the uniformizer $\pi$ yield isomorphic rings $B^+_{\mathrm{st}}$, so we can and we will identify them via these isomorphisms. The dependence on $\pi$ will then be encoded in the morphism $\iota_\pi : B^+_{\mathrm{st}} \hookrightarrow B^+_{dR}$.

We will need the following crystalline interpretation of the ring $B^+_{\mathrm{st}}$ (see [21], [33]). Let $R^\times_n$ denote the PD-envelope of the ring $W_n[x]$ with respect to the closed immersion $W_n[x] \to V_n$, $x \to \pi$, equipped with the log-structure associated to $\mathbf{N} \to R_n$, $1 \to x$. Let

$$\widehat{B}^+_{\mathrm{st}} = \operatorname{proj}\lim_n H^0_{\mathrm{cr}}(\mathrm{Spec}(\mathcal{O}_{\overline{V},n})/R^\times_n)[1/p].$$

The ring $\widehat{B}_{\mathrm{st}}^+$ has a natural action of $G_K$, Frobenius $\phi$, and a monodromy operator $N$. Kato [21, 3.7] shows that the ring $B_{\mathrm{st}}^+$ is canonically (and compatibly with all the structures) isomorphic to the subring of elements of $\widehat{B}_{\mathrm{st}}^+$ annihilated by a power of the monodromy operator $N$.

More generally, for any fine log-scheme $X^\times$, which is log-smooth and proper over $V^\times$, set

$$H^i_{\mathrm{cr}}(X^\times_{\overline{V}}/W(k)) := \operatorname{proj}\lim_n H^i_{\mathrm{cr}}(X^\times_{\overline{V},n}/W_n(k)), \qquad H^i_{\mathrm{cr}}(X^\times) := \mathbf{Q}\otimes H^i_{\mathrm{cr}}(X_0^\times/W(k)^0).$$

Assume now that the special fiber of $X^\times$ is of Cartier type. Then Kato defines [21, 4.2,4.5] canonical morphisms

$$\mathbf{Q}\otimes H^i_{\mathrm{cr}}(X^\times_{\overline{V}}/W(k)) \xrightarrow{h_\pi} (\widehat{B}_{\mathrm{st}}^+\otimes_{K_0} H^i_{\mathrm{cr}}(X^\times))^{N=0} \xleftarrow{\sim} (B_{\mathrm{st}}^+\otimes_{K_0} H^i_{\mathrm{cr}}(X^\times))^{N=0}.$$

It can be checked (cf. [33, 4.5.6-7]) that these morphisms are compatible with the product structure, Galois action, and the Frobenius.

Moreover, Hyodo and Kato [18, 5.1] have constructed a canonical $K$-isomorphism

$$\rho_\pi : K\otimes_{K_0} H^i_{\mathrm{cr}}(X^\times) \xrightarrow{\sim} H^i_{dR}(X^\times_K/K),$$

which is compatible with products [33, 4.4.13]. Hence the composition

$$\rho_\pi h_\pi : \mathbf{Q}\otimes H^i_{\mathrm{cr}}(X^\times_{\overline{V}}/W(k)) \to B_{\mathrm{st}}^+\otimes_{K_0} H^i_{dR}(X^\times_K/K)$$

is functorial in $X^\times$ and compatible with products and Galois action.

Let $X^\times$ be any fine log-scheme, which is log-smooth and proper over $V^\times$ with saturated log-structure on the generic fiber. We will need the crystalline interpretation of $B^+_{dR}\otimes_K H^i_{dR}(X^\times_K/K)$ from [21] (see also [33, 4.7]):

$$B^+_{dR}\otimes_K H^i_{dR}(X^\times_K/K) \xrightarrow{\sim} \operatorname{proj}\lim_s(\mathbf{Q}\otimes H^i_{\mathrm{cr}}(X^\times_{\overline{V}}/V^\times,\mathcal{O}/J^{[s]})) \quad [33, 4.7.6], \tag{2.2.1}$$

$$F^r(B^+_{dR}\otimes_K H^i_{dR}(X^\times_K/K)) \xrightarrow{\sim} \operatorname{proj}\lim_{s\geq r}(\mathbf{Q}\otimes H^i_{\mathrm{cr}}(X^\times_{\overline{V}}/V^\times,J^{[r]}/J^{[s]})) \quad [33, 4.7.13]. \tag{2.2.2}$$

2.2. **Syntomic regulators.** We will recall now briefly the definition and properties of syntomic regulators. For details we refer the reader to [16] and [25]. Let $X$ be a scheme of finite type, separated and flat over $W(k)$. Recall the differential definition [19] of syntomic cohomology of Fontaine-Messing [9]. Assume first that we have an immersion $i: X\hookrightarrow Z$ over $W(k)$ such that $Z$ is a smooth $W(k)$-scheme endowed with a compatible system of liftings of the Frobenius $\{F_n : Z_n\to Z_n\}$. Let $D_n = D_{X_n}(Z_n)$ be the PD-envelope of $X_n$ in $Z_n$ (compatible with the canonical PD-structure of $pW_n(k)$) and $J_{D_n}$ the ideal of $X_n$ in $D_n$. Consider the following complexes

$$s_n'(r)_X := \operatorname{Cone}(J^{[r-\cdot]}_{D_n}\otimes\Omega^{\cdot}_{Z_n/W_n(k)} \xrightarrow{p^r-\phi} \mathcal{O}_{D_n}\otimes\Omega^{\cdot}_{Z_n/W_n(k)})[-1],$$

where $\phi$ is the Frobenius. The complexes $s_n'(r)_X$ are, up to canonical quasi-isomorphisms, independent of the choice of $i$ and $\{F_n\}$.

In general, immersions as above exist étale locally, and one defines $s_n'(r)_X\in\mathbf{D}^+(X_{\text{ét}},\mathbf{Z}/p^n)$ by gluing the local complexes. Finally, one defines $s_n'(r)_{X_{\overline{V}}}\in\mathbf{D}^+((X_{\overline{V}})_{\text{ét}},\mathbf{Z}/p^n)$ as the inductive limit of $s_n'(r)_{X_{V'}}$, where $V'$ varies over the integral closures of $V$ in all finite extensions of $K$ in $\overline{K}$. Set

$$H^i(X,s_n'(r)) := H^i_{\text{ét}}(X,s_n'(r)_X), \quad H^i(X_{\overline{V}},s_n'(r)) := H^i_{\text{ét}}(X_{\overline{V}},s_n'(r)_{X_{\overline{V}}}).$$

We have the long exact sequence

$$\ldots\to H^i(X,s_n'(r))\to H^i_{\mathrm{cr}}(X/W(k),J^{[r]}_{X_n/W_n(k)}) \xrightarrow{p^r-\phi} H^i_{\mathrm{cr}}(X/W(k),\mathcal{O}_{X_n/W_n(k)})\to\ldots$$

There exists a well-behaved product

$$H^i(X,s_n'(r))\otimes H^j(X,s_n'(r'))\to H^{i+j}(X,s_n'(r+r'))$$

compatible with the crystalline products.

Let now $X^\times$ be a fine finite type and separated log-scheme over $W(k)$. Then (see [33, 2.1]) we have the logarithmic analog of the above complexes $s'_n(r)_{X^\times}$ on $X_{\text{ét}}$ and the corresponding cohomology groups

$$H^i(X^\times, s'_n(r)) := H^i_{\text{ét}}(X, s'_n(r)_{X^\times}), \quad H^i(X^\times_{\overline{V}}, s'_n(r)) := H^i_{\text{ét}}(X_{\overline{V}}, s'_n(r)_{X^\times_{\overline{V}}}).$$

We have a natural, compatible with products, map

$$\varepsilon : H^i(X, s'_n(b)) \to H^i(X^\times, s'_n(b)).$$

For a scheme $X$, let $K_*(X)$ denote the higher $K$-theory groups of $X$ as defined by Quillen [29]. Similarly, for a noetherian scheme $X$, let $K'_*(X)$ denote Quillen's $K'$-theory. For a prime $p$, the corresponding groups with coefficients $\mathbf{Z}/p^n$ [32], will be denoted by $K_i(X;\mathbf{Z}/p^n)$ and $K'_i(X;\mathbf{Z}/p^n)$. For $p>3$ there is a well-behaved product on the groups $K_i(X;\mathbf{Z}/p^n)$. For $p=2$, if $n>1$ there is a product and it is commutative and associative for $n>3$. If $p=3$, there is a product that is commutative and associative for $n>1$ ([3], [24]).

For a noetherian regular connected scheme $X$, we have the following $\gamma$-filtrations compatible with products:

$$F^k_\gamma K_0(X) = \begin{cases} K_0(X) & \text{if } k\leq 0,\\ \langle \gamma_{i_1}(x_1)\cdots\gamma_{i_n}(x_n) | \varepsilon(x_1)=\ldots=\varepsilon(x_n)=0, i_1+\cdots+i_n\geq k\rangle & \text{if } k>0,\end{cases}$$

$$\begin{aligned} F^k_\gamma K_q(X;\mathbf{Z}/p^n) &= \langle \gamma_{i_1}(x_1)\cup\cdots\cup\gamma_{i_n}(x_n) | x_i\in K_{q_i}(X;\mathbf{Z}/p^n), q_i\geq 2,\\ &\qquad i_1+\cdots+i_n\geq k\rangle,\\ \mathcal{F}^k_\gamma K_q(X;\mathbf{Z}/p^n) &= \langle a\gamma_{i_1}(x_1)\cup\cdots\cup\gamma_{i_n}(x_n) | a\in F^{i_0}_\gamma K_0(X), x_i\in K_{q_i}(X;\mathbf{Z}/p^n), q_i\geq 2,\\ &\qquad i_0+i_1+\cdots+i_n\geq k\rangle,\end{aligned}$$

where $\varepsilon$ is the augmentation on $K_0(X)$ and $p^n>2$. We will also consider another $\gamma$-filtration: $\widetilde{F}^i_\gamma = \langle \gamma^k(x) | k\geq i\rangle$, where $\langle\ldots\rangle$ denotes the subgroup generated by the given elements. These filtrations are related: by [30, 3.4] and [15, 5.4], we have

$$M(d,i,2j)\mathcal{F}^i_\gamma K_j(X;\mathbf{Z}/p^n) \subset \widetilde{F}^i_\gamma K_j(X;\mathbf{Z}/p^n) \subset F^i_\gamma K_j(X;\mathbf{Z}/p^n), \quad j\geq 2, \tag{2.2.3}$$

where $d$ is the dimension of $X$ and the integers $M(k,m,n)$ are defined by the following procedure [30, 3.4]. Let $l$ be a positive integer, and let $w_l$ be the greatest common divisor of the set of integers $k^N(k^l-1)$, as $k$ runs over the positive integers and $N$ is large enough with respect to $l$. Let $M(k)$ be the product of the $w_l$'s for $2l<k$. Set $M(k,m,n)=\prod_{2m\leq 2l\leq n+2k+1} M(2l)$.

Using Illusie's computation of the crystalline cohomology of $B.GL_m/W(k)$ [16, II], one can define universal classes $C_{i,m}\in H^{2i}(B.GL_m/W(k), s'_n(i))$. For any flat finite type scheme $X$ over $W(k)$, via the method of Gillet [14, 2.22], [25, 2.3], they yield functorial and compatible families of Chern classes

$$\begin{aligned} c^{\text{syn}}_{ij} &: K_j(X) \to H^{2i-j}(X, s'_n(i)) \quad \text{for } j\geq 0,\\ \overline{c}^{\text{syn}}_{ij} &: K_j(X;\mathbf{Z}/p^n) \to H^{2i-j}(X, s'_n(i)) \quad \text{for } j\geq 2,\end{aligned}$$

which are also compatible with the crystalline Chern classes

$$c^{\text{cr}}_{ij} : K_j(X) \to H^{2i-j}_{\text{cr}}(X_n/W_n(k), \mathcal{O}_{X_n/W_n(k)})$$

via the canonical map $H^{2i-j}(X, s'_n(i)) \to H^{2i-j}_{\text{cr}}(X_n/W_n(k), \mathcal{O}_{X_n/W_n(k)})$.

Recall the construction of the classes $\overline{c}^{\text{syn}}_{i,j}$. First, one constructs the universal classes $C_{i,m}$. Recall [16] that

$$H^*_{\text{cr}}(B.GL_m/W_n(k)) \simeq H^*_{dR}(B.GL_m/W_n(k)) \simeq W_n(k)[x_1,\ldots,x_m],$$

where the classes $x_i\in H^{2i}_{dR}(B.GL_m/W_n(k))$ are the de Rham Chern classes of the universal locally free sheaf on $B.GL_m/W_n(k)$ (defined via a projective space theorem). We have $x_i\in H^{2i}_{\text{cr}}(B.GL_m/W_n(k), J^{[i]}_{B.GL_m/W_n(k)})$ and $\phi(x_i)=p^i x_i$. Since $B.GL_m/W_n(k)$ is smooth over $W_n(k)$, it follows that we have the exact sequence

$$0\to H^{2i}(B.GL_m/W(k), s'_n(i)) \to H^{2i}_{\text{cr}}(B.GL_m/W_n(k), J^{[i]}_{B.GL_m/W_n(k)}) \xrightarrow{\phi-p^i} H^{2i}_{\text{cr}}(B.GL_m/W_n(k)).$$

For $m \geq i$, we define $C_{i,m} = x_i \in H^{2i}(B.GL_m/W(k), s'_n(i))$. By construction these classes are compatible with the crystalline classes.

The classes $C_{i,m} \in H^{2i}(B.GL_m/W(k), s'_n(i))$ yield compatible universal classes (see [14, p. 221]) $C_{i,m} \in H^{2i}(X, GL_m(\mathcal{O}_X), s'_n(i))$, hence a natural map of pointed simplicial sheaves on $X$, $C_i : B.GL(\mathcal{O}_X) \to \mathcal{K}(2i, \tilde{s}'_n(i)_X)$, where $\mathcal{K}$ is the Dold–Puppe functor of $\tau_{\geq 0}\tilde{s}'_n(i)_X[2i]$ and $\tilde{s}'_n(i)_X$ is an injective resolution of $s'_n(i)_X$. The characteristic classes $\overline{c}_{i,j}$, $j \geq 2$, are now defined [14, 2.22] as the composition

$$\begin{aligned} K_j(X, \mathbf{Z}/p^n) &\to H^{-j}(X, \mathbf{Z} \times B.GL(\mathcal{O}_X)^+, \mathbf{Z}/p^n) \to H^{-j}(X, B.GL(\mathcal{O}_X)^+, \mathbf{Z}/p^n) \\ &\xrightarrow{C_i} H^{-j}(X, \mathcal{K}(2i, \tilde{s}'_n(i)_X), \mathbf{Z}/p^n) \xrightarrow{f} H^{2i-j}(X, s'_n(i)), \end{aligned}$$

where $B.GL(\mathcal{O}_X)^+$ is the (pointed) simplicial sheaf on $X$ associated to the +- construction. Here, for a (pointed) simplicial sheaf $\mathcal{E}.$ on $X$, $H^{-j}(X, \mathcal{E}., \mathbf{Z}/p^n) = \pi_j(R\Gamma(X, \mathcal{E}.), \mathbf{Z}/p^n)$ is the generalized sheaf cohomology of $\mathcal{E}.$ [14, 1.7]: if we let $\mathcal{P}^j_X$ denote the constant sheaf of j-dimensional mod $p^n$ Moore spaces, then $H^{-j}(X, \mathcal{E}., \mathbf{Z}/p^n) = [\mathcal{P}^j_X, \mathcal{E}.]$, where, for two pointed simplicial sheaves $\mathcal{F}.$, $\mathcal{F}'.$ on $X$, $[\mathcal{F}., \mathcal{F}'.]$ denotes the morphisms from $\mathcal{F}.$ to $\mathcal{F}'.$ in the homotopy category. The map $f$ is defined as the composition

$$\begin{aligned} H^{-j}(X, \mathcal{K}(2i, \tilde{s}'_n(i)_X), \mathbf{Z}/p^n) &= \pi_j(\mathcal{K}(2i, \tilde{s}'_n(i)(X)), \mathbf{Z}/p^n)) \xrightarrow{h_j} H_j(\mathcal{K}(2i, \tilde{s}'_n(i)(X)), \mathbf{Z}/p^n)) \\ &\to H_j(\tilde{s}'_n(i)(X)[2i]) = H^{2i-j}(X, s'_n(i)), \end{aligned}$$

where $h_j$ is the Hurewicz morphism.

This gives mod $p^n$ Chern classes in $H^*(X, s'_n(*))$. The integral ones are defined in an analogous way.

**Lemma 2.1.** *The syntomic Chern classes have the following properties.*

(1) $c^{\mathrm{syn}}_{ij}$*, for* $j > 0$*, is a group homomorphism.*
(2) $\overline{c}^{\mathrm{syn}}_{ij}$*, for* $j \geq 2$ *is a group homomorphism unless* $j = 2$ *and* $p = 2$*.*
(3) $\overline{c}^{\mathrm{syn}}_{ij}$ *are compatible with the reduction maps* $s'_n(i) \to s'_m(i)$*,* $n \geq m$*.*
*Moreover, if* $X$ *is regular, then*
(4) *Let* $p$ *be odd or* $p = 2$*,* $n \geq 2$ *and* $l, q \neq 2$*. If* $\alpha \in K_l(X; \mathbf{Z}/p^n)$ *and* $\alpha' \in K_q(X; \mathbf{Z}/p^n)$*, then*
$$\overline{c}^{\mathrm{syn}}_{ij}(\alpha\alpha') = -\sum_{r+s=i} \frac{(i-1)!}{(r-1)!(s-1)!} \overline{c}^{\mathrm{syn}}_{rl}(\alpha)\overline{c}^{\mathrm{syn}}_{sq}(\alpha'),$$
*assuming that* $l, q \geq 2$*,* $l + q = j$*,* $2i \geq j$*,* $i \geq 0$*.*
(5) *If* $\alpha \in F^j_\gamma K_0(X)$*,* $j \neq 0$*, and* $\alpha' \in F^k_\gamma K_q(X; \mathbf{Z}/p^n)$*,* $q \geq 2$*, is such that* $\overline{c}^{\mathrm{syn}}_{lq}(\alpha') = 0$ *for* $l \neq k$*, then*
$$\overline{c}^{\mathrm{syn}}_{j+k,q}(\alpha\alpha') = -\frac{(j+k-1)!}{(j-1)!(k-1)!} c^{\mathrm{syn}}_{j0}(\alpha)\overline{c}^{\mathrm{syn}}_{kq}(\alpha'),$$
*assuming that* $p \neq 2$ *or* $q > 2$*.*
(6) *The above multiplication formulas hold also for* $p = 2$*,* $n \geq 4$*,* $q = 2$ *and* $\alpha'$ *such that* $\partial\alpha' \in K_1(X)$ *belongs to* $V^*$*.*
(7) *The integral Chern class maps* $c^{\mathrm{syn}}_{i0}$ *restrict to zero on* $F^{i+1}_\gamma K_0(X)$*.*
(8) *The Chern class maps* $\overline{c}^{\mathrm{syn}}_{ij}$ *restrict to zero on* $F^{i+1}_\gamma K_j(X; \mathbf{Z}/p^n)$*,* $j \geq 2$*, unless* $j = 2$*,* $p = 2$*.*

*Proof.* This lemma was proved by us in [25, 2.3] for $p \neq 2$. When $p$ is even the situation is more complicated because the Hurewicz map is not a group homomorphism for $j = 2$ (see property (2)). But with this in mind the proof goes through assuming the restrictions listed in the lemma. Property (6) follows by adding Weibel's analysis [34, 3.5.1]. An interested reader will find all the necessary information on behaving of the Hurewicz map, comultiplication, and primitivity for $p = 2$ in Weibel's article [34]. □

**Remark 2.2.** For any $K$-scheme $X$, one can argue as above to show that the étale Chern class maps

$$c^{\text{ét}}_{ij} : K_j(X) \to H^{2i-j}(X, \mathbf{Z}/p^n(i)), \quad \overline{c}^{\text{ét}}_{ij} : K_j(X; \mathbf{Z}/p^n) \to H^{2i-j}(X, \mathbf{Z}/p^n(i))$$

have analogous properties to those of the syntomic Chern classes.

2.3. **Log-étale descent.** The following proposition will be essential in our construction of the comparison morphism; it will allow us to descent the syntomic cohomology of the regular resolution to that of the original log-scheme. We are working in this section in the category of fine and saturated log-schemes. Recall that a map of log-scheme is a log-blow-up if it is log-étale and blows-up some of the closed strata. The reader will find a precise definiton in [26].

**Proposition 2.3.** *For any $n \geq 1$, $r \geq 0$, any log-smooth separated scheme of finite type $X^\times \to V^\times$, and any log-blow-up $\pi : U^\times \to X^\times$, there is a natural isomorphism*

$$H^*_{\mathrm{cr}}(X_n^\times/W_n(k), J^{[r]}_{X_n^\times/W_n(k)}) \xrightarrow[\sim]{\pi^*} H^*_{\mathrm{cr}}(U_n^\times/W_n(k), J^{[r]}_{U_n^\times/W_n(k)}).$$

*Proof.* We know that log-blow-ups do not change de Rham cohomology. The proposition follows by some flatness arguments.

By Zariski descent for log-crystalline cohomology, we may assume $X$ to be affine. There is a commutative diagram of maps of topoi

$$\begin{array}{ccc} & (U_n^\times/W_n(k))_{\mathrm{cr}} & \\ & \swarrow \pi \quad \searrow f_{U^\times} & \\ (X_n^\times/W_n(k))_{\mathrm{cr}} & \xrightarrow{f_{X^\times}} & (V_n^\times/W_n(k))_{\mathrm{cr}}. \end{array}$$

It suffices to show that the morphism $Rf_{X^\times *}J^{[r]}_{X_n^\times/W_n(k)} \xrightarrow{\pi^*} Rf_{U^\times *}J^{[r]}_{U_n^\times/W_n(k)}$ is a quasi-isomorphism.

Let $i_T : S^\times \hookrightarrow T^\times \to W_n(k), \omega : S^\times \to V_n^\times$, be a PD-thickening. Notice that the log-scheme $T^\times$ is saturated: $S^\times$ is saturated ($\omega$ being étale), $i_T$ is a nilimmersion, and, since $i_T$ is an exact morphism, we have $i_T^{-1}(M_{T^\times}/\mathcal{O}^*_{T^\times}) \simeq M_{S^\times}/\mathcal{O}^*_{S^\times}$, where, for a log-scheme $Y^\times$, $M_{Y^\times}$ denotes its log-structure. We have canonically $(Rf_{X^\times *}J^{[r]}_{X_n^\times/W_n(k)})_{T^\times} \simeq Rf_{X_S^\times/T^\times *}(J^{[r]}_{X_S^\times/T^\times})$, where $X_S^\times = X_n^\times \times_{V_n^\times} S^\times$ and $f_{X_S^\times/T^\times}$ is the composition

$$f_{X_S^\times/T^\times} : (X_S^\times/T^\times)_{\mathrm{cr}} \to (X_S^\times)_{\text{ét}} \to T^\times_{\text{ét}}.$$

We may assume that $S^\times = V(k')_n^\times$, where $k'$ is a finite field extension of $k$ and $V(k')$ denotes the unramified extension of $V$ corresponding to $k'$. Let $Y^\times := W_n(k')[x]^\times$ be the scheme $\mathrm{Spec}(W_n(k')[x])$ equipped with the log-structure associated to the map $\mathbf{N} \to W_n(k')[x]$, $1 \mapsto x$. We have an exact closed immersion $i_W : V(k')_n^\times \leftarrow W_n(k')[x]^\times$ given by sending $x$ to a uniformizer of $V(k')$. Let $I$ be the kernel of $i_W$. It is a principal ideal.

We may also assume that there exists a retraction $h : T^\times \to W_n(k')[x]^\times$ such that $hi_T = i_W$ ($W_n(k')[x]^\times$ being log-smooth over $W_n(k)$). Notice that, since $T^\times$ is equipped with divided powers, the retraction $h$ factors through a closed subscheme $Y_m^\times$ of $W_n(k')[x]^\times$ given by $I^m$.

Since $X_S^\times$ is affine and log-smooth over $S^\times$, it can be lifted to a (necessarily saturated) log-smooth scheme $X_S^\times \hookrightarrow X_{Y_m^\times}$ over $Y_m^\times$. Let $X_T^\times$ denotes the pullback of $X_{Y_m^\times}$ via $h$ to $T^\times$. Since $\pi : U^\times \to X^\times$ is log-étale, we have the following cartesian diagram of maps of log-schemes

$$\begin{array}{ccccc} U_S^\times & \longrightarrow & U_T^\times & \longrightarrow & U_{Y_m}^\times \\ \downarrow \pi & & \downarrow \pi_T & & \downarrow \pi_{Y_m} \\ X_S^\times & \longrightarrow & X_T^\times & \longrightarrow & X_{Y_m}^\times \\ \downarrow & & \downarrow g_X & & \downarrow \\ S^\times & \xrightarrow{i_T} & T^\times & \xrightarrow{h} & Y_m^\times, \end{array}$$

where $\pi_T$, $\pi_{Y_m}$ are log-étale liftings of $\pi$. Due to the fact that the morphisms $X_{Y_m}^\times \to Y_m^\times$ and $U_{Y_m}^\times \to Y_m^\times$ are integral it is a cartesian diagram of schemes as well.

The scheme $X_T^\times$ is flat over $T^\times$ (as it is log-smooth and integral over $T^\times$ – the monoid $M_{T^\times}/\mathcal{O}^*_{T^\times}$ being generated by one element) thus $J^{[r]}_{X_T^\times} \simeq J^{[r]}_{T^\times} \otimes_{\mathcal{O}_{T^\times}} \mathcal{O}_{X_T^\times}$. Similarly, the scheme $U_T^\times$ is flat over $T^\times$

and $J^{[r]}_{U_T^\times}\simeq J^{[r]}_{T^\times}\otimes_{\mathcal{O}_{T^\times}}\mathcal{O}_{U_T^\times}$. Since there are isomorphisms

$$Rf_{X_S^\times/T^\times *}(J^{[r]}_{X_S^\times/T^\times})\simeq Rg_{X*}(J^{[r-\cdot]}_{X_T^\times}\otimes\Omega^{\cdot}_{X_T^\times/T^\times}),\quad Rf_{U_T^\times/T^\times *}(J^{[r]}_{U_S^\times/T^\times})\simeq Rg_{U*}(J^{[r-\cdot]}_{U_T^\times}\otimes\Omega^{\cdot}_{U_T^\times/T^\times}),$$

where $g_U=g_X\pi_T$. it suffices to show that the natural morphism

$$J^{[s]}_{X_T^\times}\otimes\Omega^r_{X_T^\times/T^\times}\xrightarrow{\pi_T^*}R\pi_{T*}(J^{[s]}_{U_T^\times}\otimes\Omega^r_{U_T^\times/T^\times})$$

is a quasi-isomorphism.

By the above and [17, III.3.7]

$$R\pi_{T*}(J^{[s]}_{U_T^\times}\otimes\Omega^r_{U_T^\times/T^\times})=R\pi_{T*}L\pi_T^*(J^{[s]}_{X_T^\times}\otimes\Omega^r_{X_T^\times/T^\times})=J^{[s]}_{X_T^\times}\otimes\Omega^r_{X_T^\times/T^\times}\otimes^L R\pi_{T*}\mathcal{O}_{U_{T^\times}}.$$

It follows that it suffices to show that the natural morphism $\mathcal{O}_{X_T}\xrightarrow{\pi_T^*}R\pi_{T*}\mathcal{O}_{U_T}$ is a quasi-isomorphism, or, because the map $X_T\to X_{Y_m}$ is a homeomorphism of topological spaces, that the natural morphism $\mathcal{O}_{X_T}\xrightarrow{\pi_{Y_m}^*}R\pi_{Y_m*}\mathcal{O}_{U_T}=R\pi_{Y_m*}L\pi^*_{Y_m*}\mathcal{O}_{X_T}=\mathcal{O}_{X_T}\otimes^L_{\mathcal{O}_{Y_m}}R\pi_{Y_m*}\mathcal{O}_{U_{Y,m}}$ is a quasi-isomorphism (recall that $\mathcal{O}_{X_T}$ and $\mathcal{O}_{U_T}$ are flat over $\mathcal{O}_T$).

It suffices thus to show that $\mathcal{O}_{X_{Y,m}}\xrightarrow{\pi_{Y_m}^*}R\pi_{Y_m*}\mathcal{O}_{U_{Y,m}}$ is a quasi-isomorphism. We argue by induction on $m$. Assume that the statement is true for $m=1$. Since $U_{Y,m}$ is flat over $Y_m$, we have the following exact sequence

$$0\to\mathcal{O}_{U_{Y,m-1}}\to\mathcal{O}_{U_{Y,m}}\to\mathcal{O}_{U_{Y,1}}\to 0.$$

This and the induction hypothesis yield that the sequence

$$0\to\Gamma(U_{Y,m-1},\mathcal{O}_{U_{Y,m-1}})\to\Gamma(U_{Y,m},\mathcal{O}_{U_{Y,m}})\to\Gamma(U_{Y,1},\mathcal{O}_{U_{Y,1}})\to 0$$

is exact and that $H^i(U_{Y,m},\mathcal{O}_{U_{Y,m}})=0$, for $i>0$. Evoking once more the induction hypothesis, we get that $\Gamma(X_{Y,m},\mathcal{O}_{X_{Y,m}})\xrightarrow{\sim}\Gamma(U_{Y,m},\mathcal{O}_{U_{Y,m}})$. Since $X_{Y,m}$ is affine, this gives us what we wanted.

It remains to show that $\mathcal{O}_{X_S}\xrightarrow{\pi_S^*}R\pi_{S*}\mathcal{O}_{U_S}$ is a quasi-isomorphism. Since $U^\times\to X^\times$ is a log-blow-up, this is just a $\mathbf{Z}/p^n$-version of Theorem 11.3 from [22]. □

**Corollary 2.4.** *Let $X^\times\to V^\times$ be any log-smooth separated scheme of finite-type. Then for any $n\geq 1$, $r\geq 0$, and any log-blow-up $\pi:U^\times\to X^\times$, there is a natural isomorphism*

$$H^*(X^\times,s'_n(r))\underset{\sim}{\xrightarrow{\pi^*}}H^*(U^\times,s'_n(r)).$$

*Proof.* Use the long exact sequence

$$\ldots\to H^i(X^\times,s'_n(r))\to H^i_{\mathrm{cr}}(X_n^\times/W_n(k),J^{[r]}_{X_n^\times/W_n(k)})\xrightarrow{\beta}H^i_{\mathrm{cr}}(X_1^\times/W_n(k),\mathcal{O}_{X_1^\times/W_n(k)})\to\ldots,$$

where $\beta(x,y)=(p^rx-\phi(x))$ and Proposition 2.3. □

## 3. COMPARISON THEOREM

We are now ready to construct the comparison morphism. First, let's recall some facts about Bott elements. Let $Y$ be a scheme such that $\Gamma(Y,\mathcal{O}_Y)$ contains a primitive $p^n$'th root of unity. Let $\mu_{p^n}(Y)$ denote the group of $p^n$'th roots of unity in $\Gamma(Y,\mathcal{O}_Y)$. Recall [34, 2.7.2] that, for $p^n>2$, there are compatible functorial Bott element homomorphisms

$$\beta_Y:\mu_{p^n}(Y)\to K_2(Y;\mathbf{Z}/p^n).$$

In what follows, choose a sequence of nontrivial $p$-power roots of unity $\zeta=(\zeta_n)$, $\zeta_n\in\overline{\mathbf{Q}}_p$, $\zeta_n^{p^n}=1$, $\zeta_{n+1}^p=\zeta_n$, and take for $t\in B_{\mathrm{cr}}$ the element associated to this sequence. We have $t\in H^0(\overline{V},s'_n(1))\hookrightarrow B^+_{\mathrm{cr},n}$. Let $K_1$ be a finite field extension of $K$ imside $\overline{K}$ containing $\zeta_n$ and $V_1$ its ring of integers. For $p^n>2$, define the Bott classes $\beta_n\in K_2(K_1;\mathbf{Z}/p^n)$ and $\tilde{\beta}_n\in K_2(V_1;\mathbf{Z}/p^n)$ as $\beta_{K_1}(\zeta_n)$ and $\beta_{V_1}(\zeta_n)$, respectively. These classes form a compatible sequence with respect to $n$.

**Lemma 3.1.** *We have*

$$\overline{c}^{\text{ét}}_{i,2i}(\beta^i_n)=(-1)^{i-1}(i-1)!\zeta_n^{\otimes i}\in H^0(K_1,\mathbf{Z}/p^n(i));\qquad \overline{c}^{\text{ét}}_{j,2i}(\beta^i_n)=0,\quad j\neq i;$$
$$\overline{c}^{\text{syn}}_{i,2i}(\tilde{\beta}^i_n)=(-1)^{i-1}(i-1)!t^i\in H^0(\overline{V},s'_n(i));\qquad \overline{c}^{\text{syn}}_{j,2i}(\tilde{\beta}^i_n)=0,\quad j\neq i.$$

*Proof.* The Chern class maps

$$\overline{c}^{\text{syn}}_{i,2}:K_2(V_1;\mathbf{Z}/p^n)\to H^{2i-2}(V_1,s'_n(i))$$

are induced by the syntomic Chern classes $C_i\in H^{2i}(B.GL/W(k),s'_n(i))$ of the universal vector bundle on $B.GL/W(k)$ (see Lemma 2.1 in [25]). On $\tilde{\beta}_n$ they act via the universal Chern classes $C_{i,1}\in H^{2i}(B.GL_1/W(k),s'_n(i))$ [34, 2.7.2]. But those vanish for $i>1$ giving that $\overline{c}^{\text{syn}}_{i,2}(\tilde{\beta}_n)=0$, for $i\neq 1$. Similarly we get the vanishing of the étale Chern classes. We computed in [25, Lemma 4.1] that $\overline{c}^{\text{syn}}_{1,2}(\tilde{\beta}_n)=t$ and $\overline{c}^{\text{ét}}_{1,2}(\beta_n)=\zeta_n$. The rest follows from the product formulas in Lemma 2.1. □

We would like now to relate K-theory mod $p^n$ to étale cohomology. The following proposition shows that we can invert étale Chern classes modulo some constants.

**Proposition 3.2.** *Let $Y$ be a smooth scheme of dimension $d$ over $\overline{K}$, and let $p^n\geq 5$. Let $j\geq\max\{2d,2\}$, $j\geq 3$ for $d=0$ and $p=2$, and $2i-j\geq 0$. There exists an integer $T(d,i,j)$ depending only on $d$, $i$, and $j$ such that, the kernel and cokernel of the Chern classes*

$$\overline{c}^{\text{ét}}_{ij}:\operatorname{gr}^i_\gamma K_j(Y;\mathbf{Z}/p^n)\to H^{2i-j}(Y,\mathbf{Z}/p^n(i))$$

*are annihilated by $T(d,i,j)$. An odd prime $p$ divides $T(d,i,j)$ if and only if $p\leq d+j+1$.*

*Proof.* This proposition is a $K$-theory version of the following theorem.

**Theorem 3.3.** *(Suslin [31]). The change of topology map*

$$H^j(Y_{\text{Zar}},\mathbf{Z}/p^n(i))\to H^j(Y_{\text{ét}},\mathbf{Z}/p^n(i)).$$

*is an isomorphism for $i\geq d$.*

Here $\mathbf{Z}/p^n(i)$ are the complexes of motivic sheaves $\mathbf{Z}/p^n(i):=X\mapsto z^i(X,2i-*)\otimes\mathbf{Z}/p^n$ in the Zariski and étale topology, respectively. Recall how the complex $z^r(X,*)$ is defined [2]. Denote by $\triangle^n$ the algebraic n-simplex $\operatorname{Spec}\mathbf{Z}[t_0,\dots,t_n]/(\sum t_i-1)$. Let $z^r(X,i)$ be the free abelian group generated by irreducible codimension $r$ subvarieties of $X\times\triangle^i$ meeting all faces properly. Then $z^r(X,*)$ is the chain complex thus defined with boundaries given by pullbacks of cycles along face maps. We know that $H^j(X,\mathbf{Z}/p^n(i))\simeq CH^i(X,\mathbf{Z}/p^n(2i-j))$ is the Bloch higher Chow group and the cycle class defines an isomorphism of $H^j(X_{\text{ét}},\mathbf{Z}/p^n(i))$ with the corresponding étale cohomology group (hence we will use the same notation for both).

**Remark 3.4.** Suslin states this theorem for quasi-projective schemes but it is in fact true for any separated scheme (see [12]).

Let's start the proof. We know that the algebraic Chern class $\overline{c}^{\text{ét}}_{ij}$ is equal to the following composition

$$\overline{c}_{ij}:K_j(Y;\mathbf{Z}/p^n)\xrightarrow{\rho_j}K^{\text{Top}}_j(Y;\mathbf{Z}/p^n)\xrightarrow{c_{ij}}H^{2i-j}(Y,\mathbf{Z}/p^n(i)),\quad j\geq 2,$$

where $K^{\text{Top}}_j(Y;\mathbf{Z}/p^n)$ is the étale $K$-theory of Dwyer and Friedlander [4], [10]. The natural map

$$\rho_j:K_j(Y;\mathbf{Z}/p^n)\to K^{\text{Top}}_j(Y;\mathbf{Z}/p^n)$$

from algebraic to étale $K$-theory is Thomason's sheafified version [32, 4.15] of the map defined in [4] for affine schemes and $c_{ij}$ is the topological Chern class [30, 4.1.4]. The map $\rho_j$ is compatible with $\gamma$-operations. Recall [32, 4.11] that we have an isomorphism

$$K_j(Y;\mathbf{Z}/p^n)[\beta_n^{-1}]\xrightarrow[\sim]{\rho_j}K^{\text{Top}}_j(Y;\mathbf{Z}/p^n).$$

We will write $K^{\text{ét}}_j(Y;\mathbf{Z}/p^n)$ for $K_j(Y;\mathbf{Z}/p^n)[\beta_n^{-1}]$ (or for the isomorphic cohomology of $K$-theory spectra in étale topology).

We have a Dwyer-Friedlander spectral sequence

$$E_2^{s,q} = \begin{cases} H^s(Y, \mathbf{Z}/p^n(i)) & \text{if } 0 \leq s \leq q = 2i, \\ 0 & \text{otherwise} \end{cases}$$

converging to $K_{q-s}^{\mathrm{Top}}(Y; \mathbf{Z}/p^n)$, $q-s \geq 3$. Here the differential $d_r : E_r^{s,q} \to E_r^{s+r,q+r-1}$. Let $F^s K_j^{\mathrm{Top}}(Y; \mathbf{Z}/p^n)$ denote the filtration on $K_j^{\mathrm{Top}}(Y; \mathbf{Z}/p^n)$ defined by this spectral sequence. We can prove, as in Soulé [30, 3.4], that

$$M(d,i,j) F^{2i-j} K_j^{\mathrm{Top}}(Y; \mathbf{Z}/p^n) \subset \widetilde{F}_\gamma^i K_j^{\mathrm{Top}}(Y; \mathbf{Z}/p^n) \subset F^{2i-j} K_j^{\mathrm{Top}}(Y; \mathbf{Z}/p^n). \tag{3.3.1}$$

We also know [30, 4.2] that $c_{ij}$ restricts to zero on $F^{2i-j+1} K_j^{\mathrm{Top}}(Y; \mathbf{Z}/p^n)$. Hence, it induces a map

$$\overline{c}_{ij} : \widetilde{F}_\gamma^i / \widetilde{F}_\gamma^{i+1} K_j^{\mathrm{Top}}(Y; \mathbf{Z}/p^n) \xrightarrow{f} F^{2i-j}/F^{2i-j+2} K_j^{\mathrm{Top}}(Y; \mathbf{Z}/p^n) \xrightarrow{g} H^{2i-j}(Y, \mathbf{Z}/p^n(i)).$$

We claim that the kernel of $\overline{c}_{ij}$ is annihilated by $M(d,i+1,j)(i-1)!$ and its cokernel by $M(d,i,j)M(2d)^2(i-1)!$. Indeed, by the inclusions (3.3.1), the map $f$ has kernel and cokernel annihilated by $M(d,i+1,j)$, respectively $M(d,i,j)$. Concerning the map $g$, notice that, by Soulé [30, 4.2], the image of $c_{ij}$ in $H^{2i-j}(Y, \mathbf{Z}/p^n(i)) = E_2^{2i-j,2i}$ lies in the kernel $K^{2i-j,2i}$ of all higher differentials $d_r$, $r \geq 2$, in the Dwyer-Friedlander spectral sequence. Hence we have a factorization

$$g : F^{2i-j}/F^{2i-j+2} K_j^{\mathrm{Top}}(Y; \mathbf{Z}/p^n) \to K^{2i-j,2i} \hookrightarrow H^{2i-j}(Y, \mathbf{Z}/p^n(i)).$$

Since [30, 3.3.2] $M(2d)d_r = 0$ for any $r \geq 2$, the cokernel of the inclusion $K^{2i-j,2i} \hookrightarrow H^{2i-j}(Y, \mathbf{Z}/p^n(i))$ is annihilated by $M(2d)^d$. Consider now the composition

$$\begin{aligned} E_\infty^{2i-j,2i} = & F^{2i-j}/F^{2i-j+2} K_j^{\mathrm{Top}}(Y; \mathbf{Z}/p^n) \to K^{2i-j,2i} \\ & \xrightarrow{\phi_{ij}} F^{2i-j}/F^{2i-j+2} K_j^{\mathrm{Top}}(Y; \mathbf{Z}/p^n) = E_\infty^{2i-j,2i}, \end{aligned}$$

where $\phi_{ij}$ is the natural projection (a surjection). This composition is proved in [30, 4.2] to be equal to multiplication by $(-1)^{i-1}(i-1)!$. Hence the kernel of $g$ is annihilated by $(i-1)!$. Also, since $M(2d)d_r = 0$ for any $r \geq 2$, the kernel of $\phi_{ij}$ is annihilated by $M(2d)^d$. Hence the cokernel of $g$ is annihilated by $M(2d)^{2d}(i-1)!$.

Consider now the change of topology map

$$\psi_j : \widetilde{F}_\gamma^i / \widetilde{F}_\gamma^{i+1} K_j(Y; \mathbf{Z}/p^n) \xrightarrow{\tilde{\rho}_j} \widetilde{F}_\gamma^i / \widetilde{F}_\gamma^{i+1} K_j^{\text{ét}}(Y; \mathbf{Z}/p^n).$$

We claim that it is surjective and its kernel is annihilated by $M(d,i+1,2j)$. This will follow from Suslin's theorem mentioned above (comparing Zariski and étale motivic cohomologies) via spectral sequences relating motivic cohomology to algebraic $K$-theory. Recall that Levine [23, 12.2, 12.13] has constructed compatible Zariski and étale Atiyah-Hirzebruch spectral sequences from motivic cohomology to $K$-theory:

$$\begin{aligned} E_2^{s,q} &= H^s(Y_{\mathrm{Zar}}, \mathbf{Z}/p^n(q/2)) \Rightarrow K_{s-q}(Y; \mathbf{Z}/p^n) \\ E_2^{s,q} &= H^s(Y, \mathbf{Z}/p^n(q/2)) \Rightarrow K_{s-q}^{\text{ét}}(Y; \mathbf{Z}/p^n). \end{aligned}$$

Here the differential $d_r : E_r^{s,q} \to E_r^{s+r,q+r-1}$.Denote by $F_{AH}^i$ the filtration on $K$-theory groups defined by these spectral sequences. Levine shows [23, 10.8, 11.6] that $\widetilde{F}_\gamma^i K_j^{\text{ét}}(Y; \mathbf{Z}/p^p) \subset F_{AH}^i K_j^{\text{ét}}(Y; \mathbf{Z}/p^n)$ and that

$$M(d,i,2j) F_{AH}^i K_j(Y; \mathbf{Z}/p^n) \subset \widetilde{F}_\gamma^i K_j(Y; \mathbf{Z}/p^n) \subset F_{AH}^i K_j(Y; \mathbf{Z}/p^n).$$

Consider the following diagram

$$\begin{array}{ccc} \widetilde{F}_\gamma^i / \widetilde{F}_\gamma^{i+1} K_j(Y; \mathbf{Z}/p^n) & \xrightarrow{\psi_j} & \widetilde{F}_\gamma^i / \widetilde{F}_\gamma^{i+1} K_j^{\text{ét}}(Y; \mathbf{Z}/p^n) \\ \downarrow & & \downarrow \\ F_{AH}^i / F_{AH}^{i+1} K_j(Y; \mathbf{Z}/p^n) & \xrightarrow{\sim} & F_{AH}^i / F_{AH}^{i+1} K_j^{\text{ét}}(Y; \mathbf{Z}/p^n) \end{array}$$

By the above, the kernel of the left vertical map is annihilated by $M(d,i+1,2j)$. Using Suslin's theorem via an analysis of the above Atiyah-Hirzebruch spectral sequences Friedlander-Walker [11] show that the natural map

$$\psi_j : K_j(Y;\mathbf{Z}/p^n) \to K_j^{\text{ét}}(Y;\mathbf{Z}/p^n)$$

is surjective for $j \geq 2d$ and that the induced map

$$F^i_{AH}/F^{i+1}_{AH}K_j(Y;\mathbf{Z}/p^n) \xrightarrow{\sim} F^i_{AH}/F^{i+1}_{AH}K_j^{\text{ét}}(Y;\mathbf{Z}/p^n)$$

is an isomorphism for $i \geq d$, $j \geq 2d$. Our claim follows now easily.

Finally, recall (2.2.3) that the cokernel of the natural map

$$\widetilde{F}^i_{\gamma}/\widetilde{F}^{i+1}_{\gamma}K_j(Y;\mathbf{Z}/p^n) \to F^i_{\gamma}/F^{i+1}_{\gamma}K_j(Y;\mathbf{Z}/p^n)$$

is annihilated by $M(d,i,2j)$.

Combining all of the above, we get that, for $j \geq 2d$, the cokernel of

$$\overline{c}^{\text{ét}}_{ij} : \operatorname{gr}^i_{\gamma} K_j(Y;\mathbf{Z}/p^n) \to H^{2i-j}(Y,\mathbf{Z}/p^n(i))$$

is annihilated by $M(d,i,j)M(2d)^{2d}(i-1)!$, and its kernel is annihilated by $M(d,i+1,2j)M(d,i,2j)M(d,i+1,j)(i-1)!$. Take

$$T = T(d,i,j) = (i-1)!M(d,i,j)M(d,i+1,j)M(d,i+1,2j)M(d,i,2j)M(2d)^{2d}.$$

Since an odd prime $p$ divides $M(d,i,j)$ if and only if $p < (j+2d+3)/2$, and divides $M(l)$ if and only if $p < (l/2)+1$, we get the last statement of the proposition. □

We will show now that we can pass from the K-theory of the generic fiber to the K-theory of a regular model. Let $K_1$ denote a finite extension of $K$ and let $V_1$ be its ring of integers.

**Lemma 3.5.** *Let $X$ be a regular flat scheme over $V_1$ and $j : X_{K_1} \hookrightarrow X$ the open immersion. Then the restriction*

$$j^* : K_j(X;\mathbf{Z}/p^n) \xrightarrow{\sim} K_j(X_{K_1};\mathbf{Z}/p^n), \quad j > d+1,$$

*is an isomorphism and the induced map*

$$j^* : \operatorname{gr}^i_{\gamma} K_j(X;\mathbf{Z}/p^n) \to \operatorname{gr}^i_{\gamma} K_j(X_{K_1};\mathbf{Z}/p^n), \qquad j > d+1$$

*has kernel and cokernel annihilated by $M(d,i+1,2j)$ and $M(d,i,2j)$, respectively.*

*Proof.* We have the localization sequence

$$\to K'_j(X_{k_1};\mathbf{Z}/p^n) \to K'_j(X;\mathbf{Z}/p^n) \to K'_j(X_{K_1};\mathbf{Z}/p^n) \to K'_{j-1}(X_{k_1};\mathbf{Z}/p^n) \to,$$

where $k_1$ is the residue field of $V_1$. Since $X$ is regular, it suffices to show that $K'_j(X_{k_1};\mathbf{Z}/p^n) = 0$ for $j > d$. We will argue by induction on $d$. By devissage we may assume $X_{k_1}$ to be reduced. Let $S$ be the singular locus of $X_{k_1}$ and $U$ the open complement. We have the associated localization sequence

$$\to K'_j(S;\mathbf{Z}/p^n) \to K'_j(X;\mathbf{Z}/p^n) \to K'_j(U;\mathbf{Z}/p^n) \to K'_{j-1}(S;\mathbf{Z}/p^n) \to .$$

Since $U$ is nonsingular, $K'_j(U;\mathbf{Z}/p^n) \simeq K_j(U;\mathbf{Z}/p^n) = 0$ for $j > d$ by [13]. By induction $K'_j(S;\mathbf{Z}/p^n) = 0$ for $j \geq d$, hence the claim.

It follows that we have an isomorphism

$$j^* : \widetilde{F}^i_{\gamma}K_j(X;\mathbf{Z}/p^n) \to \widetilde{F}^i_{\gamma}K_j(X_{K_1};\mathbf{Z}/p^n), \qquad j > d+1,\ j > 1.$$

Take $x \in F^i_{\gamma}K_j(X_{K_1};\mathbf{Z}/p^n)$. By 2.2.3, $M(d,i,2j)x \in \widetilde{F}^i_{\gamma}K_j(X_{K_1};\mathbf{Z}/p^n)$ giving the statement about cokernel. For kernel, let $x \in F^i_{\gamma}K_j(X;\mathbf{Z}/p^n)$ be such that $j^*(x) \in F^{i+1}_{\gamma}K_j(X_{K_1};\mathbf{Z}/p^n)$. Again by 2.2.3, $M(d,i+1,2j)j^*(x) \in \widetilde{F}^{i+1}_{\gamma}K_j(X_{K_1};\mathbf{Z}/p^n)$. Hence $M(d,i+1,2j)j^*(x) = j^*(y)$ for some $y \in \widetilde{F}^{i+1}_{\gamma}K_j(X;\mathbf{Z}/p^n)$. By injectivity of $j^*$ we get that $M(d,i+1,2j)x = y$ giving injectivity on the grading. □

3.1. **The comparison morphism.** We are now ready to define our comparison map. Let $X^\times$ be a log-smooth vertical log-scheme over $V^\times$ of pure relative dimension $d$ and with Cartier type reduction. Note that the log-structure of $X^\times_K$ is trivial and the base change $X^\times_{V_1}$ for any finite extension $V_1$ of $V$ is saturated. In particular, the special fiber of $X^\times$ is of Cartier type. Assume that $p^n \geq 5$ and $b \geq 2d+1$, and if $d=0$, $p=2$ assume that $b \geq 2$. Define a transformation

$$\alpha^n_{ab} : H^a(X_{\overline{K}}, \mathbf{Z}/p^n(b)) \to H^a_{\mathrm{cr}}(X^\times_{\overline{V},n}/W_n(k), \mathcal{O}_{X^\times_{\overline{V},n}/W_n(k)})\{-b\},$$

where $\{-b\}$ denotes a twist of the filtration ($H^a_{\mathrm{cr}}(X^\times_{\overline{V},n}/W_n(k), \mathcal{O}_{X^\times_{\overline{V},n}/W_n(k)})$ is equipped with the natural crystalline Hodge filtration) and the Frobenius, in the following way.

For $x \in H^a(X_{\overline{K}}, \mathbf{Z}/p^n(b))$, take $(\overline{c}^{\text{ét}}_{b,2b-a})^{-1}(x) \in F^b_\gamma/F^{b+1}_\gamma K_{2b-a}(X_{\overline{K}}; \mathbf{Z}/p^n)$ to be any element in the preimage of $T(d,b,2b-a)x$ (this is possible by Proposition 3.2). Let $x_1 \in F^b_\gamma K_{2b-a}(X_{\overline{K}}; \mathbf{Z}/p^n)$ be a lifting of the element $(\overline{c}^{\text{ét}}_{b,2b-a})^{-1}(x)$. Take an extension $V_1$ of $V$ such that the class $x_1$ comes from $x_1 \in F^b_\gamma K_{2b-a}(X_{K_1}; \mathbf{Z}/p^n)$, where $K_1$ is the field of fractions of $V_1$. Notice that the log-scheme $X^\times_{V_1}$ is log-regular. Hence, by [26], there exists a log-blow-up $\pi : Y^\times \to X^\times_{V_1}$ such that the scheme $Y$ is (classically) regular.

By the proof of Lemma 3.5, we can now find a unique element $x_1' \in F^b_\gamma K_{2b-a}(Y; \mathbf{Z}/p^n)$ such that $j^*(x_1') = T(d,b,2b-a)x_1$, where $j^*$ is the restriction $j^* : K_{2b-a}(Y; \mathbf{Z}/p^n) \to K_{2b-a}(Y_{K_1}; \mathbf{Z}/p^n)$ (note that $Y_{K_1} = X_{K_1}$). Set

$$\alpha^n_{ab}(x) := \psi_n \mu_{V_1} (\pi^*)^{-1} \varepsilon \overline{c}^{\mathrm{syn}}_{b,2b-a}(T(d,b,2b-a)x_1').$$

Here $\psi_n \mu_{V_1} (\pi^*)^{-1} \varepsilon$ is the composition

$$H^a(Y, s_n'(b)) \xrightarrow{\varepsilon} H^a(Y^\times, s_n'(b)) \xleftarrow[\pi^*]{\sim} H^a(X^\times_{V_1}, s_n'(b)) \xrightarrow{\mu_{V_1}} H^a(X^\times_{\overline{V}}, s_n'(b))$$
$$\xrightarrow{\psi_n} H^a_{\mathrm{cr}}(X^\times_{\overline{V},n}/W_n(k), \mathcal{O}_{X^\times_{\overline{V},n}/W_n(k)})\{-b\}.$$

**Lemma 3.6.** *The transformation $\alpha^n_{ab}$ is a well-defined natural Galois equivariant group homomorphism giving the following relationship between Tate twists*

$$(-b)T(d,b,2b-a)^3 \alpha^n_{a,b+1}(\zeta_n x) = (-b)T(d,b+1,2b+2-a)^3 \alpha^n_{ab}(x)t.$$

*Proof.* We have made several choices in our construction of $\alpha^n_{ab}$. We will analyze them. Let us first assume that we have fixed the lifting $x_1 \in F^b_\gamma K_{2b-a}(X_{\overline{K}}; \mathbf{Z}/p^n)$. By functoriality of Chern classes, the choice of $K_1$ and of the element $x_1 \in F^b_\gamma K_{2b-a}(X_{K_1}; \mathbf{Z}/p^n)$ is of no importance. For two choices of a regular model $\pi : Y^\times \to X^\times_{V_1}$, we know from [26] that we can find a third regular model that dominates (via a log-blow-up) both of them. Functoriality of Chern classes now yields that the choice of the regular model is of no importance.

The ambiguity introduced by the choice of the lifting $x_1$ comes from an element $y \in F^{b+1}_\gamma K_{2b-a}(X_{\overline{K}}; \mathbf{Z}/p^n)$. After perhaps passing to a finite extension of the base ring and changing the regular model $Y^\times$, we get that the ambiguity in the choice of the corresponding element $x_1'$ comes from an element $w$ of $F^b_\gamma K_{2b-a}(Y; \mathbf{Z}/p^n)$ that maps to $T(d,b,2b-a)y \in F^{b+1}_\gamma K_{2b-a}(X_{K_1}; \mathbf{Z}/p^n)$. Lemma 3.5 now yields that $T(d,b,2b-a)w \in F^{b+1}_\gamma K_{2b-a}(Y; \mathbf{Z}/p^n)$. Lemma 2.1 shows that this ambiguity disappears after we apply the syntomic Chern class morphisms. Similarly, the ambiguity in the choice of $(\overline{c}^{\text{ét}}_{b,2b-a})^{-1}(x)$ disappears after we multiply $x_1'$ by $T(d,b,2b-a)$ and apply the syntomic Chern class morphisms.

For functoriality, use functoriality of Chern classes and K-theory with respect to pullbacks and the fact that (saturated) base change of a log-blow-up is a log-blow-up [26]. Since all the genuine maps we used were group homomorphisms (Lemma 2.1), so is the map $\alpha^n_{ab}$. That it is also Galois equivariant follows from functoriality of Chern classes and functoriality of K-theory with respect to pullbacks.

For the last statement of the lemma, let $x_1' \in F^b_\gamma K_{2b-a}(Y; \mathbf{Z}/p^n)$ and $x_1 \in F^b_\gamma K_{2b-a}(X_{K_1}; \mathbf{Z}/p^n)$ be the elements from the construction of $\alpha^n_{ab}(x)$. Since $\overline{c}^{\text{ét}}_{1,2}(\beta_n) = \zeta_n$ (see Lemma 3.1), we have by Lemma 2.1

$$\overline{c}^{\text{ét}}_{b+1,2b+2-a}(\beta_n x_1) = (-b)\overline{c}^{\text{ét}}_{1,2}(\beta_n)\overline{c}^{\text{ét}}_{b,2b-a}(x_1) = (-b)\zeta_n T(d,b,2b-a)x.$$

It follows that

$$\begin{aligned}&(-b)T(d,b,2b-a)^3\alpha^n_{a,b+1}(\zeta_n x)=\\&\qquad T(d,b,2b-a)\psi_n\mu_{V_1}(\pi^*)^{-1}(\overline{c}^{\mathrm{syn}}_{b+1,2b+2-a}(T(d,b+1,2b+2-a)^3\tilde{\beta}_n x_1'))=\\&\qquad(-b)T(d,b+1,2b+2-a)^3\psi_n\mu_{V_1}(\pi^*)^{-1}(\overline{c}^{\mathrm{syn}}_{1,2}(\tilde{\beta}_n)\overline{c}^{\mathrm{syn}}_{b,2b-a}(T(d,b,2b-a)x_1'))=\\&\qquad(-b)T(d,b+1,2b+2-a)^3\alpha^n_{a,b}(x)t,\end{aligned}$$

as wanted. □

Let $b\geq 2d+1$ and $b\geq 2$ for $d=0$, $p=2$. Assume $X$ to be proper over $V$. Define a morphism

$$\alpha_{ab}:H^a(X_{\overline{K}},\mathbf{Q}_p(b))\to H^a_{\mathrm{cr}}(X_0^\times/W(k)^0,\mathcal{O}_{X_0^\times/W(k)^0})\otimes_{W(k)}B_{\mathrm{st}}\{-b\}$$

as the composition of $\mathbf{Q}\otimes\operatorname{proj}\lim_n\alpha^n_{ab}$ with the map (see section 2.1)

$$\mathbf{Q}\otimes\operatorname*{proj\,lim}_n H^a_{\mathrm{cr}}(X^\times_{\overline{V},n}/W_n(k),\mathcal{O}_{X^\times_{\overline{V},n}/W_n(k)})\xrightarrow{h_\pi}H^a_{\mathrm{cr}}(X_0^\times/W(k)^0,\mathcal{O}_{X_0^\times/W(k)^0})\otimes_{W(k)}B_{\mathrm{st}}$$

and with the division by $T(d,b,2b-a)^3$. The morphism $\alpha_{ab}$ is functorial in $X^\times$, preserves the Frobenius, the action of $\mathrm{Gal}(\overline{K}/K)$ and the monodromy operator, and, after extension to $B_{dR}$, is compatible with filtrations (use Lemma 4.8.4 from [33]).

We would also like to know how the map $\alpha_{ab}$ behaves with respect to finite base changes. In what follows, we will add the subscript $\pi$ to $\alpha_{ab}$ to underscore the fact that in the definition of this map we made a choice of a uniformizer. Let $V_1$ be a finite extension of $V$ with fraction field $K_1$ and residue field $k_1$. Let $e$ be the ramification index of $K_1$ over $K$ and let $\pi_1$ be a uniformizer of $V_1$. Set $X_1^\times:=X^\times_{V_1}$.

**Lemma 3.7.** *The following diagrams commute*

$$\begin{array}{ccc}H^a(X_{\overline{K}},\mathbf{Q}_p(b))\otimes_{\mathbf{Q}_p}B_{\mathrm{st}}&\xrightarrow{\alpha_{ab,\pi}}&H^a_{\mathrm{cr}}(X^\times)\otimes_{W(k)}B_{\mathrm{st}}\\\|&&\wr\downarrow\\H^a(X_{1,\overline{K}_1},\mathbf{Q}_p(b))\otimes_{\mathbf{Q}_p}B_{\mathrm{st}}&\xrightarrow{\alpha_{ab,\pi_1}}&H^a_{\mathrm{cr}}(X_1^\times)\otimes_{W(k_1)}B_{\mathrm{st}},\end{array}$$

$$\begin{array}{ccc}H^a(X_{\overline{K}},\mathbf{Q}_p(b))\otimes_{\mathbf{Q}_p}B_{dR}&\xrightarrow{\alpha^{dR}_{ab,\pi}}&H^a_{dR}(X_K/K)\otimes_K B_{dR}\\\|&&\wr\downarrow\\H^a(X_{1,\overline{K}_1},\mathbf{Q}_p(b))\otimes_{\mathbf{Q}_p}B_{dR}&\xrightarrow{\alpha^{dR}_{ab,\pi_1}}&H^a_{dR}(X_{K_1}/K_1)\otimes_{K_1}B_{dR}.\end{array}$$

*In particular, the maps $\alpha_{ab}$ and $\alpha^{dR}_{ab}$ are independent of the choice of the uniformizer $\pi$.*

*Proof.* Arguing exactly like Tsuji in his proof of a similar statement [33, 4.10.4], we reduce to showing that the maps

$$\begin{aligned}\alpha^n_{ab}&:H^a(X_{\overline{K}},\mathbf{Z}/p^n(b))\to H^a_{\mathrm{cr}}(X^\times_{\overline{V},n}/W_n(k),\mathcal{O}_{X^\times_{\overline{V},n}/W_n(k)}),\\\iota_\pi\rho_\pi h_\pi&:H^a_{\mathrm{cr}}(X^\times_{\overline{V}}/W(k))\to H^a_{dR}(X_K/K)\otimes_K B^+_{dR}\end{aligned}$$

are compatible with our base changes.

In the case of the map $\alpha^n_{ab}$ this is obvious from its construction.

In the case of the map $\iota_\pi\rho_\pi h_\pi$, since from the definition [33, 4.7.3] of the isomorphism $B^+_{dR}\otimes_{K_0}H^a_{dR}(X_K/K)\xrightarrow{\sim}\operatorname{proj}\lim_s(\mathbf{Q}\otimes H^a_{\mathrm{cr}}(X^\times_{\overline{V}}/V^\times,\mathcal{O}/J^{[s]}))$ it is easy to see that it is compatible with our base change, it suffices to show that so is its composition with $\iota_\pi\rho_\pi h_\pi$. Since this composition is equal [33, 4.8.4] to the natural map

$$H^a_{\mathrm{cr}}(X^\times_{\overline{V}}/W(k))\to\operatorname*{proj\,lim}_s(\mathbf{Q}\otimes H^a_{\mathrm{cr}}(X^\times_{\overline{V}}/V^\times,\mathcal{O}/J^{[s]})),$$

this is clear. □

### 3.2. **Main theorem.**

**Theorem 3.8.** *Let $X^\times$ be a proper log-smooth saturated vertical $V^\times$-scheme with Cartier type reduction of pure relative dimension $d$. Then, assuming $b \geq 2d+1$ and $b \geq 2$ for $d=0$, $p=2$, the natural morphism*

$$\alpha_{ab} : H^a(X_{\overline{K}}, \mathbf{Q}_p(b)) \otimes_{\mathbf{Q}_p} B_{\mathrm{st}} \to H^a_{\mathrm{cr}}(X_0^\times/W(k)^0, \mathcal{O}_{X_0^\times/W(k)^0}) \otimes_{W(k)} B_{\mathrm{st}}\{-b\}$$

*is an isomorphism. Moreover, the map $\alpha_{ab}$ preserves the Frobenius, the action of $\mathrm{Gal}(\overline{K}/K)$ and the monodromy operator. It is independent of the choice of $\pi$, compatible with base changes and Tate twists, and, after extension to $B_{dR}$, induces an isomorphism of filtrations.*

*Proof.* The listed compatibilities follow from Lemma 3.6 and Lemma 3.7. The line of the argument is standard [9]. Namely, since both sides of $\alpha_{ab}$ have the same rank over $B_{\mathrm{st}}$, it suffices to show that the morphism $\alpha_{ab}$ has a $B_{\mathrm{st}}$-linear left inverse. That, in turn, would follow – by Poincaré duality – from the compatibility of $\alpha_{ab}$ with products and traces.

First, we have to check that the morphism $\alpha_{ab}$ is compatible with products. This follows from the fact that the morphism $h_\pi$ is compatible with products and from the following lemma

**Lemma 3.9.** *Let $x \in H^a(X_{\overline{K}}, \mathbf{Z}/p^n(b))$, $y \in H^c(X_{\overline{K}}, \mathbf{Z}/p^n(e))$, $2b-a>2$, $2e-c>2$, and $p^n \geq 5$. Set $K(b,e) = -(b+e-1)!/((b-1)!(e-1)!)$. Then (assuming that all the indices are in the valid range)*

$$\begin{aligned} K(b,e)T(d,b,2b-a)^3 T(d,e,2e-c)^3 \alpha^n_{a+c,b+e}(x \cup y) \\ = K(b,e)T(d,b+e,2b+2e-a-c)^3 \alpha^n_{ab}(x) \cup \alpha^n_{ce}(y). \end{aligned}$$

*Proof.* Use Lemma 2.1. □

Next, by taking a finite unramified extension of $K$, we may assume that $X_K$ is geometrically irreducible. The case of $d=0$ is handled easily: we use Lemma 3.1. For $d \geq 1$, since the domain and the target satisfy Poincaré duality, to show that $\alpha_{a,b}$ has a left inverse, it suffices to verify that the map

$$\alpha_{2d,2b} : H^{2d}(X_{\overline{K}}, \mathbf{Q}_p(2b)) \otimes_{\mathbf{Q}_p} B_{\mathrm{st}} \to H^{2d}_{\mathrm{cr}}(X_0^\times/W(k)^0, \mathcal{O}_{X_0^\times/W(k)^0}) \otimes_{W(k)} B_{\mathrm{st}}\{-2b\}$$

is an isomorphism. Notice that

$$\dim_{\mathbf{Q}_p} H^{2d}(X_{\overline{K}}, \mathbf{Q}_p(2b)) = \dim_{K_0} H^{2d}_{\mathrm{cr}}(X_0^\times/W(k)^0, \mathcal{O}_{X_0^\times/W(k)^0}) \otimes_{W(k)} K_0 = \dim_K H^{2d}_{dR}(X_K/K) = 1.$$

By taking a finite unramified extension of $K$, we may take a rational point $P$ of the smooth locus of $X$ over $V$ (note that the special fiber of $X$ is reduced [33, 2.7.7]). Since $H^{2d}(X_{\overline{K}}, \mathbf{Q}_p(2b))$ is generated by $\mathrm{cl}^{\mathrm{\acute{e}t}}(P_K)\zeta^{2b-d}$, it suffices to show that $\alpha_{2d,2b}$ maps $\mathrm{cl}^{\mathrm{\acute{e}t}}(P_K)\zeta^{2b-d} \otimes t^{-2b+d}$ to a nontrivial element of $H^{2d}_{\mathrm{cr}}(X_0^\times/W(k)^0, \mathcal{O}_{X_0^\times/W(k)^0}) \otimes K_0$.

For that, we will first show that the extension of $\alpha_{2d,2b}$ to $B_{dR}$

$$\iota_\pi \rho_\pi \alpha_{2d,2b} : H^{2d}(X_{\overline{K}}, \mathbf{Q}_p(2b)) \otimes_{\mathbf{Q}_p} B_{dR} \longrightarrow H^{2d}_{dR}(X_K/K) \otimes_K B_{dR}$$

maps $\mathrm{cl}^{\mathrm{\acute{e}t}}(P_K)\zeta^{2b-d} \otimes t^{-2b+d}$ to $\mathrm{cl}^{dR}(P_K)$. Or that, by $B_{dR}$-linearity,

$$\iota_\pi \rho_\pi \alpha_{2d,2b}(\mathrm{cl}^{\mathrm{\acute{e}t}}(P_K)\zeta^{2b-d}) = \mathrm{cl}^{dR}(P_K) t^{2b-d}.$$

Let $\pi : Y^\times \to X^\times$ be a (saturated) log-blow-up that does not modify the regular locus of $X^\times$ and such that $Y$ is regular. Denote by $P'$ the unique $V$-point of $Y$ lying over $P$ (note that $Y_K \simeq X_K$ and $P'_K = P_K$). Let $[\mathcal{O}_{P'}]$ and $[\mathcal{O}_{P_K}]$ denote the class of $\mathcal{O}_{P'}$ and $\mathcal{O}_{P_K}$ in $K_0(Y)$ and $K_0(X_K)$, respectively ( $Y$ is regular !). Recall (see the proof of Lemma 4.2 in [25]), that there exists a constant $s(d)$ (depending only on the dimension $d$) such that $s(d)[\mathcal{O}_{P'}] \in F^d_\gamma K_0(Y)$ and that we have $c^{\mathrm{\acute{e}t}}_{d,0}(s(d)[\mathcal{O}_{P_K}]) = s(d)c^{\mathrm{\acute{e}t}}_{d,0}([\mathcal{O}_{P_K}])$. By exactly the same argument, $c^{dR}_{d,0}(s(d)[\mathcal{O}_{P_K}]) = s(d)c^{dR}_{d,0}([\mathcal{O}_{P_K}])$. We also know that

$$c^{\mathrm{\acute{e}t}}_{d,0}([\mathcal{O}_{P_K}]) = (-1)^{d-1}(d-1)!\,\mathrm{cl}^{\mathrm{\acute{e}t}}(P_K), \qquad c^{dR}_{d,0}([\mathcal{O}_{P_K}]) = (-1)^{d-1}(d-1)!\,\mathrm{cl}^{dR}(P_K).$$

We now pass to torsion coefficients. Take a field extension $K_1/K$ containing $\zeta_n$. We have the following commutative diagram (not necessarily cartesian)

$$\begin{array}{ccccc} & & Y_1^{\times} & \xrightarrow{f_Y} & Y^{\times} \\ & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ X_{\overline{V}}^{\times} & \xrightarrow{\mu_{V_1}} & X_{V_1}^{\times} & \xrightarrow{f} & X^{\times}, \end{array}$$

where $V_1$ is the ring of integers of $K_1$ and $\pi_1$ is a log-blow-up with regular model $Y_1$.

Since $s(d)f_K^*[\mathcal{O}_{P_K}]\beta_n^{2b-d}\in\mathcal{F}^{2b}K_{2(2b-d)}(X_{K_1};\mathbf{Z}/p^n)$, inclusions 2.2.3 imply that

$$M(d,2b,4(2b-d))s(d)f_K^*[\mathcal{O}_{P_K}]\beta_n^{2b-d}\in F^{2b}K_{2(2b-d)}(X_{K_1};\mathbf{Z}/p^n).$$

Set $s(b,d)=M(d,2b,4(2b-d))s(d)$. It suffices to show that

$$\iota_\pi\rho_\pi\alpha_{2d,2b}(c_{d,0}^{\text{ét}}(s(b,d)[\mathcal{O}_{P_K}])\zeta^{2b-d})=c_{d,0}^{dR}(s(b,d)[\mathcal{O}_{P_K}])t^{2b-d}.$$

Now, the product formulas for the étale and the syntomic Chern classes (see Lemma 2.1) yield that for $p^n\geq 5$

$$(d-1)!\overline{c}_{2b,2(2b-d)}^{\text{ét}}(s(b,d)f_K^*[\mathcal{O}_{P_K}]\beta_n^{2b-d})=(-1)^{2b-d}(2b-1)!c_{d,0}^{\text{ét}}(s(b,d)f_K^*[\mathcal{O}_{P_K}])\zeta_n^{2b-d},$$
$$(d-1)!\overline{c}_{2b,2(2b-d)}^{\text{syn}}(s(b,d)f_Y^*[\mathcal{O}_{P'}]\tilde{\beta}_n^{2b-d})=(-1)^{2b-d}(2b-1)!c_{d,0}^{\text{syn}}(s(b,d)f_Y^*[\mathcal{O}_{P'}])\overline{c}_{1,2}^{\text{syn}}(\tilde{\beta}_n)^{2b-d}.$$

Since the class $f_Y^*[\mathcal{O}_{P'}]$ restricts to $f_K^*[\mathcal{O}_{P_K}]$, we compute

$$\begin{aligned}(-1)^{2b-d}&(2b-1)!\alpha_{2d,2b}^n(c_{d,0}^{\text{ét}}(s(b,d)[\mathcal{O}_{P_K}])\zeta_n^{2b-d})\\ &=(-1)^{2b-d}(2b-1)!\alpha_{2d,2b}^n(c_{d,0}^{\text{ét}}(s(b,d)f_K^*[\mathcal{O}_{P_K}])\zeta_n^{2b-d})\\ &=T(d,2b,4b-2d)^3(d-1)!\psi_n\mu_{V_1}^*(\pi_1^*)^{-1}\varepsilon\overline{c}_{2b,2(2b-d)}^{\text{syn}}(s(b,d)f_Y^*[\mathcal{O}_{P'}]\tilde{\beta}_n^{2b-d})\\ &=(-1)^{2b-d}T(d,2b,4b-2d)^3(2b-1)!\psi_n\mu_{V_1}^*(\pi_1^*)^{-1}\varepsilon(c_{d,0}^{\text{syn}}(s(b,d)f_Y^*[\mathcal{O}_{P'}])\overline{c}_{1,2}^{\text{syn}}(\tilde{\beta}_n)^{2b-d})\\ &=(-1)^{2b-d}T(d,2b,4b-2d)^3(2b-1)!\psi_n\mu_{V_1}^*(\pi_1^*)^{-1}\varepsilon f_Y^*(c_{d,0}^{\text{syn}}(s(b,d)[\mathcal{O}_{P'}]))t^{2b-d}\\ &=(-1)^{2b-d}T(d,2b,4b-2d)^3(2b-1)!\psi_n\mu^*(\pi^*)^{-1}\varepsilon(c_{d,0}^{\text{syn}}(s(b,d)[\mathcal{O}_{P'}]))t^{2b-d},\end{aligned}$$

where $\mu=f\mu_{V_1}$.

Passing to the limit and tensoring with $\mathbf{Q}$ we see that it suffices to show the following compatibility of the crystalline and the de Rham Chern classes

$$\iota_\pi\rho_\pi h_\pi\mu^*(\pi^*)^{-1}c_{d,0}^{\text{cr}}(s(b,d)[\mathcal{O}_{P'}])=c_{d,0}^{dR}(s(b,d)[\mathcal{O}_{P_K}]),$$

where $c_{d,0}^{\text{cr}}:K_0(Y)\to H_{\text{cr}}^{2d}(Y^{\times}/W(k))$ denotes the composition of the crystalline Chern class map $c_{d,0}^{\text{cr}}:K_0(Y)\to H_{\text{cr}}^{2d}(Y/W(k))$ with the canonical map $H_{\text{cr}}^{2d}(Y/W(k))\to H_{\text{cr}}^{2d}(Y^{\times}/W(k))$. Or that, for any $j\geq 1$, the following diagram commutes

$$\begin{array}{ccccc} K_0(Y) & \xrightarrow{c_{j,0}^{dR}} & H_{dR}^{2j}(X_K/K) & \longrightarrow & B_{dR}^+\otimes_K H_{dR}^{2j}(X_K/K) \\ \downarrow{\scriptstyle c_{j,0}^{\text{cr}}} & & & & \uparrow{\scriptstyle \iota_\pi\rho_\pi h_\pi} \\ \mathbf{Q}\otimes H_{\text{cr}}^{2j}(Y^{\times}/W(k)) & \xleftarrow[\pi^*]{\sim} & \mathbf{Q}\otimes H_{\text{cr}}^{2j}(X^{\times}/W(k)) & \xrightarrow{\mu^*} & \mathbf{Q}\otimes H_{\text{cr}}^{2j}(X_{\overline{V}}^{\times}/W(k)). \end{array}$$

Recall now [33, 4.7.6,4.8.4] that the composition

$$H_{\text{cr}}^{2j}(X_{\overline{V}}^{\times}/W(k))\xrightarrow{\iota_\pi\rho_\pi h_\pi}B_{dR}^+\otimes_K H_{dR}^i(X_K/K)\xrightarrow{\sim}\operatorname{proj\,lim}_s(\mathbf{Q}\otimes H_{\text{cr}}^{2j}(X_{\overline{V}}^{\times}/V^{\times},\mathcal{O}/J^{[s]}))$$

is equal to the natural map

$$H_{\text{cr}}^{2j}(X_{\overline{V}}^{\times}/W(k))\to\operatorname{proj\,lim}_s(\mathbf{Q}\otimes H_{\text{cr}}^{2j}(X_{\overline{V}}^{\times}/V^{\times},\mathcal{O}/J^{[s]})).$$

Hence, using the canonical isomorphism

$$\mathbf{Q}\otimes H^*_{\mathrm{cr}}(X^\times/V^\times)\simeq H^*_{dR}(X_K/K)$$

proved by Kato (see Theorem 6.4 in [20]), the functoriality of the de Rham Chern classes, and the fact that $Y$ is proper, we just need to check that the following diagram commutes

$$\begin{array}{ccc} K_0(Y_K) & \xrightarrow{c^{dR}_{j,0}} & H^{2j}_{dR}(Y_K/K) \\ \uparrow & & \wr\uparrow \\ K_0(Y) & \xrightarrow{c^{\mathrm{cr}}_{j,0}} & \mathbf{Q}\otimes H^{2j}_{\mathrm{cr}}(Y^\times/V^\times). \end{array}$$

Here the Chern class $c^{\mathrm{cr}}_{j,0}$ is defined as the composition

$$K_0(Y)\xrightarrow{c^{\mathrm{cr}}_{j,0}}\mathbf{Q}\otimes H^{2j}_{\mathrm{cr}}(Y/V)\to\mathbf{Q}\otimes H^{2j}_{\mathrm{cr}}(Y^\times/V^\times).$$

By the splitting principle (we have the projective space theorem in de Rham cohomology), it suffices to check that the crystalline and the de Rham Chern classes of line bundles are compatible. The crystalline Chern class $c^{\mathrm{cr}}_1: H^1(Y,\mathcal{O}^*_Y)\to H^2_{\mathrm{cr}}(Y^\times/V^\times)$ is defined via the exact sequence

$$0\to 1+J_{Y^\times/V^\times}\to\mathcal{O}^*_{Y^\times/V^\times}\to\mathcal{O}^*_{Y^\times}\to 0$$

and the map $\log: 1+J_{Y^\times/V^\times}\to J_{Y^\times/V^\times}$. Since $Y^\times/V^\times$ is log-smooth its compatibility with the de Rham Chern class can be shown paraphrasing the proof of an analogous fact in the situation with trivial log-structures due to Berthelot and Ogus ([1, Lemma 3.3]).

Now, to finish the proof of the fact that $\alpha_{2d,2b}$ is an isomorphism, note that if

$$x=\alpha_{2d,2b}(\mathrm{cl}^{\text{ét}}(P_K)\zeta^{2b-d}\otimes t^{-2b+d}),$$

then, by the above,

$$\begin{aligned} x &\in (H^{2d}_{\mathrm{cr}}(X^\times_0/W(k)^0,\mathcal{O}_{X^\times_0/W(k)^0})\otimes_{W(k)}B_{st})\cap H^{2d}_{dR}(X_K/K)\\ &\subset (H^{2d}_{\mathrm{cr}}(X^\times_0/W(k)^0,\mathcal{O}_{X^\times_0/W(k)^0})\otimes_{W(k)}B_{st})^{G_K}=H^{2d}_{\mathrm{cr}}(X^\times_0/W(k)^0,\mathcal{O}_{X^\times_0/W(k)^0})\otimes K_0. \end{aligned}$$

Hence $x$ is a generator of the one dimensional vector space $H^{2d}_{\mathrm{cr}}(X^\times_0/W(k)^0,\mathcal{O}_{X^\times_0/W(k)^0})\otimes K_0$ (it is nontrivial since its image in $H^{2d}_{dR}(X_K/K)$ is $\mathrm{cl}^{dR}(P_K)$), as wanted.

To finish the proof of the theorem, it remains to show that $\alpha^{dR}_{ab}$ ($\alpha_{ab}$ extended to $B_{dR}$) induces an isomorphism on filtrations. Passing to the associated grading, one reduces to showing that the induced map

$$\overline{\alpha}^{dR,l}_{ab}:\mathbf{C}_p(l)\otimes_{\mathbf{Q}_p}H^a(X_{\overline{K}},\mathbf{Q}_p(b))\to\bigoplus_{j\in\mathbf{Z}}\mathbf{C}_p(b+l-j)\otimes_K H^{a-j}(X_K,\Omega^j_{X_K/K}),\quad l\in\mathbf{Z},$$

is injective. Since the domain of $\overline{\alpha}^{dR,l}_{ab}$ satisfies Poincaré duality and $\overline{\alpha}^{dR,l}_{ab}$ is compatible with products, for $\overline{\alpha}^{dR}_{ab}$ to be injective, it suffices to show that

$$\overline{\alpha}^{dR,0}_{2d,2b}:\mathbf{C}_p\otimes_{\mathbf{Q}_p}H^{2d}(X_{\overline{K}},\mathbf{Q}_p(2b))\to\mathbf{C}_p(2b-d)\otimes_K H^d(X_K,\Omega^d_{X_K/K})=\mathbf{C}_p(2b-d)\otimes_K H^{2d}_{dR}(X_K/K)$$

is an isomorphism. Since both the target and the domain are one-dimensional and, by the above, $\overline{\alpha}^{dR,0}_{2d,2b}(\mathrm{cl}^{\text{ét}}(P_K)\zeta^{2b-d})=\mathrm{cl}^{dR}(P_K)t^{2b-d}\neq 0$, we are done. □

## References


[1] P. Berthelot, A. Ogus, *F-isocrystals and de Rham cohomology. I.* Invent. Math. 72 (1983), no. 2, 159–199.

[2] S. Bloch, *Algebraic cycles and higher K-theory*, Adv. in Math. 61 (1986), no. 3, 267–304.

[3] W. Browder, *Algebraic K-theory with coefficients* $\mathbf{Z}/p$, Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), I, pp. 40–84, Lecture Notes in Math., 657, Springer, Berlin, 1978.

[4] W. Dwyer, E. Friedlander, *Algebraic and étale K-theory*, Trans. Amer. Math. Soc. **292** (1985), no. 1, 247–280.

[5] G. Faltings, *p-adic Hodge-theory*, J. of the AMS **1** (1988), 255-299.

[6] G. Faltings, *Almost étale extensions*, Cohomologies $p$-adiques et applications arithmétiques, II. Astrisque No. 279 (2002), 185–270.

[7] J.-M. Fontaine, *Sur certains types de représentations p-adiques du groupe de Galois d'un corps local, construction d'un anneau de Barsotti-Tate*, Ann. of Math. **115** (1982), 529–577.
[8] J.-M. Fontaine, *Le corps des périodes p-adiques*, Astérisque **223**, Soc. Math. de France, 1994, 59–111.
[9] J.-M. Fontaine and W. Messing, *p-adic periods and p-adic étale cohomology*, Current Trends in Arithmetical Algebraic Geometry (K. Ribet, ed.), Contemporary Math., vol. 67, Amer. Math. Soc., Providence, 1987, 179–207.
[10] E. Friedlander, *Étale K-theory. II. Connections with algebraic K-theory*, Ann. Sci. École Norm. Sup., IV. Ser., **15** (1982), 231–256.
[11] E. Friedlander, M. Walker, *Some remarks concerning mod-n K-theory*, Invent. Math. 145 (2001), no. 3, 545–555.
[12] T. Geisser, *Arithmetic homology, and Kato's conjecture over finite fields*, preprint, 2005.
[13] T. Geisser, M. Levine, *The K-theory of fields in characteristic p*, Invent. Math. **139** (2000), no. 3, 459–493.
[14] H. Gillet, *Riemann-Roch theorems for higher algebraic K-theory*, Adv. Math. **40** (1981), 203–289.
[15] H. Gillet, C. Soulé, *Filtrations on higher algebraic K-theory*, Algebraic $K$-theory (Seattle, WA, 1997), 89–148, Proc. Sympos. Pure Math., **67**, Amer. Math. Soc., Providence, RI, 1999.
[16] M. Gros, *Régulateurs syntomiques et valeur de fonctions L p-adiques I*, Invent. Math. **99** (1990), 293–320.
[17] *Théorie des intersections et théoréme de Riemann-Roch. Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6).* Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre. Lecture Notes in Mathematics, Vol. 225. Springer-Verlag, Berlin-New York, 1971.
[18] O. Hyodo and K. Kato, *semistable reduction and crystalline cohomology with logarithmic poles*, Astérisque **223**, Soc. Math. de France, 1994, 221–268.
[19] K. Kato, *On p-adic vanishing cycles (application of ideas of Fontaine-Messing)*, Algebraic Geometry, Sendai, 1985, Adv. Stud. Pure Math. **10**, North-Holland, Amsterdam-New York, 1987, 207–251.
[20] K. Kato, *Logarithmic structures of Fontaine-Illusie*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 191–224, Johns Hopkins Univ. Press, Baltimore, MD, 1989.
[21] K. Kato, *Semistable reduction and p-adic étale cohomology*, Astérisque **223**, Soc. Math. de France, 1994, 269–293.
[22] K. Kato, *Toric singularities*, Amer. J. Math. **116** (1994), 1073–1099.
[23] M. Levine, *K-theory and motivic cohomology of schemes*, preprint, 1999.
[24] J. Neisendorfer, *Primary homotopy theory*, Mem. Amer. Math. Soc. 25 (1980), no. 232, iv+67 pp.
[25] W. Nizioł, *Crystalline Conjecture via K-theory*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 5, 659–681.
[26] W. Nizioł, *Toric singularities: log-blow-ups and global resolutions*, J. Algebraic Geom. 15 (2006), 1-29.
[27] W. Nizioł, *p-adic motivic cohomology and arithmetic*, Proceedings of ICM2006.
[28] W. Nizioł, *On uniqueness of p-adic period morphisms*, preprint, 2006.
[29] D. Quillen, *Higher algebraic K-theory I*, Algebraic K-theory I, Lecture Notes in Math. **341**, Springer-Verlag, Berlin-Heidelberg-New-York, 1973, pp. 85–147.
[30] C. Soulé, *Operations on étale K-theory. Applications.*, Algebraic K-theory I, Lect. Notes Math. **966**, Springer-Verlag, Berlin, Heidelberg and New York, 1982, 271–303.
[31] A. Suslin, *Higher Chow groups and étale cohomology*, Cycles, transfers, and motivic homology theories, 239–254, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000.
[32] R. Thomason, *Algebraic K-theory and étale cohomology*, Ann. Scient. École Norm. Sup. **18** (1985), 437–552.
[33] T. Tsuji, *p-adic étale and crystalline cohomology in the semistable reduction case*, Invent. Math. **137** (1999), no. 2, 233–411.
[34] C. Weibel, *Étale Chern classes at the prime* 2, Algebraic $K$-theory and algebraic topology (Lake Louise, AB, 1991), 249–286, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 407, Kluwer Acad. Publ., Dordrecht, 1993.

*E-mail address*: `niziol@math.utah.edu`