The shortest disjoint paths problem

William Lochet

1University of Bergen, Norway

Abstract

For any fixed \( k \), we show the existence of a polynomial-time algorithm deciding if, given a graph \( G \) and a set of pairs of vertices \((s_1, t_1), \ldots, (s_k, t_k)\), there exist \( k \) vertex-disjoint paths from \( s_i \) to \( t_i \) such that each of these paths is a shortest path.

1 Introduction

Given a graph \( G \) and a set of pairs of vertices \((s_1, t_1), \ldots, (s_k, t_k)\), the Vertex-Disjoint Paths Problem asks whether there exists a set of vertex-disjoint paths \( P_1, \ldots, P_k \) such that every \( P_i \) is an \((s_i, t_i)\) path. This is a classical problem in graph theory which has been extensively studied. While it is NP-hard [6] when \( k \) is part of the input, the problem has been proved to admit a solution in \( O(n^3) \) for any fixed \( k \) by Robertson and Seymour [8] using the tools from the graph-minor project. The running time has later been improved to \( O(n^2) \) [7]. In the directed case, the problem is NP-hard even for \( k = 2 \) [5], but some results are known for special class of digraphs like acyclic digraphs [5], planar digraphs [9] or tournaments [3].

One natural question is, given an instance of the Vertex-Disjoint Path Problem, to find a solution which minimises the sum of the lengths of the \( P_i \). This problem appears to be much harder, as only the case \( k = 2 \) was recently solved by Björklund and Husfeldt [2]. In fact, even deciding if the problem admits an optimal solution, i.e where every \( P_i \) is a shortest path between \( s_i \) and \( t_i \) is open for \( k \geq 3 \).

This problem was first considered by Eilam-Tzoreff [4] 20 years ago. In the same paper, he gave an algorithm for the case \( k = 2 \) and conjectured that a polynomial algorithm exists for any fixed \( k \), both in the directed and undirected setting. This problem has received some attention lately, in particular Bérczi and Kobayashi [7] proved the directed case when \( k = 2 \). The goal of this paper is to solve this problem for any \( k \) in the undirected case.

2 Preliminaries

For any integer \( k \), \([k]\) denote the set of integers between 1 and \( k \), and for any integer \( j \leq k \), \([j..k]\) denote the set of integers between \( j \) and \( k \).

A graph \( G \) is said to be a \( k\)-shortest graph if there exists \( k \) partitions of \( G \) \((V_1^1, \ldots, V_1^{l_1}), (V_2^2, \ldots, V_2^{l_2}), \ldots, (V_k^k, \ldots, V_k^{l_k})\) such that \( xy \) is an edge of \( G \) implies that there exists \( i \in [k] \) and \( j \in [l_i - 1] \) such that \( x \in V_j^i \) and \( y \in V_{j+1}^i \). Moreover, if \( x \in V_j^i \) and \( y \in V_l^i \), for \( i \in [k], j, l \in [l_i] \) and \( |j - l| > 1 \) then \( xy \notin E(G) \). Intuitively, one way to obtain a \( k\)-shortest graph is to start from
any graph, doing \( k \) breath-first searches from different vertices and removing all the edges which are not between two consecutive levels of at least one of the BFS. For a \( k \)-shortest graph \( G \), we associate naturally \( k \) colours to the edges of \( G \) as follows: any edge between \( x \in V_j^i \) and \( y \in V_{j+1}^i \) is said to be of colour \( i \). Note that the same edge can be of different colours. Moreover, each colour \( i \) defines a partial order on the vertices of \( G \) \( \leq \), as follows: \( x \leq y \) if \( x \in V_j^i \) and \( y \in V_j^i \) with \( j \leq r \). This naturally defines an orientation of the edges of colour \( i \). Note that the same edge can have two different orientations for two different colours. Let \( G \) be a \( k \)-shortest graph and \( r \) and \( i \) be two indices. We say that a path \( P_1 = x_1, \ldots, x_r \) is a path of colour \( i \) if for every \( j \in [r-1] \), \( x_jx_{j+1} \) is an edge of colour \( i \) and \( x_j \leq x_{j+1} \). Note that, since whenever \( u \in V_j^i \) and \( v \in V_j^i \) with \( |j-l| > 1 \) there is no edge between \( u \) and \( v \), any path of colour \( i \) between \( x \) and \( y \) is also a shortest path in \( G \). Moreover, concatenating two paths of colour \( i \) also gives a path of colour \( i \). By convention, we consider the paths of colour \( i \) to be oriented from the endpoint which belongs to the part of \( (V_1^i, \ldots, V_l^i) \) with the lowest index to the endpoint with the largest one. In particular, an \((x, y)\)-path of colour \( i \) is a path of colour \( i \) between \( x \) and \( y \) oriented from \( x \) to \( y \). For a directed path \( P \) and two vertices \( x \) and \( y \) belonging to this path, \( P[x, y] \) denote the subpath of \( P \) from \( x \) to \( y \). By convention, if \( y \) is before \( x \) along \( P \), then \( P[x, y] \) will be the empty path. The length of a path is its number of edges. A path-partition of a path \( P \) is a set of internally vertex-disjoint subpaths of \( P \) such that concatenation of all the paths gives \( P \). Let \( Q_1, Q_2 \) be two different path partitions of the same path \( P \), the intersection of \( Q_1 \) and \( Q_2 \) is the path partition of \( P \) obtained as follows: If \( S \) is the set of vertices which are endpoints of paths of either \( Q_1 \) or \( Q_2 \), then the intersection of \( Q_1 \) and \( Q_2 \) consists of all the subpaths of \( P \) between vertices of \( S \) which are consecutive along \( P \). Note that the number of paths in the intersection of \( Q_1 \) and \( Q_2 \) is at most the sum of the number of paths in \( Q_1 \) and \( Q_2 \). Moreover, every path in the intersection is a subpath of some path in \( Q_1 \) and some path in \( Q_2 \). For an oriented edge \( e = xy \), \( x \) is called the tail of \( e \) and is denoted as \( t(e) \) and \( y \) the head, denoted as \( h(e) \).

Let \( G \) be a \( k \)-shortest graph, \((s_1, t_1), \ldots, (s_l, t_l)\) a set of \( l \) pairs of vertices and \( c \) a function from \([l] \) to \([k] \), the \( k\)-SDP defined by \( G \), the \((s_i, t_i)\) and \( c \) is the problem of finding a set of internally vertex-disjoint paths \( P_1, \ldots, P_l \) such that for any \( i \in [l] \), \( P_i \) is a path of colour \( c(i) \) between \( s_i \) and \( t_i \). The \((s_i, t_i)\) will be referred to as requests. The following lemma shows that we can reduce Eilam-Tzoreff’s question to solving an instance of \( k\)-SDP.

**Lemma 1.** Let \( G \) be a graph and \((s_1, t_1), \ldots, (s_k, t_k)\) be a set of \( k \) pairs of vertices in \( G \). Let \( G' \) be the \( k\)-shortest graph obtained from \( G \) by taking for each \( i \) \((V_1^i, \ldots, V_l^i)\) the partition obtained by doing a breath-first-search from \( s_i \), and removing the edges which are not between two consecutive levels of some BFS. There exists a set of internally vertex-disjoint paths \( P_1, \ldots, P_k \) such that for every \( i \in [k] \), \( P_i \) is a shortest path between \( s_i \) and \( t_i \) if and only if the \( k\)-SDP problem defined by \( G' \), the \((s_i, t_i)\) and the identity function \( c : [k] \to [k] \) has a solution.

**Proof.** The proof follows from the fact that, for every \( i \) and \( l \), the set \( V_l^i \) corresponds to the set of vertices at distance \( l \) from \( s_i \). Therefore, a shortest path in \( G \) between \( s_i \) and some vertex \( x \in V_l^i \) is a path of colour \( i \) from \( s_i \) to \( x \) in \( G' \) and vice versa.

The main contribution of this paper is to prove the following result.

**Theorem 2.** Let \( G \) be a \( k\)-shortest graph, \((s_1, t_1), \ldots, (s_l, t_l)\) a set of \( l \) pairs of vertices and \( c \) a function from \([l] \) to \([k] \). There exists an algorithm running in time \( n^{O(l^4k)} \) deciding if the problem of \( k\)-SDP defined by \( G \), the \((s_i, t_i)\) and \( c \) has a solution.
An interesting case is when \( k = 1 \). The problem then reduces to the problem of directed disjoint paths in acyclic digraphs by orienting all the edges of \( G \) from each set \( V_i \) to \( V_{i+1} \). Therefore, the algorithm of Fortune et al. [5] gives a solution in \( n^{O(1)} \). As noted in [1], we can also reduce the problem of directed-disjoint paths in acyclic digraphs to 1-SDP. In particular, it implies that we cannot hope to remove the dependence of \( l \) in the exponent as disjoint paths in acyclic digraphs is \( W[1]-\text{hard} \) [10].

The main idea behind the proof is to reduce to a set of \( O(l^{tk}) \) requests such that for each pair of requests of different colours, no pair of shortest paths solving these requests can intersect. Once we have achieved this, it means that the only potential conflicts arise for pairs of requests of the same colour. However, the edges of each colour class can be seen as an acyclic digraph, and we can adapt the algorithm of Fortune et al. to that case. The main difficulty lies in reducing to these \( O(l^{tk}) \) requests. To achieve this, we need to look at a potential solution to the original \( k\)-DSP problem and say that, for each pair of paths of different colours in this solution, there is a way to partition each of these paths into a finite number of subpaths, such that the endpoints of each pair of subpaths now correspond to requests that can never intersect. The next two sections are devoted to this task. In particular, the next section tries to understand the structure of bi-coloured edges.

3 Bi-coloured components

Let \( G \) be a \( k\)-shortest graph and \( i, j \) two integers in \([k]\). Consider \( G_{i,j}^+ \) (resp. \( G_{i,j}^- \)), the graph induced by the edges \( xy \) of \( G \) of colour \( i \) and \( j \) such that \( x \leq i \) and \( x \leq j \) (resp. \( x \leq i \) and \( y \leq j \)). A bi-coloured component of colours \( i, j \) is a connected component of \( G_{i,j}^+ \) and \( G_{i,j}^- \). Note that \( G_{i,j}^+ \) and \( G_{i,j}^- \) play identical roles, as reversing the order of the partition (\( i, j \)) transforms the \( k\)-shortest graph \( G \) into a \( k\)-shortest graph \( G' \) where every component of \( G_{i,j}^- \) becomes a component of \( G_{i,j}^+ \) and vice-versa.

**Lemma 3.** Let \( G \) be a \( k\)-shortest graph, \( i, j \) two indices in \([k]\), and \( S \) some component of \( G_{i,j}^+ \). There exists a constant \( C_S \) such that for any vertex \( x \in S \), if \( x \in V_r^i \), then \( x \in V_r^j + C_S \).

**Proof.** Let \( x \) be any vertex belonging to \( S \). Let \( r \) and \( t \) be the constants such that \( x \in V_r^i \) and \( x \in V_t^i \) and define \( C_S = t - r \). Let \( y \) be another vertex of \( S \). By definition of \( S \), there exists a path \( P \) in \( G_{i,j}^- \) between \( x \) and \( y \). Let \( s_1 \) be the number of edges of \( P \) which are used positively for the order induced by the colour \( i \) when going from \( x \) to \( y \), and \( s_2 \) the number of edges used negatively. By definition of this order, we have that \( y \in V_{r+s_1-s_2}^j \).

Because the orders induced by the colours \( i \) and \( j \) are the same on \( S \), we also have that \( y \in V_{t+s_1-s_2}^j \), which ends the proof. \( \square \)

Let us now show the following properties of paths of colour \( i \).

**Proposition 4.** Let \( G \) be a \( k\)-shortest graph. Suppose \( x \in V_r^i \) and \( y \in V_t^i \) for some \( i \in [k] \) and \( r, t \in [l] \) with \( r > t + 1 \). If there exists a path in \( G \) of length \( r - t \) between \( x \) and \( y \), then this path is a path of colour \( i \) from \( y \) to \( x \).

**Proof.** Let \( P = x_1, \ldots, x_s \) with \( x_1 = y, x_s = x \) and \( s = r - t + 1 \) be a path of length \( r - t \) between \( x \) and \( y \). For every \( j \in [s] \), let \( i_j \) be the integer such that \( x_j \in V_{i_j}^i \). We know that for any \( j \in [2..s], i_j \leq i_{j-1} + 1 \) as \( x_j \) and \( x_{j-1} \) are adjacent. However, \( i_1 = t, i_s = r \) and \( s = r - t + 1 \). This means that \( i_j = i_{j-1} + 1 \) for every \( j \in [2..s] \) and all the edges of \( P \) are edges of colour \( i \). \( \square \)
Proposition 5. Let $G$ be a $k$-shortest graph, $i, j$ two indices of $[k]$ and $x$ and $y$ two vertices of $G$. If there exists a path $P_i$ of colour $i$ between $x$ and $y$ and a path $P_j$ of colour $j$ between $x$ and $y$, then $P_j$ is also a path of colour $i$ and $P_i$ is also a path of colour $j$.

Proof. We know that $P_i$ and $P_j$ are shortest paths between $x$ and $y$, and in particular have the same length. The result follows by applying Proposition 4 to the paths $P_j$ and $P_i$. □

We are ready to prove the following lemma, which shows how paths of colour $i$ interact with $G^+_{i,j}$.

Lemma 6. Let $G$ be a $k$-shortest graph, $i, j$ two indices in $[k]$, $P_i$ a path of colour $i$ and $S$ some bi-coloured component of colours $i, j$. The intersection of $P_i$ and $S$ is a subpath of $P_i$.

Proof. Let $S$ be a component of $G^+_{i,j}$ and suppose that $P_i$ does not intersect $S$ along a single subpath. This means that we can find a path $P'$ of colour $i$ between two vertices $x, y$ of such that $P'$ uses no edge of $S$. Suppose $x$ is the first endpoint of this path, $y$ the last and $l$ denote the length of $P'$. Then $x \in V_i^l$ and $y \in V_j^{l+1}$. However, by Lemma 3, we know that there exists a constant $C_S$ such that, since both $x$ and $y$ belong to $S$, $x \in V^j_{r+c_S}$ and $y \in V^j_{r+l+c_S}$. By Proposition 4, this implies that $P'$ is also a path of colour $j$, and thus $P' \in S$.

The case where $S$ is a component of $G^-_{i,j}$ is entirely symmetrical. □

Let $G$ be a $k$-shortest graph, $i, j$ two indices in $[k]$, $P_i$ a path of colour $i$ and $P_j$ a path of colour $j$. We say that $P_i$ and $P_j$ are in conflict if there exists a bi-coloured component $S$ of colour $i, j$ such that the intersection of $P_i$ with $S$ is a $(s_1', t_1')$-path and the intersection of $P_j$ with $S$ is an $(s_2', t_2')$-path for some $s_1', s_2', t_1', t_2' \in S$, with the property that there exists a $(s_1', t_1')$-path $P_i'$ of colour $i$ and an $(s_2', t_2')$-path $P_j'$ of colour $j$ using at least one vertex outside of $s_1', s_2', t_1', t_2'$ in common. The component $S$ will be called a conflicting component for $P_i$ and $P_j$. By convention, two paths of the same colour are never conflicting. The following lemma is the main ingredient of our proof. It shows that for two paths of colour $i$ and $j$, there is at most one conflicting component.

Lemma 7. Let $G$ be a $k$-shortest graph, $i, j$ two indices in $[k]$, $P_i$ a path of colour $i$ and $P_j$ a path of colour $j$. Suppose $S$ is a conflicting component for the paths $P_i$ and $P_j$, then $P_i$ and $P_j$ do not have any vertex in common outside $S$.

Proof. Again, we can assume that $S$ is a component of $G^+_{i,j}$, by potentially reversing the order $i$. Suppose that the intersection of $P_i$ with $S$ is an $(s_1', t_1')$-path and the intersection of $P_j$ with $S$ is an $(s_2', t_2')$-path for some $s_1', s_2', t_1', t_2' \in S$. We prove the lemma by contradiction, distinguishing several cases depending on which part of $P_i$ and $P_j$ (compared to $S$) the intersection lies on.

Suppose first that $P_i$ and $P_j$ lie after $S$ for both paths. By definition of conflicting components, we know that there exists a vertex $x \in S$ such that there exists a path $P_1$ of colour $i$ from $x$ to $t_1'$ and a path $P_2$ of colour $j$ from $x$ to $t_2'$. Let $z$ be any vertex belonging to the intersection of $P_1 \cap P_2$. Suppose now that the intersection lies on $S$. By applying Proposition 5 to the path obtained by concatenating $P_1$ and $P_2[t_1', z]$ and the one obtained by concatenating $P_2$ and $P_2[t_2', z]$, we get that these two paths are both of colour $i$ and $j$. In particular this implies that the edges of these paths belong to $G^+_{i,j}$ and $z \in S$, which is a contradiction.

Suppose now that the intersection of $P_i$ and $P_j$ lies before $S$ on $P_i$ and after $S$ on $P_j$. By definition of conflicting components, we know that there exists a vertex $x \in S \setminus \{s_1', s_2', t_1', t_2'\}$ such that there exists a path $P_1$ of colour $i$ from $s_1'$ to $x$ and a path $P_2$ of colour $j$ from $x$ to $t_2'$. Because
the two paths $P_1$ and $P_2$ have both length at least 1. Moreover, they are both paths of colour $i$ and $j$, and with the same orientation associated to these colours. Let $z$ denote the intersection of $P_i$ and $P_j$ before $S$ on $P_i$ and after $S$ on $P_j$ and consider the path $H_1 = P_i[z, s_i'] | P_1 P_2 z].$ $H_1$ is a path of colour $i$ between $z$ and $t_2'$, and thus $|H_1| = |P_2[t_2', z]|$. Likewise, we can show that $|P_1[z, s_i']|P_1| = |P_2 P_3[t_2', z]|$, which gives us a contradiction.

The other cases are symmetrical. □

Let us now explain how the previous lemma will be used. Remember that our goal is to reduce an original instance of $k$-SDP with $l$ requests to one with $O(l^k)$ requests such that for every pair of requests of different colours, no pair of shortest paths solving these requests can intersect. Let $S$ denote the conflicting component. Because of Lemma 6, we know that the intersection of $P_i$ and $P_j$ with $S$ are subpaths. For every $a \in \{i, j\}$, consider the path partition $(P_i^1, P_i^2, P_a^3)$ of $P_a$, where $P_a^2$ is the subpath of $P_a$ on $S$, $P_i^1$ the part of $P_a$ before this component, and $P_a^3$ the part after. What the Lemma 7 roughly says is that the endpoints of $P_i^1, P_i^2, P_i^3$ and $P_j$ correspond to requests such that no pair of shortest paths solving these requests can intersect, which is exactly what we wanted. This is not true for the requests associated to $P_i^2$ and $P_j$, however since they both belong to a bi-coloured component $S$, these two requests can be considered of the same colour.

Surprisingly, the case where $P_i$ and $P_j$ are not in conflict is harder to handle. This is the goal of the next section, but let us first show sufficient conditions to guarantee the existence of a conflicting component for a pair of paths.

**Lemma 8.** Let $G$ be a $[k]$-shortest graph and $i, j$ two different indices in $[k]$, $P_i$ a path of colour $i$ and $P_j$ a path of colour $j$. If $P_i$ and $P_j$ have three common vertices, then they have a conflicting component.

**Proof.** Let $x_1$, $x_2$ and $x_3$ be three vertices in $P_i \cap P_j$. We claim that they belong to the same bi-coloured component. Indeed, consider the subpaths of $P_i$ and $P_j$ between $x_1$ and $x_2$. By Proposition 5 they are both paths of colour $i$ and $j$ and belong to the same component $S$. Without loss of generality, suppose $S$ is a component of $G_{i,j}^+$ and $x_1 \leq x_2$. If $x_3$ belongs to $P_i[x_1, x_2]$ or $P_j[x_1, x_2]$, we have that $x_3 \in S$, which ends the proof of the claim. Assume now $x_3$ appears after $x_2$ on $P_i$, the other case being symmetrical. If it appears after $x_2$ on $P_j$, then the same argument shows that $P_i[x_2, x_3]$ is also a path of $S$.

Suppose now that $x_3$ appears before $x_1$ on $P_j$. In that case we have that $P_j[x_3, x_2]$ and $P_i[x_2, x_3]$ are both shortest path, and thus have the same size. However, this implies that $P_j[x_3, x_1]$ is strictly shorter than $P_i[x_1, x_3]$, which is a contradiction.

Now that we know that $x_1$, $x_2$ and $x_3$ belong to the same bi-coloured component, let us show that this component is a conflicting component for $P_i$ and $P_j$. Indeed, since $x_1$, $x_2$ and $x_3$ belong to the same component, they either appear in the same order on the paths $P_i$ and $P_j$ if $S$ is a component of $G_{i,j}^+$ or in reverse order if $S$ is a component of $G_{i,j}^-$. In both cases, the vertex in the middle is the same in both paths, and this implies that $P_i$ and $P_j$ are conflicting on this component. □

## 4 Blind Paths

Let $G$ be a $k$-shortest graph and $i, j$ two different indices in $[k]$. Let $P_i$ be some $(s_i, t_i)$-path of colour $i$ and $P_j$ some $(s_j, t_j)$-path of colour $j$ which are internally vertex-disjoint. We say that $P_i$
sees \( P_j \) if there exists an internal vertex \( x \) of \( P_i \) such that there exists a path of colour \( i \) from \( x \) to \( t_i \) which intersects \( P_j \setminus \{s_j, t_j\} \). We say that the pair \( P_i \) and \( P_j \) is blind if \( P_j \) does not see \( P_i \) and \( P_i \) does not see \( P_j \). Note that if \( |P_i| = 2 \), then \( P_i \) does not see, or is not seen, by any other path \( P_j \).

The following lemma shows how to use Lemma 7 to find blind paths from conflicting paths.

**Lemma 9.** Let \( G \) be a \( k \)-shortest graph, \( i, j \) two integers in \([k]\), \( P_i \) a path of colour \( i \) and \( P_j \) a path of colour \( j \) which are internally vertex-disjoint. There exists a path partition \( L_i \) of \( P_i \) and a path partition \( L_j \) of \( P_j \), both of size at most 9 with the following properties:

- All the paths of \( L_a \) are paths of colour \( a \) for \( a \in \{i, j\} \)
- For any pair of paths \( H_i \in L_i \) and \( H_j \in L_j \), then either \( H_i \) and \( H_j \) are paths of the same bi-coloured component of colour \( i, j \) or \( H_i \) does not see \( H_j \).

**Proof.** Suppose \( P_i \) is an \((s_i, t_i)\)-path and \( P_j \) is an \((s_j, t_j)\) path. If \( P_i \) does not see \( P_j \), then \( L_i = \{P_i\} \) and \( L_j = \{P_j\} \) satisfy the properties of the lemma. Suppose now \( P_i \) sees \( P_j \) and let \( x_1 \) denote the last vertex of \( P_i \) from which there exists a path \( P_1^j \) of colour \( i \) to \( t_i \) which uses some vertex of \( P_j \setminus \{s_j, t_j\} \). Because \( P_i \) and \( P_j \) are internally vertex disjoint, \( x_1 \neq t_i \). Let \( x_1' \) denote the vertex just after \( x_1 \) on \( P_i \). Note that \( P_i[x_1', t_i] \) does not see \( P_j \).

Now let \( x_2 \) denote the last vertex of \( P_i[s_i, x_1] \) from which there exists a path of colour \( i \) to \( x_1 \) which uses some vertex of \( P_j \setminus \{s_j, t_j\} \). Again, if this vertex does not exist, then \( L_i = \{P_i[x_1, x_2], (x_1, x_1'), P_i[x_1', t_i]\} \) and \( L_j = \{P_j\} \) satisfy the properties of the lemma. Suppose from now on that \( x_2 \) exists and let \( x_2' \) be the vertex just after \( x_2 \) on \( P_i \). Since \( P_i \) and \( P_j \) are internally vertex-disjoint, \( x_2 \neq x_1 \) and thus \( x_2' \in P_i[s_i, x_1] \). Again, note that \( P_i[x_2', x_1] \) does not see \( P_j \).

Let \( x_3 \) denote the last vertex of \( P_i[s_i, x_2] \) from which there exists a path \( Q_1^i \) of colour \( i \) to \( x_2 \) which uses some vertex of \( P_j \setminus \{s_j, t_j\} \). Again, we can assume that this vertex exists or \( L_i = \{P_i[s_i, x_2], (x_2, x_2'), P_i[x_2', x_1], (x_1, x_1'), P_i[x_1', t_i]\} \) and \( L_j = \{P_j\} \) satisfy the properties of the lemma. Let \( x_3' \in P_i[s_i, x_2] \) denote the vertex just after \( x_3 \) on \( P_i \). Again, note that \( P_i[x_3', x_2] \) does not see \( P_j \).

Note that for any internal vertex \( x \in Q_1^i \) and \( y \in Q_2^i \), \( y <_i x \). This implies that the intersection of \( Q_1^i \) and \( Q_2^i \) is equal to \( x_1 \), and the same argument applies for \( Q_1^i \cap Q_2^i \) and \( Q_1^i \cap Q_1^j \). This means that the paths \( P_j \) and \( P_j' = P_i[s_i, x_3]Q_1^iQ_2^iQ_1^j \) intersect on at least 3 vertices and thus are conflicting by Lemma 8. Let \( S \) denote the conflicting component of \( P_j' \) and \( P_j \).

Suppose first that none of the \( s_i, t_i, s_j, t_j \) belong to \( S \) and denote by \( e_i \) the last edge of \( P_j' \) before \( S \), \( e_j \) the last edge of \( P_j \) before \( S \), \( h_i \) the first edge of \( P_j' \) after \( S \) and \( h_j \) the first edge of \( P_j \) after \( S \).

**Claim 9.1.** All the pairs of paths among \( P_j'[s_i, t(e_i)], e_i, P_j'[h(e_i), t(h_i)], h_i, P_j'[h(h_i), t_i], P_j[s_j, t(e_j)], e_j, P_j[h(e_j), t(h_j)], h_j \) and \( P_j[h(h_j), t_j] \) are blind, except from \( P_j[h(e_j), t(h_j)] \) and \( P_j'[h(e_i), t(h_i)] \) which belong to the same bi-coloured component.

**Proof.** Since \( e_i \) and \( h_i \) are not edges of \( S \) and there exists a path of colour \( i \) in this component from \( h(e_i) \) to \( t(h_i) \), then by Lemma 6 no path of colour \( i \) from \( s_i \) to \( t(e_i) \) or from \( h(h_i) \) to \( t_i \) can use any vertex of \( S \). However, any path of colour \( j \) from \( h(e_j) \) to \( t(h_j) \) is a path of \( S \), so it cannot intersect any path of colour \( i \) from \( s_i \) to \( t(e_i) \) or from \( h(h_i) \) to \( t_i \). This means that \( (P_j'[s_i, t(e_i)], P_j[h(e_j), t(h_j)]) \) and \( (P_j'[h(h_i), t_i], P_j[h(e_j), t(h_j)]) \) are blind pairs. By reversing the role of \( i \) and \( j \), it also means that \( (P_j[s_j, t(e_j)], P_j'[h(e_i), t(h_i)]) \) and \( (P_j[h(h_j), t_j], P_j'[h(e_i), t(h_i)]) \) are blind pairs.
By the definition of conflicting and Lemma 7, we can show that no path of colour $i$ from $s_i$ to $t(e_i)$ or from $h(h_i)$ to $t_i$ can intersect a path of colour $j$ from $s_j$ to $t(e_j)$ or $h(h_j)$ to $t_j$. Indeed, suppose for example that there exists a path $H_i$ of colour $i$ from $s_i$ to $t(e_i)$ that intersects a path $H_j$ of colour $j$ from $s_j$ to $t(e_j)$. In that case the paths $H_iP'_i[t(e_i), h(h_i)]$ and $H_jP_j[t(e_j), h(h_j)]$ contradict Lemma 7 as $S$ is a conflicting component for these two paths, but they also intersect outside of $S$. The other cases are symmetrical and thus all pairs of paths among $P'_i[s_i, t(e_i)]$, $P'_i[h(h_i), t_i]$, $P_j[s_j, t(e_j)]$ and $P_j[h(h_j), t_j]$ are blind.

This ends the proof of the claim as the other pairs contain an edge and are blind by definition and $P_j[h(e_j), t(h_j)]$ and $P'_i[h(e_i), t(h_i)]$ are paths of $S$.

Let $L_j = \{P_j[s_j, t(e_j)], e_j, P_j[h(e_j), t(h_j)], h_j, P_j[h(h_j), t_j]\}$. Suppose first that $t(e_i)$ appears after $x_3$ on $P'_i$. It means that $P_i[s_i, x_3]$ is a subpath of $P'_i[s_i, t(e_i)]$, and in particular $P_i[s_i, x_3]$ does not see $P_j$. Setting $L_i = \{P_i[s_i, x_3], (x_3, x_3'), P_i[x_3', x_2'], (x_2, x_2'), P_i[x_2', x_1], (x_1, x_1'), P_i[x_1', t_i]\}$, we then have that no path of $L_i$ sees $P_j$ and thus any path of $L_j$.

Suppose now that $t(e_i)$ appears before $x_3$ on $P'_i$. Note that $t(h_i)$ has to appear after or there is no path from $x_3$ to $t_i$ intersecting $P_j$, which contradicts the choice of $x_3$. In that case, setting $L_i = \{P_i[s_i, t(e_i)], e_i, P_i[h(e_i), x_3], (x_3, x_3'), P_i[x_3', x_2], (x_2, x_2'), P_i[x_2', x_1], (x_1, x_1'), P_i[x_1', t_i]\}$, we also have that the only path of $L_i$ that sees a path of $L_j$ is $P_i[h(e_i), x_3]$. Moreover, it can only see $P_j[h(e_j), t(h_j)]$, but these paths belong to the same bi-coloured component $S$.

The cases where some of the $s_i, t_i, s_j, t_j$ belong to $S$ are treated exactly the same, except that some of the $e_i, e_j, h_j, h_i$ might not exist, which means we have fewer paths to consider.

By applying the previous lemma several times, we obtain the following:

**Lemma 10.** There exists a constant $C$ such that if $G$ is a $k$-shortest graph, $i, j$ two integers in $[k]$, $P_i$ a path of colour $i$ and $P_j$ a path of colour $j$ which are internally vertex-disjoint, then there exists a path partition $L_i$ of $P_i$ and a path partition $L_j$ of $P_j$, both of size at most $C$ with the following properties:

- Each $L_a$ consists of at most $C$ paths of colour $a$.
- For any pair of path $H_i \in L_i$ and $H_j \in L_j$ which are not blind, then $H_i$ and $H_j$ are paths of the same bi-coloured component.

**Proof.** Let $Q_i, Q_j$ be the path partitions obtained by applying Lemma 9 to $P_i$ and $P_j$. We know that for any pair of paths $H_i \in Q_i$ and $H_j \in Q_j$, then either $H_i$ and $H_j$ are paths of the same bi-coloured component of colours $i, j$, or $H_i$ does not see $H_j$.

Now as long as there exists a path in $H_i \in Q_i$ such that there exists some path $H_j \in Q_j$, such that $H_j$ sees $H_i$ and is not a path of the same bi-coloured component as $H_i$, we do the following. Let $H_{j1}, \ldots, H_{jr}$ denote all the paths of $Q_j$ which sees $H_i$ and do not belong to the same bi-coloured component. For any $a \in [r]$, let $Q_{ja,a}$ and $Q_{i,a}$ denote the set of path partitions obtained by applying Lemma 9 to $H_{ja,a}$ and $H_i$. Let $Q'_i$ be the intersection of all the partitions $Q_{i,a}$ of $P_i$. Because every path of $Q'_i$ is a subpath of some $Q_{i,a}$ for any $a \in [r]$, it means that this path is not seen by any path in $Q_{ja,a}$ which is not a path of the same bi-coloured component. Let us update $Q_i$ by replacing $H_i$ by $Q'_i$ and update $Q_j$ by replacing each of the $H_{ja,a}$ by $Q_{ja,a}$. By doing that, the number of paths in $Q_i$ which is seen by some path $H_j \in Q_j$ which is not a path of the same bi-coloured component
decreases strictly as none of the paths of $Q_j'$ satisfy these properties. At each step, we multiply the number of paths in $Q_j$ by at most 9 and the number of paths in $Q_i$ by at most $9|Q_j|$. However, we only have to do this 9 times as initially the sets $Q_i$ and $Q_j$ have size at most 9. At that time, all the properties are satisfied and the size of both $Q_i$ and $Q_j$ is smaller than $C = 9^{55}$.

\section{Proof of the main theorem}

\subsection{Proof of the blind case}

The next lemma shows how to decide if there is a solution where all the pairs of paths are blind and we add for every request a list of bi-coloured components which are forbidden.

\textbf{Lemma 11.} Let $G$ be a $k$-shortest graph, $(s_1, t_1), \ldots, (s_l, t_l)$ a set of pairs and $c$ a function from $[l]$ into $[k]$. Moreover, suppose that for every $i$, there is a list $F_i$ of bi-coloured components where one of the colours being $c(i)$. There exists an algorithm running in time $n^{O(k)}$ that either returns a solution $P_1, \ldots, P_l$ to the $k$-SDP defined by $G$, the $(s_i, t_i)$ and $c$ or shows that no solution is such that each $P_i$ does not use any vertex of any component in $F_i$ and moreover, for any indices $i$ and $j$, either $P_j$ and $P_i$ are blind or $P_i$ is a path of some component of $F_j$ or $P_j$ is a path of some component of $F_i$.

To prove this lemma, we will build an auxiliary digraph $D$ such that a solution satisfying the properties of the lemma exists if and only if there exists a directed path in $D$ between two specified vertices.

First note that, by potentially replacing some vertices with an independent set with the same neighbourhood, we can assume that all the $s_i$ and $t_i$ are disjoint.

The vertices of $D$ will correspond to $l$-tuples $(x_1, \ldots, x_l)$ of vertices of $G$. Intuitively, we are trying to build the paths $P_i$ starting from $s_i$ and $x_i$ is the last vertex of a prefix of $P_i$ we are considering. For any pair of vertices $(x_1, \ldots, x_l)$ and $(y_1, \ldots, y_l)$, $D$ contains the arc from $(x_1, \ldots, x_l)$ to $(y_1, \ldots, y_l)$ if the following are satisfied:

- There exists $i \in [l]$ such that $x_j = y_j$ for all $j \in [l]$, $j \neq i$.
- $x_i y_i$ is an edge of colour $c(i)$ such that there exists a path of colour $c(i)$ from $y_i$ to $t_i$ avoiding the components in $F_i$.
- For all $j \in [l]$ different from $i$, $y_i \neq x_j$ and either there is no path of colour $c(j)$ from $x_j$ to $t_j$ that uses the vertex $x_i$, or $x_i$ is a vertex of a component of $F_j$.

Let $S = (s_1, \ldots, s_l)$ and $T = (t_1, \ldots, t_l)$. The next two claims finishes the proof of Lemma 11.

\textbf{Claim 11.1.} If there exists a solution $P_1, \ldots, P_l$ to the $k$-SDP defined by $G$, $(s_i, t_i)$ and $c$ such that each $P_i$ does not use a vertex of any component in $F_i$ and moreover, for any indices $i$ and $j$, either $P_j$ and $P_i$ are blind, $P_i$ is a path of some component of $F_j$ or $P_j$ is a path of some component of $F_i$, then there is a path in $D$ from $S$ to $T$.

\textbf{Proof.} Let $P_1, \ldots, P_l$ denote such a solution in $G$. Let $X$ be the set of vertices of $D$ corresponding to $l$-tuples obtained by taking one vertex per path $P_i$. We can define a natural order on $X$ by considering for each $P_i$ the order induced by the path and taking the lexicographic order. Note that $T$ is the maximal element of $X$. 8
Consider now the largest element \( A = (x_1, \ldots, x_l) \) of \( X \) which is reachable in \( D \) from \( S \) and suppose, in order to reach a contradiction, that this element is not \( T \). Consider some colour \( c_1 \) and \( I \) the set of indices of \( i \) of \([l]\) such that for \( c(i) = c_1 \) and \( x_i \neq t_i \). Because the edges of colour \( c_1 \) induce an acyclic digraph, there exists an index \( i \in I \) such that for every \( j \in I \) with \( j \neq i \), there is no path of colour \( c_1 \) from \( x_j \) to \( x_i \). Now for any \( j \in [l] \) such that \( c(j) \neq c_1 \), then either the path \( P_i \) and \( P_j \) are blind, in which case there is no path of colour \( c(j) \) from \( x_j \) to \( t_j \) that uses \( x_i \), or \( P_i \) and thus \( x_i \) is in some component of \( F_j \), or \( P_j \) is a path of some component of \( F_i \). Note that in the last case, any path from \( x_j \) to \( t_j \) is a path of some component of \( F_i \), but \( x_i \) cannot be a vertex of this component, and thus no such path can use \( x_i \). Therefore, if we note \( x'_i \) the vertex just after \( x_i \) on \( P_i \) and \( A' \) the vertex of \( D \) obtained from \( A \) by only changing \( x_i \) into \( x'_i \), then there exists an arc from \( A \) to \( A' \). However, this means that \( A' \) is reachable from \( S \) in \( D \), which contradicts the maximality of \( A \).

And the opposite direction.

**Claim 11.2.** If there exists a path from \( S \) to \( T \) in \( D \), then there exists a solution \( P_1, \ldots, P_l \) to the SDP defined by \( G' \), the \((s_i, t_i)\) and \( c \) such that \( P_i \) does not use any vertex of any component in \( F_i \).

**Proof.** Suppose there exists a path \( P = X_1, \ldots, X_r \) from \( S \) to \( T \) in \( D \). For every \( j \in [r] \), note \( X_j = (x'_1, \ldots, x'_l) \). For every \( i \) and \( j \), consider the graph \( P_i^j \) induced by the vertices \( x_i^t \), for \( t \leq j \). By definition of \( D \), \( P_i^j \) is a path of colour \( c(i) \) from \( s_i \) to \( x_i^t \) avoiding the components of \( F_i \). We will prove by induction on \( j \), that the paths \( P_i^1, \ldots, P_i^j \) are such that

- All the \( P_i^t \) are internally disjoint.
- For any \( i \) and \( r \), there is no path of colour \( c(i) \) from \( x_i^1 \) to \( t_j \) avoiding the components in \( F_i \) that uses any vertex of \( P_i^t \) outside of possibly \( x_i^t \).

Since the path starts at \( S \), all the properties are satisfied when \( j = 1 \). Suppose now that this is true for some \( j \in [r-1] \) and let us show that the properties hold for \( j + 1 \). By definition of the arcs of \( D \), there exists an index \( i \) such that \( x_i^j x_i^{j+1} \) is an edge of colour \( c(i) \) and for every other index \( s \), \( x_s^j x_s^{j+1} \). This means that \( P_i^{j+1} \) is the concatenation of \( P_i^j \) with \( x_i^{j+1} \) and all the \( P_s^{j+1} \) are equal to \( P_s^j \) for \( s \neq i \). Moreover, by definition of \( D \), we know that for any \( s \neq i \), \( x_i^{j+1} \) is disjoint from all the \( x_s^j \) and by induction hypothesis \( x_i^{j+1} \) does not belong to any of the \( P_s^j \). This implies that the \( P_s^j \) for \( s \in [l] \), are disjoint.

Any path of colour \( c(i) \) from \( x_i^{j+1} \) to \( t_i \) avoiding the components of \( F_i \) is a subpath of a path of colour \( c(i) \) from \( x_i^j \) to \( t_i \) avoiding the components of \( F_i \). This means that no such path can use any vertex of \( P_i^t = P_i^{j+1} \) outside of possibly \( x_i^t \) for all \( s \in [l] \) different from \( i \).

Finally, for \( s \in [l] \) different from \( i \) we know that no path of colour \( c(s) \) from \( x_t^j = x_i^{j+1} \) to \( t_s \) avoiding the components in \( F_s \) can use any vertex of \( P_s^t \) outside of \( x_i^t \) by induction hypothesis. Moreover, these paths can also not use \( x_i^{j+1} \) by definition of the arcs of \( D \), which ends our induction.

This means that each \( P_i^r \) is a path of colour \( c(i) \) avoiding the components in \( F_i \) from \( s_i \) to \( t_i \), and all these paths are disjoints, which ends the proof.

Therefore, the problem reduces to deciding the existence of a path in \( D \). As \(|D| = n^l\), this can be done in \( n^{O(l)} \).
5.2 Reducing to the blind case

The next lemma shows how to reduce to the blind case with some forbidden lists.

**Lemma 12.** Let $G$ be a $k$-shortest graph, $(s_1,t_1),\ldots,(s_l,t_l)$ a set of pairs and $c$ a function from $[l]$ to $[k]$. Let $P_1,\ldots,P_l$ be a solution to the k-DSP defined by $G$, the $(s_i,t_i)$ and $c$. There exists a constant $C(k,l)$ depending only on $k$ and $l$, a set of path partitions $L_1,\ldots,L_l$, a function $a$ that associates to each path of the $L_i$ a colour in $[k]$ and a function $b$ that associates to each path of the $L_i$ a set of bi-coloured components with the following properties:

- For every $i \in [l]$, $L_i$ is a path partition of $P_i$ of at most $C(k,l)$ paths.
- If $P$ is a path of some $L_i$ with $a(P) = c_1$, then $P$ is a path of colour $c_1$ and $b(P)$ consists of a set of at most $C(k,l)$ bi-coloured components where one of the colours is $c_1$ and such that $P$ does not use any vertex in these components.
- For any pair of paths $H_i \in L_i$ and $H_j \in L_j$ such that $a(H_i) \neq a(H_j)$, then either $H_i$ and $H_j$ are blind, $H_j$ is a path contained in one component of $b(H_i)$ or $H_i$ is a path contained in one component of $b(H_j)$.

Note that if $P_i$ is a path of colour $j$, then any path partition of $P_i$ consists of paths of colour $j$. The function $a$ is there to reassign the colour of some paths belonging to bi-coloured components in order to achieve the last property of the lemma.

**Proof.** Let $C$ be the constant from Lemma 10. We will prove by induction on $k$ that the lemma is true with $C(k,l) \leq (7Cl)^l$. When $k = 1$, there is only one colour and setting $L_i = \{P_i\}$ for all $i$ satisfies the properties, and thus $C(1,l) \leq l$.

Suppose now that $k > 1$, and without loss of generality, that $c(1) = 1$. For every pair of indices $i,j \in [l]$, let $Q_{i,j}, Q_{j,i}$ denote the path partitions of $P_i, P_j$ obtained by applying Lemma 10 and for every $i \in [l]$, let $Q_i$ denote the intersection of all the $Q_{i,j}$. Note that, since every $Q_{i,j}$ has size $C$, this implies that the $Q_i$ have size at most $C(l)$. For the moment, we consider the paths of $Q_i$ only as paths of colour $c(i)$. Let $Q'_1$ denote the set of paths of $Q_1$, for which there exists some other path among the $Q_j$ such that the pair is not blind. By Lemma 10, all these paths belong to some bi-coloured component of colour 1 and some other colour $j$. Let $B$ denote the set of all these bi-coloured components and let $R = Q_1 \setminus Q'_1$.

For any path $R_t \in R$, we know that the intersections of $R_t$ with any component $C_j \in B$ is a subpath by Lemma 6. Let $e_j$ be the last edge of $R_t$ before $C_j$ and $h_j$ the first edge after. Let $a_t$ and $b_t$ denote the first and last vertex of $R_t$, and consider $R_{t,j} = \{ R_t[a_t,t(e_j)], e_j, R_t[h(e_j),h_j], h_j, R_t[h(h_j),b_t]\}$ a path partition of $R_t$. Note that, except the two edges $e_j$ and $h_j$, each path of $R_{t,j}$ is either disjoint from $C_j$ or a path of this component. Let $L(R_t)$ denote the path partition of $R_t$ obtained by taking the intersection of all the $R_{t,j}$. We know that, since $|B| \leq ClL(R_t) \leq 5Cl$. Moreover, we know that for any path $P' \subseteq L(R_t)$, and any $C_j \in B$, there is a path $r_{t,j} \subseteq R_{t,j}$ such that $P'$ is a subpath of $r_{t,j}$. In particular it means that $P'$ is either an edge, disjoint from $C_j$, or a path of $C_j$. Let $R_{t,1}^2$ be the set of paths of $L(R_t)$ which belong to one of the component of $B$, and $R_{t,1}' = L(R_t) \setminus R_{t,1}^2$. Note that every path in $R_{t,1}'$ is either an edge or a path disjoint from all the components of $B$.

Let $H_1$ denote the set of paths in $Q'_1$ and all the $R_{t,1}'$. Note that for every path $H' \in H_1$, $H'$ is path of a bi-coloured component of $B$. Denote by $c'(H')$ the colour of this component which is
not 1. We will now consider \( H' \) as a path of colour \( c'(H') \) (possibly reversing the endpoints if the component is a component of \( C_{1_c}(H') \)). Let \( G_1 \) be the \((k-1)\)-shortest graph obtained from \( G \) by removing the partition associated to colour 1 and removing all the edges which are edges of colour 1 only. Consider now the \((k-1)\)-SDP problem defined on \( G_1 \) by all the endpoints of the paths in \( Q_i \) for \( i > 1 \) considered as path of colour \( c(i) \), and all the paths in \( H' \in H_1 \) considered as path of colour \( c'(H') \). Note that the set of paths in \( Q_i \) and \( H_1 \) is a solution to this problem and moreover, there is at most \( 7(|C_1|)^2 \) requests, as each \( Q_i \) is smaller than \( C_1 \) and \( H_1 \) is smaller than \( 6|C_1|^2 \).

By induction hypothesis, there exists a path partition of all the paths of the \( Q_i \) for \( i > 1 \) and \( H_1 \) as well as functions \( a' \) and \( b' \) defined on these paths such that each of these path partitions consists of at most \( C(k-1, 7(|C_1|)^2) \) paths, and \( b' \) associates to each path at most \( C(k-1, 7(|C_1|)^2) \) bi-coloured components. Let us define the path partitions \( L_1 \), as well as \( a \) and \( b \) as follows: \( L_1 \) is the union of all the paths in some \( R_1 \), as well as all the paths in the path partitions for every path in \( H_1 \) obtained by applying induction. For every path \( P \) of \( R_1 \), let \( a(P) = 1 \) and \( b(P) = B \). For all the other paths of \( L_1 \), \( a \) and \( b \) correspond to the value of \( a' \) and \( b' \) on this path. Likewise, for every \( i > 1 \), \( L_i \) consists of the union of the path partitions for the paths in \( Q_i \) obtained by applying induction, and the function \( a \) and \( b \) correspond to the \( a' \) and \( b' \) on these paths.

Let us now show that \( L_i \) and functions \( a \) and \( b \) satisfy the required properties. First, it is clear that the \( L_i \) thus defined are path partitions, as they are obtained by replacing paths of some path partitions by their own path partition. Moreover, \(|L_i| \leq 7(|C_1|)^2 \cdot C(k-1, 7(|C_1|)^2) \leq 7(|C_1|)^2 \cdot (7(|C_1|)^2)^{2^{k-1}} \leq (7|C_1|)^k \). Likewise, for any paths \( P \) in these partitions, \( b(P) \) is smaller than \( C(k-1, |C_1|^2) \leq (7|C_1|)^k \). Now suppose \( H_i \) is a path of \( L_i \) and \( H_j \) is a path of \( L_j \) such that \( a(H_i) \neq a(H_j) \). If none of these paths belong to some \( R_1 \), then the last property of the lemma is satisfied for \( H_i \) and \( H_j \) by induction and because \( a \) and \( b \) correspond to \( a' \) and \( b' \) on these paths. Suppose now that one of the paths, say \( H_i \) belongs to \( R_1 \) for some \( R \in R_1 \). Because \( R_1 \) is a subpath of \( R \), it means that if \( H_i \) is not a subpath of some path of \( Q'_1 \), then \( H_i \) and \( H_j \) are blind. If \( H_j \) is a subpath of some path of \( Q'_1 \), then \( H_j \) is a path belonging to some component of \( B \). However, \( b(H_i) = B \), which ends the proof.

Finally, we can prove our main result.

**Proof of Theorem 2.** Suppose there exists a solution \( P_1, \ldots, P_l \) to the \( k \)-SDP problem defined by \( G \), the \((s_i, t_i)\) and \( c \). Let \( L_1, \ldots, L_l \), \( a \) and \( b \) be the path partitions and functions obtained by applying Lemma 12 to \( P_1, \ldots, P_l \). For every \( i \in [l] \), let \( P_{i,1}, \ldots, P_{i,l}, \) denote the paths of \( L_i \) and \((s_{i,1}, t_{i,1}), \ldots, (s_{i,l}, t_{i,l})\) the endpoints of these paths. Remember that by Lemma 12, \( l \leq C(k,l) \).

Suppose we guess all the \((s_{i,j}, t_{i,j})\) as well as the functions \( a \) and \( b \) for each of the \( P_{i,j} \), and consider the \( k \)-SDP problem defined by all the remaining pairs \((s_{i,j}', t_{i,j}')\), then the set of paths \( P_{i,j} \) is a solution to this problem such that, for any pair of paths \( P_{i,j}, P_{i,j}' \) such that \( a(P_{i,j}) \neq a(P_{i,j}') \), either \( P_{i,j} \) and \( P_{i,j}' \) are blind, \( P_{i,j} \) is a path contained in one component of \( b(P_{i,j}) \) or \( P_{i,j}' \) is a path contained in one component of \( b(P_{i,j}) \). This means that we can apply the algorithm of Lemma 11 to find a solution of the \( k \)-SDP defined by \((s_{i,j}', t_{i,j}')\) in \( n^{O(C(k,l))} \). By concatenating for each \( i \) all the paths of this solution corresponding to the paths of \( L_i \), we obtain a solution to the initial \( k \)-SDP problem.

As there is at most \( n^{O(C(k,l))} \) choices for the \((s_{i,j}', t_{i,j}')\), \( a(P_{i,j}) \) and \( b(P_{i,j}) \) this gives an algorithm running in time \( n^{O(C(k,l))} \), which ends the proof.
Acknowledgements

The author wishes to thank Frédéric Havet and Saket Saurabh for their useful comments on the manuscript.

References


