Choice Trees (draft)

Representing Nondeterministic, Recursive, and Impure Programs in Coq

NICOLAS CHAPPE, ENS Lyon, France
PAUL HE, University of Pennsylvania, USA
LUDOVIC HENRIO, CNRS, France
YANNICK ZAKOWSKI, Inria, France
STEVE ZDANCEWIC, University of Pennsylvania, USA

This paper introduces CTrees, a monad for modeling nondeterministic, recursive, and impure programs in Coq. Inspired by Xia et al.'s ITrees, this novel data structure embeds computations into coinductive trees with three kind of nodes: external events, and two variants of nondeterministic branching. This apparent redundancy allows us to provide shallow embedding of denotational models with internal choice in the style of ccs, while recovering an inductive LTS view of the computation. CTrees inherit a vast collection of bisimulation and refinement tools, with respect to which we establish a rich equational theory.

We connect CTrees to the ITrees infrastructure by showing how a monad morphism embedding the former into the latter permits to use CTrees to implement nondeterministic effects. We demonstrate the utility of CTrees by using them to model concurrency semantics in two case studies: ccs and cooperative multithreading.

Additional Key Words and Phrases: Nondeterminism, Formal Semantics, Interaction Trees, Concurrency

ACM Reference Format:
Nicolas Chappe, Paul He, Ludovic Henrio, Yannick Zakowski, and Steve Zdancewic. 2018. Choice Trees (draft): Representing Nondeterministic, Recursive, and Impure Programs in Coq. *J. ACM* 37, 4, Article 111 (August 2018), 29 pages. https://doi.org/10.1145/1122445.1122456

1 INTRODUCTION

Reasoning about and modeling nondeterministic computations is important for many purposes. Formal specifications use nondeterminism to abstract away from the details of implementation choices. Accounting for nondeterminism is crucial when reasoning about the semantics of concurrent and distributed systems, which are, by nature, nondeterministic due to races between threads, locks, or message deliveries. Consequently, precisely defining nondeterministic behaviors and developing the mathematical tools to work with those definitions has been an important research endeavor, and there has been much work on the subject, including formalisms like nondeterministic automata, labeled transition systems and relational operational semantics [6], powerdomains [44], or game semantics [2, 39], among others, all of which have been used to give semantics to nondeterministic programming language features such as concurrency [16, 31, 42].

In this paper, we are interested in developing tools for modeling nondeterministic computations in a dependent type theory such as Coq's CIC [45]. Although any of the formalisms mentioned above

Authors' addresses: Nicolas Chappe, ENS Lyon, France, nicolas.chappe@ens-lyon.fr; Paul He, University of Pennsylvania, Philadelphia, PA, USA, paulhe@cis.upenn.edu; Ludovic Henrio, CNRS, Lyon, France, ludovic.henrio@cnrs.fr; Yannick Zakowski, Inria, France, yannick.zakowski@inria.fr; Steve Zdancewic, University of Pennsylvania, USA, stevez@cis.upenn.edu.

0004-5411/2018/8-ART111 $15.00
https://doi.org/10.1145/1122445.1122456

could be used for such purposes (and many have been [20, 23, 25, 34, 43]), those techniques offer various tradeoffs when it comes to the needs of formalization: automata, and labeled transitions systems, while offering powerful bisimulation proof principles, are not easily made modular (except, perhaps, with complex extensions to the framework [17]). Relationally-defined operational semantics are flexible and expressive, but again suffer from issues of compositionality, which makes it challenging to build general-purpose libraries that support constructing complex models. Conversely, powerdomains and game semantics are more denotational approaches, aiming to ensure compositionality by construction; however, the mathematical structures involved are themselves very complex, typically involving many relations and constraints [2, 30, 39] that are not easy to implement in constructive logic (though there are some notable exceptions [23, 34]). Moreover, in all of the above-mentioned approaches, there are other tensions at play. For instance, how "deep" the embedding is affects the amount of effort needed to implement a formal semantics—"shallower" embeddings typically allow more re-use of metalanguage features, e.g., meta-level function application can obviate the need to define and prove properties about a substitution operation; "deeper" embeddings can side-step meta-level limitations (such as Coq's insistence on pure, total functions) at the cost of additional work to define the semantics. Moreover, these tradeoffs can have significant impact on how difficult it is to use other tools and methodologies: for instance, to use QuickChick [24], one must be able to extract an executable interpreter from the semantics, something that isn't always easy or possible.

This paper introduces a new formalism designed specifically to facilitate the definition of and reasoning about nondeterministic computations in Coq's dependent type theory. The key idea is to update Xia, et al.'s *interaction trees* (ITrees) framework [46] with native support for nondeterminisic "choice nodes" that represent existential choices made during computation. The main technical contributions of this paper are to introduce the definition of these CTrees ("choice trees") and to develop the suitable metatheory and equational reasoning principles to accommodate that change.

We believe that CTrees offer an appealing, and novel, point in the design space of formalisms for working with nondeterministic specifications within type theory. Unlike purely relational specifications, CTrees build nondeterminism explicitly into a datatype, as nodes in a tree, and the nondeterminism is realized propositionally at the level of the equational theory, which determines when two CTree computations are in bisimulation. This means that the user of CTrees has more control over how to represent nondeterminism and when to apply the incumbent propositional reasoning. By reifying the choice construct into a data structure, one can write meta-level functions that manipulate CTrees, rather than working entirely within a relation on syntax. This design allows us to bring to bear the machinery of monadic interpreters to refine the nondeterminism into an (executable) implementation. While ITrees can represent such choice nodes, in our experience, using that feature to model "internal" nondeterminism is awkward: the natural equational theory for ITrees is too fine, and other techniques, such as interpretation into `Prop`, don't work out neatly.

At the same time, the notion of bisimulation for CTrees is still connected to familiar definitions like those from labeled transition systems (LTS), meaning that much of the well-developed theory from prior work can be imported whole-sale. Indeed, we define bisimilarity for CTrees by viewing them as LTSs and applying standard definitions. The fact that the definition of bisimulation ends up being subtle and nontrivial is a sign that we gain something by working with the CTrees: like their ITree predecessors, CTrees have compositional reasoning principles, the type is a monad, and the useful combinators for working with ITrees, namely sequential composition, iteration and recursion, interpretation, etc., all carry over directly. CTrees, though, further allow us to conveniently, and flexibly, define nondeterminstic semantics, ranging from simple choice operators to various flavors of parallel composition. The benefit is that, rather than just working with a "raw" LTS directly, we can construct one using the CTrees combinators—this is a big benefit because, in practice, the

LTS defining the intended semantics of a nondeterministic programming language cannot easily be built in a compositional way without using some kind of intermediate representation, which is exactly what CTrees provides (see the discussion about Figure 3). A key technical novelty of our CTrees definition is that it makes a distinction between *stepping* choices (which correspond to τ transitions and introduce new LTS states) and *delayed* choices (which don't correspond to a transition and don't create a state in the LTS). This design allows for compositional construction of the LTS and generic reasoning rules that are usable in any context.

The net result of our contributions is a library, entirely formalized in Coq, that offers flexible building blocks for constructing nondeterministic, and hence concurrent, models of computation. To demonstrate the applicability of this library, we use it to implement the semantics from two different formalisms: `ccs` [32] and a language with cooperative threads inspired from the literature [1]. Crucially, in both of these scenarios, we are able to define the appropriate parallel composition combinators such that the semantics of the programming language can be defined fully compositionally (i.e., by straightforward induction on the syntax). Moreover, we recover the classic definition of program equivalence for `ccs` directly from the equational theory induced by the encoding of the semantics using CTrees; for the language with cooperative threading, we prove some standard program equivalences.

To summarize, this paper makes the following contributions:

- We introduce CTrees, a novel data structure for defining nondeterministic computations in type theory, along with a set of combinators for building semantic objects using CTrees.
- We develop the appropriate metatheory needed to reason about strong and weak bisimilarity of CTrees, connecting their semantics to concepts familiar from labeled transition systems.
- We show that CTrees admit appropriate notions of refinement and that we can use them to construct monadic interpreters; we show that ITrees can be faithfully embedded into CTrees.
- We demonstrate how to use CTrees in two case studies: (1) to define a semantics for Milner's classic `ccs` and prove that the resulting derived equational theory coincides with the one given by the standard operational semantics, and (2) to model cooperative multithreading with support for `fork` and `yield` operations and prove nontrivial program equivalences.

All of our results have been implemented in Coq, and all claims in this paper are fully mechanically verified. For expository purpose, we stray away from Coq's syntax in the body of this paper, but systematically link our claims to their formal counterpart via hyperlinks represented as (🐓).[1]

The remainder of the paper is organized as follows. The next section gives some background about interaction trees and monadic interpreters, along with a discussion of the challenges of modeling nondeterminism, laying the foundation for our results. We introduce the CTrees data structure and its main combinators in Section 3. Section 4 introduces several notions of equivalences over CTrees—(coinductive) equality, strong bisimilarity, weak bisimilarity, and trace equivalence—and describes its core equational theory. Section 5 describes how to interpret uninterpreted events in an ITree into "choice" branches in a CTree, as well as how to define the monadic interpretation of events from CTrees. Section 6 describes our first case study, a model for `ccs`. Section 7 describes our second case study, a model for the `imp` language extended with cooperative multithreading. Finally, Section 8 discusses related work and concludes.

2 BACKGROUND

[1] For reviewers: all hyperlinks point to an anonymous clone of our development, they are safe to follow. Should our paper be accepted, we will redirect the links to the main repository, and submit a (cleaned up and better documented) version of our development for artifact evaluation.

```
CoInductive itree (E: Type → Type) (R: Type) : Type :=
| Ret (r: R)                            (* computation terminating with value r *)
| Vis {A: Type} (e : E A) (k : A → itree E R) (* event e yielding an answer in A *)
| later (t: itree E R).                 (* "silent" tau transition with child t *)
```

Fig. 1. Interaction trees: definition

2.1 Interaction trees and monadic interpreters

Monadic interpreters have grown to be an attractive way to mechanize the semantics of a wide class of computational systems in dependent typed theory, such as the one found in many proof assistants, for which the host language is purely functional and total. In the Coq ecosystem, interaction trees [46] provide a rich library for building and reasoning about such monadic interpreters. By building upon the free(r) monad [22, 28], one can both design highly reusable components, as well as define modular models of programming languages more amenable to evolution. By modeling recursion coinductively, in the style of Capretta's delay monad [5, 9], such interpreters can model non-total object languages while retaining the ability to *extract* correct-by-construction, executable, reference interpreters. By generically lifting monadic implementations of effects into a monad homomorphism, complex interpreters can be built by stages, starting from an initial structure where all effects are free and incrementally introducing their implementation. Working in a proof assistant, these structures are well suited for reasoning about program equivalence and program refinement: each monadic structure comes with its own notion of refinement, and the layered infrastructure gives rise to increasingly richer equivalences [47], starting from the free monad, which comes with no associated algebra.

Interaction trees are coinductive data structures for representing (potentially divergent) computations that interact with an external environment through *visible events*. A definition of the ITree datatype is shown in Figure 1.[2] The datatype takes as its first parameter a signature—described as a family of types `E : Type → Type`—that specifies the set of interactions the computation may have with the environment. The `Vis` constructor builds a node in the tree representing such an interaction, followed by a continuation indexed by the return type of the event. The second parameter, `R`, is the *result type*, the type of values that the entire computation may return, if it halts. The constructor `Ret` builds such a pure computation, represented as a leaf. Finally, the `later` constructor models an internal, non-observable step of computation, allowing the representation of silently diverging computations; it is also used for guarding corecursive definitions.[3]

To illustrate the approach supported by ITrees, and motivate the contributions of this paper, we consider how to define the semantics for a simple imperative programming language, `imp`:

$$\mathtt{comm} \triangleq \mathtt{skip} \mid x ::= e \mid c1; c2 \mid \mathtt{while}\ b\ \mathtt{do}\ c$$

The language contains a `skip` construct, assignments, sequential composition, and loops—we assume a simple language of expressions, e, that we omit here. Consider the following `imp` programs:

$$p_1 \triangleq \mathtt{while}\ \mathit{true}\ \mathtt{do\ skip} \qquad p_2 \triangleq x ::= 0; x ::= y \qquad p_3 \triangleq x ::= y$$

Following a semantic model for `imp` built on ITrees in the style of Xia et al. [46], one builds a semantics in two stages. First, commands are represented as monadic computations of type `itree MemE unit`: commands do not return values, so the return type of the computation is the trivial `unit` type; interactions with the memory are (at first) left uninterpreted, as indicated by the

[2] The signature of ITrees is presented with a positive coinductive datatype for expository purposes. The actual implementation is defined in the negative style.

[3] The ITree library uses `Tau` to represent `later` nodes. `Tau` and τ are overloaded in our context, so we rename it to `later` here to avoid ambiguity. CTrees will replace the `later` (i.e. `Tau`) constructor with a more general construct anyway.

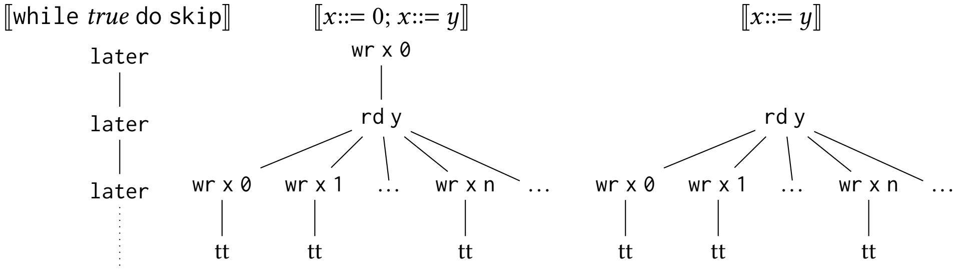


Fig. 2. Example ITrees denoting the imp programs p_1, p_2, and p_3.

event signature MemE. This signature encodes two operations: rd yields a value, while wr yields only the acknowledgment that the operation took place, which we encode again using unit.

```
Variant MemE : Type → Type :=
  | rd (x : var)             : MemE value
  | wr (x : var) (v : value) : MemE unit
```

Indexing by the value type in the continuation of rd events gives rise to non-unary branches in the tree representing these programs. For instance, the programs p_1, p_2, p_3 are, respectively, modeled at this stage by the trees shown in Figure 2. These diagrams omit the Vis and Ret constructors, because their presence is clear from the picture. For example, the second tree would be written as

$$p_2 = \texttt{Vis (wr x 0) (fun \_ ⇒ Vis (rd y) (fun ans ⇒ (Vis (wr x ans) (fun \_ ⇒ Ret tt))))}.$$

The later nodes in the first tree are the guards from Capretta's monad: because the computation diverges silently, it is modeled as an infinite sequence of such guards. The equivalence used for computations in the ITree monad is a *weak bisimulation*, dubbed *equivalence up-to taus* (eutt), which allows one to ignore finite sequences of later nodes when comparing two trees. It remains termination-sensitive: the silently diverging computation is not equivalent to any other ITree.

With ITrees, no assumptions about the semantics of the uninterpreted memory events is made. Although one would expect p_2 and p_3 to be equivalent as imp programs, their trees are not eutt since the former starts with a different event than the latter. This missing algebraic equivalence is concretely recovered at the second stage of modeling: imp programs are given a semantics by interpreting the trees into the state monad, by *handling* the MemE events. This yields computations in stateT mem (itree voidE) unit, or, unfolding the definition, mem → itree voidE (mem * unit). Here, voidE is the "empty" event signature, such that an ITree at that type either silently diverges or deterministically returns an answer. For p_2 and p_3, assuming an initial state m, the computations become (writing the trees horizontally to save space):

$$\begin{aligned}\texttt{interp}\ h_{mem}\ [\![p_2]\!]\ \texttt{m} &= \texttt{later} - \texttt{later} - \texttt{later} - (\texttt{m}\{x \leftarrow 0\}\{x \leftarrow \texttt{m}(y)\}, tt)\\ \texttt{interp}\ h_{mem}\ [\![p_3]\!]\ \texttt{m} &= \texttt{later} - \texttt{later} - (\texttt{m}\{x \leftarrow \texttt{m}(y)\}, tt)\end{aligned}$$

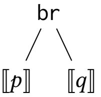
The later nodes are introduced by the interpretation of the memory events. More precisely, an interp combinator applies the handler h_{mem} to the rd and wr nodes of the trees, implementing their semantics in terms of the state monad. Assuming an appropriate implementation of the memory, one can show that $\texttt{m}\{x \leftarrow 0\}\{x \leftarrow \texttt{m}(y)\}$ and $\texttt{m}\{x \leftarrow \texttt{m}(y)\}$ are extensionally equal, and hence p_2 and p_3 are eutt after interpretation.

2.2 Nondeterminism

While the story above is clean and satisfying for stateful effects, nondeterminism is much more challenging. Suppose we extend `imp` with a branching operator `br p or q` whose semantics is to nondeterministically pick a branch to execute. This new feature is modeled very naturally using a boolean-indexed `br` event, creating a binary branch in the tree. The new event signature, a sample use, and the corresponding tree are shown below:

```
Variant BrE : Type → Type :=
| br : BrE bool.
```

```
Vis br (fun b ⇒ if b then p else q)
```



Naturally, as with memory events, `br` does not come with its expected algebra: associativity, commutativity and idempotence. To recover these necessary equations to establish program equivalences such as $p_3 \equiv$ `br` p_2 `or` p_3, we need to find a suitable monad to interpret `br` into.

Zakowski et al. used this approach in the Vellvm project [48] for formalizing the nondeterministic features of the LLVM IR. Their model consists of a propositionally-specified set of computations: ignoring other effects, the monad they use is `itree E _ → Prop`. The equivalence they build on top of it essentially amounts to a form of bijection up-to equivalence of the contained monadic computations. However, this approach suffers from several drawbacks. First, one of the monadic laws is broken: the `bind` operation does not associate to the left.[4] Second, the approach is very much akin to identifying a communicating system with its set of traces, except using a richer structure, namely monadic computations, instead of traces: it forgets all information about *when* nondeterministic choices are made. As has been well identified by the process calculi tradition, such a model leads to equivalences of programs that are too coarse to be compositional in general, in contrast to bisimulations [7]. Third, because the set of computations is captured propositionally, this interpretation is incompatible with the generation of an *executable* interpreter by extraction, losing one of the major strengths of the ITree framework. Zakowski et al. work around this difficulty by providing two interpretations of their nondeterministic events, and formally relating them.

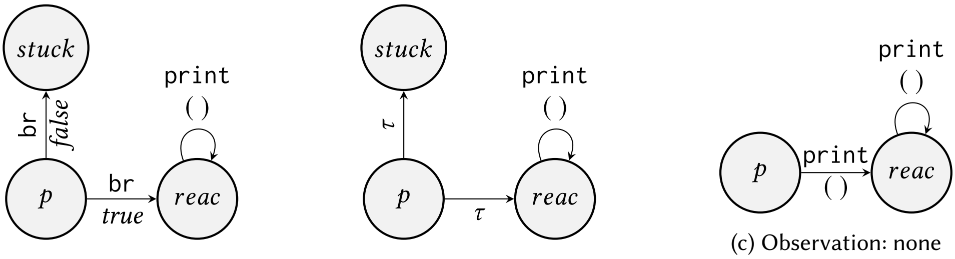
This difficulty with properly tackling nondeterminism extends also to concurrency. Lesani et al. used ITrees to prove the linearizability of concurrent objects [27]. Here too, they rely on sets of linearized traces and consider their linear interleavings. While a reasonable solution in their context, that approach strays from the monadic interpreter style and fails to capture bisimilarity.

This paper introduces CTrees, a suitable monad for modeling nondeterministic effects. As with ITrees, the structure is compatible with divergence, external interaction through uninterpreted events, extraction to executable reference interpreters, and monadic interpretation. The core intuition is based on the observation that the tree-like structure from ITrees is indeed the right one for modeling nondeterminism. The problem arises from how the ITrees definition of `eutt` observes which branch is taken, requiring that *all* branches be externally visible: while appropriate to model nondeterminism that results from a lack of information, it does not correspond to true *internal* choice. Put another way, thinking of the trees as labeled transition systems, ITrees are *deterministic*. With CTrees, we therefore additionally consider truly branching nodes, explicitly build the associated nondeterministic LTS, and define proper bisimulations on the structure.

The resulting definitions are very expressive. As expected, they allow us to establish desired `imp` equations such as $p_3 \equiv$ `br` p_2 `or` p_3, but also to model `ccs` and cooperative multithreading.

Before getting to that, and to better motivate our definitions, let us further extend our toy language with a `block` construction that cannot reduce, and a `print` instruction that simply prints

[4] This problem is known in the literature [29] as acknowledged by Zakowski et al..



(a) Observation: branches

(b) Observation: branching

(c) Observation: none

Fig. 3. Three possible semantics for the program p, from an LTS perspective

a dot. We will refer to this language as `ImpBr`.

$$\texttt{comm} \triangleq \texttt{skip} \mid x ::= e \mid c1; c2 \mid \texttt{while}\ b\ \texttt{do}\ c \mid \texttt{br}\ c1\ \texttt{or}\ c2 \mid \texttt{block} \mid \texttt{print}$$

Consider the program $p \triangleq$ `br` (`while` *true* `do` `print`) `or` `block`. Depending on the intended operational semantics associated with `br`, this program can have one of two behaviors: (1) either to always reactively print an infinite chain of dots, or (2) to become nondeterministically either similarly reactive, or completely unresponsive.

When working with (small-step) operational semantics, the distinction between these behaviors is immediately apparent in the reduction rule for `br` (we only show rules for the left branch here).

$$\frac{}{\texttt{br}\ c1\ \texttt{or}\ c2 \longrightarrow c1}\ \textsc{BrInternal} \qquad \frac{c_1 \longrightarrow c_1'}{\texttt{br}\ c_1\ \texttt{or}\ c_2 \longrightarrow c_1'}\ \textsc{BrDelayed}$$

BrInternal specifies that `br` may simply reduce to the left branch, while BrDelayed specifies that `br` can reduce to any state reachable from the left branch. From an observational perspective, the former situation describes a system where, although we do not observe which branch has been taken, we do observe that *a* branch has been taken. On the contrary, the latter only progresses if one of the branches can progress, we thus directly observe the evolution of the chosen branch.

In order to design the right monadic structure allowing for enough flexibility to model either behavior, it is useful to look ahead and anticipate how we will reason about program equivalence, as described in detail in Section 4. The intuition we follow is to interpret our computations as labeled transition systems and define bisimulations over those, as is generally done in the process algebra literature. From this perspective, the `imp` program p may correspond to three distinct LTSs depending on the intended semantics, as shown in Figure 3.

Figure 3a describes the case where picking a branch is an unknown external event, hence where taking a specific branch is an *observable* action with a dedicated label: this situation is naturally modeled by a `Vis` node in the style of ITrees, that is $[\![\texttt{br}\ p\ \texttt{or}\ q]\!] \triangleq Vis\ \texttt{br}\ \lambda b \cdot \text{if } b \text{ then } [\![p]\!] \text{ else } [\![q]\!]$.

Figure 3b corresponds to BrInternal: both the stuck and the reactive states are reachable, but we do not observe the label of the transition. This transition exactly corresponds to the internal τ step of process algebra. This situation is captured by introducing a new kind of node in our data structure, a Br_S branch, that maps in our bisimulations defined in Section 4 to a nondeterministic *internal step*. For this semantics, we thus have $[\![\texttt{br}\ p\ \texttt{or}\ q]\!] \triangleq Br_S^2\ \lambda b \cdot \text{if } b \text{ then } [\![p]\!] \text{ else } [\![q]\!]$.[5]

Figure 3c corresponds to BrDelayed but raises the questi n: how do we build such a behavior? A natural answer can be to assume that it is the responsibility of the model, i.e., the function mapping `imp`'s syntax to the semantic domain, CTrees, to explicitly build this LTS. Here, $[\![p]\!]$ would be an infinite sequence of `Vis print` nodes, containing no other node. While that would be convenient for

[5] The 2 indicates the arity of the branching.

developing the meta-theory of CTrees, this design choice would render them far less compositional (and hence less useful) than we want them to be. Indeed, we want our models to be defined as computable functions by recursion on the syntax, whenever possible. But to build this LTS directly, the model for `br p or q` needs to introspect the models for $[\![p]\!]$ and $[\![q]\!]$ to decide whether they can take a step, and hence whether it should introduce a branching node. But, in general, statically determining whether the next reachable instruction is `block` is intractable, so that introspection will be hard (or impossible) to implement. We thus extend CTrees with a third category of node, a Br_D node, which does not directly correspond to states of an LTS. Instead, Br_D nodes aggregate sub-trees such that the (inductively reachable) Br_D children of a Br_D node are "merged" in the LTS view of the CTree. This design choice means that, for the BrDelayed semantics, the model is again trivial to define: $[\![\texttt{br}\ p\ \texttt{or}\ q]\!] \triangleq Br^2_D\ \lambda b \cdot \text{if } b \text{ then } [\![p]\!] \text{ else } [\![q]\!]$, but the definition of bisimilarity for CTrees ensures that the behavior of $[\![p]\!]$ is exactly the LTS in Figure 3c.

3 CTREES: DEFINITION AND COMBINATORS

3.1 Core definitions

We are now ready to define our core datatype, displayed in the upper part of Figure 4. The definition remains close to a coinductive implementation of the free monad, but hardcodes support for an additional effect: unobservable, nondeterministic branching. The CTree datatype, much like an ITree, is parameterized by a signature of (external) events `E` encoded as a family of types, and a return type `R`. It is defined as a coinductive tree[6] with four kind of nodes: pure computations (`Ret`), external events (`Vis`), internal branching with implicitly associated τ step (`brS`), and delayed internal branching (`brD`). The continuation following external events is indexed by the return type specified by the emitted event. For the sake of simpler exposition, we restrict both internal branches to be of finite width: their continuation are indexed by finite types `fin`, but this is not a fundamental limitation.[7] When using finite branching, we abuse notation and write, for instance, $Br^2_S\ t\ u$ for the computation branching with two branches, rather than explicitly spelling out the continuation which branches on the choice index: `fun i ⇒ match i with 0 ⇒ t | 1 ⇒ u end`.

The remainder of Figure 4 displays (superficially simplified) definitions of the core combinators. As expected, `ctree E` forms a monad for any interface `E`: the `bind` combinator simply lazily crawls the potentially infinite first tree, and passes the value stored in any reachable leaf to the continuation. The `iter` combinator is central to encoding looping and recursive features: it takes as argument a body, `body`, intended to be iterated, and is defined such that the computation returns either a new index over which to continue iterating, or a final value; `iter` ties the recursive knot. Its definition is analogous to the one for ITrees, except that we need to ask ourselves how to guard the **`cofix`**: if `body` is a constant, pure computation, unguarded corecursion would be ill-defined. ITrees use a `later` node for this purpose. Here, we instead use a unary delayed branch as a guard, written `Guard`. We additionally write `Step` for the unary stepping branch—we will discuss how they relate in Section 5.3. The minimal computations respectively triggering an event `e`, generating observable branching, or delaying a branch, are defined as `trigger`, Br^n_S, and Br^n_D.

Convenience in building models comes at a cost: many CTrees represent the same LTS. Figure 5 illustrates this by defining several CTrees implementing the stuck LTS and the silently spinning one. We define formally the necessary equivalences on computations to prove this informal statement in

[6] The actual implementation uses a negative style with primitive projections. We omit this technical detail in the presentation.

[7] We actually also support another branch that defines *enhanced CTrees*, with arbitrary branching over indexes specified in the type of the data structure by an interface akin to `E`. It is not linked here for anonymity reasons, but should the paper be accepted, we will add it to the artifact. It is only used for the material described in this paper for Lemma 5.2.

```
(* Core datatype *)
CoInductive ctree (E : Type → Type) (R : Type) :=
| Ret (r : R)                                   (* a pure computation *)
| Vis {X : Type} (e : E X) (k : X → ctree)      (* an external event *)
| brS (n : nat) (k : fin n → ctree)             (* stepping branching *)
| brD (n : nat) (k : fin n → ctree)             (* delayed branching *)

(* Bind, sequencing computations *)
CoFixpoint bind {E T U} (t : ctree E T) (k : T → ctree E U) : ctree E U :=
match u with
| Ret r   ⇒ k r
| Vis e h ⇒ Vis e (fun x ⇒ bind (h x) k)
| brS n h ⇒ brS n (fun x ⇒ bind (h x) k)
| brD n h ⇒ brD n (fun x ⇒ bind (h x) k)
end

(* Unary guards *)
Definition Guard (t : ctree E R) : ctree E R := brD 1 (fun _ ⇒ t)
Definition Step  (t : ctree E R) : ctree E R := brS 1 (fun _ ⇒ t)

(* Main fixpoint combinator *)
CoFixpoint iter {I: Type} (body : I → ctree E (I + R)) : I → ctree E R :=
  bind (body i) (fun lr ⇒ match lr with
                              | inr r ⇒ Ret r
                              | inl i ⇒ Guard (iter body i)
                              end)

Notation "E ⇝ F" := (∀ X, E X → F X)
(* Atomic ctrees triggering a single event *)
Definition trigger E : E ⇝ ctree E := fun R e ⇒ Vis e (fun x ⇒ Ret x)
(* Atomic branching ctrees *)
Definition Br_S^n: ctree E (fin n) := fun n ⇒ brS n (fun x ⇒ Ret x)
Definition Br_D^n: ctree E (fin n) := fun n ⇒ brD n (fun x ⇒ Ret x)
```

Fig. 4. CTrees: definition and core combinators ()

Section 4. The astute reader may wonder whether both kind of branching nodes really are necessary. While convenient, we show in Section 4.5 that Br_S can be expressed in terms of Br_D and Step.

3.2 A hint of introspection: heads of computations

Br_D nodes avoid the need for introspection on trees to model something as generic as a delayed branching construct such as the one specified by BrDelayed. However, introspection becomes necessary to build a tree that depends on the reachable external actions of the sub-trees. This is the case for example for ccs’s parallel operator that we model in Section 6. The set of reachable external actions is not computable in general, as we may have to first know if the computation to the left of a sequence terminates before knowing if the events contained in the continuation are reachable. We are, however, in luck, as we have at hand a semantic domain able to represent potentially divergent computations: CTrees themselves!

The head combinator, described on Figure 6, builds a pure, potentially diverging computation *only made of delayed choices*, and whose leaves contain all reachable subtrees starting with an

```
(* Stuck processes *)
Definition stuckE (e : E void) : ctree E void := trigger e
Definition stuckS : ctree E void := BrS 0
Definition stuckD : ctree E void := BrD 0
CoFixpoint spinD : ctree E R := Guard spinI
CoFixpoint spinD_nary n : ctree E R := Br^n_D;; spinV_nary n

(* Spinning processes *)
CoFixpoint spin : ctree E R := Step spinV
CoFixpoint spin_nary n : ctree E R := Br^n_S;; spinI_nary n
```

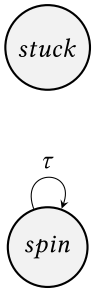


Fig. 5. Concrete representations of stuck and spinning LTSs

```
Variant action E R :=
  | ARet (r : R)
  | ABr {n} (k : fin n → ctree E R)
  | AVis {X} (e : E X) (k : X → ctree E R).

CoFixpoint head {E X} (t : ctree E X) : ctree E (action E X) :=
  match t with
  | Ret x ⇒ Ret (ARet x) | Vis e k ⇒ Ret (AVis e k) | brS n k ⇒ Ret (ABr k)
  | BrD n k ⇒ BrD n (fun i ⇒ head (k i))
  end.
```

Fig. 6. Lazily computing the set of reachable observable nodes ()

observable node. These "immediately" observable trees are captured in an action datatype, which is used as the return type of the built computation. The head combinator simply crawls the tree by reconstructing all delayed branches, until it reaches a subtree with any other node at its root; it then returns that subtree as the corresponding action.

4 EQUIVALENCES AND EQUATIONAL THEORY FOR CTREES

Section 3 introduced CTrees, the domain of computations we consider, as well as a selection of combinators upon it. We now turn to the question of comparing computations represented as CTrees for notions of equivalence and refinement. In particular, we formalize the LTS representation of a CTree that we have followed to justify our definitions. This section first introduces a (coinductive) syntactic equality of CTrees, then it recovers the traditional notions of strong and weak bisimilarity of processes, as well as trace equivalence, for an LTS derived from the tree structure. We equip these notions with a primitive equational theory for CTrees; this theory provides the building blocks necessary for deriving domain-specific equational theories, such as the ones established in Section 6 for ccs, and in Section 7 for cooperative scheduling.

4.1 Coinductive proofs and up-to principles in Coq

Working with CTrees requires the use of several coinductive predicates and relations to describe equivalences, refinements and invariants. Doing so at scale in Coq would be highly impractical if only using its native support for coinductive proofs, but it is nowadays more feasible thanks to library support [18, 36, 49]. Our development relies on Pous's coinduction library [37] to define and reason about the various coinductive predicates and relations we manipulate. We briefly recall the essential facilities, based on Pous's *companion* [36], that the library provides.

$$\textsc{refl}^{up} R \triangleq \{(x,x)\} \qquad \textsc{sym}^{up} R \triangleq \{(y,x) \mid R\ x\ y\} \qquad \textsc{trans}^{up} R \triangleq \{(x,z) \mid \exists y,\ R\ x\ y \wedge R\ y\ z\}$$
$$\textsc{Vis}^{up} R \triangleq \{(Vis\ e\ k, Vis\ e\ k') \mid \forall v,\ R\ (k\ v)\ (k'\ v)\}$$
$$\textsc{bind}^{up}(equiv)\ R \triangleq \{(x \gg\!= k, y \gg\!= l) \mid equiv\ x\ y \wedge \forall v, R\ (k\ v)\ (l\ v)\}$$
$$\textsc{upto}^{up}(equiv)\ R \triangleq \{(x,y) \mid \exists x'\ y',\ equiv\ x\ x' \wedge R\ x'\ y' \wedge\ equiv\ y'\ y\}$$

Fig. 7. Main generic up-to principles used for relations of CTrees where `R : rel (ctree E X)`

The core construction provided is a greatest fixpoint operator (`gfp` b):`X` for any complete lattice `X`, and monotone endofunction b : `X` ⟶ `X`. In particular, the sort of Coq propositions `Prop` forms a complete lattice, as do any function from an arbitrary type into a complete lattice—coinductive relations, of arbitrary arity, over arbitrary types, can therefore be built using this combinator. We most often instantiate `X` with the complete lattice of binary relations over CTrees: `X := ctree E A ⟶ ctree E A ⟶ Prop` for fixed parameters `E` and `A`. We write such binary relations as `rel(A,B)` for `A ⟶ B ⟶ Prop`, and `rel(A)` for `rel(A,A)`; for instance: `X := rel(ctree E A)`.

The library provides tactic support for coinductive proofs based on Knaster-Tarski's theorem: any post fixpoint is below the greatest fixpoint. Specialized to relations on CTrees, the proof method consists in exhibiting a relation `R`, that can be thought of as a set of pairs of trees, providing a "coinduction candidate", and proving that `(∀ t u, R t u ⟶ b R t u) ⟶ ∀ t u, R t u ⟶ gfp b t u`. The core benefit of the companion is, however, to further provide support for proofs by *enhanced coinduction*.

Given an endofunction b, a (sound) enhanced coinduction principle, also known as an *up-to principle*, relies on an additional function f: `X ⟶ X` allowing one to work with `bf` (the composition of b with f) instead of b: any post fixpoint of bf is below the greatest fixpoint of b. Significant effort has been invested in identifying classes of sound up-to principles, and developing ways to combine them [35, 40, 41]. The companion relies on one particular sub-class of sound principles that forms a complete lattice, the *compatible* functions: we hence can conduct proofs systematically up-to the greatest compatible function, dubbed the *companion* and written $\mathtt{t}_b$. In practice, this means that a coinductive proof can, on-the-fly, use any valid up-to principle drawn from the companion. We refer the interested reader to the literature [36] and our formal development for further details, and specialize things to our use-case to describe the up-to functions we use in our development.

Figure 7 describes the main generic up-to principles we use for our relations on CTrees. The validity of $\textsc{refl}^{up}$, $\textsc{sym}^{up}$ and $\textsc{trans}^{up}$ for a given `gfp` b entails respectively the reflexivity, symmetry and transitivity of the relation ($b\mathtt{t}_b\ R$), whose post fixpoints constitute one's goal during a (up-to companion) coinductive proof for `gfp` b, as well as of ($\mathtt{t}_b\ R$), that our bisimulation game seeks to reach—where R captures the bisimulation candidate. The $\textsc{Vis}^{up}$ and $\textsc{bind}^{up}(\_)$ up-to functions help when reasoning structurally, respectively allowing to cross through `Vis` nodes and `bind` constructs. Finally, validity of the $\textsc{upto}^{up}(equiv)$ principle allows for rewriting via the *equiv* relation during coinductive proofs for b.

4.2 Coinductive equality for CTree

Coq's equality, `eq`, is not a good fit to express the structural equality of coinductive structures—even the eta-law for a coinductive data structure does not hold up-to `eq`. We therefore define, as is standard, a structural equality[8] by coinduction `equ: rel(ctree E A)` (written ≅ in infix). The endofunction simply matches head constructors and behaves extensionally on continuations.

[8] Note that for ITrees, this relation corresponds to what Xia et al. dub as *strong bisimulation*, and name `eq_itree`. We carefully avoid this nomenclature here to reserve this term for the relation we define in Section 4.4

$$\texttt{label} \triangleq \texttt{tau} \mid \texttt{obs}\ e\ v \mid \texttt{val}\ v$$

$$\frac{k\ v \xrightarrow{l} t}{Br_D^n\ k \xrightarrow{l} t} \qquad \frac{}{Br_S^n\ k \xrightarrow{\texttt{tau}} k\ v} \qquad \frac{}{Vis\ e\ k \xrightarrow{\texttt{obs}\ e\ v} k\ v} \qquad \frac{}{\texttt{Ret}\ v \xrightarrow{\texttt{val}\ v} \emptyset}$$

Fig. 8. Inductive characterisation of the LTS induced by a CTree (🐓)

Definition 4.1. Structural equality (🐓)

$$\begin{aligned}\texttt{equ} \triangleq \texttt{gfp}\ \lambda R \cdot \{(\texttt{Ret}\ v,\ \texttt{Ret}\ v)\} \cup \{(Vis\ e\ k,\ Vis\ e\ k') \mid \forall v,\ R\ (k\ v)\ (k'\ v)\} \cup \\ \{(Br_D^n\ k,\ Br_D^n\ k') \mid \forall v,\ R\ (k\ v)\ (k'\ v)\} \cup \{(Br_S^n\ k,\ Br_S^n\ k') \mid \forall v,\ R\ (k\ v)\ (k'\ v)\}\end{aligned}$$

The equ relation raises no surprises: it is an equivalence relation, and is adequate to prove all eta-laws—for the CTree structure itself and for the cofixes we manipulate. Similarly, the usual monadic laws are established with respect to equ.

Lemma 4.2. *Monadic laws (🐓)*

$$\texttt{Ret}\ v \gg\!= k \cong k\ v \qquad x \leftarrow t\ ;;\ \texttt{Ret}\ x \cong t \qquad (t \gg\!= k) \gg\!= l \cong t \gg\!= (\lambda x \Rightarrow k\ x \gg\!= l)$$

Of course, formal equational reasoning with respect to an equivalence relation other than eq comes at the usual cost: all constructions introduced over CTrees must be proved to respect equ (in Coq parlance, they must be Proper), allowing us to work painlessly with setoid-based rewriting.

Finally, we establish some enhanced coinduction principles for equ.

Lemma 4.3. *Enhanced coinduction for* equ *(🐓)*
$\textsc{refl}^{up}$, $\textsc{sym}^{up}$, $\textsc{trans}^{up}$, $\textsc{bind}^{up}(\cong)$ *and* $\textsc{upto}^{up}(\cong)$ *provide valid up-to principles for* equ.

While equ, being a structural equivalence, is very comfortable to work with, it, naturally, is much too stringent. To reason semantically about CTrees, we need a relation that remains termination sensitive but allows for differences in internal steps, that still imposes a tight correspondence over external events, but relaxes its requirement for nondeterministically branching nodes. We achieve this by drawing from standard approaches developed for process calculi.

4.3 Looking at CTrees under the lens of labeled transition systems

To build a notion of bisimilarity between CTree computations, we associate a labeled transition system to a CTree, as defined in Figure 8. This LTS exhibits three kinds of labels: a tau witnesses a stepping branch, an obs e x observes the encountered event together with the answer from the environment considered, and a val v is emitted when returning a value. Interestingly, there is a significant mismatch between the structure of the tree and the induced LTS: the states of the LTS correspond to the nodes of the CTree *that are not immediately preceded by a delayed choice.* Accordingly, the definition of the transition relation between states inductively iterates over delayed branches. Stepping branches and visible nodes map immediately to a set of transitions, one for each outgoing edge; finally a return node generates a single ret transition, moving onto a stuck state, encoded as a nullary branching node and written $\emptyset$. These rules formalize the intuition we gave in Section 2.2 that allowed us to derive the LTSs of Figure 3 from the corresponding ImpBr terms.

Defining the property of a tree to be *stuck*, that is: $t \nrightarrow \triangleq\ \forall l\ u,\ \neg(t \xrightarrow{l} u)$, we can make the depictions from Figure 5 precise: nullary nodes are stuck by construction, since stepping would require a branch, while $\texttt{spin}_\texttt{D}$ is proven to be stuck by induction, since it cannot reach any step.

$$Br_S^0 \nrightarrow \qquad\qquad Br_D^0 \nrightarrow \qquad\qquad \texttt{spin}_\texttt{D}\ n \nrightarrow$$

$$\frac{t \xrightarrow{l} u \quad l \neq \mathtt{val}\ v}{t \gg\!= k \xrightarrow{l} u \gg\!= k} \qquad\qquad \frac{t \xrightarrow{\mathtt{val}\ v} \emptyset \quad k\ v \xrightarrow{l} u}{t \gg\!= k \xrightarrow{l} u}$$

$$\frac{t \gg\!= k \xrightarrow{l} u}{(l \neq \mathtt{val}\ v \wedge \exists t', t \xrightarrow{l} t' \wedge u \cong t' \gg\!= k) \vee (\exists v, t \xrightarrow{\mathtt{val}\ v} \emptyset \wedge k\ v \xrightarrow{l} u)}$$

Fig. 9. Transitions under bind (🐓)

The stepping relation interacts slightly awkwardly with `bind`: indeed, although a unit for `bind`, the `Ret` construct is not inert from the perspective of the LTS. Non `val` transitions can therefore be propagated below the left-hand-side of a `bind`, while a `val` transition in the prefix does not entail the existence of a transition in the `bind`—Figure 9 describes the corresponding lemmas.

We additionally define the traditional *weak transition* $s \overset{l}{\Rightarrow} t$ on the LTS that can perform tau transitions before or after the l transition (and possibly be none if $l = \tau$). This part of the theory is so standard that we can directly reuse parts of the development for `ccs` that Pous developed to illustrate the companion [38], with the exception that we need to work in a Kleene Algebra with a model closed under equ rather than `eq`.

4.4 Bisimilarity

Having settled on the data structure and its induced LTS, we are back on a well-traveled road: strong bisimilarity (referred simply as bisimilarity in the following) is defined in a completely standard way over the LTS view of CTrees.

Definition 4.4 (Bisimulation for CTrees (🐓)). The progress function sb for bisimilarity maps a relation $\mathcal{R}$ over CTrees to the relation such that $sb\ R\ s\ t$ holds if and only if:

$$\forall l\ s',\ s \xrightarrow{l} s' \implies \exists t'.\ s'\ \mathcal{R}\ t' \wedge t \xrightarrow{l} t'$$

and conversely

$$\forall l\ t',\ t \xrightarrow{l} t' \implies \exists s'.\ s'\ \mathcal{R}\ t' \wedge s \xrightarrow{l} s'$$

i.e.

$$\begin{tikzcd} s \arrow[r, phantom, "sb\,\mathcal{R}"] \arrow[d, "l"'] & t \arrow[d, "l"] \\ s' \arrow[r, phantom, "\mathcal{R}"] & t' \end{tikzcd}$$

Bisimilarity, written $s \sim t$, is defined as the greatest fixpoint of sb: `sbisim` $\triangleq$ `gfp` sb.

All the traditional tools surrounding bisimilarity can be transferred to our setup. We omit the details for spacing concerns, but additionally provide:

- weak bisimilarity, written $s \approx t$, derived from the definition of the weak transition (🐓);
- a characterization of the traces (represented as colists) of a CTree, used to define trace-equivalence (written $\equiv_{\mathtt{tr}}$) (🐓);
- strong simulations (written $\lesssim$), defined as the greatest fixpoint of the half game for strong bisimulation (🐓).

4.4.1 *Core equational theory.*

Bisimilarity forms an equivalence relation satisfying a collection of primitive laws for CTrees summed up in Figure 10. We use simple inference rules to represent an implication from the premises to the conclusion, and double-lined rules to represent equivalences. Each rule is proved as a lemma with respect to the definitions above.

The first four rules recover some structural reasoning on the syntax of the trees from its semantic interpretation. These rules are much closer to what `eutt` provides by construction for ITrees: leaves are bisimilar if they are equal, and computations performing the same external interaction must

$$\frac{x = y}{\texttt{Ret}\ x \sim \texttt{Ret}\ y} \qquad \frac{\forall x,\ h\ x \sim k\ x}{Vis\ e\ h \sim Vis\ e\ k} \qquad \frac{(\forall x, \exists y,\ h\ x \sim k\ y) \land (\forall y, \exists x,\ h\ x \sim k\ y)}{Br_S^n\ h \sim Br_S^m\ k}$$

$$\frac{(\forall x, \exists y,\ h\ x \sim k\ y) \land (\forall y, \exists x,\ h\ x \sim k\ y)}{Br_D^n\ h \sim_R Br_D^m\ k} \qquad \frac{t \sim u \qquad (\forall x,\ g\ x \sim k\ x)}{t \gg= g \sim u \gg= k}$$

$$\texttt{Guard}\ t \sim t \qquad \frac{u \not\to}{Br_D^2\ t\ u \sim t} \qquad Br_D^2\ t\ (Br_D^2\ u\ v) \sim Br_D^2\ (Br_D^2\ t\ u)\ v \qquad Br_D^2\ t\ u \sim Br_D^2\ u\ t$$

$$Br_D^2\ t\ t \sim t \qquad Br_D^2\ (Br_D^2\ t\ u)\ v \sim Br_D^3\ t\ u\ v \qquad Br_S^2\ t\ u \sim Br_S^2\ u\ t \qquad Br_S^2\ t\ t \sim \texttt{Step}\ t$$

$$\texttt{spin}_{\texttt{D}}\ n \sim \texttt{spin}_{\texttt{D}}\ m \qquad \frac{(n > 0 \land m > 0) \lor (n = m = 0)}{\texttt{spin}_{\texttt{S}}\ n \sim \texttt{spin}_{\texttt{S}}\ m}$$

Fig. 10. Elementary equational theory for sbisim (🐓)

remain point-wise bisimilar. Stepping branches, potentially of distinct arity, can be matched one against another if and only if both domains of indexes can be injected into the other to reestablish bisimilarity. In contrast, this condition is sufficient but not necessary for delayed branches, since the points of the continuation structurally immediately accessible do not correspond to accessible states in the LTS. Finally, bisimilarity is a congruence for bind.

Another illustration of this absence of equivalence for delayed branching nodes as head constructor is that such a computation may be strongly bisimilar to a computation with a different head constructor. The simplest example is that sbisim can ignore (finite numbers of) Guard nodes. Another example is that stuck processes behave as a unit for delayed branching nodes. We furthermore obtain the equational theory that we expect for nondeterministic effects: delayed branching is associative, commutative, idempotent, and can be merged into delayed branching nodes of larger arity.[9] In contrast, stepping branches are only commutative, and almost idempotent, provided we introduce an additional Step. This Step can in turn be ignored by moving to weak bisimilarity.

Finally, two delayed spins are always bisimilar (neither process can step) while two stepping spins are bisimilar if and only if they are both nullary (neither one can step), or both non-nullary.

Naturally, this equational theory gets trivially lifted at the language level for ImpBr (🐓).

4.4.2 Proof system for bisimulation proofs. As is usual, the laws in Figure 10, enriched with domain-specific equations, allow for deriving further equations purely equationally. But to ease the proof of these primitive laws, as well as new nontrivial equations requiring explicit bisimulation proofs, we provide proof rules that are valid during bisimulation proofs.

Given a bisimulation candidate R, we write $t \sim_R u$ for $sb\ (\mathtt{t}_{sb}\ R)$: one needs to "play the bisimulation game" by crossing sb, and may rely on R to conclude by coinduction, while furthermore being able to exploit the companion $\mathtt{t}_{sb}$ to leverage sound up-to principles. We depict the main rules we use in Figure 11. These proof rules are notably convenient because they avoid an exponential explosion in the number of cases in our proofs, which otherwise would arise due to the systematic binary split entailed by the natural way to play the bisimulation game. These rules essentially match up counterpart CTree constructors at the level of bisimilarity, but additionally make a distinction as to whether applying the rule soundly acts as playing the game—i.e., the premises refer to R, allowing to conclude using the coinduction hypothesis—or whether they do not—, i.e., the premises

[9] Stating these facts generically in the arity of branching is quite awkward, we hence state them here for binary branching, but adapting them at other arities is completely straightforward.

$$\texttt{Ret}\ v \sim_R \texttt{Ret}\ v$$

$$\frac{\forall v,\ R\ (k\ v)\ (k'\ v)}{\mathit{Vis}\ e\ k \sim_R \mathit{Vis}\ e\ k'} \qquad \frac{(\forall x, \exists y,\ R\ (k\ x)\ (k'\ y)) \land (\forall y, \exists x,\ R\ (k\ x)\ (k'\ y))}{\mathit{Br}_S^n\ k \sim_R \mathit{Br}_S^m\ k'} \qquad \frac{\forall v,\ R\ (k\ v)\ (k'\ v)}{\mathit{Br}_S^n\ k \sim_R \mathit{Br}_S^n\ k'}$$

$$\frac{(\forall x, \exists y,\ (k\ x) \sim_R (k'\ y)) \land (\forall y, \exists x,\ (k\ x) \sim_R (k'\ y))}{\mathit{Br}_D^n\ k \sim_R \mathit{Br}_D^m\ k'} \qquad \frac{t \sim_R u}{\texttt{Guard}\ t \sim_R \texttt{Guard}\ u} \qquad \frac{t\ R\ u}{\texttt{Step}\ t \sim_R \texttt{Step}\ u}$$

Fig. 11. Proof rules for coinductive proofs of sbisim (🐓)

still refer to $\sim_R$. The latter situation arises when using the proof rules that strip off delayed branches from the structure of our trees: on either side, they do not entail any step in the corresponding LTSs, but rather correspond to recursive calls to its inductive constructor.

Furthermore, we provide a rich set of valid up-to principles:

Lemma 4.5. *Enhanced coinduction for* sbisim (🐓) *The functions* $\textsc{refl}^{up}$, $\textsc{sym}^{up}$, $\textsc{trans}^{up}$, $\textsc{Vis}^{up}$, $\textsc{bind}^{up}(\sim)$, $\textsc{upto}^{up}(\cong)$ *and* $\textsc{upto}^{up}(\sim)$ *provide valid up-to principles for* sbisim.

In particular, the equation Guard $t \sim t$ can be used for rewriting during bisimulation proofs, allowing for asymmetric stripping of guards.

4.5 Eliminating nondeterministic stepping choices

Out of programming convenience, CTrees offer both stepping and delayed branching of arbitrary arity. However, as hinted at in Section 3, this is superfluous: a stepping branch can always be simulated by delaying the same choice, and guarding each branch by a unary Step. We make this intuition formal by proving that the following combinator, precisely performing this transformation, preserves the behavior of the computation with respect to sbisim.

Definition 4.6 (Elimination of nondeterministic stepping choice (🐓)).

```
Definition BrSElim {E X} (t : ctree E X) : ctree E X :=
  iter (fun t ⇒ match t with
                | Ret r   ⇒ Ret (inr r)
                | brD k   ⇒ x  ← Br_D^n;; Ret (inl (k x))
                | brS k   ⇒ x  ← Br_D^n;; Step (Ret (inl (k x)))
                | Vis e k ⇒ x  ← trigger e;; Ret (inl (k x))
                end) t.
```

Lemma 4.7. *Stepping nondeterministic branches can be eliminated (🐓)* $\quad \forall t,$ BrSElim $t \sim t$

The proof of this statement is quite illustrative of bisimulation proofs over CTrees. We provide its sketch in the appendix as a means to illustrate the layers of facilities put in place in the formal development to allow for the translation of paper intuitions, most notably the enhanced coinduction principles that the companion allows us to build. It also illustrates the difficulties that the structure raises when bridging the distance between the syntax (the tree) and its implicit semantics (the LTS).

5 INTERPRETATION FROM AND TO CTREES

The ITree ecosystem fundamentally relies on the incremental interpretation of effects, represented as external events, into their monadic implementations. Through this section, we show how CTrees fit into this narrative both by supporting the interpretation of their own external events, and by being a suitable target monad for ITrees, for the implementation of nondeterministic branching.

```
Definition interp (h : E ⤳ M) : ctree E ⤳ M := fun R ⇒
  iter (fun t ⇒ match t with
  | Ret r   ⇒ ret (inr r)
  | BrD n k ⇒ bind (mBrD n) (fun x ⇒ ret (inl (k x)))
  | BrS n k ⇒ bind (mBrS n) (fun x ⇒ ret (inl (k x)))
  | Vis e k ⇒ bind (h e)     (fun x ⇒ ret (inl (k x)))
  end).
```

Fig. 12. Interpreter for CTrees (class constraints omitted) ()

5.1 Interpretation

ITrees support interpretation: provided a *handler* `h:E ⤳ M` implementing its signature of events `E` into a suitable monad `M`, the `(interp h):itree E ⤳ M` combinator provides an implementation of any computation into `M`. The only restriction imposed on the target monad `M` is that it must support its own `iter` combinator, i.e., be iterative, so that the full coinductive nature of the tree can be internalized in `M`. For this implementation to be sensible and amenable to verification in practice, one must, however, check an additional property: `interp h` should form a monad morphism—in particular, it should map `eutt` ITrees to equivalent monadic computations in `M`.

Unsurprisingly, given their structure, CTrees enjoy their own `interp` combinator. Its definition, provided in Figure 12, is very close to its ITree counterpart. The interpreter relies on the CTree version of `iter`, chaining the implementations of the external events in the process. The target monad must naturally still be iterative, but must also explain how it internalizes branching nodes through the `mBrD` and `mBrS` operations.

Perhaps more surprisingly, the requirement that `interp h` define a monad morphism unearths interesting subtleties. Let us consider the elementary case where the interface `E` is implemented in terms of (possibly pure) uninterpreted computations, that is when `M := ctree F`. The requirement becomes: $\forall t\ u, t \sim u \longrightarrow$ `interp h` $t \sim$ `interp h` u. But this result does not hold for an arbitrary `h`: intuitively, our definition for `sbisim` has implicitly assumed that implementations of external events may eliminate reachable states in the computation's induced LTS—through pure implementations—but should not be allowed to introduce new ones.

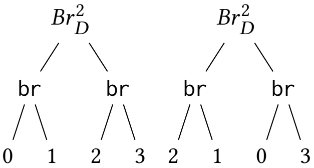


Fig. 13. Two strongly bisimilar trees

The counter-example in Figure 13, where `br` is the binary event introduced in Section 2.2, fleshes out this intuition. Indeed, both trees are strongly bisimilar: each of them can emit the label obs `br` *false* by stepping to either the `Ret` 0 or `Ret` 2 node, or emit the label obs `br` *true* by stepping to either the `Ret` 1 or `Ret` 3 node. However, they are strongly bisimilar because the induced LTS processes the question to the environment—`br`— and its answer—*false*/*true*—in a single step, such that the computations never observe that they have had access to distinct continuations. However, if one were to introduce in the tree a `Step` node before the external events, for instance using the handler `h := fun e ⇒ Step (trigger e)`, a new state allowing for witnessing the distinct continuations would become available in the LTSs, leading to non-bisimilar interpreted trees.

We hence say a handler `h : E ⤳ ctree F` is *simple* if it implements each event `e` either as a pure leaf, i.e., `h e = Ret x`, or simply translates it, i.e., `h e = trigger f`. Similarly, we say that `h : E ⤳ stateT S (ctree F)` is simple if it is point-wise simple. We show that we recover the desired property for the subclass of simple handlers:

Lemma 5.1 (Simple handlers interpret into monad morphisms ()).

- *If* `h : E ⤳ ctree F` *is simple, then* $\forall t\ u,\ t \sim u \longrightarrow$ `interp h` $t \sim$ `interp h` u.

```
Definition refine (h : bool → ∀(n:nat), M (fin n)) : ctree E ⇝ M := fun R ⇒
iter (fun t ⇒ match t with
| Ret r   ⇒ ret (inr r)
| BrD n k ⇒ bind (h false n) (fun x ⇒ ret (inl (k x)))
| BrS n k ⇒ bind (h true n) (fun x ⇒ ret (inl (k x)))
| Vis e k ⇒ bind (mtrigger e) (fun x ⇒ ret (inl (k x)))
end).
```

Fig. 14. Refiner for CTrees (class constraints omitted) (🐓)

```
Definition refine_cst (h : bool → ∀(n:nat), fin (S n)) : ctree E ⇝ ctree E :=
refine (fun b n ⇒ match n with
| O ⇒ stuckD
| S n ⇒ if b then Step (Ret (h b n)) else Guard (Ret (h b n))
end).

Definition refine_state (h : St → ∀(b:bool) (n:nat), St * fin (S n))
: ctree E ⇝ stateT St (ctree E) :=
refine (fun b n s ⇒ match n with O ⇒ stuckD | S n ⇒ Ret (h s b n) end).
```

Fig. 15. Constant and stateful refinements (🐓)

- *If* `h : E ⇝ stateT (ctree F)` *is simple, then* $\forall t\ u,\ t \sim u \rightarrow \forall s,$ `interp h` $t\ s \sim$ `interp h` $u\ s$.

The latter result is, in particular, sufficient to transport `ImpBr` equations established before interpretation—such as the theory of `br _ or _`—through interpretation (🐓). More generally, we can reason after interpretation to establish equations relying both on the nondeterminism and state algebras, for instance to establish the equivalence $p_3 \equiv$ `br` p_2 `or` p_3 mentioned in Section 2.2 (🐓).

5.2 Refinement

Interpretation provides a general theory for the implementation of external events. Importantly, CTrees also support an analogous facility for the *refinement* of its internal branches: one can shrink the set of accessible paths in a computation—and, in particular, determinize it.

We provide, to this end, a new combinator, `refine`, defined in Figure 14. Similar to `interp`, it takes as an argument a handler specifying how to implement our object of interest into a monad `M`. Since here we are interested in the implementation of branching nodes, the handler describes how to monadically choose a branch—the boolean flag allows for different behaviors between delayed and stepping branches.[10] As usual, the implementation monad must be iterative, but this time it must also specify how to re-embed the external events using an operation called `mtrigger` (a concept already used in Yoon et al. [47]).

As hinted at by the combinator's name, the source program should be able to simulate the refined program. Fixing `M` to `ctree F`, this is expressed as $\forall t$, `refine h` $t \lesssim t$. However, one cannot hope to obtain such a result for an arbitrary `h`, because it could implement internal branches with an observable computation that can't be simulated by t. We hence provide two illustrative families of well-behaved refinements, depicted on Figure 15. Constant refinements, implemented into `ctree E`, lift a function systematically, returning the same branch given the nature and arity of the branching node. Stateful refinements carry a memory into a piece of state `St` and implement the refinement into `stateT St (ctree E)` by using the current state to pick a branch, and update the state. A particular example of such a stateful refinement is a round-robin scheduler.

[10] This definition becomes even more uniform with `interp` once we allow for parameterized, arbitrary indexing of branching nodes, as in the enhanced CTrees branch of our development.

```
Definition inject {E} : itree E ⇝ ctree E := interp (fun e ⇒ trigger e).
Definition internalize {E} : ctree (Choose +' E) ⇝ ctree E :=
  interp (fun e ⇒ match e with | inl1 (choose n) ⇒ Br_S^n| inr1 e ⇒ trigger e).
Definition embed {E} : itree (Choose +' E) ⇝ ctree E :=
  fun _ t ⇒ internalize (inject t).
```

Fig. 16. Implementing external branching events into the CTree monad (🐓)

Lemma 5.2. *The constant and stateful refinements are proper refinements (🐓)* [11]

$$\forall h\ t,\ \texttt{refine\_cst}\ h\ t \lesssim t \qquad \text{and} \qquad \forall h\ t\ s,\ \texttt{refine\_state}\ h\ t\ s \lesssim t$$

Finally, the shallow nature of CTrees also offers testing opportunities. Xia et al. [46] describe how external events such as IO interactions can alternatively be implemented in `OCaml` and linked against at extraction. Similarly, we demonstrate on `ImpBr` how to execute a CTree by running an impure refinement implemented in `OCaml` by picking random branches along the execution.

5.3 ITree embedding

We have used CTrees directly as a domain to represent the syntax of `ImpBr`, as well as in our case studies (see Section 6 and 7). CTrees can, however, fulfill their promise sketched in Section 2, and be used as a domain to host the monadic implementation of external representations of nondeterministic events in an ITree.

To demonstrate this approach, we consider the family of events `choose (n: nat) : Choose (fin n)`, and aim to define an operator `embed` taking an ITree computation modeling nondeterministic branching using these events, and implementing them as stepping branches into a CTree. This operator, defined in Figure 16, is the composition of two transformations. First, we `inject` ITrees into CTrees by (ITree) interpretation. This injection rebuilds the original tree as a CTree, where `Tau` nodes have become `Guard` nodes, and an additional `Guard` has been introduced in front of each external event. Second, we `internalize` the external branching contained in a CTrees implementing a `Choose` event, using the isomorphic stepping branch.

We show that this embedding is a monad morphism in the sense that it transports `eutt` ITrees into `sbisim` CTrees:

Lemma 5.3. `embed` *respects* `eutt` *(🐓)* $\quad \forall t\ u, \texttt{eutt}\ t\ u \implies embed(t) \sim embed(u)$

The proof of this theorem highlights how `Tau` nodes in ITrees collapse two distinct concepts that nondeterminism forces us to unravel in CTrees. The `eutt` relation is defined as the greatest fixpoint of an inductive endofunction `euttF`. In particular, one can recursively and *asymmetrically* strip finite amounts of `Tau`s—the corecursion is completely oblivious to these nodes in the structures. Corecursively, however, `Tau` nodes can be matched *symmetrically*—a construction that is useful in exactly one case, namely to relate the silently spinning computation, $\texttt{Tau}^{\omega}$, to itself. From the CTrees perspective, recursing in `euttF` corresponds to recursion in the definition of the LTS: `Tau` nodes are `Guard` nodes. But corecursing corresponds to a step in the LTS: $\texttt{Tau}^{\omega}$ corresponds to $\texttt{Step}^{\omega}$. ITrees' `Tau` thus corresponds to either a `Guard` or a `Step`. Nondeterminism forces us to separate both concepts, as whether a node in the tree constitutes an accessible state in the LTS becomes semantically relevant. We obtain two "dual" notions that have no equivalent in ITrees: we may

[11] The second theorem actually cannot be stated in the main development provided as artifact: the return types of the trees are not identical — the refined tree computes additionally a final state. We prove it in the enhanced CTrees branch with support for heterogeneous relations and arbitrary relations on labels.

$$l ::= \tau \mid c \mid \bar{c} \qquad\qquad P ::= 0 \mid l \cdot P \mid P \oplus Q \mid P \parallel Q \mid \nu c \cdot P \mid\, !P$$

$$\frac{}{a \cdot P \xrightarrow{a}_{\text{ccs}} P} \qquad \frac{P \xrightarrow{l}_{\text{ccs}} P'}{P \oplus Q \xrightarrow{l}_{\text{ccs}} P'} \qquad \frac{Q \xrightarrow{l}_{\text{ccs}} Q'}{P \oplus Q \xrightarrow{l}_{\text{ccs}} Q'} \qquad \frac{P \xrightarrow{l}_{\text{ccs}} P'}{P \parallel Q \xrightarrow{l}_{\text{ccs}} P \parallel' Q}$$

$$\frac{Q \xrightarrow{l}_{\text{ccs}} Q'}{P \parallel Q \xrightarrow{l}_{\text{ccs}} P \parallel Q'} \qquad \frac{P \xrightarrow{c}_{\text{ccs}} P' \quad Q \xrightarrow{\bar{c}}_{\text{ccs}} Q'}{P \parallel Q \xrightarrow{\tau}_{\text{ccs}} P' \parallel Q'} \qquad \frac{P \xrightarrow{l}_{\text{ccs}} P' \quad l \notin \{c, \bar{c}\}}{\nu c \cdot P \xrightarrow{l}_{\text{ccs}} \nu c \cdot P'} \qquad \frac{P \parallel !P \xrightarrow{l}_{\text{ccs}} P'}{!P \xrightarrow{l}_{\text{ccs}} P'}$$

Fig. 17. Syntax for ccs (🐓) and its operational semantics (🐓)

have finite amounts of asymmetric corecursive choices, i.e., finitely many `Steps`, and structurally infinite stuck processes, i.e., `Guard`$^{\omega}$.

In the proof of Lemma 5.3, this materializes by the fact that an induction on `euttF` leaves us disappointed in the symmetric `Tau` case: we have no applicable induction hypothesis, but expose in our embedding a `Guard`, which does not allow us to progress in the bisimulation. We must resolve the situation by proving that being able to step in `embed t` implies that `t` is not `Tau`$^{\omega}$, i.e., that we can inductively reach a `Vis` or a `Ret` node (🐓).

6 CASE STUDY: A MODEL FOR CCS

We claim that CTrees form a versatile tool for building semantic models of nondeterministic systems, concurrent ones in particular. In this section, we illustrate the use of CTrees as a model of concurrent communicating processes by providing a semantics for Milner's Calculus of Communicating Systems (ccs) [31]. The results we obtain—the usual algebra, up-to principles, and precisely the same equivalence relation as the usual operational-based strong bisimulation—are standard, per se, but they are all established by exploiting the generic notion of bisimilarity of CTrees. The result is a shallowly embedded model for ccs in Coq that could be easily, and modularly, combined with other language features.

6.1 Syntax and operational semantics

The syntax and operational semantics of ccs are shown in Figure 17. The language assumes a set of *names*, or communication channels, ranged over by c. For any name c, there is a co-name $\bar{c}$ satisfying $\bar{\bar{c}} = c$. An *action* is represented by a label l; it is either a communication label c or $\bar{c}$, representing the sending/reception of a message on a channel, or the reserved action τ, which represents an internal action.

The standard operational semantics, shown in the figure, is expressed as a labeled transition system, where states are terms P and labels are actions l. The ccs operators are the following: 0 is the process with no behavior. A prefix process $l \cdot P$ emits an action l and then becomes the process P. The internal choice $P \oplus Q$ behaves either like the process P or like the process Q, in the same fashion as the BrInternal semantics for `br` in Section 2.2. The parallel composition of two processes $P \parallel Q$ interleaves the behavior of the two processes, while allowing the two processes to communicate. If the process P emits a name c and the process Q emits its co-name $\bar{c}$, then the two processes can progress simultaneously and the parallel composition emits an internal action τ. Channel restriction $\nu c \cdot P$ prevents the process P from emitting an action c or $\bar{c}$: the operational rule states that any emission of another action is allowed. Finally the replicated process $!P$ behaves as an unbounded replication of the process P. Operationally, $!P$ has the behavior of $P \parallel !P$.

$$\overline{0} \triangleq \emptyset \qquad a\bar{\cdot}p \triangleq \texttt{trigger}\ a\ ;;\ p \qquad p \overline{\oplus} q \triangleq Br_D^2\ p\ q \qquad \bar{!}p \triangleq p \overline{\|} \bar{!}p$$

$$\nu c\bar{\cdot}P \triangleq \texttt{interp}\ \mathsf{h\_new}\ c\ P \qquad \text{where } \mathsf{h\_new}\ c\ e = \begin{cases} \emptyset & \text{if } e = act\ c \text{ or } e = act\ \bar{c} \\ \texttt{trigger}\ e & \text{otherwise} \end{cases}$$

$$\begin{aligned} p \overline{\|} q \triangleq \texttt{cofix}\ F\ p\ q \cdot Br_D^3\ & (p' \leftarrow \mathsf{head}\ p\ ;;\ \mathsf{actL}\ F\ q\ p') \\ & (q' \leftarrow \mathsf{head}\ q\ ;;\ \mathsf{actR}\ F\ p\ q') \\ & (p' \leftarrow \mathsf{head}\ p\ ;;\ q' \leftarrow \mathsf{head}\ q\ ;;\ actLR\ F\ p'\ q') \end{aligned}$$

$$\begin{aligned} \mathsf{actL}\ F\ q\ (Br_S^n\ k) &\triangleq Br_S^n\ (\lambda i \cdot F\ (k\ i)\ q) & \qquad \mathsf{actR}\ F\ p\ (Br_S^n\ k) &\triangleq Br_S^n\ (\lambda i \cdot F\ p\ (k\ i)) \\ \mathsf{actL}\ F\ q\ (Vis\ e\ k) &\triangleq Vis\ e\ (\lambda i \cdot F\ (k\ i)\ q) & \qquad \mathsf{actR}\ F\ p\ (Vis\ e\ k) &\triangleq Vis\ e\ (\lambda i \cdot F\ p\ (k\ i)) \end{aligned}$$

$$\mathsf{actLR}\ F\ r\ r' \triangleq \begin{cases} \mathsf{Step} \cdot F\ (k\ ())\ (k'\ ()) & \text{if } \exists a.\ r = Vis\ (act\ a)\ k \wedge r' = Vis\ (act\ \bar{a})\ k' \\ \emptyset & \text{otherwise} \end{cases}$$

Fig. 18. Denotational model for ccs using ccs$^\#$ as a domain (🐓)

6.2 Model

We define a denotational model for ccs using `ctree actE void` as domain, written ccs$^\#$ in the following. As witnessed by this type, processes do not return any value, but may emit actions modeled as external events expecting `unit` for answer: `Inductive actE ::= | act a : actE unit`. Figure 18 defines the semantic operators associated with each construct of the language. They are written as over-lined versions of their syntactic counterparts, and defined over ccs$^\#$.

The empty process is modeled as a stuck tree—we cannot observe it. Actions are directly defined as visible events, and thus the prefix triggers the action, and continues with the remaining of the process. As discussed in Section 2.2, the delayed branching node fits exactly with the semantics of the choice operator in ccs, only progressing if one of the composed terms progresses. Restriction raises a minor issue: the compositional definition implies that the CTree for the restricted term has already been produced when we encounter the restriction and, a priori, that tree might contain visible actions on the name being restricted. We enforce scoping by replacing those actions by a stuck tree, $\emptyset$, effectively cutting these branches. This is done using the `interp` operator from CTrees, with h_new, a handler that does the substitution.

Parallel composition is more intricate, as the operator requires significant introspection of the composed terms. The traditional operational semantics of ccs is not explicitly constructive: each of the three reduction rules depends on the existence of specific transitions in the sub-processes. We perform this necessary introspection, in a constructive way, defining the tree as an explicit cofixpoint, and using the `head` operator introduced in Section 3.2. While the operation head p precisely captures the desired set of actions that p may perform, computing this set could, in general, silently diverge. We therefore cannot bluntly initiate the computation by sequencing the heads of p and q,

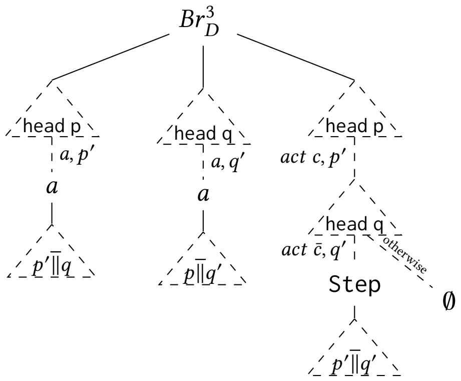


Fig. 19. Depiction of the tree resulting from $p \overline{\|} q$

as divergence in the former may render inaccessible valid transitions in the latter.[12] Instead, we initiate the computation with a ternary delayed choice: the left (resp. middle) branch captures the behaviors starting with an interaction by p (resp. q), while the right branch captures the behaviors starting with a synchronisation between p and q. Essentially, the tree nondeterministically explores the set of applicable instances of the three operational rules for parallel composition. In particular, if the operational rule to step in the left (resp. right) process is non-applicable, the left (resp. middle) branch of the resulting tree silently diverges. The right branch silently diverges if neither process can step, but, in general, it also contains branches considering the interaction of incompatible actions; we cut these branches by inserting $\emptyset$. In all cases, the operator continues corecursively, having progressed in either or both processes. Figure 19 shows the CTree resulting from $p \overline{\|} q$.

The last operator to consider is the replication $\overline{!}$. In theory, it could be expressed in terms of parallel composition directly, as the cofix $\overline{!}p \triangleq p \overline{\|}\ \overline{!}p$. Unfortunately, although it is sound, defining the $\overline{!}$ operator in this way is too involved for Coq's syntactic criterion on cofixes to recognize that the corecursive call is guarded under $\overline{\|}$. To circumvent this difficulty, we use an auxiliary operator, $p \overline{\|!} q$, capturing the parallel composition of a process p with a replicated process $!q$. It nondeterministically explores the four kinds of interactions that such a process could exhibit: a step in p; the creation of a copy of q performing a step to p', before being composed in parallel with p; the creation of a copy of q synchronizing with p; or the emission of two copies of q synchronizing one with another. We finally define the replication operator as $\overline{!}p \triangleq p \overline{\|!} p$. We omit the formal definition here, since it is similar to that for $\overline{\|}$, but point out the interested reader to its formal counterpart (🐓) or two the appendix.

With these tools at hand, the model $[\![\cdot]\!] : \texttt{ccs} \to \texttt{ccs}^\#$ is defined by recursion on the syntax.

Equational Theory. We provide a first validation of our model by proving that it satisfies the expected equational theory with respect to CTrees's notion of strong bisimulation, enabling the usual algebraic reasoning advocated for process calculi. In particular, we prove that our definition for the replication is sane in that it validates equationally the expected definition: $\overline{!}p \sim \overline{!}p \overline{\|} p$ (🐓). We also prove an illustrative collection of expected equations satisfied by our operators (🐓):

$$p \overline{\oplus} q \sim q \overline{\oplus} p \qquad p \overline{\oplus} (q \overline{\oplus} r) \sim (p \overline{\oplus} q) \overline{\oplus} r \qquad p \overline{\oplus} \overline{0} \sim p \qquad p \overline{\oplus} p \sim p$$

$$p \overline{\|} \overline{0} \sim p \qquad p \overline{\|} q \sim q \overline{\|} p \qquad p \overline{\|} (q \overline{\|} r) \sim (p \overline{\|} q) \overline{\|} r$$

To facilitate these proofs, we first prove sound up-to principles at the level of `ccs` for each constructor: strong bisimulation up-to $c\bar{\cdot}[\cdot]$, $[\cdot]\overline{\oplus}[\cdot]$, $[\cdot]\overline{\|}[\cdot]$, $\overline{!}[\cdot]$, and $\nu c\bar{\cdot}[\cdot]$ are all valid principles, allowing us to rewrite `sbisim` under semantic contexts during bisimulation proofs. Additionally to these language-level up-to principles, we inherit the ones generically supported by `sbisim` (Lem. 4.5).

6.3 Equivalence with the operational strong bisimilarity

In addition to proving that we recover in our semantic domain the expected up-to principles and the right algebra, we furthermore show that the model is sound and complete with respect to strong bisimulation compared to its operational counterpart. We do so by first establishing an asymmetrical bisimulation between `ccs` and `ccs`$^\#$, matching operational steps over the syntax to semantic steps in the CTree. We write $\bar{l}$ for the obvious translation of labels between both LTSs.

[12] Technically, the variant of `ccs` considered here actually cannot generate such a computation, so we could therefore rule this case out extensionally. The case could, however, easily arise in a variant of `ccs` relying on recursive processes rather than replication, and in other calculi, so we therefore favor this more general, reusable, approach.

Definition 6.1 (Strong bisimulation between ccs *and* ccs$^\#$*.).* A relation $\mathcal{R} : rel(\text{ccs}, \text{ccs}^\#)$ is a strong bisimulation if and only if, for any label l, ccs term P, and ccs$^\#$ tree q.

$$P \,\mathcal{R}\, q \wedge P \xrightarrow{l}_{\text{ccs}} P' \implies \exists q',\ P' \,\mathcal{R}\, q' \wedge q \xrightarrow{\bar{l}} q'$$

and conversely

$$P \,\mathcal{R}\, q \wedge q \xrightarrow{\bar{l}} q' \implies \exists P'.\ P' \,\mathcal{R}\, q' \wedge P \xrightarrow{l} P'$$

$$\begin{tikzcd} P \arrow[r, phantom, "\mathcal{R}"] \arrow[d, "l"', "\text{ccs}"] & {} & q \arrow[d, "\bar{l}"] \\ P' \arrow[r, phantom, "\mathcal{R}"] & {} & q' \end{tikzcd}$$

Lemma 6.2 (🐓). *The relation $R \triangleq \{(P, q) \mid [\![P]\!] \sim q\}$ is a strong bisimulation.*

We derive from this result that the operational and semantic strong bisimulations define exactly the same relation over ccs:

Lemma 6.3 (🐓). *$\forall P\ Q,\ [\![P]\!] \sim [\![Q]\!]$ iff $P \sim_{\text{ccs}} Q$*

7 CASE STUDY: MODELING COOPERATIVE MULTITHREADING

As a second case study, we consider cooperative multithreading. Cooperative scheduling is used in languages such as Javascript, async Rust, or Akka/Scala actors, but is also a very general model, as preemptive multi-tasking is equivalent to cooperative scheduling where threads are willing to yield (i.e., let other threads run) at any time. We extend the syntax of imp with two constructs:

$$\texttt{comm} \triangleq \texttt{skip} \mid x ::= e \mid c1; c2 \mid \texttt{while}\ b\ \texttt{do}\ c \mid \texttt{fork}\ c1\ c2 \mid \texttt{yield}$$

The command fork forks two copies of the currently running program, the first of which first runs $c1$ and the second of which first runs $c2$, before they each proceed with the rest of the program. Importantly, $c2$ retains control, and $c1$ will only execute when $c2$ voluntarily yields control or ends. This yielding of control is achieved by the yield statement, which signals that the current thread stops and lets a new thread run—possibly the same one again.

This semantics implements a mechanism like the fork system call, which duplicates the current process, but without a possibility for joining threads. For example, the program

$$(\texttt{fork}\ (x ::= 1)\ (\texttt{yield}; x ::= 2)); y ::= x$$

forks two copies of the program, with the "main" thread immediately yielding, allowing for either thread to run next. Its semantics is to first spawn a thread for $x ::= 1$, then have the main thread reach the yield, giving $x ::= 1$ a chance to run. Assuming the spawned thread goes next, it runs in sequence $x ::= 1$ and $y ::= x$, after which the main thread recovers control and finishes its execution. $y ::= x$ is part of both threads and is thus executed twice, after each assignment to x.

Alternatively, some cooperative scheduling languages [1] consider a spawn operator that simply spawns an independent thread: we can encode this behavior in several ways. Notably, if spawn always occurs in tail position, there is no continuation to duplicate. For instance, the program

$$\texttt{fork}\ x ::= 1\ (\texttt{fork}\ (x ::= 2)\ \texttt{skip})$$

spawns two threads that set x to different values, and terminates. The two spawned threads can then be scheduled in either order, resulting in $x = 1$ or $x = 2$ in the final state. This constraint could be syntactically enforced in the language if relevant. Alternatively, one can use the command while *true* do yield to "terminate" a thread; for instance to prevent the first thread above from reaching $y ::= x$. With fancier encodings using reserved shared-variables, nested "joins" and other synchronization operations can be modeled. In this case study, we do not concern ourselves with such extensions, and restrict ourselves to the formalization of the syntax described above.

$$
\begin{array}{rcll}
\text{schedule } v_0 \ \lfloor\text{-}\rfloor & \triangleq & \texttt{ret}\ (\,) & \\
\text{schedule } v_n \ \lfloor\text{-}\rfloor & \triangleq & Br_S^n\ (\lambda i \cdot \text{schedule } v_n\ \lfloor i \rfloor) & \text{for } n > 0 \\
\text{schedule } v_n \ \lfloor i \rfloor & \triangleq & & \\
\text{if } v[i] = & & \text{then} & \\
\texttt{ret}\ (\,) & & Br_D^1\ (\lambda\_ \cdot \text{schedule } v[-i]_{n-1}\ \lfloor\text{-}\rfloor) & \\
Br_S^m\ k & & Br_S^m\ (\lambda i' \cdot \text{schedule } v[i \mapsto k\ i']_n\ \lfloor i \rfloor) & \\
Br_D^m\ k & & Br_D^m\ (\lambda i' \cdot \text{schedule } v[i \mapsto k\ i']_n\ \lfloor i \rfloor) & \\
Vis\ \texttt{Yield}\ k & & Br_D^1\ (\lambda\_ \cdot \text{schedule } v[i \mapsto k\ (\,)]_n\ \lfloor\text{-}\rfloor) & \\
Vis\ \texttt{Spawn}\ k & & Br_S^1\ (\lambda\_ \cdot \text{schedule } (k\ \mathit{true} :: v[i \mapsto k\ \mathit{false}])_{n+1}\ \lfloor i+1 \rfloor) & \\
Vis\ \mathsf{e}\ k & & Vis\ e\ (\lambda x \cdot \text{schedule } (v[i \mapsto k\ x])_n\ \lfloor i \rfloor) &
\end{array}
$$

Fig. 20. The definition of schedule ()

7.1 Model

As with the approach described in Section 2, the semantics of the language is defined in two stages. First, we represent statements as computations of type `ctree (YieldE + SpawnE + MemE) unit`, where `YieldE` and `SpawnE` are used to represent `yield` and `fork` respectively. The remaining statements of `imp` are defined in a standard way—we omit them. The new event families are defined as follows:

```
Variant YieldE : Type → Type :=          Variant SpawnE : Type → Type :=
| Yield : YieldE unit.                   | Spawn : SpawnE bool.
```

`Yield` carries no additional information and acts purely as a signal to yield control, and `Spawn` introduces a binary branch in the CTree, allowing us to store the asynchronous thread in one branch and the main thread that continues running in the other branch. Formally, these events are used to denote the corresponding statements:

$$
[\![\texttt{yield}]\!] \triangleq \texttt{trigger}\ (\texttt{Yield}) \qquad [\![\texttt{fork}\ c1\ c2]\!] \triangleq \quad b \leftarrow \texttt{trigger}\ (\texttt{Spawn})\ ;;\ \text{if } b \text{ then } [\![c1]\!] \text{ else } [\![c2]\!]
$$

The second stage is more complex than the simple stateful interpretation of standard `imp`: we need to interpret the concurrency-related events. In particular, since the semantics of these events involves scheduling other threads that work together, we cannot hope to use `interp`, as is done with memory events—this situation is similar to $\overline{\|}$ for `ccs`, in both cases the combinator cannot be implemented as a simple fold. Instead, we define a custom interpreter for `Spawn` and `Yield` events, namely a scheduler operating over a thread pool of threads, each of which is a CTree that may contain `Spawn` and `Yield` events, and producing a final CTree devoid of these concurrency events.

As is usual with cooperative scheduling, the thread pool, encoded as a vector, contains a designated *active* thread, i.e., the currently running thread. We write v_n for a vector of size n, and vector operations for removing the i-th element as $v[-i]$, updating the i-th element to x as $v[i \mapsto x]$, and adding an element x to the front as $x :: v$ (so x is the new 0-th element in the resulting vector).

The scheduler, schedule, is defined formally in Figure 20. References to schedule in its body should be interpreted as corecursive calls—we abuse notations to lighten the presentation. It takes two arguments: the thread pool, and the index of the active thread in the pool. If no thread is active, the second argument is empty, written $\lfloor\text{-}\rfloor$ (we use an option datatype in Coq). If there is an active thread, $\lfloor i \rfloor$, the schedule makes progress in that thread. Otherwise, schedule nondeterministically chooses the next thread to run. If the thread pool is empty, then schedule emits a returning CTree. In more detail, there are several ways the active thread may progress. If the thread performs a branching or a memory operation, the schedule returns a CTree that also performs the same operation, and in each branch replaces the active thread (index i) in the thread pool with its continuation. If the active thread returns, it is removed from the thread pool; there is no active thread after the removal. If the thread yields, then schedule removes the `Yield` and starts scheduling with no active thread (which will nondeterminstically pick a new active thread, potentially the same one again). If the thread spawns a new thread, then schedule adds the *true* branch of that thread to the thread pool, updates the active thread to the *false* branch, and continues running on the *false* branch (note that this shifts the active thread from index i to $i + 1$).

Using schedule, we can interpret the `Yield` and `Spawn` events in concert, allowing us to eliminate them from the CTrees. We denote this "scheduled" semantics by $\mathcal{S}[\![p]\!] \triangleq$ schedule $[[\![p]\!]]_1\ \lfloor\text{-}\rfloor$: `ctree MemE unit`: it denotes a CTree with only memory events. As described in Section 2.1, the final step of the second stage of modeling our `imp` programs can use a handler for the memory events to obtain a computation in the state monad of type `stateT mem (ctree voidE) unit`.

7.2 Equational theory

The model described in Section 7.1 allows us to derive some program equivalences at source-level w.r.t. weak bisimilarity of their models. For example, the following programs all just run c, though some of them first perform some "invisible" steps related to concurrency (🐓):

$$\mathcal{S}[\![\texttt{fork}\ c\ \texttt{skip}]\!] \approx \mathcal{S}[\![\texttt{yield};\ c]\!] \approx \mathcal{S}[\![c]\!]$$

We use weak rather than strong bisimilarity since schedule can introduce stepping branches when interpreting the two concurrency events.

Other equations, especially ones that make use of multiple threads in nontrivial ways, rely on the stability of schedule under `sbisim`:

Lemma 7.1 (schedule preserves $\sim$ (🐓)). *If the elements of vectors v_n and w_n are deterministic and strongly bisimilar up to a permutation ρ, then* schedule v_n *(Some i)* $\sim$ schedule w_n *(Some $(\rho\ i)$).*

The requirement that individual threads be deterministic means that all delayed branches have arity less than 2, which is satisfied by all denotations of programs in this language. This condition greatly simplifies the proof by constraining the shape that the strongly bisimilar CTrees can take.

Lemma 7.1 allows us to permute the thread pool, which is useful in examples such as (🐓):

$$\mathcal{S}[\![\texttt{fork}\ c1\ (\texttt{fork}\ c2\ \texttt{skip})]\!] \approx \mathcal{S}[\![\texttt{fork}\ c2\ (\texttt{fork}\ c1\ \texttt{skip})]\!]$$

This program spawns two asynchronous threads then yields control to one of the two. This equivalence captures the natural fact that it does not matter which thread is spawned first, since neither can run until both are spawned.

Furthermore, Lemma 7.1 allows us to validate some simple optimizations that do not directly involve reasoning about concurrency or memory, such as (🐓):

$$\mathcal{S}[\![\texttt{fork}\ c1\ (\texttt{fork}\ (\texttt{while}\ \mathit{true}\ \texttt{do}\ \texttt{yield})\ \texttt{skip})]\!] \approx \mathcal{S}[\![\texttt{fork}\ (\texttt{yield};\texttt{while}\ \mathit{true}\ \texttt{do}\ \texttt{yield})(\texttt{fork}\ c1\ \texttt{skip})\ ]\!]$$

This example is similar to the previous one, except that one of the spawned threads is a while loop, which we wish to unroll by one iteration. Crucially, the loop and its unrolled form are strongly bisimilar, so this equivalence follows from Lemma 7.1 just as in the previous example. Other optimizations that can be done before interpreting events, such as constant folding or dead code elimination, can be proven sound similarly.

Finally, equivalences involving memory operations are still valid as well (🐓):

$$\texttt{fork}\ (x ::= 2)\ (x ::= 1) \equiv x ::= 2$$

where $\equiv$ here refers to equivalence ($\approx$ in this case) after interpreting both concurrency and memory events. This result follows from the result in Section 5.1, which allows us to transport equations made before interpreting state events into the computations in the state monad after interpetation.

8 RELATED WORK

Since Milner's seminal work on CCS [32] and the π-calculus [33], process algebras have been the topic of a vast literature [6]. We mention only a few parts of it that are most relevant to our work. In the Coq realm, Lenglet et al. [26] have formalized HOπ, a minimal π-calculus, notably exhibiting the difficulty inherent to the formal treatment of name extrusion. Beyond its formalization, dealing with scope extrusion as part of a compositional semantics is known to be a challenging problem [11, 12]. By restricting to CCS in our case study, we have side-stepped this difficulty. Foster et al. [15] formalize in Isabelle/HOL a semantics for CSP and the Circus language using a variant implementation of ITrees, where continuations to external events are partial functions. They, however, only model deterministic processes, leaving support for nondeterministic ones to future work. This paper introduces the tools to address that problem. CSP has also been extensively studied by Brookes [8] by providing a model based on the compositional construction of infinite sets of traces: CTrees offers a complementary coinductive model to this more set-theoretic approach. Furthermore, Brookes tackles questions of fairness, an avenue that we have not yet explored in our setup.

Formal semantics for nondeterminism are especially relevant when dealing with low-level concurrent semantics. In shared-memory-based programming languages, rather than message passing ones, concurrency gives rise to the additional challenge of modeling weak memory models, a topic that has received considerable attention. Lesani et al. [27] prove correct concurrent objects implemented using ITrees, assuming a sequentially consistent model of shared memory. They relate the ITrees semantics to a trace-based one to reason about refinement, something that we conjecture would not be necessary when starting from CTrees. On the operational side, increasingly relaxed models have been captured and sometimes formalized. Major landmarks include the work by Sevcík et al. on modeling TSO using a central synchronizing transition system linking the program semantics to the memory model in the CompCertTSO compiler [43]; or Kang et al.'s promising semantics [20, 25] that have captured large subsets of the C++11 concurrency model without introducing out-of-thin-air behaviors. On the other side of the spectrum, Alglave et al. have worked with axiomatic semantics [3, 4] to build an expressive, unifying framework to model, compare, and reason about weak memory models. Recently, and although not formalized, Jagadeesan et al. [19] have introduced a denotational model of elegant simplicity modeling weak memory models using posets and preconditions.

Our model for cooperative multithreading is partially reminiscent of Abadi and Plotkin's work [1]: they define a denotational semantics based on partial traces that they prove fully abstract, and satisfying an algebra of stateful concurrency. The main difference between the two approaches is that partial traces use the memory state explicitly to define the composition of traces, where CTrees can express the semantics of a similar language independently of the memory model. The formal model we describe here tackles a slightly different language than theirs, but we should be able to adapt it reasonably easily to obtain a formalization of their work. More recently, Din et al. [13, 14] have suggested a novel way to define semantics based on the composition of symbolic traces, partially inspired by symbolic execution [21]. They use it, in particular, to formalize actor languages, which rely on cooperative scheduling, with a similar modularity as the one we achieve (orthogonal semantic features can be composed), but not in a compositional way.

Our work brings proper support for nondeterminism to monadic interpreters in Coq. As with ITrees however, the tools we provide are just right to conveniently build denotational models of first order languages, such as CCS, but have difficulty retaining compositionality when dealing with higher-order languages. In contrast, on paper, game semantics has brought a variety of techniques lifting this limitation. In particular in a concurrent setup, event structures have spawned a successful line of work [10, 39] from which inspiration could be drawn for further work on CTrees.

9 CONCLUSION AND PERSPECTIVES

We have introduced CTrees, a model for nondeterministic, recursive, and impure programs in Coq. Inspired by ITrees, we have introduced two kinds of nondeterministic branching nodes, and designed a toolbox to reason about these new computations. In addition to the Coq library presented in this paper, we have developed an enhanced implementation (with heavier notations, but more expressive) supporting arbitrary indexing of branches and heterogeneous bisimulations between CTrees. We have illustrated the expressiveness of the framework through two significant case studies. Both nonetheless offer avenues for further work, notably through an extension of ccs to name passing *à la* π-calculus, and to further extend the equational theory for cooperative multithreading that we currently support.

The restriction imposed on handlers in Section 5.1 suggests another perspective: rather than requiring handlers to implement events in a non-observable way, we could prevent the programs in Figure 13 from being bisimilar in the first place. We conjecture that this could be achieved by tweaking the LTS so that external events induce two steps: the question to the environment first, and the answer second, resulting in a finer grained bisimilarity. Whether such a finer bisimilarity would still allow us to equate all the programs we want needs further investigation.

REFERENCES

[1] Martín Abadi and Gordon D Plotkin. 2010. A model of cooperative threads. *Logical Methods in Computer Science* 6, 4 (2010), 1–39. https://doi.org/10.2168/LMCS-6(4:2)2010

[2] Sampson Abramsky and Paul-Andreé Melliès. 1999. Concurrent Games and Full Completeness. In *Proceedings of the 14th Annual ACM/IEEE Symposium on Logic in Computer Science*. IEEE Computer Society Press.

[3] Jade Alglave, Will Deacon, Richard Grisenthwaite, Antoine Hacquard, and Luc Maranget. 2021. Armed Cats: Formal Concurrency Modelling at Arm. *ACM Trans. Program. Lang. Syst.* 43, 2 (2021), 8:1–8:54. https://doi.org/10.1145/3458926

[4] Jade Alglave, Luc Maranget, and Michael Tautschnig. 2014. Herding cats: modelling, simulation, testing, and data-mining for weak memory. In *ACM SIGPLAN Conference on Programming Language Design and Implementation, PLDI '14, Edinburgh, United Kingdom - June 09 - 11, 2014*, Michael F. P. O'Boyle and Keshav Pingali (Eds.). ACM, 40. https://doi.org/10.1145/2594291.2594347

[5] Thorsten Altenkirch, Nils Anders Danielsson, and Nicolai Kraus. 2017. Partiality, Revisited. In *Foundations of Software Science and Computation Structures*, Javier Esparza and Andrzej S. Murawski (Eds.). Springer Berlin Heidelberg, Berlin, Heidelberg, 534–549.

[6] J.A. Bergstra, A. Ponse, and S.A. Smolka (Eds.). 2001. *Handbook of Process Algebra*. Elsevier Science. https://doi.org/10.1016/B978-044482830-9/50038-2

[7] B. Bloom, S. Istrail, and A. R. Meyer. 1988. Bisimulation Can't Be Traced. In *Proceedings of the 15th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages* (San Diego, California, USA) *(POPL '88)*. Association for Computing Machinery, New York, NY, USA, 229–239. https://doi.org/10.1145/73560.73580

[8] Stephen D. Brookes. 2002. Traces, Pomsets, Fairness and Full Abstraction for Communicating Processes. In *Proc. 13th Intl. Conf. on Concurrency Theory (CONCUR 2002) (LNCS, Vol. 2421)*, Lubos Brim, Petr Jancar, Mojmír Kretínský, and Antonín Kucera (Eds.). Springer, Berlin Heidelberg, 466–482.

[9] Venanzio Capretta. 2005. General Recursion via Coinductive Types. *Logical Methods in Computer Science* 1, 2 (2005), 1–18. https://doi.org/10.2168/LMCS-1(2:1)2005

[10] Simon Castellan, Pierre Clairambault, Silvain Rideau, and Glynn Winskel. 2017. Games and Strategies as Event Structures. *Log. Methods Comput. Sci.* 13, 3 (2017). https://doi.org/10.23638/LMCS-13(3:35)2017

[11] Silvia Crafa, Daniele Varacca, and Nobuko Yoshida. 2012. Event structure semantics of parallel extrusion in the pi-calculus. In *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics)*, Vol. 7213 LNCS. 225–239. https://doi.org/10.1007/978-3-642-28729-9_15

[12] Ioana Cristescu, Jean Krivine, and Daniele Varacca. 2013. A Compositional Semantics for the Reversible π-Calculus. In *Proceedings - Symposium on Logic in Computer Science*. 388–397. https://doi.org/10.1109/LICS.2013.45

[13] Crystal Chang Din, Reiner Hähnle, Einar Broch Johnsen, Ka I. Pun, and Silvia Lizeth Tapia Tarifa. 2017. Locally Abstract, Globally Concrete Semantics of Concurrent Programming Languages. In *Automated Reasoning with Analytic Tableaux and Related Methods*, Renate A. Schmidt and Cláudia Nalon (Eds.). Springer International Publishing, Cham, 22–43.

[14] Crystal Chang Din, Reiner Hähnle, Ludovic Henrio, Einar Broch Johnsen, Violet Ka I Pun, and Silvia Lizeth Tapia Tarifa. 2022. LAGC Semantics of Concurrent Programming Languages. https://doi.org/10.48550/ARXIV.2202.12195

[15] Simon Foster, Chung-Kil Hur, and Jim Woodcock. 2021. Formally Verified Simulations of State-Rich Processes Using Interaction Trees in Isabelle/HOL. In *32nd International Conference on Concurrency Theory, CONCUR 2021, August 24-27, 2021, Virtual Conference (LIPIcs, Vol. 203)*, Serge Haddad and Daniele Varacca (Eds.). Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 20:1–20:18. https://doi.org/10.4230/LIPIcs.CONCUR.2021.20

[16] Robert Harper. 2013. *Practical Foundations for Programming Languages.* Cambridge University Press.

[17] Ludovic Henrio, Eric Madelaine, and Min Zhang. 2016. A Theory for the Composition of Concurrent Processes. In *Formal Techniques for Distributed Objects, Components, and Systems*, Elvira Albert and Ivan Lanese (Eds.). Springer International Publishing, Cham, 175–194.

[18] Chung-Kil Hur, Georg Neis, Derek Dreyer, and Viktor Vafeiadis. 2013. The Power of Parameterization in Coinductive Proof. In *Proceedings of the 40th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages* (Rome, Italy) *(POPL '13)*. ACM, New York, NY, USA, 193–206. https://doi.org/10.1145/2429069.2429093

[19] Radha Jagadeesan, Alan Jeffrey, and James Riely. 2020. Pomsets with preconditions: a simple model of relaxed memory. *Proc. ACM Program. Lang.* 4, OOPSLA (2020), 194:1–194:30. https://doi.org/10.1145/3428262

[20] Jeehoon Kang, Chung-Kil Hur, Ori Lahav, Viktor Vafeiadis, and Derek Dreyer. 2017. A promising semantics for relaxed-memory concurrency. In *Proceedings of the 44th ACM SIGPLAN Symposium on Principles of Programming Languages, POPL 2017, Paris, France, January 18-20, 2017*, Giuseppe Castagna and Andrew D. Gordon (Eds.). ACM, 175–189. https://doi.org/10.1145/3009837.3009850

[21] James C. King. 1976. Symbolic Execution and Program Testing. *Commun. ACM* 19, 7 (July 1976), 385–394.

[22] Oleg Kiselyov and Hiromi Ishii. 2015. Freer monads, more extensible effects. In *Proceedings of the 8th ACM SIGPLAN Symposium on Haskell, Haskell 2015, Vancouver, BC, Canada, September 3-4, 2015.* 94–105. https://doi.org/10.1145/2804302.2804319

[23] Jérémie Koenig and Zhong Shao. 2020. Refinement-Based Game Semantics for Certified Abstraction Layers. In *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science* (Saarbrücken, Germany) *(LICS '20)*. Association for Computing Machinery, New York, NY, USA, 633–647. https://doi.org/10.1145/3373718.3394799

[24] Leonidas Lampropoulos and Benjamin C. Pierce. 2018. *QuickChick: Property-Based Testing in Coq.* Electronic textbook. https://softwarefoundations.cis.upenn.edu/qc-current/index.html

[25] Sung-Hwan Lee, Minki Cho, Anton Podkopaev, Soham Chakraborty, Chung-Kil Hur, Ori Lahav, and Viktor Vafeiadis. 2020. Promising 2.0: global optimizations in relaxed memory concurrency. In *Proceedings of the 41st ACM SIGPLAN International Conference on Programming Language Design and Implementation, PLDI 2020, London, UK, June 15-20, 2020*, Alastair F. Donaldson and Emina Torlak (Eds.). ACM, 362–376. https://doi.org/10.1145/3385412.3386010

[26] Sergueï Lenglet and Alan Schmitt. 2018. HOπ in Coq. In *Proceedings of the 7th ACM SIGPLAN International Conference on Certified Programs and Proofs, CPP 2018, Los Angeles, CA, USA, January 8-9, 2018*, June Andronick and Amy P. Felty (Eds.). ACM, 252–265. https://doi.org/10.1145/3167083

[27] Mohsen Lesani, Li-yao Xia, Anders Kaseorg, Christian J. Bell, Adam Chlipala, Benjamin C. Pierce, and Steve Zdancewic. 2022. C4: verified transactional objects. *Proc. ACM Program. Lang.* 6, OOPSLA (2022), 1–31. https://doi.org/10.1145/3527324

[28] Thomas Letan, Yann Régis-Gianas, Pierre Chifflier, and Guillaume Hiet. 2018. Modular Verification of Programs with Effects and Effect Handlers in Coq. In *Formal Methods - 22nd International Symposium, FM 2018, Held as Part of the Federated Logic Conference, FloC 2018, Oxford, UK, July 15-17, 2018, Proceedings.* 338–354. https://doi.org/10.1007/978-3-319-95582-7_20

[29] Kenji Maillard, Cătălin Hrițcu, Exequiel Rivas, and Antoine Van Muylder. 2020. The next 700 Relational Program Logics. *Proceedings of the ACM on Programming Languages* 4, POPL, Article 4 (2020), 33 pages. https://doi.org/10.1145/3371072

[30] Paul-André Melliès and Samuel Mimram. 2007. Asynchronous Games: Innocence Without Alternation. In *CONCUR 2007 – Concurrency Theory*, Luís Caires and Vasco T. Vasconcelos (Eds.). Springer Berlin Heidelberg, Berlin, Heidelberg, 395–411.

[31] R. Milner. 1989. *Communication and Concurrency.* Prentice-Hall, Inc., USA.

[32] Robin Milner. 1989. *Communication and Concurrency.* Prentice-Hall, Inc., USA.

[33] Robin Milner, Joachim Parrow, and David Walker. 1992. A calculus of mobile processes, I. *Information and Computation* 100, 1 (1992), 1–40. https://doi.org/10.1016/0890-5401(92)90008-4

[34] Arthur Oliveira Vale, Paul-André Melliès, Zhong Shao, Jérémie Koenig, and Léo Stefanesco. 2022. Layered and Object-Based Game Semantics. *Proc. ACM Program. Lang.* 6, POPL, Article 42 (jan 2022), 32 pages. https://doi.org/10.1145/3498703

[35] Damien Pous. 2007. Complete Lattices and Up-To Techniques. In *Programming Languages and Systems*, Zhong Shao (Ed.). Springer Berlin Heidelberg, Berlin, Heidelberg, 351–366.

[36] Damien Pous. 2016. Coinduction All the Way Up. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science* (New York, NY, USA) *(LICS '16)*. Association for Computing Machinery, New York, NY, USA, 307–316. https://doi.org/10.1145/2933575.2934564

[37] Damien Pous. 2022. *The coq-coinduction library.* https://github.com/damien-pous/coinduction

[38] Damien Pous. 2022. *The coq-coinduction library: examples.* https://github.com/damien-pous/coinduction-examples

[39] Silvain Rideau and Glynn Winskel. 2011. Concurrent Strategies. In *Proceedings of the 26th Annual IEEE Symposium on Logic in Computer Science, LICS 2011, June 21-24, 2011, Toronto, Ontario, Canada*. IEEE Computer Society, 409–418. https://doi.org/10.1109/LICS.2011.13

[40] Davide Sangiorgi. 1998. On the bisimulation proof method. *Mathematical Structures in Computer Science* 8, 5 (1998), 447–479. https://doi.org/10.1017/S0960129598002527

[41] Davide Sangiorgi and Jan Rutten. 2012. *Advanced Topics in Bisimulation and Coinduction* (2nd ed.). Cambridge University Press, USA.

[42] Davide Sangiorgi and David Walker. 2001. *The π-calculus* (1st ed.). Cambridge University Press, USA.

[43] Jaroslav Sevcík, Viktor Vafeiadis, Francesco Zappa Nardelli, Suresh Jagannathan, and Peter Sewell. 2013. CompCertTSO: A Verified Compiler for Relaxed-Memory Concurrency. *J. ACM* 60, 3 (2013), 22:1–22:50. https://doi.org/10.1145/2487241.2487248

[44] M.B. Smyth. 1976. Powerdomains. In *Mathematical Foundations of Computer Science (Lecture Notes in Computer Science, Vol. 4)*. Springer.

[45] The Coq Development Team. 2022. *The Coq Proof Assistant.* https://doi.org/10.5281/zenodo.5846982

[46] Li-yao Xia, Yannick Zakowski, Paul He, Chung-Kil Hur, Gregory Malecha, Benjamin C. Pierce, and Steve Zdancewic. 2020. Interaction Trees. *Proceedings of the ACM on Programming Languages* 4, POPL (2020). https://doi.org/10.1145/3371119

[47] Irene Yoon, Yannick Zakowski, and Steve Zdancewic. 2022. Formal Reasoning About Layered Monadic Interpreters. *Proceedings of the ACM on Programming Languages* 6, ICFP (2022).

[48] Yannick Zakowski, Calvin Beck, Irene Yoon, Ilia Zaichuk, Vadim Zaliva, and Steve Zdancewic. 2021. Modular, Compositional, and Executable Formal Semantics for LLVM IR. *Proc. ACM Program. Lang.* 5, ICFP, Article 67 (aug 2021), 30 pages. https://doi.org/10.1145/3473572

[49] Yannick Zakowski, Paul He, Chung-Kil Hur, and Steve Zdancewic. 2020. An Equational Theory for Weak Bisimulation via Generalized Parameterized Coinduction. In *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP)*. https://doi.org/10.1145/3372885.3373813

A ELIMINATION OF STEPPING BRANCHES: A PROOF SKETCH

Lemma A.1. *Stepping nondeterministic branches can be eliminated (🐓)*

$$\forall t,\ \texttt{BrSElim}\ t \sim t$$

Proof. We proceed by coinduction, with the natural candidate: $\{(\texttt{BrSElim}\ t, t)\ \mid\ t\texttt{:}\ \texttt{ctree E X}\}$. We then case-analyse on the structure of t, and partially reduce a step of iteration in each case (formally, this reduction relies on the eta-unfolding of `unary_stepping_form`, which only holds with respect to equ, and hence is made possible by the validity of strong bisimulation up-to equ).

- Case `Ret` x: we can simply conclude by reflexivity (validity of $\textsc{refl}^{up}$).
- Case $Vis\ e\ k$: the left-hand-side is of the shape $\texttt{trigger}\ e\ \ggg= \lambda x \Rightarrow \texttt{Guard}\ (\texttt{BrSElim}\ (k\ x))$. It rewrites to $Vis\ e\ \lambda x \Rightarrow \texttt{Guard}\ (\texttt{BrSElim}\ (k\ x))$ ($\textsc{upto}^{up}$(equ)), from which we can use the `Vis` structural rule (Figure 11) to step without having to explicitly consider each process stepping. At this point, we can almost conclude by coinductive hypothesis, except for the trailing `Guard` on the left-hand side; however, we can simply eliminate it by rewriting the equation $\texttt{Guard}\ t \sim t$ ($\textsc{upto}^{up}(\texttt{sbisim})$).
- Case $Br_S^n\ k$: We need to play explicitly the game in both directions—taking the branch x of the Br_S^n node is the same as taking both x in Br_D^n and crossing the `Step`. We then conclude by coinduction by simply swiping off a `Guard` as in the previous case ($\textsc{upto}^{up}(\texttt{sbisim})$).
- Case $Br_D^n\ k$: Things are more involved in this case. Indeed, the case analysis on t has exposed a bit of syntax, a Br_D^n node, that appears on both side of the equation. However, as highlighted in Figure 11, processing these nodes does *not* give access to the coinduction hypothesis.

Rather, we introduce auxiliary lemmas to capture the state of the trees that correspond to the next state of the LTS: one to transport a step in `BrSElim` to one in its argument, and one the other way around.

$$\frac{\texttt{BrSElim}\; t \xrightarrow{l} u}{\exists t',\ t \xrightarrow{l} t' \wedge u \sim \texttt{BrSElim}\; t'}\ \text{L1} \qquad \frac{t \xrightarrow{l} u}{\exists u',\ \texttt{BrSElim}\; t \xrightarrow{l} u' \wedge u' \sim \texttt{BrSElim}\; u}\ \text{L2}$$

Each of these lemmas are proved by reinforcing them to structurally specify the shape of the arguments with respect to equ, and then proceeding by induction on the transition. Once we have these lemmas at hand, the main proof is straightforward: we transport a step in the function to its argument by L1, use the hypothesis of bisimilarity to transport it to the argument on the other side of the game, and transport it back to the function by L2, allowing us to conclude by coinduction hypothesis.

□

B A SEMANTIC REPLICATION OPERATOR IN CCS$^{\#}$

For completeness, we provide below the denotational model of the $\overline{!}$ operator. The principles of this definition has been provided in the body of the article. Recall that we use an auxiliary operator $\overline{\|!}$ to ensure syntactically that the coinductive definition is well-guarded.

$$\overline{!}p \triangleq p\overline{\|!}p$$

$$\begin{aligned} p\overline{\|!}q \triangleq \texttt{cofix}\; F\; p\; q \cdot Br_D^4\ \ & (p' \leftarrow \mathsf{head\ p}\ ;;\ \texttt{actL}\; F\; q\; p') \\ & (q' \leftarrow \mathsf{head\ q}\ ;;\ \mathsf{pbR}\; F\; p\; q') \\ & (p' \leftarrow \mathsf{head\ p}\ ;;\ q' \leftarrow \mathsf{head\ q}\ ;;\ \mathsf{pbLR}\; F\; p'\; q') \\ & (q' \leftarrow \mathsf{head\ q}\ ;;\ q'' \leftarrow \mathsf{head\ q}\ ;;\ \mathsf{pbRR}\; F\; q'\; q'') \end{aligned}$$

$$\begin{aligned} &\texttt{actL}\; F\; q\; (Br_S^n\; k) \triangleq Br_S^n\; (\lambda i \cdot F\; (k\; i)\; q) & \quad & \mathsf{pbR}\; F\; p\; (Br_S^n\; k) \triangleq Br_S^n\; (\lambda i \cdot F\; (p\overline{\|}k\; i)\; q) \\ &\texttt{actL}\; F\; q\; (Vis\; e\; k) \triangleq Vis\; e\; (\lambda i \cdot F\; (k\; i)\; q) & & \mathsf{pbR}\; F\; p\; (Vis\; e\; k) \triangleq Vis\; e\; (\lambda i \cdot F\; (p\overline{\|}k\; i)\; q) \end{aligned}$$

$$\mathsf{pbLR}\; F\; r\; r' \triangleq \begin{cases} \texttt{Step} \cdot F\; (k\; ()\overline{\|}k'\; ())\; q & \text{if } \exists a.\, r = Vis\; (act\; a)\; k \wedge r' = Vis\; (act\; \bar{a})\; k' \\ \emptyset & \text{otherwise} \end{cases}$$

$$\mathsf{pbRR}\; F\; r\; r' \triangleq \begin{cases} \texttt{Step} \cdot F\; (p\overline{\|}k\; ()\overline{\|}k'\; ())\; q & \text{if } \exists a.\, r = Vis\; (act\; a)\; k \wedge r' = Vis\; (act\; \bar{a})\; k' \\ \emptyset & \text{otherwise} \end{cases}$$