Group Actions on Lorentz Spaces, Mathematical Aspects: a Survey

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Abstract. From a purely mathematical viewpoint, one can say that most recent works in Lorentz geometry, concern group actions on Lorentz manifolds. For instance, the three major themes: space form problem of Lorentz homogeneous spacetimes, the completeness problem, and the classification problem of large isometry groups of Lorentz manifolds, all deal with group actions. However, in the first two cases, actions are "zen" (e.g., proper), and in the last, the action is violent (i.e., with strong dynamics). We will survey recent progress in these themes, but we will focus attention essentially on the last one, that is, on Lorentz dynamics.

1. Introduction

Let Γ be a topological group acting continuously on a topological space X. Recall the notion of properness of such an action, it will be one key word in this text. So, the action is proper, if for any compact subset C of X, the set of return transformations $R_C = \{g \in \Gamma, gC \cap C \neq \emptyset\}$ is compact. In other words, if $x_i \in X$ converge, and $g_i x_i$ converge, then $\{g_i\}$ is confined in a compact subset of Γ.

1.1. Properness domain?

In general, it is not possible to define in a natural way a "maximal" properness domain D_Γ, i.e., an open Γ-invariant set where the action is proper, and such that on $L_\Gamma = X - D_\Gamma$, the dynamics is strong and far from being proper.

This explains why intermediate notions similar to properness are considered, as example, the notion of wandering…

There is however, at least one case, when this does work perfectly: X is the conformal sphere, and Γ is a discrete subgroup of the Möbius group of conformal transformations. In this case, X is a disjoint union $D_\Gamma \cup L_\Gamma$, where the action on

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D_Γ is proper, and in opposite the action on L_Γ is minimal: any orbit in L_Γ is dense (in L_Γ) (see for instance [79]).

Here, we will deal (even if not so stated) with properties of such a domain in a Lorentz situation (also, even if this is only roughly defined).

1.2. Causality domain?

In this paper, X will be a Lorentz manifold, and Γ acts always isometrically on it (expect the last part on conformal actions). We have a causality (partial) pre-order: $p < q$, which means, q is in the causal future of p.

Definition 1.1. *The action of Γ on X is causal, if for any $p \in X$, and $g \in \Gamma$, gp is not comparable to p, that is, neither $gp < p$, nor $p < gp$ are satisfied (for this notion one can take X just a pre-ordered space, not necessarily a Lorentz manifold).*

In fact, for $g \in \Gamma$, one can associate its causality set $C(g)$ which consists of points p such that p is not comparable to gp. The causality domain $C(\Gamma)$ of Γ equals $\cap_{g\in\Gamma} C(g)$. These notions seem to appear, for the first time, and as efficient tools in [17]. Therefore, the Γ-action is causal, if $C(\Gamma) = X$. If X is causal, that is, $<$ is actually an order, then the Γ-action is causal, iff, the quotient space X/Γ is causal.

As a dynamical condition, causality for actions is to be compared with properness. We think, from a purely mathematical point of view that it deserves to be considered (on general ordered spaces) for its own interest.

On the other hand, in physics, causality (or variants) is a realistic condition on a spacetime X.

1.3. Co-compactness

Mathematicians love compact manifolds! They are (unfortunately) never causal. For this reason, they are treated by physicists as non realistic. But, who knows, for instance, if causality like other physical notions are not violated, near the big bang? Why is non-causality not the right "physical" answer to the metaphysical question, what happened before the big bang?

Recall here that the Γ-action on X is co-compact if there is a compact set L in X whose iterates under Γ cover the whole X.

1.4. Content

We will essentially consider dynamics of group actions on Lorentz manifolds, with respect to the three notions above: properness, causality, cocompactness. They are the three key words which unify the content of the paper, even if they are not explicitly involved.

2. Geodesic flow

Let M be a Lorentz manifold. Its geodesic flow is a "local" flow on the tangent bundle TM. It is better to call it the geodesic vector field, since in general it is non-complete. For $c \in \mathbb{R}$, let T_cM be the subset of vectors v_x, with $\| v_x \|^2 = c$. For example T_0M is the light cone bundle on M. Each level T_cM is invariant under the geodesic vector field.

The Lorentz manifold M is complete if its geodesic vector field is complete (as a vector field).

In the analogous Riemannian case, only one T_1M $(c = 1)$ is relevant. It has compact fibers, and hence this unit tangent bundle is compact when M is. In particular a compact Riemannian manifold is complete.

In the Lorentz case, one can fix -1, 0 and 1 as sufficient set of relevant values, that is one has to consider the three different dynamics of geodesic vector field on the bundles $T_{-1}M, T_0M$, and T_1M. None of them have compact fibers, hence these bundles are never compact (even if the basis M is compact). It is a purely Riemannian heritage to think that compactness automatically implies completeness! In fact, a priori, completeness is very special. Indeed, roughly speaking, one deals with quadratic like differential equations, which notoriously present, generically, explosions. In the sequel, we will recall completeness results which may be interpreted as a posteriori estimates.

Example 2.1. *The "simplest" non complete Lorentz metric is the Bohl metric on the torus. Endow $\mathbb{R}^2 - \{0\}$ with the metric $\frac{dxdy}{x^2+y^2}$. Any line $\{x = Constant \neq 0\}$ is an isotropic non-complete geodesic.*

2.1. Completeness

Let us notice that for Lorentz manifolds there are many interesting notions of partial completeness: future (or past) completeness, lightlike (timelike, spacelike) completeness...

2.1.1. Projectivized geodesic foliation. In the complete case, we have a geodesic flow which is a $\mathbb{R}$-action. It is therefore, a kind of a group action related to Lorentz geometry, which however will not be considered in our survey here. No systematic investigation of this dynamics, exist in the literature. Maybe, the mathematical (and psychological) difficulty comes from the noncompactness of the ambient manifold to this flow, even when the basis M is assumed compact. However, the projectivized tangent bundle $\mathbb{P}TM$ is compact in this case, and is endowed with a one-dimensional geodesic foliations. This seems to be a most tame object to study (see [29] for the 2-dimensional case).

2.2. b-Completeness

Usual non-completeness, means that some geodesic reaches "infinity" with finite energy. However, completeness does not prevent existence of non-geodesic curves reaches "infinity" by using only "finite energy".

The notion of b-completeness (b stands for bundle), implies a stronger "physical" completeness, which prohibits finite total curvature curves to reach infinity.

It also admits "nice" (at least coherently defined) completions and compactification (see [39, 76]). We also believe here that the extended group actions to the completed spaces, are interesting objects of study, although, we do not consider them here.

The definition goes as follows. It generalizes in fact to any manifold M endowed with a connection. Indeed, in this case the frame bundle P_M has an associated "canonical" parallelism (i.e., a trivialization of TP_M). Let's recall how to construct it. The connection induces a splitting of TP_M into horizontal and vertical bundles. The horizontal is tautologically parallelizable: it has a canonical frame field, obtained by identifying it with the tangent space of M. Any choice of a basis of the Lie algebra of the structural group (here $gl(n, \mathbb{R})$) determines a vertical parallelism. Therefore, we have a parallelism on P_M defined up to the choice of a basis of the Lie algebra. Any such parallelism determines a Riemannian metric, by letting it to be orthonormal. A change of the basis induces bi-Lipschitz equivalence between metrics. One says the connection is b-complete if (any) such a metric is complete.

Observe that a quotient of a b-complete manifold by an isometry group of the connection, acting properly discontinuously and freely, is b-complete.

2.3. "Bounded completeness"

One "dramatic" fashion which ensures completeness of a Lorentz compact manifold M, is to suppose that, each geodesic is bounded in TM, (by means of any fiberwise norm on TM, it does not matter since M is compact), or equivalently the geodesic is contained in a compact set of TM. Such a condition allows standard dynamical study of the geodesic flow. Obviously, there are weaker and also stronger (uniform) variants. It is worth investigating theses notions, and showing how much are different they are. Some examples of flat and anti de Sitter compact manifolds, given below are complete but not "boundly complete".

In contrast with b-completeness, a quotient of a boundly complete is not necessarily boundly complete.

2.4. Hopf-Rinow

The Lorentz Hopf-Rinow Theorem is false, in all its formulations. In particular Lorentz "geodesic connectedness" and completeness are different notions (see for instance [53]).

3. Completeness results

3.1. Geometric structures

Let (G, X) be a homogeneous "geometric structure", i.e., where X is a homogeneous space G/H of a Lie group G. A (G, X)-structure on M is an atlas on M with charts taking value in (open subsets of) X and such that chart transitions (defined on open subsets of X) are restriction of elements of G (seen as transformation of X). A manifold endowed with a (G, X)-structure is called (G, X)-manifold. If M is simply connected, the analytic continuation principle implies the existence of a local diffeomorphism $\mathcal{D} : M \to X$, which expresses in the chart of the structures as restrictions of elements of G. It is unique up to composition by automorphisms of X.

If M is not simply connected, then the developing map $\mathcal{D} : \tilde{M} \to X$, is defined on its universal cover. By the (essential) uniqueness of $\mathcal{D}$, there exists a holonomy homomorphism: $\rho : \pi_1(M) \to G$, such that $\mathcal{D}\gamma = \rho(\gamma)\mathcal{D}$, for any $\gamma \in \pi_1(M)$.

In the general case, $\mathcal{D}$ and ρ can be very pathological, for instance, $\mathcal{D}$ (resp. Γ, the image $\Gamma = \rho(\pi_1(M)))$ is not necessarily a covering from $\widetilde{M}$ to its image (resp. a discrete subgroup of G).

When there exists an open subset $\Omega \subset X$, on which Γ acts properly freely, and $\mathcal{D}$ is a covering from $\widetilde{M}$ to Ω the structure is called Kleinian. This is the most regular property of the (G, X)-structure. For instance, we have:

Proposition 3.1. *Suppose M is compact.*

- *If the structure is Kleinian, then Ω is the unique maximal connected open set containing the developing image on which Γ acts properly.*
- *Assume that the G action preserves a complete connection on X. This induces in a natural way a connection on M. This connection is complete, iff, the (G, X)-structure is Kleinian with $\Omega = X$.*

3.2. Manifolds with constant (sectional) curvature

Fix a dimension n, and let $X(c)$ be the complete simply connected Lorentz space of constant curvature c. One can normalize c to be -1, 0 or 1. $X(0)$ is the Minkowski space Min_n, $X(1)$ is the de Sitter space dS_n defined as the set of vectors in Min_{n+1} with norm 1.

In order to define Anti de Sitter space AdS_n, consider $\mathbb{R}^{2,n-1}$, the linear space $\mathbb{R}^{1+n}$ equipped with a quadratic form of signature $(2, n-1)$: AdS_n is the the domain of $\mathbb{R}^{2,n-1}$ where the the quadratic form takes value -1, the Lorentzian metric being the restriction of the ambient quadratic form to the tangent spaces of this domain. Observe that AdS_n is not simply connected; $X(-1)$ is the universal covering of AdS_n (see [83] for details).

Remark 3.2. *In the sequel, we denote by AdS_n some cyclic quotient of $\widetilde{AdS_n}$: this*

A Lorentz manifold M^n of constant curvature c is modeled on $(G, X(c))$, where G is the isometry group of $X(c)$. Therefore, M is geodesically complete means that its universal cover $\tilde{M}$ is (globally) isometric to $X(c)$.

We believe the most revolutionary result in the subject is that proved by Y. Carrière in the flat case, and then adapted by B. Klingler to the general case (this result in the de Sitter case was also independently proved by M. Morrill in her thesis):

Theorem 3.3. [27, 56] *A* compact Lorentz *manifold of constant curvature is complete.*

In the flat case, the proof proceeds by checking the geodesic connectedness of $\tilde{M}$. In the non flat case, the universal space $X(c)$ itself is not geodesically connected. The point is to show that $\tilde{M}$ is "as geodesically connected as" $X(c)$ itself.... In all cases, the goal is achieved by a clever analysis of the dynamics of the holonomy group.

We stress out that this result is false for *locally homogeneous* Lorentz geometric structures, i.e., a Lorentz manifold M modeled on (G, X), where G acts transitively on X, and preserves a Lorentz metric on X, but not necessarily with constant curvature, may be non-complete. See for instance [54] for a construction of left invariant *non-complete* Lorentz metrics on the group $X = SL(2, \mathbb{R})$. Any (metric) quotient X/Γ, where Γ is a co-compact lattice of $SL(2, \mathbb{R})$ is a compact non-complete locally homogeneous Lorentz manifold. Observe here that non-completeness of M simply follows from that of X. In some sense, M is as complete as it could be! We dare ask:

Question 3.4. *If (G, X) is a homogeneous Lorentz space, is any compact (G, X)-manifold M* Kleinian *(in fact with $\Omega = X$)?*

3.2.1. Singular structures. As this will be done in the sequel, the next step towards understanding compact manifolds of constant curvature, is to consider the holonomy group (or equivalently because of completeness, the fundamental group). The de Sitter case is "hyper-rigid" due to the so-called Calabi-Markus phenomenon [25] which states that only finite groups can act properly on the (full) de Sitter space dS_n (assume here $n \geq 3$ to avoid complication with the non-simply connected dS_2). Therefore, there is no compact Lorentz manifold of positive constant curvature!

Here, we want to emphasize the importance of compactness. Indeed, let M^n be a compact (Riemannian) hyperbolic (i.e., of constant curvature -1) manifold. Let $x \to H(x)$ be a hyperplane field on M. Lift it as a hyperplane field $x \to \tilde{H}(x)$ on the hyperbolic space $\mathbb{H}^n$. To a tangent hyperplane of $\mathbb{H}^n$, corresponds a geodesic hyperplane which is interpreted as a point of dS_n. Thus, we get a mapping $\mathcal{D}_H : \mathbb{H}^n \to dS_n$. It is equivariant with respect to the $\pi_1(M)$ action (seen as a subgroup of $O(1, n)$) on both $\mathbb{H}^n$ and dS_n. The previous results implies in particular that $\mathcal{D}_H$ can never be the developing map of a $(O(1, n), dS_n)$-structure, that is $\mathcal{D}_H$ can not be a local diffeomorphism. In the generic case, $\mathcal{D}_H$ will have "tame" singularities. Therefore, we get a singular de Sitter structure on M.

However, we will see in Section 5.2.3 how this construction provide fair regular geometric structures in interesting (non-compact) cases.

3.3. Completeness in presence of Killing fields

Riemannian homogeneous manifolds are complete, even if they are non-compact. This is false in the Lorentz case, as mentioned in the case of $SL(2, \mathbb{R})$ above. Marsden proved the first general completeness theorem, for homogeneous and compact pseudo-Riemannian manifolds [63]. In fact, he proved "boundly completeness". For instance, in the Lorentz case, from an everywhere *timelike* Killing field, one construct a Clairaut first integral of the geodesic flow, with compact levels. Homogeneity does not lead to existence existence of such Killing fields, but a one concludes after a little bit work (see [53] for related results).

Remark 3.5. *It seems interesting to get extension of Marsden's Theorem to other classes of (non pseudo-Riemannian) connections.*

4. The π_1-action, algebraic classification

At this stage we know that a compact Lorentz manifold of constant curvature is a quotient $M = X/\Gamma$, where X is the Minkowski space or the anti de Sitter space (the case of de Sitter space was excluded above 3.2.1), and Γ is a discrete group of X acting properly co-compactly and freely. The question is to "classify" such Γ. In the Euclidean case, the similar question is the classical classification of crystallographic groups, for which some aspects still remain fascinating problems for geometers. However, the first step of the classification was the celebrated Bieberbach Theorem (for crystallographic groups). The fundamental question that we will ask in our Lorentz case is in fact in the same vein as Bieberbach Theorem[1].

4.1. Bieberbach rigidity

Consider the general case of a homogeneous space I/H, quotient of a *connected* Lie group I by a *connected* Lie subgroup H. One central problem about homogeneous (non-Riemannian) spaces, is the study of discrete subgroups $\Gamma \subset I$ acting properly co-compactly and freely on I/H (so that $M = \Gamma \setminus I/H$ is a compact manifold).

One may start considering a radically simpler and soft problem which is, first, to find a connected Lie subgroup $G \subset I$ acting co-compactly (or say, transitively) and *properly* on I/H, and next to find a co-compact lattice Γ in G.

One says that I/H satisfies a " Bieberbach rigidity" if all its compact quotients are of this type (say, up to finite covers to avoid obvious trivial counterexamples).

One says to have a "unique Bieberbach rigidity", if up to conjugacy, there is only one group G as above (for all Γ's).

4.1.1. Flat manifolds. As example, after many works during the last decade, the structure of compact flat Lorentz manifolds, was elucidated, as in the following Theorem:

Theorem 4.1. ([44], [48], [51],...) *Let $M = Min_n/\Gamma$ be a compact Lorentz flat manifold. Then there is a solvable group G acting isometrically and simply transitively on the Minkowski space Min_n and a lattice Γ in G such that up to finite covers, $M = Min_n/\Gamma$ $(= G/\Gamma)$.*

[1] As a matter of fact; Bieberbach's Theorem, as formulated in [28], is a fundamental ingredient of the study in [17].

Actually this structure theorem for the fundamental groups of compact Minkowskian manifolds, was proved before the completeness Theorem, that is we only deal at this time with complete manifolds.

In other words, the result says that M is a quotient I/Γ, where I is a solvable group endowed with a Lorentz left invariant metric which is complete and flat. There are many examples of such solvable Lie groups, see [43] for concrete constructions in the case of the 3-dimensional Heisenberg and SOL groups and [51] and [52] for a general study.

4.1.2. Anti de Sitter manifolds. A compact anti de Sitter manifold must have odd dimension, since according to Gauss-Bonnet formula, for even-dimensional anti de Sitter manifolds, the Euler number equals the volume, up to a non-trivial multiplicative constant. But, any compact Lorentz manifold has a vanishing Euler number, since it possesses a direction field.

Conversely, for any odd dimension, there are closed anti de Sitter manifolds. This was mentioned for the first time by R. Kulkarni [59]. They are just obtained by taking $G = U(1,d) \subset O(2,2d)$, in the introduction of the Bieberbach rigidity above. Indeed, one verifies that $U(1,d)$ acts isometrically on AdS_{2d+1} (this is the meaning of the inclusion $U(1,d) \subset O(2,2d)$) transitively and properly, the isotropy group being $U(d)$. As said above, any co-compact lattice in $U(1,d)$ gives rise to a compact anti de Sitter manifold of dimension $2d+1$. To fix ideas, let us introduce as in [60] the next terminology:

Definition 4.2. *An anti de Sitter manifold of dimension $2d+1$ is called standard (resp. special standard) if up to conjugacy, its holonomy group Γ is contained in $U(1,d)$ (resp. $SU(1,d)$).*

The special anti de Sitter manifolds have the following Riemannian description. Let $\mathbb{H}^d_{\mathbb{C}}$ be the hyperbolic complex space of (complex) dimension d. It is nothing but the homogeneous space $= U(1,d)/S^1 \times U(d)$. Therefore, AdS_{2d+1} is a circle fiber bundle over $\mathbb{H}^d_{\mathbb{C}}$. In fact, AdS_{2d+1} is the circle bundle associated to the canonical line bundle (in the complex meaning) of $\mathbb{H}^d_{\mathbb{C}}$.

4.1.3. Anti de Sitter, Dimension > 3. The results of [84] leads one to hope that a unique Bieberbach rigidity phenomenon holds, for compact anti de Sitter manifolds of dimension ≥ 5. In other words, we dare ask:

Conjecture 4.3. *Up to finite coverings, every compact anti de Sitter manifold of dimension ≥ 5, is standard.*

4.1.4. Anti de Sitter, dimension $= 3$. For $d = 1$, the circle fibration $AdS_3 \to \mathbb{H}^1_{\mathbb{C}}$ is just the usual fibration over the hyperbolic plane $\mathbb{H}^2$ of its unit tangent space. Also, (an index 2 quotient of) AdS_3 is identified to the group $PSL(2,\mathbb{R})$ and $O(2,2)$ is identified to $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$. The action of this last group on $X = PSL(2,\mathbb{R})$ (seen at the same time as a group and the anti de Sitter space) is given by $(g,h).x = gxh^{-1}$.

Up to switch of factors, a special standard quotient is such that

$$\Gamma \subset PSL(2,\mathbb{R}) \times \{1\},$$

and special means that $\Gamma \subset SL(2,\mathbb{R}) \times S^1$

In their pioneering work, Kulkarni and Raymond [60] showed that any co-compact holonomy group $\Gamma \subset PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$, is (up to switch of factors) a graph: there is a closed hyperbolic surface S, and two homomorphisms $\rho_L, \rho_R : \pi_1(S) \to PSL(2,\mathbb{R})$, such that ρ_L is fuchsian (discrete and injective), and such that Γ is the image of $\rho_L \times \rho_R$:

$$\Gamma = \{(\rho_L(\gamma), \rho_R(\gamma) \in PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})), \gamma \in \pi_1(S)\}.$$

The non-standard case corresponds to the fact that the image of ρ_R is not contained in a circle (in $PSL(2,\mathbb{R})$). The first examples were observed by Ghys and Goldman [47, 49], by just taking ρ_R small enough. F. Salein showed in particular that ρ_R can be "very" big, that is to say not homotopic to the trivial representation, or equivalently, with non-vanishing Euler number. For instance, let $f : S \to S'$ be a (non-trivial) ramified covering, where S' is another hyperbolic surface, which is holomorphic, when S is endowed with the structure given from ρ_L. Let ρ_L the homomorphism induced by f, and Γ determined by (ρ_L, ρ_R). F. Salein [73] proved that Γ acts *properly* freely and co-compactly on AdS_3. (Exercise: all these closed manifolds are complete and b-complete. For which holonomy Γ is the manifold boundedly complete?)

4.2. Margulis spacetimes

In the flat case, according to Theorem 4.1, compact complete flat manifolds have (virtually) solvable fundamental groups. The difficulty to find non-solvable proper actions on Min led J. Milnor to wonder: *does the free group admit a proper action on Min_3?* In [66], when Milnor addressed this question, he suggested a way to produce an example. In [62], following this (partial) hint, Margulis answers affirmatively to the question. Since [62], proper quotients of Minkowski by free groups of isometries are called *Margulis spacetimes.*

Afterwards, T. Drumm introduced the notion of *crooked planes,* giving a more intuitive geometric vision on these spacetimes ([36]), and extended considerably the list of Margulis's spacetimes by proving that *every discrete free subgroup of $SO_0(1,2)$ is the linear part of the holonomy of a Margulis spacetime* ([37]).

In his work [62], G. Margulis associates to every hyperbolic element g of $\mathrm{Isom}(Min_3)$ a real number $\alpha(g)$ – the so-called *Margulis invariant* – and proved that a discrete purely hyperbolic subgroup of $\mathrm{Isom}(Min_3)$ can act properly on Minkowski space only if all the Margulis invariants have the same sign (see the good survey [1] giving in particular a lucid account of why this sign condition is necessary). On the other hand, it is not true that the positivity of all Margulis invariants ensure the properness of the action. The correct reverse statement was recently proved by Goldman, Labourie and Margulis in a forthcoming preprint "Proper affine actions and geodesic flows of hyperbolic surfaces": once the linear part $F \subset SO_0(1,2)$ is fixed, we consider the geodesic flow on the surface $F\backslash\mathbb{H}^2$, and denote by $\mathcal{P}(F)$ the space of all invariant probability measures for this geodesic flow. For any affine deformation $\rho : F \to \mathrm{Isom}(Min_3)$, and for any closed orbit c of the geodesic flow, define $A(c) = \alpha(\rho(g))/l(c)$ where g is the element of F corresponding to c, $\alpha(\rho(g))$ is the Margulis invariant of $\rho(g)$, and $l(c)$ the period of c (the length of the corresponding closed geodesic). Closed orbits can be considered as particular elements of $\mathcal{P}(F)$ (a kind of Dirac measures), and the map A extends uniquely to a continuous map $A : \mathcal{P}(F) \to \mathbb{R}$. Then, the affine deformation ρ is proper (i.e., $\rho(\Gamma)$ acts properly on Mink_3 if and only if $A(m)$ vanishes nowhere.

Let us also mention [38], and, more recently, [30], where it is proven that Margulis invariant characterize completely the free group of isometries in the purely hyperbolic case. The survey [32] covers the topics discussed in this section.

In Section 5.6, we will add some comments on Margulis spacetimes.

Remark 4.4. *Fundamental domains for Lorentz groups in the anti de Sitter case, and in general, polyhedra in Lorentz manifolds, were recently investigated by for A. Pratoussevitch and J.M. Schlenker, respectively* [71, 77].

5. Global hyperbolicity

Let M be an (open) Lorentz manifold with a closed (compact without boundary) spacelike Cauchy hypersurface. By general theory [33], there is an abstract maximal globally hyperbolic (abbreviation MGH) extension $\bar{M}$ of M.

Assume now that M has a (G,X)-structure, hence so does $\bar{M}$ (by analyticity). The question is how to describe this geometric structure of $\bar{M}$ (say by means of that of M)?

The most natural case is that of 3-dimensional manifolds of constant curvature, since they are the only solutions of the vacuum Einstein equations (with cosmological constant) in dimension $2+1$. One may then ask the general question for all constant curvature manifolds of any dimension.

The first work in this direction is [65], even if the results there are not stated in the terminology presented here. This celebrated preprint completely solves the negative curvature case in dimension $2+1$.

K. Scannell, Mess student, solves the positive curvature case in any dimension in his thesis [75], where he established a natural $1-1$ correspondence between $n+1$-dimensional spatially closed MGH de-Sitter spacetimes with flat conformal Riemannian structures on closed n-manifolds.

Remark 5.1. *We have to pay attention to the meaning of "flat conformal Riemannian structure". Here, we mean a (G,X) structure where X is the conformal sphere and G the Möbius group. This is a quite lazy convention that we maintain from* [75]*: it is maybe more usual to define flat conformal structures as the conformal classes of Riemannian metrics which can be written everywhere locally as scalar multiples of local flat Riemannian metrics.*

According to Liouville's Theorem, in dimension $n \geq 3$, these two notions coincide, but this is dramatically false in dimension 2. Nevertheless, their is a huge mathematical literature on Möbius structures on closed manifolds, particularly in the 2-dimensional case, which is fairly well understood.

Finally, and maybe quite surprisingly, the flat case has been systematically studied only recently by one of the authors ([17]). However, some fundamental observations appeared in [65] where the $2+1$-dimensional case is treated, and in [9], the classification is performed in any dimension, but in a particular case, assuming that the spacelike Cauchy hypersurface admits some Riemannian metric with negative constant curvature. We should also indicate [74], specifying the possible geometric character in the Thurston's terminology of spacelike hypersurfaces of MGH flat $3+1$-spacetimes.

5.1. The Anti de Sitter case

5.1.1. The 3-dimensional case. We present here briefly Mess results, and address some questions. We consider as in 4.1.4 the model $PSL(2, \mathbb{R})$, endowed with the Lorentzian metric defined by its Killing form, and admitting as isometry group the product $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ acting by left and right translations.

Let S be a closed surface, and Γ be its fundamental group. Denote by $Teich(\Gamma)$ the Teichmüller space of S: we consider it as the space of discrete injective representations of Γ in $PSL(2, \mathbb{R})$ with cocompact image, modulo inner automorphisms of $PSL(2, \mathbb{R})$. Any pair (ρ_L, ρ_R) of elements of $Teich(\Gamma)$ thus defines a representation $\rho : \Gamma \to Iso(AdS_3)$, well defined up to conjugacy. The causality domain of $\rho(\Gamma)$ in AdS_3, as defined in Section 1.2, is a (convex) domain $C(\rho)$. Actually, the action of $\rho(\Gamma)$ on $C(\rho)$ is proper, and the quotient manifold $M(\Gamma) = \rho(\Gamma)\backslash C(\rho)$ is MGH, admitting a Cauchy surface homeomorphic to S. Moreover, Mess proved that any spatially closed GH spacetime locally modelled on AdS_3 can be isometrically embedded in such a $M(\Gamma)$, in particular, they are Kleinian.

In a more precise way, Mess did not consider the case where S is a torus. But this case is quite simple: it is the Torus Universe as presented in [26]. In particular, the left and right holonomy groups are necessarily contained in 1-parameter hyperbolic subgroups of $PSL(2, \mathbb{R})$ (see [20]).

Mess results involves actually many interesting geometric and physically relevant features as real trees, earthquakes, and so on, that we cannot pretend to develop here further.

5.1.2. Higher dimensions. Many propositions in [65] still apply in higher dimensions: for example, MGH spacetimes with constant negative curvature are still Kleinian, and thus uniquely defined by their holonomy groups. However, the complete classification remains an open question. A natural way to define GH AdS spacetimes in dimension $n+1$ is the following: let Γ be a cocompact lattice of $SO_0(1, n)$, the identity component of $O(1, n)$. The quotient space $S = \Gamma\backslash\mathbb{H}^n$ is a closed hyperbolic manifold. Let ρ_0 be the composition of the inclusion $\Gamma \subset$

$SO_0(1, n)$ with the natural embedding $SO_0(1, n) \subset SO_0(2, n)$. Since $SO_0(2, n)$ is the isometry group of AdS, we can define the causality domain $C(\rho_0)$, which actually has a simple geometrical description. The quotients $M(\rho_0) = \rho_0(\Gamma)\backslash C(\rho_0)$ are then MGH spacetimes with Cauchy hypersurfaces homeomorphic to S. These spacetimes are *static,* meaning that they admit a timelike vector field with integrable orthogonal plane fields.

This construction still applies for small deformations $\rho : \Gamma \to SO_0(2, n)$ of ρ_0, providing nonstatic MGH spacetimes $M(\rho)$. But some rigidity aspect appears here: it is not a really trivial task to exhibit such a deformation; the only known procedure is the deformation along codimension 1 geodesic hypersurfaces of S given in [55]. Anyway, these deformations exist, and a first question is the following: is the space of holonomy representations of spatially closed AdS spacetimes with Cauchy hypersurface homeomorphic to S, connected?

There is another natural question: what are the possible topologies for closed Cauchy hypersurfaces of AdS GH spacetimes? Are they necessarily homeomorphic (up to finite coverings and products by flat tori) to hyperbolic manifolds or to the product of two hyperbolic manifolds?

5.2. The de Sitter case

We now present Scannell's work ([75]) relating spatially closed MGH $n+1$-spacetimes locally modelled on dS with Cauchy hypersurface S with flat conformal structures on S. Unfortunately, the description is quite delicate, since, in general, these spacetimes are *not* Kleinian.

Actually, up to time reversing isometries, these spacetimes are all geodesically complete in the future (it is a nontrivial by-product of Scannell's classification), so we will restrict to this case for our description. The fundamental group of S is still denoted Γ.

5.2.1. A quick presentation of dS. We use the projective model of dS. More precisely, it will be more convenient to lift the usual projective model in the sphere, the double covering of the projective space. Here, the projective model is the space of vectors $(x_0, x_1, \ldots, x_{n+1})$ in $Mink_{n+2}$ with positive norm contained in the sphere $\mathcal{S}$ of equation $\sum x_i^2 = 1$. In order to distinguish this model from the usual definition of de Sitter space given in 3.2, we denote it by ds. The boundary of ds is $x_0^2 = x_1^2 + \cdots + x_{n+1}^2$: this is the union of two n-dimensional subspheres ∂ds_+ (where $x_0 > 0$) and ∂ds_- (where $x_0 < 0$). Each $\partial ds_\pm$ is also the boundary of a copy of the Klein model of the hyperbolic space. Geodesics are intersections of round circles in $\mathcal{S}$ with ds; a geodesic is spacelike if this circle avoids $\partial ds_\pm$, lightlike if it is tangent to $\partial ds_\pm$, and timelike if the circle intersects each $\partial ds_\pm$ at exactly two points.

The choice of a chronological orientation of ds is equivalent to the choice of one of the $\partial ds_\pm$ – let say, $\partial ds_\pm$ – as the ideal boundary *in the future*. Now, for any point x in ds, the timelike future-complete geodesic rays starting from x hit ∂ds_+ in a bounded region, which, when we identify $\partial ds_+ \approx \partial\mathbb{H}^{n+1}$ with the conformal sphere $\mathbb{S}^n \approx \partial\mathbb{H}^{n+1}$, is an open n-ball $B(x)$. Conversely, every round n-ball B is the "visible domain of infinity" $B(x)$ of an unique element x of ds.

5.2.2. Scannell's results. We first consider the Kleinian case: consider a Kleinian flat conformal structure on S, i.e., a simply connected domain D of $\partial ds_+ \approx \mathbb{S}^n$ and a representation $\rho : \Gamma \to \mathrm{Conf}(\partial ds_+) = SO_0(1,n)$ with image preserving D, and acting freely and properly discontinuously on it, such that S is homeomorphic to the quotient space $\rho(\Gamma)\backslash D$. Then, the points x of ds such that $B(x)$ is included in D form an open domain Ω (in ds) on which $\rho(\Gamma) \subset SO_0(1,n)$ acts freely and properly discontinuously. Therefore, the quotient space $M(\rho) = \rho(\Gamma)\backslash\Omega$ is well defined. Observe that it is obviously geodesically complete in the future.

When D is the entire sphere ∂ds_+, we obtain by this construction (finite quotients of) the entire de Sitter space. When D is the complement of a single point x_0 of ∂ds_+, we obtain the parabolic case: Γ is then necessarely abelian (up to finite index) and only contains parabolic elements of $SO_0(1,n)$; the n-dimensional spheres in $\mathcal{S}$ tangent to ∂ds_+ at x_0 foliates Ω and they induce on the quotient n-dimensional tori which are Cauchy hypersurfaces

These two cases being from now excluded, the boundary $\partial\Omega$ in ds is a null-hypersurface. For any point x in Ω, we can define its "maximal time of existence" $\tau(x)$: this is the supremum of total proper times of timelike curves starting from $\partial\Omega$ and ending at x. Then, τ is a $\rho(\Gamma)$-invariant C^1-function. It induces on the quotient a "cosmological time" function whose levels sets are spacelike hypersurfaces which are Cauchy hypersurfaces, proving that $M(\Gamma)$ is globally hyperbolic.

In general, a conformal structure on S is given precisely by:

– a local homeomorphism $\overline{\mathcal{D}} : \widetilde{S} \to \partial ds_+$,
– a morphism $\rho : \Gamma \to SO_0(1,n)$ such that $\rho(\gamma) \circ \overline{\mathcal{D}} = \overline{\mathcal{D}} \circ \rho(\gamma)$.

To such a data, associate the space $\mathcal{M}$ formed by closed subsets $\tilde{B}$ of $\widetilde{S}$ on which $\overline{\mathcal{D}}$ restrict as a homeomorphism with image a closed round n-ball. Equip $\mathcal{M}$ with its obvious topology: it is a n-manifold admitting a natural action of Γ which is properly discontinuous. Denote by M the quotient space. There is a natural map $\mathcal{D} : \mathcal{M} \to ds$: define $\mathcal{D}(\tilde{B})$ as the unique point x in ds for which $B(x) = \overline{\mathcal{D}}(\tilde{B})$. This map is a local homeomorphism, respecting the respective Γ-actions. It follows that $\mathcal{D}$ is the developing map of a de Sitter structure on M. K. Scannell proved that the cosmological time function on M is well defined, admitting as level sets Cauchy hypersurfaces for M. Moreover, this construction provides all spatially closed MGH de Sitter spacetimes.

To be complete, we must give a flavor of the non-Kleinian case, this particular flavor arising from the "pathologies" of flat conformal structures: restrict to $n = 2$, and take as surface S the 2-torus. The most obvious flat conformal structures on S are the conformal class of flat metrics on the torus: i.e., the quotient of $\mathbb{C}$ by lattices. But it is far from exhausting the entire list of flat conformal structures on the torus! Indeed, consider the torus as the quotient of $\mathbb{C}$ by a lattice Λ. Take as developing map the exponential $\exp : \mathbb{C} \to \mathbb{C}^*$. The translations by elements of Λ

correspond to homotheties by elements of $\exp(\Lambda)$. This simple family of examples prove that:

- the holonomy can be noninjective: it happens when Λ contains a integer multiple of $2i\pi$,
- the developing map can be non-injective,
- two non-isometric MGH spacetimes can admit the same holonomy.

Actually, “worse” situations arise: when the surface S has higher genus, any irreducible representation $\Gamma \to SL(2, \mathbb{C})$ is the holonomy of one (or maybe an infinite number of) flat conformal structure on S (see [45]). For example, the holonomy group can be dense on $SL(2, \mathbb{C})$.

Remark 5.2. *Anyway, we have to indicate here that we gave above the complete description of conformal (Möbius) structures on the* 2*-torus, i.e., of globally hyperbolic* dS_3*-spacetimes admitting a Cauchy surface homeomorphic to the* 2*-torus.*

5.2.3. The dual Riemannian version: hyperbolic ends. There is a dual Riemannian version, which in the 2-dimensional case is much more popular in the mathematical community, although a bit more delicate to define than the Lorentzian version: the theory of hyperbolic ends (see [80, 61]).

First of all, we must state clearly that this construction does not apply in the particular case of finite quotients of dS-spacetimes: indeed, finite subgroups of $SO(1, n)$ never act freely on the hyperbolic space!

Constructing directly from the conformal structure on S the hyperbolic structure on $S \times \mathbb{R}$ is a delicate task, but is easier when the Lorentzian version is available: indeed, the construction of the hyperbolic end from the globally hyperbolic is essentially given by the (inverse of) process described at Section 3.2.1 above. More precisely: the dS-spacetime M associated to the given conformal structure is foliated by the level sets of the function τ defined above. The tangent planes of these spacelike hypersurfaces form a field $y \mapsto L(y)$ of spacelike hyperplanes on M. But the pairs (y, L) where y is an element of the spacelike hyperplane L of dS correspond bijectively and naturally with the pairs (x, H) where x is an element of $\mathbb{H}$ and H a totally geodesic hyperplane containing x: take H as the geodesic hypersurface admitting $\partial B(y)$ as boundary at infinity, and x as the intersection point between H and the great circle in $\mathcal{S}$ orthogonal to L at y. This procedure composed with the field $y \mapsto L(y)$ produces then a Γ-equivariant map $\mathcal{D}$ from the universal covering $\widetilde{M}$ to $\mathbb{H}$. Moreover, Scannell proved that the level sets of τ are strictly convex. *In fine,* this strict convexity property ensures that $\mathcal{D}$ is a local homeomorphism. It can thus be interpreted as the developing map of a hyperbolic structure on M: this is precisely the associated hyperbolic end M_{hyp}.

The procedure described above actually defines for every strictly convex spacelike hypersurface Σ in M a map f from Σ into M_{hyp}, with image a strictly convex hypersurface. There is an inverse procedure defining from a strictly convex hypersurface in M_{hyp} a spacelike hypersurface in M. We will use this remark later;

we will actually need there the following fact: the eigenvalues of the second fundamental form[2] of $f(\Sigma)$ at $f(x)$ are the inverses of the eigenvalues of the second fundamental form of Σ at x.

5.3. The flat case

The most obvious examples of MGH flat manifolds are quotients of Min_{n+1} by abelian discrete groups of rank n of spacelike translations; we call these examples *translation spacetimes.* They admit as Cauchy hypersurfaces flat tori.

The second natural family of examples are what we call *Misner spacetimes.* Let v_1, v_2 be two lightlike vectors in Min_{n+1}, and denote by $v_i^\perp$ their orthogonal. Let L be the 1-parameter group formed by pure (i.e., without elliptic part) loxodromic isometries (or "boosts") preserving the directions v_i.

Vectors v in Min_{n+1} for which the Minkowski scalar products $\langle v \mid v_i \rangle$ are negative form a domain which has two connected components $\Omega^\pm$, one being geodesically complete in the future and the other, in the past. These domains are "quarter of space". Isometries of Min_n preserving $\Omega^\pm$ form a group which is a compact extension of the abelian group A of dimension n: elements of A are compositions of loxodromic elements in L with translations by vectors in $v_1^\perp \cap v_2^\perp$. This group A acts properly discontinuously on $\Omega^\pm$, and the orbits of this action are product of hyperbolae with euclidean spaces: these orbits are spacelike hypersurfaces isometric to the euclidean space.

For any lattice Λ of A, the quotient spaces $\Lambda \backslash \Omega^\pm$ are globally hyperbolic spacetimes geodesically complete in the future or in the past, and the orbits of A project in these quotients as toroidal Cauchy hypersurfaces.

The last family are the so-called *standard spacetimes* that G. Mess has already described in [65]: the simplest members of this family of examples are constructed from cocompact lattices of $SO_0(1,n)$. Let Γ be such a lattice. The set of timelike vectors of Min_n admits two connected components $\Omega^\pm$ respectively geodesically complete in the future and in the past. The action of Γ on $\Omega^\pm$ is free and properly discontinuous: we denote by $M^\pm(\Gamma)$ the quotient manifold.

Every level set $\{Q = -t^2\} \cap \Omega^\pm$ is Γ-invariant; it induces in $M^\pm(\Gamma)$ a hypersurface with induced metric of constant sectional curvature $-\frac{1}{t^2}$. Since $\{Q = -1\}$ is the usual representant of the hyperbolic space, the flat Lorentzian metric on $M^\pm(\Gamma)$ admits the warped product form $-dt^2 + t^2 g_0$, where g_0 is the hyperbolic metric on $\Gamma \backslash \mathbb{H}^n$. We call these examples *radiant standard spacetimes.* Observe that $M^+(\Gamma)$ (resp. $M^-(\Gamma)$) is geodesically complete in the future (resp. in the past), and that there is a time reversing isometry between them.

New examples are obtained by adding translation parts (see [65] or [9]): any representation of a Γ in $\mathrm{Isom}(Min_n)$ admitting as linear part an embedding onto a cocompact lattice of $SO(1,n)$ is the holonomy group of a flat spatially closed globally hyperbolic spacetime. Actually, there are two such globally hyperbolic

[2] Maybe physicists are more acquainted with the notion of shape operator: it has the same eigenvalues than the second fundamental form.

spacetimes, one being geodesically complete in the future, and the other, geodesically complete in the past. Moreover, they are Kleinian, more precisely, they are quotients by the holonomy group of convex domains of Min_n. Of course, if M is a flat GH spacetime with Cauchy hypersurface S, and N a flat euclidean torus, then the product $M \times N$, equipped with the product metric, is still a flat GH manifold, with Cauchy hypersurface $S \times N$.

In [17], we prove that any flat globally hyperbolic spacetime admitting a closed Cauchy hypersurface is finitely covered by a globally hyperbolic spacetime which can be isometrically embedded in a translation spacetime, in a Misner spacetime or in a (twisted) product of a standard spacetime by an euclidean torus.

A key point of this analysis, entering in the spirit of the present survey, is that standard spacetimes are precisely the quotients by the holonomy group of the causality domain defined in Section 1.2. In some way, we can say that the properness of the action is ensured by the causality restriction. This viewpoint is developed in [18].

Remark 5.3. *Actually, motivated by some examples appearing in the literature, we do not reduce in* [17] *to the case where the Cauchy hypersurface are closed, but we also consider the case where the Cauchy hypersurfaces are complete for their induced metric. This case is considerably more difficult: for example, it is not clear that any subgroup of* $Isom(Min_n)$ *admitting as linear part a Kleinian group, i.e., a discrete subgroup of* $SO(1,n)$*, is the holonomy group of globally hyperbolic spacetime. We proved that this statement is true for* convex cocompact *Kleinian groups, i.e., geometrically finite (admitting a finite sided polyhedral fundamental domain) and containing only hyperbolic elements*[3]*. This problem when the Kleinian group is geometrically finite is an open interesting question.*

5.4. Absolute time, CMC foliations

Everybody knows that from the general relativity point of view, there is no natural global time function on spacetime. Globally hyperbolic spacetimes do have many time functions –, i.e., strictly increasing along timelike paths – but *a priori,* none of them has a preferred status. However, some of them have a special interest, at least from the mathematical point of view. We present here two of them.

5.4.1. Cosmological time function. We have already mentioned previously this function when discussing de Sitter spacetimes. The cosmological time function is defined in any spacetime as follows: $\tau(x)$ is the supremum of the proper times of future oriented timelike curves ending at x. In all examples of globally hyperbolic spacetimes discussed in this section, except for the cases of translation spacetimes, finite quotients of de Sitter spacetimes and parabolic de Sitter spacetimes, this function, if the chronological orientation of spacetime is well chosen, has only finite values and is C^1. This function provides quick proofs for the global hyperbolicity of these examples. Moreover, it is a "gauge-invariant" intrinsic feature of spacetime. On the other hand, it has a poor level of differentiability: even if in all

[3]For $n = 2$, a Kleinian group is geometrically finite if and only if it is finitely generated.

circumstances considered here it is C^2 almost everywhere (for the Lebesgue measure), in general, it is *not* C^2 everywhere – for example, in the case of standard spacetimes, it is C^2 only in the radiant case.

Spacetimes with regular cosmological time function are defined and studied in [11]. We also mention [21], where the cosmological function of $2+1$-dimensional standard spacetimes is geometrically studied, discussing the link between these notions and measured foliations, real trees, etc. In particular, there is a remarkable discussion on the striking fact that standard spacetimes geometrically realize the well-known and extensively studied correspondence between measured geodesic laminations and measured foliations. There is also the description of an interesting (mainly suggested) gluing operation obtaining from two standard spacetimes a globally hyperbolic spacetime with constant curvature -1, and relying on the cosmological function.

Let's also mention the very up-to-date [22], where the geometrical notion of *geodesic stratification* is associated to flat higher-dimensional standard spacetimes.

5.4.2. CMC foliations. The constant mean curvature of a spacelike hypersurface at a point is the trace of the shape operator at this point. A CMC hypersurface is a spacelike hypersurface with constant mean curvature. It is well known that Einstein's equations has a considerably most tractable form when considered in the neighborhood of a CMC hypersurface. Of course, this viewpoint is perfectly suited for the local study of Einstein equations, since local pieces of CMC hypersurfaces always exist; but the global existence of CMC hypersurfaces is a strong hypothesis. Let's mention here the survey [72] presenting similar questions for nonvacuum Einstein equations under strong or weak conditions.

We will call CMC foliation a foliation of spacetimes admitting as leaves CMC hypersurfaces. A function f is a CMC time function if:

- it is a time function, i.e., is increasing along timelike curves oriented towards the future,
- its value at a point x is the mean curvature of the level set $f^{-1}(f(x))$.

If we reverse the time orientation, we change the chronological orientations of timelike curves, but we also change the sign of shape operator: the CMC time function changes its sign, but is still a CMC time function.

Actually, the maximum principle of CMC hypersurfaces implies the uniqueness of CMC time function on a given spacetime – but not its existence! More precisely, for right conventions of sign on the shape operator, if S' is a spacelike hypersurface contained in the future of another spacelike hypersurface S, then, at every common point of tangency, the mean curvature of S is greater than the mean curvature of S'. Therefore, in a spacetime admitting a CMC time function τ, every closed CMC hypersurface is a fiber of τ. Indeed, its constant mean curvature value has to be greater (respectively less) than the maximal (respectively minimal) value of τ on itself.

Thus, a CMC time function, when it exists, is an intrinsic feature of the spacetime.

In [9], L. Andersson proved that every flat standard spacetime admits a CMC time function. The proof follows from the observation, already pointed out in [65], that these standard spacetimes can always be considered as small deformations of radiant standard spacetimes. In the radiant case, the CMC time function is obvious – and coincide with the CT function – and the level sets of this radiant CT function persist in the deformations as spacelike with controlled mean curvature. This control on the curvature enables the successful use of barrier methods of [46].

We should mention that CMC time functions of future (resp. past) geodesically complete standard spacetimes take value in $]-\infty, 0[$ (resp. $]0, +\infty[$)).

Observe that according to [17], the same conclusion holds more generally for every flat MGH spacetime with closed Cauchy hypersurfaces, except for translations spacetimes.

The authors, jointly with F. Béguin, established the existence of CMC time functions for locally AdS_3 globally hyperbolic spacetimes with closed spacelike surfaces. ([19, 20]). The proof is similar in many points with the Andersson's proof in the flat case.

As a comment for the genus 1 case[4] (the Torus Universe), we only mention that the proof is quite easy, the fibers of the CMC time function being the orbits of a 2-dimensional abelian Lie group.

The proof for the higher genus case is based as in the flat case on the exhibition of barriers to which are applied Gerhard's criteria [46]. But there is a fundamental difference: the AdS_3-spacetimes admitting obvious CMC functions are the static ones, but it is not possible to consider general AdS_3-spacetimes as small deformations of the static ones. The construction of barriers is thus undertaken with another method: basically, these barriers are constructed as smooth approximations of level sets of the CT function. More precisely, some level sets of the CT time function are strictly convex, other level sets are strictly concave; and in [20] we approximate these C^1 spacelike surfaces by smooth spacelike surfaces which are still respectively (strictly) convex and concave. In particular, their constant mean curvature values are respectively negative and positive everywhere. According to [46], the existence of a maximal hypersurface –, i.e., with null mean curvature – follows. The proof is then achieved thanks to the main result of [12]: in dimension $2+1$, the existence of a single CMC hypersurface ensures the existence of a CMC time function. This CMC time function in this context is a surjection onto $\mathbb{R}$.

The proof should presumably extend to higher dimensions, but it requires approximation of convex (or concave) hypersurfaces by smooth convex (concave) hypersurfaces – this is not so easy a task – and, more seriously, to supply an alternative to the use of [12] which has been established only in dimension $2+1$. Anyway, this draft of proof would be successful only through a better knowledge of the topological type of GH AdS-spacetimes.

[4]The genus 0 does not occur: the sphere cannot be a spacelike surface in a AdS_3 spacetime.

Finally, the existence of CMC time functions in globally hyperbolic spacetimes locally modelled on dS_3 with closed surfaces has not been published or announced anywhere. Here we give a sketchy proof, using the dual theory of hyperbolic ends.

First of all, we must point out that de Sitter spacetime itself does *not* admit CMC time functions! Of course, it admits (many) CMC foliations, but the leaves of these foliations are totally geodesic, and thus, this CMC foliation violates the increasing hypothesis for CMC time functions. More generally, this observation applies for finite quotients of de Sitter space, i.e., the elliptic case.

The same phenomena applies for the parabolic case: the n-dimensional spheres we mentioned earlier as Cauchy surfaces have all constant mean curvature -2, thus the argument above apply here also, proving that parabolic spacetimes do not admit CMC time functions.

However:

Theorem 5.4. *Cauchy-closed MGH dS_3-spacetimes which are not elliptic or parabolic all admit CMC time functions.*

Remark 5.5. *In this context, the CMC time function takes value in $]-\infty, -2[$ when the time orientation is selected so that the spacetime is future geodesically complete.*

Proof. Let S be a Cauchy surface of the spacetime M under study. Since M is nonelliptic, S is not a sphere. Moreover, since M is not parabolic, and if S has genus 1, it follows from remark 5.2 that the image of the developing map of the flat conformal structure is the complement of 2 points. These points are the extremities of a geodesic in $\mathbb{H}^3$ that we will use later; let's denote it by c.

In [61], it is proved that, hyperbolic ends always admit a *smooth* foliation by closed surfaces with constant *scalar* curvature. More precisely, the constant scalar curvature values of leaves vary between 0 and 1, and the principal eigenvalues are negative (here, the reader must trust our unexpressed sign conventions). Actually, [61] only deals with the case where S has genus bigger than 2. In the case of the 2-torus, the fibers of the distance function to the geodesic c defined above are the leaves of the required foliation.

This foliation provides a dual foliation in M (cf. Section 5.2.3). Moreover, the leaves of this foliation have constant scalar curvature too! Last but not least, the scalar curvatures of the leaves increases in time between 1 and $+\infty$. If L_t denote the leaf of this foliation with constant scalar curvature t, at any point x of L_t,and the principal eigenvalues λ and μ are both negative. and satisfy of course $\lambda\mu = t$. Hence, once of them is less than $-\sqrt{t}$, and the same is true for the mean curvature value $\lambda + \mu$. This is uniform on x: *the mean curvature of L_t is everywhere less than $-\sqrt{t}$.* On the other hand, if L_t^s denote the image of L_t under the Gauss flow which pushes every point during the proper time s along the normal of L_t, the principal eigenvalues of L_t^s converges uniformly to -1. Hence, for $t > 4$, and for s sufficiently big the mean curvature of L_t^s is everywhere greater that the maximal mean curvature value on L_t. Therefore, L_t and L_t^s form a pair of barriers to which [46] can be applied: M contains a CMC hypersurface. We then conclude as in the AdS_3 case by application of [12]. □

Remark 5.6. *In* [10]*, answering a question in* [21]*, L. Andersson proves that in standard spacetimes, the cosmological time functions and the CMC time functions have the same asymptotic properties.*

5.5. BTZ Black holes and wormholes

BTZ black holes was defined in [15, 16]: these are $2+1$-dimensional spacetimes locally modeled on AdS_3 presenting common features with realistic Schwarzschid (for the non-rotating case) and Kerr black holes (for the rotating case). Loosely speaking, they admit a natural conformal boundary at infinity and there is an open domain – the "black hole" – of spacetime which cannot be "seen" from this conformal boundary. In other words, future oriented lightlike rays starting from the "black hole" domain are incomplete, whereas outside the "black hole", there are complete lightlike rays reaching the conformal boundary.

Our quick presentation is very poor, however, these geometric objects have been extensively studied from the beginning of the 90's as toy models to which questions about the interaction of quantum phenomena and gravity can be tested. Let us mention [26] or [82] as cautions for the physical interest of these examples. And let's mention as good texts presenting BTZ black holes: [7, 6, 24, 8].

From (our) mathematical point of view, these black holes, and their multi-connected versions called "multiblack holes" and "wormholes" are the natural generalizations of globally hyperbolic geometric spacetimes. It has been understood from the beginning that all of them can be defined a the quotient by a discrete group Γ of isometries of a domain $\Omega(\Gamma)$ of AdS_3, the domains being defined in some way as the causality domain of the discrete group. Actually, there is some nontrivial result here: these domains of causality is not precisely $C(\Gamma)$. Elements of Γ are exponentials of elements $X(\gamma)$ of the Lie algebra $sl(2,\mathbb{R}) \times sl(2,\mathbb{R})$, and the domain $\Omega(\Gamma)$ is defined as the open domain where all the right invariant vector fields defined by the $X(\gamma)$ are spacelike. As a matter of fact, the action of Γ is proper and causal. In the rotating cyclic case, $C(\Gamma) \neq \Omega(\Gamma)$. But, as it is proved in [6] in a particular case, and that we will generalize in a forthcoming paper, the equality $C(\Gamma) = \Omega(\Gamma)$ holds as soon as the group Γ is not virtually cyclic.

Let us also mention that the so-called *angular momenta* of BTZ (multi) black holes (which is a physical notion) is the perfect analog in the AdS context of the Margulis invariant discussed in 4.2. It is an interesting challenge to evaluate how far this analogy can be continued. In particular, the validity of the results of [38, 30] in the BTZ context would be very interesting.

5.6. Causal properties of Margulis spacetimes

In his book, S. Carlip, refering to Margulis spacetimes, wrote: "The resulting geometries are fairly bizarre…, and they could potentially serve as counterexamples for a number of plausible claims about $(2+1)$-dimensional gravity" ([26], p. 11).

Here, we mainly want to stress out that globally hyperbolic parts of Margulis spacetimes are fairly well identified: according to [17], any finitely generated purely hyperbolic subgroup Γ of $\mathrm{Isom}(Min_3)$ is the holonomy group of two MGH spacetimes. More precisely, there are two (disjoint) Γ-invariant convex open domains $C^{\pm}$ in Min_3 whose quotients $U^{\pm}$ are MGH spacetimes containing every globally hyperbolic spacetimes with holonomy group Γ. This is true in particular when Γ acts properly, i.e., when $M = \Gamma \backslash Min_3$ is a Margulis spacetime. In other words, M contains disjoint isometric copies of $U^{\pm}$, and the complement of these two subdomains is a region N where causality properties are dramatically violated. Let's now state some properties of M:

- for every timelike geodesic $t \mapsto \gamma(t)$, there are two values $t_{\pm}$ such that for $t < t_-$ (resp. $t > t_+$) $\gamma(t)$ is in U^- (resp. U^+),
- every point in the interior of N belongs to a closed timelike curve (CTC), and CTC are all contained in N,
- $\partial U^{\pm}$ are null-surfaces covered by future or past (depending on the sign $\pm$) null geodesic rays,
- M does not contained closed lightlike geodesics,
- (Penrose boundary) M can be naturally "completed" by future and past conformal ideal boundaries $\mathcal{J}^+$, $\mathcal{J}^-$, each of them being a finite number of annular components.

Naïvely, one is tempted at first glance to consider N as a black hole since, at least, it could be the feeling of observers inside U^-: they cannot observe any singularity, since their past cone is complete, enjoy the nice sensation to be part of a globally hyperbolic spacetime, but all of them are promised to a dramatic issue: the entrance in N.

On the other hand, the convexity of C^- implies that the volume of the intersection between its boundary and spacelike planes of Min_3 decreases when the plane is moved in the future: this is not compatible with the increasing entropy property of black holes. Anyway, this phenomena does not contradict Hawking's Theorem, since for this theorem, the black hole is the region which cannot be observed from the future conformal boundary $\mathcal{J}^+$. Here, the conformal boundary of C^- is contained in $\mathcal{J}^-$. Thus, to follow the classical treatment of black holes, we must be concerned with C^+: then, the singularity region N is observed by everybody: this is a naked singularity, that all relativists reject as physically unrealistic.

6. Isometric actions

Now we investigate isometric actions on Lorentz manifolds. (To begin with, let us mention that, even if in setting problems the compactness of the Lorentz manifold is not needed, the whole results of the present section, except Paragraph 6.7, concern compact manifolds.)

Question 6.1. *Let (M, g) be a Lorentz manifold, and $G = Isom(M, g)$ its isometry group. When is the action of G on M* essential*?*

Firstly, by "essential", we mean that it is really a (pure) Lorentz isometry group, i.e., it cannot preserve a Riemannian metric on M?

Here, one may ask, why in this notion, we are comparing Lorentz metrics with Riemannian ones, and not with any other structures. The point is that the comparison is from a dynamical point of view. A Riemannian metric is in a dynamical sense a structure of lower order: Riemannian isometries are equicontinuous, and therefore have no "chaotic" dynamics. In contrast (as it is well known, and we will recall below), Lorentz isometries can be, for instance of Anosov type (this is reminiscent to the revolutionary facts of special Relativity, asserting possibility of contraction of local time, and dilation of lengths). It is thus natural to call inessential a Lorentz metric having an isometry group which coincides with that of (an auxiliary) Riemannian metric.

In fact, one can try to generalize a notion of "essentiality" to other geometric structures. This is not so easy to formulate, but the idea is to find a good notion of dynamical hierarchy between geometric structures, that is to decide how stronger is the dynamics (i.e., isometry group) generated by someone with respect to the other. As a paradigmatic example, a conformal pseudo-Riemannian structure is essential, when its conformal group does not preserve a pseudo-Riemannian metric (in the same conformal class) (see §7.)

Now, it is known that preserving a Riemannian metric is equivalent to acting properly (all objects are smooth). Therefore, the question becomes

When is the action of G on M non proper?

If furthermore the manifold is compact, then the G-action is proper iff G is compact, hence our question becomes:

When is the isometry group of a compact *Lorentz manifold non-compact?*

6.0.1. Sub-question: Lorentz homogeneous spaces. Here we specialize the question to the homogeneous case. Therefore, the question is to classify with the following (algebraic) conditions:

- $M = G/H$ (G a Lie group, and H a closed subgroup of G).

(To simplify, We suppose that G acts faithfully on M, i.e., we cannot simplify G/H to a smaller G'/H'.)

- The left G-action $((g, xH) \in G \times M \to (gx)H \in M)$ preserves a Lorentz metric.
- The isotropy group H is not compact (this means non-properness).

If M is compact, the last condition becomes: G is not compact.

6.0.2. Super-question: stable properness. Let $\mathrm{Diff}^k(M)$ be the group of diffeomorphisms of class k of M. It acts on $Lor^{k-1}(M)$, the space of C^{k-1} Lorentz metrics on M. Endow $Lor^{k-1}(M)$ with the Banach or Fréchet topology (Fréchet for $k = \infty$). For the sake of simplicity, we will not note k, and assume that M is compact.

- It is known that $\mathrm{Diff}(M)$ acts properly on $Riem(M)$, the space of Riemannian metrics. In particular, the quotient $Riem(M)/Diff(M)$ is Hausdorff, it is the modular space of M.
- Notice that, any function on $Riem(M)/Diff(M)$ is a Riemannian invariant: e.g., volume, diameter...

Super-question 6.2. *When is the Diff(M)-action on Lor(M) proper?*

For $g \in Lor(M)$, $\mathrm{Stabilizer}(g) = \mathrm{Isom}(g)$. If the $\mathrm{Diff}(M)$-action is proper then, $\forall\ g \in Lor(M)$, $\mathrm{Isom}(g)$ is compact, that is the super-question is stronger than the question! We quote from [35] that the difficulty in the global studying of Lorentz manifolds lies in the fact that $Lor(M)/Diff(M)$ is not Hausdorff.

6.0.3. Some motivations. The question we are asking is reminiscent to the (former) Lichnérowitch conjecture, "Conformal groups of Riemannian manifolds", solved by Ferrand and Obata [40, 70]. It starts by the observation that, although $\mathrm{Conf}(M, g)$, the conformal group of a compact Riemannian manifold (M, g), is not, a priori, compact, the only known examples for which the group is indeed non-compact are the Euclidean spheres. The result, which was actually proved, in its final form by J. Ferrand, confirms this fact: only the usual spheres have non-compact conformal group (among compact Riemannian manifolds).

Ferrand-Obata Theorem and our present question are in fact particular cases of a rigidity phenomenon in geometric dynamics (see for instance [35]).

Our sub-question concerns classification of a small class in the wide world of compact homogeneous spaces. The homogeneous Riemannian problem, is "trivial": take $M = G/H$, where G is any compact Lie group and H is a closed subgroup of it. In contrast, we know very little information about general compact non-Riemannian homogeneous spaces. The interest of the Lorentz case (that is, our sub-question) is that it seems to be the easiest non-Riemannian homogeneous problem.

The case where H is discrete is special. Indeed, in this case, G covers G/H, and one can pull back the G-invariant geometric structure on G itself. Therefore, the nature of this geometric structure can be seen on G, in fact at its Lie algebra level $\mathcal{G}$.

Fact 6.3. *The problem of closed homogeneous Lorentz manifolds with discrete isotropy is equivalent to find a co-compact lattice H in a Lie group G, and a Lorentz scalar product on $\mathcal{G}$ preserved by the adjoint action of H.*

Proof. Left translate on G the Lorentz scalar product on $\mathcal{G}$ which is $Ad(H)$-invariant. The Lorentz metric on G is: G-left-invariant, and H-right invariant. Therefore, it passes to a G-invariant Lorentz metric on G/H.

In particular, if $\mathcal{G}$ admits a bi-invariant (i.e., $Ad(G)$-invariant) Lorentz scalar product, then any quotient G/H, where H is discrete is a homogeneous Lorentz manifold. □

6.1. Examples: Lie algebra with bi-invariant Lorentz metrics

We will start here by giving examples of compact homogeneous Lorentz manifolds. Obviously, the only interesting non-trivial cases are when the isometry group is neither compact nor abelian.

6.1.1. "Baby" example: $PSL(2,\mathbb{R})$. The Killing form of $PSL(2,\mathbb{R})$ is non-degenerate (as the "simplest" semi-simple Lie group). Since the dimension is 3, the Killing form has a type $-++$ or $--+$. Anyway, up to a change of sign it is Lorentzian. It is bi-invariant as for any group. Therefore $PSL(2,\mathbb{R})$ has a bi-invariant Lorentz metric, and any compact quotient $M = PSL(2,\mathbb{R})/H$ where H is a co-compact lattice (a surface group) is a compact homogeneous Lorentz manifold. In fact, the isometry group of M is essentially (i.e., up to finite index) $PSL(2,\mathbb{R})$. Actually, as already discussed 4.1.4, these examples are locally modelled on AdS_3.

6.2. Oscillator groups

The oscillator groups (sometimes called warped Heisenberg groups as in [85]) is a family of "sympathetic groups": they are *solvable* but look like $SL(2,\mathbb{R})$ (we use the adjective "sympathetic" for these guys because we find they are so, and also this adjective is used in some literature to refer to groups which enjoy many properties of semi-simple Lie groups). They admit bi-invariant Lorentz metrics (which are of course different from the Killing form, which is degenerate for these groups). Also, they do have co-compact lattices (co-compact is superfluous, since any lattice in a solvable Lie group is co-compact). Let us anticipate here and say that an oscillator group has essentially one lattice (all of them are commensurable).

6.2.1. The simplest example. The semi-direct product $G = S^1 \ltimes Heis$.

Recall the definition of $Heis$, the Heisenberg group of dimension 3:

$$Heis = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R}\}.$$

$Heis$ is characterized essentially, by the existence of a non-split exact sequence: $1 \to \mathbb{R} \to Heis \to \mathbb{R}^2 \to 1$.

The circle S^1 acts automorphically on $Heis$, where the action is trivial on the center $\mathbb{R}$, and it is by rotation on $\mathbb{R}^2$.

The simplest oscillator group is the semi-direct product $G = S^1 \ltimes Heis$.

G can also be characterized as a non-trivial central extension of Ec, the group of Euclidean isometries of the plane: S^1: $1 \to \mathbb{R} \to G \to Ec \to 1$.

6.2.2. Generalization: "canonical" oscillator groups. Recall the construction of Heisenberg algebras $\mathcal{HE}_d$ (dim $= 2d+1$). Consider $\mathbb{R} \bigoplus \mathbb{C}^d$, with basis $Z, e_1, \ldots, e_d$ The only non-vanishing brackets are: $[e_k, ie_k] = Z$ (here $i = \sqrt{-1}$). Equivalently, $[X, Y] = \omega(X, Y)Z$, where ω is the symplectic form $\omega(X, Y) = \langle X, iY \rangle_0$, where, $\langle , \rangle_0$ is the Hermitian product.

- Canonical oscillator algebras are obtained by adding an exterior element t, such that: $[t, e_k] = ie_k, [t, ie_k] = -e_k$, and $[t, Z] = 0$. Denote by $\mathcal{HE}_d^t$ the resulting Lie algebra.
- Define on it a scalar product $\langle , \rangle$ as follows. Endow $\mathbb{C}^d$ with its Hermitian structure $\langle , \rangle_0$. Decree: $\mathbb{C}^d$ orthogonal to $Span\{t, Z\}$, $\langle t, t \rangle = \langle Z, Z \rangle = 0$ and $\langle t, Z \rangle = 1$.
- It turns out that $\langle , \rangle$ is a $Ad(\mathcal{HE}_d^t)$-invariant Lorentz scalar product. In other words, for every u in $\mathcal{HE}_d^t$, ad_u is antisymmetric with respect to $\langle , \rangle$. (Exercise: why this does not work for the Heisenberg algebras themselves?)
- Consider $\tilde{G} = \tilde{He_d^t}$ the simply connected Lie group generated by $\mathcal{HE}_d^t$.
- $\tilde{He_d^t}$ is a semi-direct product of $\mathbb{R}$ by He_d: the action of $\mathbb{R}$ on the center is trivial, and its action on $\mathbb{C}^d$ is via multiplication by $\exp is$.
- This is in fact an action of S^1. Consider then the semi-direct product $G = He_d^t = \tilde{He_d^t}/\mathbb{Z} = S^1 \ltimes He_d$ (here $\mathbb{Z}$ is simply the subgroup of integers of $\mathbb{R}$)
- Any lattice in the Heisenberg group He_d is also a lattice in He_d^t (since He_d is co-compact in He_d^t). As example of lattice in He_1, we have:

$$Heis_{\mathbb{Z}} = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{Z}\}.$$

6.2.3. General construction of oscillator groups. The most general oscillator groups are defined as above as semi-direct product $\mathbb{R} \ltimes He_d$, where $\mathbb{R}$ acts on $\mathbb{C}^d$ via a homomorphism $s \to \exp(2\pi s A) \in U(d)$ such that: $\exp(2\pi A) = 1$, and A diagonalizable with eigenvalues say, $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}$ having the same sign. This last condition on signs guarantees that the obtained Lie algebra admits a bi-invariant Lorentz metric. The arithmetic (or say quantum) condition, λ_i integers, implies that the $\mathbb{R}$-action factors via an action of S^1. An oscillator group is any such semi-direct product $G = S^1 \ltimes He_d$. It enjoys the same property as that of canonical oscillator groups, i.e., when the matrix A is scalar.

6.2.4. Further remarks.

1. Remember that the Lorentz scalar product was defined, among other conditions, by the fact that $\langle t, t \rangle = 0$. In fact, one can take $\langle t, t \rangle =$ Constant $\neq 0$, and multiply the other given products by any constant ($\neq 0$), and gets another bi-invariant Lorentz metric.
2. However, up to automorphism, there exists only one bi-invariant Lorentz metric on a oscillator algebra. In particular a metric is isometric to any multiple of itself. This follows from existence of homotheties. (This is true for $\mathbb{R}^n$ but not for $PSL(2, \mathbb{R})$.)

3. Oscillator groups are (locally) symmetric Lorentz spaces of non reductive type, that is they have non-reductive holonomy. This means they have a codimension 1 *parallel* foliation which has no supplementary parallel direction field. This foliation is nothing but determined by translates of the Heisenberg group (on the left or the right, it is the same thing since Heisenberg group is normal in the oscillator group).
4. The Ricci curvature of an oscillator group equals its Killing form (up to constant).

6.2.5. Historical comments.

1. Actually, it is the He_1^t, the 4-dimensional canonical oscillator example (the simplest example) which was named in the literature as the oscillator group [78], see justification below. It is also known as the diamond group in Representation Theory.
2. The bi-invariant Lorentz metrics were known to Medina-Revoy, and "partially" to Zimmer [89] and Gromov [50].
   This seems folkloric in relativistic literature: some gravitational plane waves spacetimes...
   Also, Witten and Nappi [69] used the oscillator group to built "a WZW model based on a non semi-simple group".

6.2.6. Justification of the name "oscillator". The Lie algebra $\mathcal{HE}_1^t$ has the following representation in the algebra of operators of the Hilbert space $E = L^2(\mathbb{R})$:

$$Z \to 1, X \to q, Y \to p, t \to p^2 + q^2$$

where the operators q and p are given by:

$$q(f) = xf, f \in L^2(\mathbb{R}), p(f) = \frac{\partial f}{\partial x}$$

(1 is the Identity (operator), q the position, p the momentum, and $p^2 + q^2$ the energy).

To show that this gives a homomorphism, one verifies in particular: $[q, p] = 1$, which is the Heisenberg uncertainty principle.

Finally, $p^2 + q^2$ is the energy of the harmonic oscillator, which explains the origin of the terminology, that is this representation gives a quantification of the harmonic oscillator.

6.3. Other examples: discrete isometry groups, general constructions

6.3.1. Discrete case. So far, our examples have the form $M = G/H$, where G is a Lie group group, which is non-compact, and (implicitly) connected. However, it might happen that G is not the full isometry group of M. However, this fact causes no loss for us. The true difficulty, is when a homogeneous Lorentz space has the form G/H, where G is *compact* (and connected), so at first glance, M looks like inessential, but it might happen that $\mathrm{Isom}(M)$ is not compact. For instance, G could be the identity component of the isometry group, which has "a discrete part" Isom$(M)/G$ non-compact. Let's give an example illustrating this phenomenon (which might be the general one?).

On $\mathbb{R}^n$, consider a lorentz scalar product g, and let $O(g)$ be its orthogonal group. The essential point is that we consider $\mathbb{R}^n$ together with its lattice $\mathbb{Z}^n$. Hence, $O(g)$ is isomorphic to $O(1, n-1)$, but in general, $O(g, \mathbb{Z}) = O(g) \cap GL(n, \mathbb{Z})$ is not isomorphic to $O(1, n-1; \mathbb{Z}) = O(1, n-1) \cap GL(n, \mathbb{Z})$.

Consider the flat torus $(T^n, g) = (\mathbb{R}^n, g)/\mathbb{Z}^n$. Then, Isom $(T^n, g) = T^n \rtimes O(g, \mathbb{Z})$. Therefore, the identity component is compact, and the "discrete part" is $O(g, \mathbb{Z})$.

Let us consider the simplest example, apparently firstly observed by Avez: let A be a hyperbolic element of $SL(2, \mathbb{Z})$ hyperbolic, that is, with real eigenvalues of norm $\neq 1$, e.g.,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $\{\omega^s, \omega^u\}$ be a diagonalisation basis for the dual of A: ω^s and ω^u are linear forms on $\mathbb{R}^2$ defined up to scalars. Finally, $g = c\omega^u\omega^s$, where c is an arbitrary constant ($\neq 0$). Then, A preserves g. In fact, up to a finite index, Isom$(T^2, g) = T^2 \rtimes \mathbb{Z}$ ($\mathbb{Z}$ is generated by A).

In dimension > 2, the discrete part $O(g, \mathbb{Z})$ may be much bigger. Indeed, by a Harisch-Chandra Borel Theorem [23], if g is rational (that is up to a multiplicative constant, the coefficients of g in the canonical basis, are all rational), then $O(g, \mathbb{Z})$ is a lattice in $O(g)$ (in particular $O(g, \mathbb{Z})$ is isomorphic to the fundamental group of a *finite volume* hyperbolic manifold, which can be compact).

One remarks here that a non co-compact lattice in $O(g)$, or say $O(1, n-1)$ since we are allowed to identify them at this stage, contains hyperbolic and parabolic elements (in the sense of $O(1, n-1)$). We then see, from a dynamical point of view, that Lorentz isometries may have, Anosov (in dimension 2), partially hyperbolic, and also horocycle-like behaviors. This contrasts with Riemannian isometries which are built up from "blocks", on which dynamics is equivalent to a translation on a torus.

6.3.2. Warped products. Let M be an essential Lorentz manifold. Observe that its (direct) product by a Riemannian manifold is also essential. If both are homogeneous, then the same is true for the product.

In fact, local products may also preserve essentiallity. To define them (in our Lorentz context), let $M = \tilde{N}$ be Lorentz and $\tilde{L}$ be Riemannian. Consider $M = \tilde{N} \times \tilde{L}/\Gamma$, where Γ is a subgroup of $Isom(\tilde{N}) \times Isom(\tilde{L})$. Assume Γ is *non split* (or in other terms irreducible), that is, M is not a product. If the centralizer of Γ acts non-properly on $\tilde{N} \times \tilde{L}$, then M is essential.

We recall now another way to preserve essentiallity, the warped product construction. Let (N, g) be Lorentz, and (L, h) Riemannian and $w : L \to R^+$ a (warping) function. The warped product $M = L \times_w N$, is the topological product $L \times N$, endowed with the metric $h \bigoplus wg$.

The crucial property for us here is that, if $f : N \to N$, is an isometry then, its trivial extension: $\bar{f} : (x, y) \in L \times N \to (x, f(y)) \in L \times N$, is an isometry of $L \times_w N$

In particular, as above, in the class of essential Lorentz manifolds , one can perform warped products by (any) Riemannian manifolds. Similarly to local products, one can define local warped products, which may preserve essentiallity.

6.3.3. "Counter-examples". Consider the "hyperbolic torus" T^3_A, where $A \in SL(2, \mathbb{Z})$ is hyperbolic (see §6.3.1). So, T^3_A is the suspension of A seen as a diffeomorphism (of Anosov type) of the 2-torus T^2. As was said in §6.3.1, A preserves a Lorentz metric g on T^2. Endow $T^2 \times \mathbb{R}$ with the flat product metric $g \bigoplus dt^2$. The mapping $\phi : (x, t) \to (Ax, t + 1)$ is isometric. Therefore, T^3_A inherits a flat Lorentz metric, with the suspension flow acting isometrically. It is an Anosov flow, and hence T^3_A is in particular essential. Nevertheless, T^3_A is not Lorentz homogeneous. Its isometry group is (up to a finite cover) generated by the suspension flow. On the other hand this is a sol-manifold: $T^3_A = SOL/\mathbb{Z} \times_A \mathbb{Z}^2$. Summarizing, T^3_A is an essential Lorentz manifold, topologically homogeneous (i.e., the isometry group admits a dense orbit), but not Lorentz homogeneous.

This "contrasting" fact is also valid for 3-Nil manifolds, i.e., compact quotients of the (3-dimensional) Heisenberg group (these manifolds are however not essential, since their isometry group are cyclic).

6.4. Classification of Killing algebras

We have the following result which answers, at least at the Lie algebra level our question 6.1. It was proved independently in its final form in [3] and [85] (at the same month!). Partial steps were done in [50] and [89].

Before announcing it, recall that the Lie algebra of a group acting isometrically on a compact Riemannian manifold is a sum of an abelian Lie algebra with a semi-simple Lie algebra of *compact* type (i.e., the Lie algebra of a compact semi-simple Lie group).

The result says that, in the Lorentz case, the new factor that might occur, is a subalgebra of $\mathcal{S}$, where $\mathcal{S}$ is the Lie algebra of $SL(2, \mathbb{R})$ or an oscillator group.

Theorem 6.4. *Let G be a* connected *Lie group acting isometrically on a compact Lorentz manifold M. Then, up to compact objects, G is a subgroup of $PSL(2, \mathbb{R})$ or of an oscillator group.*

More precisely, the Lie algebra $\mathcal{G}$ is isomorphic to a direct algebra sum:

$$\mathcal{K} + \mathbb{R}^k + \mathcal{S},$$

where $\mathcal{K}$ is the Lie algebra of a compact semi-simple Lie group, $k \geq 0$ is an integer and $\mathcal{S}$ is a subalgebra of:

- *$sl(2, \mathbb{R})$.*
- *an oscillator algebra.*

Furthermore, the group S associated to $\mathcal{S}$ acts on M locally freely, i.e., stabilizer in S are discrete

Corollary 6.5. *The stabilizer of any point of M is* "almost discrete"*: its connected component is compact.*

Remark 6.6. *The Corollary is far from being a priori obvious. It is false for non-compact homogeneous spacetimes, and for general homogeneous pseudo-Riemannian manifolds, even compact.*

6.4.1. The full Killing algebra. We dealt above with a group acting on M, i.e., a subgroup of $\mathrm{Isom}(M)$. One may ask about the structure of the full $\mathrm{Isom}(M)$ itself. In other words, can any group as described above (at the Lie algebra level) be *exactly* the (full) isometry group of some compact Lorentz manifold? For example, can the S-factor for the full $\mathrm{Isom}(M)$ be the affine group $\mathrm{Aff}(\mathbb{R})$, or in contrary, once $\mathrm{Aff}(\mathbb{R})$ acts isometrically, then does its action automatically extend (isometrically) to $SL(2,\mathbb{R})$ (always at a Lie algebra level)? The answer to this precise example is that extension indeed exists. In general, the answer was given in [86] and [4] independently (in the same season!)

By the Killing algebra of M we mean the Lie algebra of $\mathrm{Isom}(M)$.

Theorem 6.7. [86, 4] *The Killing Lie algebra of a compact Lorentz manifold is isomorphic to a direct sum*

$$\mathcal{K} + \mathbb{R}^k + \mathcal{S},$$

where $\mathcal{K}$ is the Lie algebra of a compact semi-simple Lie group, $k \geq 0$ is an integer and $\mathcal{S}$ is trivial or isomorphic to:

- *a Heisenberg algebra,*
- *an oscillator algebra, or*
- *$sl(2,\mathbb{R})$.*

Conversely, any such algebra is isomorphic to the Lie algebra of the isometry group of some compact Lorentz manifold.

6.5. Sub-question: Homogeneous case

6.5.1. Algebraic classification.

Theorem 6.8. [3, 4, 85, 86] *Let $M = G/H$ be a compact homogeneous Lorentz manifold. Then, up to compact objects: G is $SL(2,\mathbb{R})$ or an oscillator group.*

More precisely, there is a subgroup $S \subset G$, such that:

- *S is normal, and the Lie algebra of S is a factor in $\mathcal{G}$*
- *S is co-compact in G (i.e., G/S is compact)*
- *S is isomorphic to $PSL_k(2,\mathbb{R})$ the k-folded cover of $PSL(2,\mathbb{R})$, or*
- *S is an oscillator group*
- *S acts on M locally freely, that is H is "almost discrete", in the sense that its identity component is compact.*

6.5.2. Geometric classification.

Theorem 6.9. [85] *Let $M = G/H$ be a compact homogeneous Lorentz manifold. Then, up to compact objects, it is isometric to S/H, where H is a co-compact lattice (in particular discrete) in S, where S is $PSL(2,\mathbb{R})$ or an oscillator group.*

- *Roughly, M is a "local product" modeled on $S \times \tilde{L}$, where $\tilde{L}$ is a homogeneous Riemannian manifold*

• *The case $S = PSL_k(2,\mathbb{R})$ (due to Gromov* [50]*):*

- *$M = S \times \tilde{L}/H$:*
- *$\tilde{L}$ is a compact homogeneous Riemannian manifold*
- *There is H_0 a lattice in S, such that H is the graph of a homomorphism $\rho : H_0 \to Isom(\tilde{L})$*
- *The centralizer of $\rho(H_0)$ acts transitively on $\tilde{L}$.*
- *The metric on $S \times \tilde{L}$ equals: $c.Killing \bigotimes r_{\tilde{L}}$, for some constant c, where $r_{\tilde{L}}$ denotes the Riemannian metric of $\tilde{L}$.*
- *Conversely, with these data, one constructs a compact homogeneous space-time.*

• *In the case where S is an oscillator group, the geometric description is a little bit complicated: the (local) product structure can be somehow "twisted" (see* [85]*).*

6.6. Super-question

Remember that the super-question concerns the action of Diff(M) on $Lor(M)$, the space of Lorentz metrics on M. Consider two converging sequences, g_n and h_n of Lorentz metrics (in the C^2 topology). Suppose they are isometric, that is, there is a sequence of diffeomorphisms ϕ_n, such that $h_n = (\phi_n)_* g_n$. Properness, means that, after passing to a subsequence, the sequence ϕ_n must converge (the limit will be an isometry between the two limit metrics).

6.6.1. Main ingredient: actions of discrete groups. So far, only actions of connected Lie groups were considered. However, as was seen in the case of flat tori 6.3.1, the essentiality may come from the discrete part. It seems that the only works which deal with Lorentz isometries, without, a priori, connectedness hypothesis, are firstly that of D'Ambra [34] (where actually connectedness is proved at an intermediate step), and [87, 88], which investigate dynamics of Lorentz isometries in a systematic way. In fact, also sequences of isometries were considered there (seen as generalized dynamical systems). It was also observed that the approach can be adapted to sequences of isometries between two different Lorentz metrics, or even sequences of Lorentz metrics, which is exactly the situation you meet, as above, when dealing with properness. The philosophy of this work is to see how such sequences of isometries degenerate. One consider their graphs in $M \times M$, which are totally geodesic (and isotropic) for the product metric. The limits are geodesic laminations in a suitable space. It turns out however, that, by projecting on M, one gets a codimension one foliation, with geodesic and lightlike leaves (the metric on them is degenerate). At least in dimension 3, one knows many obstructions to the existence of such foliations. As a corollary, we get:

Theorem 6.10. [88] *For M the 3-sphere, the Diff(M)-action on $Lor(M)$ is proper.*

6.6.2. Case of compact surfaces. P. Mounoud pushed forward the analogy between the case of isometries of a fixed metric, and that between sequences of metrics (as in the definition of properness). He applied this for Lorentz compact surfaces which must be (topologically) a Klein Bottle or a torus. New ideas are needed here since lightlike (in fact isotropic) foliations always exist. He firstly proved:

Theorem 6.11. [68] *For $M =$ Klein bottle, the Diff(M)-action on $Lor(M)$ is proper.*

Let M be now a 2-torus, and $\mathcal{F}$ the space of flat Lorentz metrics on it. Any such a metric is linear on the universal cover $\mathbb{R}^2$ (up to a diffeomorphism). One observed that the $\mathrm{Diff}_0(M)$-action on $\mathcal{F}$ is proper, where $\mathrm{Diff}_0(M)$ is the group of diffeomorphisms isotopic to the identity. The quotient is the Lorentz Teichmüller space of the 2-torus. It is identified to the de Sitter space:

$$dS_2 = SL(2,\mathbb{R})/\{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, t \in \mathbb{R}\}.$$

The action of $\mathrm{Diff}(M)$ on $\mathcal{F}/Diff_0(M)$ is identified to the action of $SL(2,\mathbb{Z})$ on dS_2. This action is dual to the action of $\{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, t \in \mathbb{R}\}$ (the geodesic flow) on the unit tangent bundle $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ of the modular surface $\mathbb{H}^2/SL(2,\mathbb{Z})$. Therefore the $\mathrm{Diff}(M)$-action on $\mathcal{F}$ is in particular ergodic.

The main result of [68] is that in contrast:

Theorem 6.12. *The Diff(M)-action on $Lor(M) - \mathcal{F}$ (the space of non-flat metrics) is proper.*

Therefore, the $\mathrm{Diff}(M)$ dynamics on $Lor(M)$ is fuchsian-like: strong (ergodic...) on its "limit set " $\mathcal{F}$, and proper on its "discontinuity domain" $Lor(M) - \mathcal{F}$.

Among other beautiful ideas, the proof uses an amazing lemma, which states that, if a Lorentz metric on the torus has its curvature constant along one isotropic foliation, then this metric is flat.

6.7. Non-compact manifolds

We consider here our question (or its variants, sub, or super) in the case of non-compact Lorentz manifolds. Despite its natural both mathematical and physical interest (only non-compact spacetimes are realistic in physics), the question here is far from being elucidated, and sufficiently investigated. Maybe, the reason is that no prompt answer seems to be available.

From a relativistic point of view, one observes that only few classical exact solutions have essential isometry groups. One may dare ask:

Question 6.13. *(for relativists) Classify physical solution (i.e., an exact solution, a spacetime with a natural energy-momentum tensor, a spacetime satisfying suitable causality conditions) having an essential isometry group.*

As example, spaces of constant curvature, dS_n, Min_n and AdS_n have essential groups. They are homogeneous, with non-compact isotropy.

As celebrated exact solutions, *pp*-waves have essential isometry groups. Our question above asks for a rigidity of essential "physical" spacetimes, that is they must belong to a (small) list to be founded and enumerated.

From a purely mathematical point of view, it seems that Nadine Kowalsky was the first to consider this problem, in her thesis (supervised by Zimmer). As the general question looks too waste to be systematically investigated, she made an algebraic hypothesis on the acting group. It was with a little bit variation of this same hypothesis that other authors contribute. So, N. Kowalsky asked, and solved the following:

Question 6.14. *When a* simple *Lie group acts isometrically non-properly on a Lorentz manifold?*

Observe that the de Sitter and de Sitter spaces, are examples of such spaces. Their isometry groups are $O(1, n)$ and $O(2, n)$ (they are simple except for $O(2, 2)$). The answer of Kowalsky is that these are the only examples, at an algebraic level.

Theorem 6.15. [57] *If a simple Lie group acts isometrically non-properly on a Lorentz manifold, then it is isomorphic to $O(1, n)$ or $O(2, n)$ (for some n).*

This was the principal result of [57], and was also announced in [58], together with another announcements of results. Unfortunately, Kowalsky died prematurely, before publishing half of the announced results. One of the announced results is on a geometric description of the Lorentz manifolds as in the theorem above. Here again dS, and AdS appear (essentially) as unique examples.

Theorem 6.16. *If a simple Lie group G acts isometrically non-properly on a Lorentz manifold M, then M is a warped product of dS_n or AdS_n with some Riemannian metric (here we must assume G not locally isomorphic to $SL(2, \mathbb{R})$ in order to avoid consideration of local warped products...).*

D. Witte [81] proved this result, assuming Theorem 6.15 and that the action is transitive. Let us observe that, even with these assumptions, the result is by no means obvious! In [13], the authors introduce a new geometric approach allowing a unified proof of both the previous two theorems. They also consider some generalizations. Let us notice here that S. Adams was the first and principal "investigator" on Kowalsky's heritage. In particular, he relaxed in many ways the algebraic condition (simplicity) on the Lie group (the conclusions are different). He also yields another proof of Theorem 6.15 (see for instance [2]). Notice however that all approaches, except in [13], are deeply algebraic.

6.8. Idea of proof of Theorem 6.4

Consider the L^2 bilinear form on the Lie algebra $\mathcal{G}$:

$$\kappa(X,Y) = \int_M \langle X(x), Y(x)\rangle_x dx$$

where X, Y are Killing fields and $\langle , \rangle$ is the Lorentz metric.

6.8.1. Steps.

1) κ is a *bi-invariant* quadratic form on $\mathcal{G}$. This is a general fact: an action of $\mathcal{G}$ means, a homomorphism (of Lie brackets) $X \in \mathcal{G} \to \bar{X} \in$ Vector-fields on M.
   So, to $Y \in \mathcal{G}$, is associated:
   - ϕ^t a one-parameter subgroup of G, and
   - $\overline{\phi}^t$ a one parameter group of diffeomorphisms on M.

   Naturally:
   $$\overline{\phi}^t_* \overline{X} = \overline{Ad(\phi^t)X}$$
   It then follows, if G preserves a volume dx and a q-covariant tensor T, then, the formula:
   $$\kappa^T(X_1, \ldots, X_q) = \int_M T(\overline{X_1}(x), \ldots, \overline{X_q}(x))dx$$
   determines a *bi-invariant* q-tensor on $\mathcal{G}$.
2) However, κ might be trivial! For instance, if $G = SO(n), n > 2$, let T be any left invariant quadratic form (degenerate or not, positive or not...). Then necessarily, κ^T is a multiple of the Killing form, by simplicity. In particular, it may happen that $\kappa^T = 0$ for T Lorentz.
3) The point is thus to show that, in our situation, κ is sufficiently non-trivial.... A major step in the proof will be to show that κ satisfies a condition (*), which roughly speaking means that κ is between being a Lorentz and a Euclidean scalar product!
4) Theorems 6.4 follows from an "Algebraic Lemma" classifying" those Lie algebras admitting *Ad*-invariant scalar product satisfying (*). This classification is similar (but of higher-order difficulty) to the lemma saying that a Lie algebra with an *Ad*-invariant positive scalar product is a sum of an abelian algebra and compact one.

6.8.2. Condition (*). Behind the condition (*) is the following:

Lemma 6.17. *(Fundamental non-degeneracy Fact) Let M be a compact Lorentz manifold, $\phi^t \subset Isom(M)$ a one parameter group with infinitesimal generator (a Killing field) X. Suppose ϕ^t is non-precompact (i.e., non-equicontinuous, or equivalently the closure of $\{\phi^t, t \in \mathbb{R}\}$ in $Isom(M)$ is not compact). Then, X is everywhere non-timelike: $\langle X(x), X(x)\rangle \geq 0$, $\forall x$.*

Corollary 6.18. *(Condition (*)): Let $\mathcal{P}$ a linear subspace of $\mathcal{G}$ containing a dense set of non-precompact Killing fields. Then, $\kappa|\mathcal{P} \geq 0$, and dim $Ker(\kappa|\mathcal{P}) \leq 1$.*

6.8.3. Proof of Lemma 6.17. The proof of the Fundamental non-degeneracy Fact is based on two uniformity facts:

Fact 6.19. *Let $\{\phi^t\} \subset Isom(M)$ be a one parameter group of isometries. If for some $t_i \to \infty$, $\{\phi^{t_i}\}$ is precompact (i.e., equicontinuous), then $\{\phi^t\}$ is precompact.*

Sketch. Let $L = \overline{\{\phi^t\}} \subset \mathrm{Isom}(M)$. Then, L is an abelian Lie group, and hence it is a cylinder $T^k \times \mathbb{R}^d$ (where T^k is a torus).

But, L has a dense one parameter group ($\{\phi^t\}$ itself), i.e., a dense geodesic (when L is seen as a flat Euclidean cylinder). It then follows that $L = T^k$, or $L = \mathbb{R}$. Now, if there $\exists$ $\{\phi^{t_i}\}$ equicontinuous, then $L \neq \mathbb{R}$, and hence $L = T^k$, i.e., $\{\phi^t\}$ is equicontinuous. □

Fact 6.20. *If for some $x_i \in M, t_i \to \infty$, $\{D_{x_i}\phi^t\}$ is equicontinuous (i.e., $\| D_{x_i}\phi^t \|$ and $\| (D_{x_i}\phi^t)^{-1} \|$ bounded), then $\{\phi^{t_i}\}$ is equicontinuous (and therefore by the fact above $\{\phi^t\}$ is equicontinuous)*

Sketch. By definition of its Lie group structure, $\mathrm{Isom}(M)$ acts properly (and freely) on the frame bundle $P(M)$. □

These two facts are true in affine dynamics, i.e., for $\{\phi^t\}$ preserving any linear connection.

We need a third fact special to the Lorentz case:

Fact 6.21. *If a Killing field X is somewhere timelike (i.e., $\langle X(x_0), X(x_0)\rangle < 0$), then it generates an equicontinuous flow $\{\phi^t\}$.*

Sketch. Let U be a neighborhood of x_0 where X is timelike. By Poincaré recurrence Lemma, there exist x_i near x_0, $t_i \to \infty$, such that $\phi^{t_i}x_i$ is near x_0.

Now, near x_0, $D_{x_i}\phi^{t_i}$ behave as Riemannian isometries, and are then equicontinuous. Apply the second fact, and then the first one to deduce that $\{\phi^t\}$ is equicontinuous. □

7. Conformal actions

We will be very succinct at this §. We essentially take the opportunity to mention recent works on the domain, and quote references for detailed and complete exposition.

As for Riemannian geometry, one defines, conformal Lorentz manifolds, conformal actions, conformally flat structures.... The conformal group is essential when it cannot be reduced to the isometry group of some Lorentz metric in the conformal class. The "vague" conjecture is that it is possible, and anyway interesting, to classify essential conformal Lorentz manifolds. To be precise one may ask a kind of Lorentz conformal Lichnérowicz conjecture (see [35]). Recall for this that the universal substratum of conformal Lorentz geometry, is the (static) Einstein cosmos Ein_n. Its conformal structure is obtained (up to a 2 folded cover) as follows. Consider on $\mathbb{R}^{2+n}$ a quadratic form of type $- - +, \dots +$, and let $C^{2,n}$

be its isotropic cone. Then, Ein_n is the projectivization of $C^{2,n}$, endowed with its natural conformal Lorentz structure. Its conformal group is $O(2,n)$, which acts essentially.

C. Frances [41] exhibited a huge class of conformally flat manifolds, which are essential. As amazing fact, the Einstein cosmos itself as a topological manifold, which is $S^1 \times S^n$, has a big Teichmüller space of conformally flat (Lorentz) structures, some of which are essential.

Unrelated to essentiallity, C. Frances [42] also studied closed conformally flat Lorentz manifolds, for themselves. Their holonomy groups are in some sense the Lorentz parallel of Kleinian groups, i.e., discrete groups of the Möbius group. As the usual (Riemannian) sphere is the boundary at infinity of the hyperbolic space, the Einstein cosmos is the conformal boundary. Therefore, we find ourselves here in the heart of an "equivariant" AdS/CFT correspondence.

Coming back to essentiallity, the geometrical ingredient of the Lorentzian Lichnérowicz's conjecture, still stands up. That is, it seems that a compact essential Lorentz manifold is conformally flat. The non-compact case is false (see for instance [5]). This contrasts with the Riemannian non-compact case, since (even if Lichnérowicz did not dare to ask it) the Euclidean space is the unique essential Riemannian manifold.

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References

[1] H. Abels, *Properly discontinuous groups of affine transformations, A survey,* Geometriae Dedicata **87** (2001), 309–333.

[2] S. Adams, *Dynamics on Lorentz manifolds,* World Scientific Publishing Co., Inc., River Edge, NJ.

[3] S. Adams, G. Stuck, *The isometry group of a compact Lorentz manifold, I,* Invent. Math. **129** (1997), 239–261.

[4] S. Adams, G. Stuck, *The isometry group of a compact Lorentz manifold, II,* Invent. Math. **129** (1997), 263–287.

[5] D. Alekseevski, *Self-similar Lorentzian manifolds,* Ann. Global Anal. Geom. **3** (1985), no. 1, 59–84.

[6] S. Aminneborg, I. Bengtsson, S. Holst, *A Spinning Anti-de Sitter Wormhole,* Class. Quant. Grav. **16** (1999) 363–382, gr-qc/9805028.

[7] S. Aminneborg, I. Bengtsson, D. Brill, S. Holst, P. Peldan, *Black Holes and Wormholes in 2+1 Dimensions,* Class. Quant. Grav. **15** (1998) 627–644, gr-qc/9707036.

[8] S. Aminneborg, I. Bengtsson, S. Holst, P. Peldan, *Making Anti-de Sitter Black Holes,* Class. Quant. Grav. **13** (1996), 2707–2714, gr-qc/9604005.

[9] L. Andersson, *Constant mean curvature foliations of flat space-times,* Comm. Anal. Geom. **10** (2002), no. 5, 1125–1150.

[10] L. Andersson, *Constant mean curvature foliations of simplicial flat space-times,* math.DG/0307338.

[11] L. Andersson, G.J. Galloway, R. Howard *The cosmological time function,* Classical Quantum Gravity **15** (1998), 309–322.

[12] L. Andersson, V. Moncrief, A. Tromba *On the global evolution problem in* $2+1$ *gravity,* J. Geom. Phys. **23** (1997), no. 3–4, 191–205.

[13] A. Arouche, M. Deffaf, Y. Raffed, *A geometric approach of groups actions on Lorentz non-compact manifolds,* To appear.

[14] U. Bader, A. Nevo, *Conformal actions of simple Lie groups on compact pseudo-Riemannian manifolds,* J. Differential Geom. **60** (2002), no. 3, 355–387.

[15] M. Banados, M. Henneaux, C. Teitelboim, J. Zanelli, *Geometry of the* $2+1$ *Black Hole,* Phys. Rev. D **48** (1993) 1506–1525, gr-qc/9302012.

[16] M. Banados, C. Teitelboim, J. Zanelli, *The Black Hole in Three-Dimensional Space Time,* Phys. Rev. Lett. **69** (1992) 1849–1851, hep-th/9204099.

[17] T. Barbot, *Flat globally hyperbolic spacetimes,* preprint, math.GT/0402257.

[18] T. Barbot, *Limit sets of discrete Lorentzian groups,* in preparation.

[19] T. Barbot, F. Béguin, A. Zeghib, *Feuilletages des espaces temps globalement hyperboliques par des hypersurfaces à courbure moyenne constante,* C.R. Acad. Sci. Paris, Ser. I **336** (3) (2003), 245–250.

[20] T. Barbot, F. Béguin, A. Zeghib, *CMC foliations on globally hyperbolic spacetimes,* in preparation.

[21] R. Benedetti, E. Guadagnini, *Cosmological time in* $(2+1)$*-gravity,* Nuclear Phys. B **613** (2001), no. 1–2, 330–352.

[22] F. Bonsante, *Flat Spacetimes with Compact Hyperbolic Cauchy Surfaces,* math.DG/0311019.

[23] A. Borel, Harish-Chandra, *Arithmetic subgroups of algebraic groups,* Ann. of Math. (2) **75** (1962), 485–535.

[24] D. Brill, *Multi-Black-Hole Geometries in (2+1)-Dimensional Gravity,* Phys.Rev. D **53** (1996) 4133–4176, gr-qc/9511022.

[25] E. Calabi., L. Markus, *Relativistic space forms,* Ann. Math. **75** (1962), 63–76

[26] S. Carlip, *Quantum gravity in* $2+1$ *dimensions,* Cambridge Monographs on Math. Phys. (1998), Cambridge University Press.

[27] Y. Carrière, *Autour de la conjecture de L. Markus sur les variétés affines,* Invent. Math. **95** (1989), 615–628

[28] Y. Carrière, F. Dal'bo, *Généralisations du* 1^{er} *Théorème de Bieberbach sur les groupes cristalographiques,* Enseignement Math. **35** (1989), 245–262

[29] Y. Carrière, L. Rozoy, *Complétude des métriques lorentziennes de* T^2 *et difféomorphismes du cercle,* Bol. Soc. Brasil. Mat. (N.S.) **25** no. 2 (1994), 223–235.

[30] V. Charette, T.A. Drumm, *Strong marked isospectrality of affine Lorentzian groups,* math.DG/0310464.

[31] V. Charette, T.A. Drumm, D. Brill, *Closed time-like curves in flat Lorentz spacetimes.* J. Geom. Phys. **46** (2003), no. 3–4, 394–408.

[32] V. Charette, T. Drumm, W. Goldman, M. Morrill, *Complete flat affine and Lorentzian manifolds,* Geometriae Dedicata **97** (2003), 187–198.

[33] Y. Choquet-Bruhat, R. Geroch, *Global aspects of the Cauchy problem in general relativity,* Comm. Math. Phys. **14**, 1969 329–335.

[34] G. D'Ambra, *Isometry groups of Lorentz manifolds,* Invent. Math. **92** (1988), 555–565.

[35] G. D'Ambra and M. Gromov, *Lectures on transformation groups: geometry and dynamics, Surveys in Differential Geometry,* (Supplement to the Journal of Differential Geometry), 1 (1991) 19–111.

[36] T.A. Drumm, *Fundamental polyhedra for Margulis space-times,* Topology **31** (4) (1992), 677–683.

[37] T.A. Drumm, *Linear holonomy of Margulis space-times,* J. Differential Geom. **38** (3) (1993), 679–690.

[38] T.A. Drumm, W. Goldman, *Isospectrality of flat Lorentz* 3*-manifolds,* J. Diff. Geom. **58** (2001), 457–465.

[39] G. Ellis and S. Hawking Hawking, *The large scale structure of space-time,* Cambridge Monographs on Mathematical Physics, No. 1. Cambridge University Press, London-New York, 1973.

[40] J. Ferrand, *The action of conformal transformations on a Riemannian manifold,* Math. Ann. **304** (1996) 277–291.

[41] C. Frances, Thesis, ENS-Lyon 2002,
www.umpa.ens-lyon.fr/∼cfrances/these2-frances.pdf

[42] C. Frances, *Sur les groupes kleiniens lorentziens,*
www.umpa.ens-lyon.fr/∼cfrances/kleinlorentz.pdf

[43] D. Fried, W. Goldman, *Three-dimensional affine crystallographic groups,* Adv. Math. **47** (1983), 1–49.

[44] D. Fried, W. Goldman, M. Hirsch, *Affine manifolds with nilpotent holonomy,* Comment. Math. Helv. **56** (1981), 487–523.

[45] D. Gallo, M. Kapovitch, A. Marden, *The monodromy groups of Schwarzian equations on closed Riemann surfaces,* Ann. Math. **151** (2000), 625–704.

[46] C. Gerhardt, *H-surfaces in Lorentzian manifolds,* Comm. Math. Phys. **89** (1983), no. 4, 523–553.

[47] E. Ghys, *Flots d'Anosov dont les feuilletages stables et instables sont différentiables,* Ann. Sc. Ec. Norm. Sup., **20** (1987) 251–270.

[48] W. Goldman, Y. Kamishima, *The fundamental group of a compact Lorentz space is virtually polycyclic,* J. Differential. Geom. **19** (1984), 233–240.

[49] W. Goldman, *Nonstandard Lorentz space forms,* J. Differential. Geom. **21** (1985), 301–308.

[50] M. Gromov, *Rigid transformation groups,* "Géométrie différentielle ", D. Bernard et Choquet-Bruhat. Ed. Travaux en cours 33. Paris. Hermann (1988).

[51] F. Grunewald, G. Margulis, *Transitive and quasi-transitive actions of affine groups preserving a generalized Lorentz structure,* J. Geom. Phys. **5** (1988), 493–530.

[52] M. Guediri, *Compact flat spacetimes,* J. Diff. Geom. Appli., to appear. available at: www.mpim-bonn.mpg.de/html/preprints/preprints.html.

[53] M. Guediri, *On the geodesic connectedness of simply connected Lorentz surfaces,* Ann. Fac. Sci. Toulouse Math. (6) 6 (1997), no. 3, 499–510.

[54] M. Guediri, J. Lafontaine, *Sur la complétude des variétés pseudo-riemanniennes,* J. Geom. Phys. **15** (1995), no. 2, 150–158.

[55] D. Johnson, J. Millson, *Deformation spaces associated to compact hyperbolic manifolds,* in "Discrete Groups in geometry and analysis" (New Haven), 48–106, Progr. Math. **67** (1987).

[56] B. Klingler, *Complétude des variétés lorentziennes à courbure constante,* Math. Ann. **306** (1996), 353–370.

[57] N. Kowalsky, *Noncompact simple automorphism groups of Lorentz manifolds,* Ann. Math. **144** (1997), 611–640.

[58] N. Kowalsky, *Actions of non-compact simple groups of Lorentz manifolds,*C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), no. 5, 595–599.

[59] R. Kulkarni, *Proper actions and pseudo-Riemannian space forms,* Adv. Math. **40** (1981), 10–51.

[60] R. Kulkarni, F. Raymond, 3-*dimensional Lorentz space-forms and Seifert fiber spaces,* J. Diff. Geom. **21** (1985), 231–268.

[61] F. Labourie, *Problème de Minkowski, et surfaces à courbure constante dans les variétés hyperboliques,* Bull. Soc. Math. Fr. **119** (1991), 307–325.

[62] G. Margulis, *Free properly discontinuous groups of affine transformations,* Dokl. Akad. Nauk. SSSR **272** (1983), 937–940.

[63] J. Marsden, *On completeness of homogeneous pseudo-Riemannian manifolds,* Ind. Univ. Math. J, Vol **22** (1973) 1065–1066.

[64] A. Medina, Ph. Revoy, *Les groupes oscillateurs et leurs réseaux,* Manuscripta. Math. **52** (1985), 81–95.

[65] G. Mess, *Lorentz spacetimes of constant curvature,* preprint IHES/M/90/28 (1990).

[66] J. Milnor, *On fundamental groups of complete affinely flat manifolds,* Adv. Math. **25** (1977), 178–187.

[67] M. Morrill, UCLA thesis (1996).

[68] P. Mounoud, *Dynamical properties of the space of Lorentzian metrics,* Comment. Math. Helv. **78** (2003), no. 3, 463–485.

[69] C. Nappi, E. Witten, *Wess-Zumino-Witten model based on a nonsemisimple group* Phys. Rev. Lett. **71** (1993), no. 23, 3751–3753.

[70] M. Obata, Morio *The conjectures on conformal transformations of Riemannian manifolds* J. Differential Geometry **6** (1971/72), 247–258.

[71] A. Pratoussevitch, *Fundamental domains in Lorentzian geometry,* available at: www.math.uni-bonn.de/people/anna/publications.html.en.

[72] A.D. Rendall, *Constant mean curvature foliations in cosmological spacetimes,* Helv. Phys. Acta **69** (1996), no. 4, 490–500, gr-qc/9606049.

[73] F. Salein, *Variétés anti-de Sitter de dimension* 3 *exotiques,* Ann. Inst. Fourier **50** (2000), no. 1, 257–284.

[74] K. Scannell, 3-*manifolds which are spacelike slices of flat spacetimes,* Classical Quantum Gravity **18** (2001), no. 9, 1691–1701.

[75] K. Scannell, *Flat conformal structures and the classification of de Sitter manifolds,* Comm. Anal. Geom. **7** (1999), no. 2, 325–345.

[76] B. Schmidt, *The local b-completeness of space-times,* Comm. Math. Phys. **29** (1973), 49–54.

[77] J.M. Schenker, *Convex polyhedra in Lorentzian space-forms,* Asian J. Math. **5** (2001), no. 2, 327–363.

[78] R. F. Streater, *The representations of the oscillator group,* Commun. Math. Phys. **4** (1967), 217–236.

[79] W. Thurston, *Three-dimensional geometry and topology,* Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ.

[80] W. Thurston, in: D.B.A. Epstein (Ed.), London Math. Soc. Lecture Notes, vol 111, Cambridge University Press, 1987.

[81] D. Witte, *Homogeneous Lorentz manifolds with simple isometry group,* Beiträge Algebra Geom. **42**, no. 2 (2001) 451–461.

[82] E. Witten, $2+1$-*dimensional gravity as an exactly soluble system,* Nucl. Phys. B **311** (1988), 46–78.

[83] J. Wolf, *Spaces of constant curvature,* New York, McGraw-Hill, 1967

[84] A. Zeghib, *On closed anti de Sitter spacetimes,* Math. Ann., **310** (1998) 695–716.

[85] A. Zeghib, *Sur les espaces-temps homogènes,* Geometry and Topology Monographs **1** (1998) 531–556: http://www.maths.warwick.ac.uk/gt/GTMon1/paper26.abs.html

[86] A. Zeghib, *The identity component of the isometry group of a compact Lorentz manifold,* Duke Math. J., **92** (1998) 321–333.

[87] A. Zeghib, *Isometry groups and geodesic foliations of Lorentz manifolds. Part I: Foundations of Lorentz dynamics.* GAFA, **9** (1999) 775–822.

[88] A. Zeghib, *Isometry groups and geodesic foliations of Lorentz manifolds. Part II: Geometry of analytic Lorentz manifolds with large isometry groups.* GAFA, **9** (1999) 823–854.

[89] R. Zimmer, *On the automorphism group of a compact Lorentz manifold and other geometric manifolds,* Invent. Math. **83** (1986) 411–426.

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