Transformation Groups
https://doi.org/10.1007/s00031-025-09934-y

Transformation Groups

ORIGINAL PAPER

On Homogeneous Holomorphic Conformal Structures

Mohamed Deffaf[1] · Mehdi Belraouti[1] · Yazid Raffed[1] · Abdelghani Zeghib[2]

Received: 22 May 2024 / Accepted: 25 August 2025
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2025

Abstract
We study compact complex manifolds M admitting a conformal holomorphic Riemannian structure, invariant under the action of a complex semisimple Lie group G. We classify all such manifolds on which G acts transitively and essentially. In particular, we show that they are conformally flat.

1 Introduction

Throughout this paper, M will denote a compact connected complex manifold of dimension n. A holomorphic Riemannian metric g on M is a holomorphic field of non-degenerate complex quadratic forms on TM. Locally, it can be written as $\sum g_{ij}(z)dz_i dz_j$, where $(g_{ij}(z))$ is an invertible symmetric complex matrix depending holomorphically on z. It is the complex analogue of a pseudo-Riemannian metric. Unlike the real case, there are only a few compact complex manifolds admitting a holomorphic Riemannian metric. A first natural example is given by the flat standard model $\sum dz_i^2$ on $\mathbb{C}^n$. Since this metric is invariant under translations, any complex

✉ Mehdi Belraouti
mbelraouti@usthb.dz

Mohamed Deffaf
mdeffaf@usthb.dz

Yazid Raffed
yazidsaid.raffed@usthb.edu.dz

Abdelghani Zeghib
abdelghani.zeghib@ens-lyon.fr

1 Faculté de Mathématiques, USTHB, BP 32, El-Alia, 16111, Bab-EzzouarAlger, Algeria

2 UMPA, ENS de Lyon, Lyon, France

Published online: 13 September 2025

torus admits a holomorphic Riemannian metric. Actually, up to finite cover, complex tori are the only compact K hler manifolds admitting such a structure (see [13]).

Consider a cover $\{U_i\}$ of M, along with holomorphic Riemannian metrics g_i on each U_i, such that $g_i = f_{ij} g_j$ on $U_i \cap U_j$, for some holomorphic maps $f_{ij} : U_i \cap U_j \longrightarrow \mathbb{C}$. Two such covers $(\{U_i\}, g_i)_i$ and $(\{V_j\}, h_j)_j$ on M are said to be conformally equivalent if, for every i, j, there is a holomorphic map $\phi_{ij} : U_i \cap V_j \longrightarrow \mathbb{C}$ such that, $g_i = \phi_{ij} h_j$ on $U_i \cap V_j$. A conformal holomorphic structure on M is then a conformal class of a cover $(\{U_i\}, g_i)_i$. It is said to be conformally flat if it is locally conformally diffeomorphic to $\mathbb{C}^n$. Contrary to the real case, conformal holomorphic Riemannian structures do not necessarily derive from holomorphic Riemannian ones. For instance, the complex projective space $\mathbb{CP}^1$ admits a conformal holomorphic Riemannian structure, but has no holomorphic Riemannian metric. Another example is provided by the Einstein complex space $\mathrm{Eins}_n(\mathbb{C})$ (see Example 1.1.1 below). Indeed, the Fubini-Study metric induces a K hler metric on $\mathrm{Eins}_n(\mathbb{C})$ (see [17, Example 10.6]). Thus, by [13] (see also [5, 7]), it does not admit a holomorphic Riemannian metric.

Let G be a Lie group acting on M and preserving some conformal holomorphic Riemannian structure.

Definition 1.1 The action is said to be essential if G does not preserve any holomorphic Riemannian metric on M.

This paper aims to classify pairs (M, G), where G is a complex semisimple Lie group acting essentially and transitively on M. Before going any further, let us start by giving some examples of such pairs.

1.1 Constructions

1.1.1 The Complex Einstein Universe $\mathrm{Eins}_n(\mathbb{C})$

On $\mathbb{C}^{n+2}$, consider the standard holomorphic Riemannian metric $q = dz_0^2 + ... + dz_{n+1}^2$ and let $\mathrm{Co}_{n+1}(\mathbb{C}) = \{z \in \mathbb{C}^{n+2} : q(z,z) = 0\}$ be its light-cone. The complex quadric $\mathrm{Q}_n(\mathbb{C}) = (\mathrm{Co}_{n+1} - \{0\})/\mathbb{C}^* \subset \mathbb{C}P^{n+1}$ is the projectivization of the light-cone [17, Example 10.6]. The geometry of complex quadrics was amply studied in the literature in [11, 12, 14, 15, 19].

The induced metric on Co_{n+1} is degenerate, with kernel the tangent space of $\mathbb{C}^*$ -orbits. It follows that the metric becomes non-degenerate on $\mathrm{Q}_n(\mathbb{C})$, but it is defined up to a constant. Therefore, a holomorphic conformal structure is well defined on $\mathrm{Q}_n(\mathbb{C})$. The group $\mathsf{PSO}(n+2, \mathbb{C})$, which acts transitively on $\mathrm{Q}_n(\mathbb{C})$, preserves naturally this holomorphic conformal structure. In fact, it is the unique holomorphic conformal structure on $\mathrm{Q}_n(\mathbb{C})$ preserved by $\mathsf{SO}(n+2, \mathbb{C})$. Moreover, the action of $\mathsf{PSO}(n+2, \mathbb{C})$ is essential. It is called the complex Einstein universe, and denoted $\mathrm{Eins}_n(\mathbb{C})$. A conformally flat holomorphic conformal structure is then equivalent to giving a $(\mathsf{PSO}(n+2, \mathbb{C}), \mathrm{Eins}_n(\mathbb{C}))$-structure.

The stabilizer (of some point) is a parabolic group P_1. In fact, $\mathsf{PSO}(n+2, \mathbb{C})$ acts transitively on Gr_k^0, the space of isotropic k-planes. This requires $k \leq$ the integer part of $n/2+1$. Let P_k be the stabilizer of this action. The parabolic groups P_k are exactly the maximal parabolic subgroups of $\mathsf{PSO}(n+2,\mathbb{C})$ (maximal meaning that only one root space corresponding to a simple root is not contained in such a subgroup). In our investigation in Section 6.2.2, we will, in particular, see that only $Gr_1^0 = \mathrm{Q}_n(\mathbb{C})$ admits a $\mathsf{PSO}(n+2,\mathbb{C})$-invariant holomorphic conformal structure.

1.1.2 The $\mathsf{Sp}(2n,\mathbb{C})$-case

The symplectic group $\mathsf{Sp}(2n,\mathbb{C})$ preserves the (complex) symplectic form $\omega((x_1,\ldots,x_{2n}),(y_1,\ldots,y_{2n})) = \sum_{i=1}^n x_i y_{n+i} - \sum_{i=1}^n y_i x_{n+i}$. So its diagonal action on $\mathbb{C}^{2n}\times\mathbb{C}^{2n}$ preserves the quadratic form on $\mathbb{C}^{4n}$:

$$q((x_1,\ldots,x_{2n}),(y_1,\ldots,y_{2n})) = \sum_{i=1}^n x_i y_{n+i} - \sum_{i=1}^n y_i x_{n+i}$$

This determines an embedding $\mathsf{Sp}(2n,\mathbb{C})\to\mathsf{SO}(4n,\mathbb{C})$.

Observe that $\mathsf{GL}(2,\mathbb{C})$ acts on $\mathbb{C}^{2n}\times\mathbb{C}^{2n}$ by $(x,y)\to(ax+by,cx+dy)$. This action commutes with the $\mathsf{Sp}(2n,\mathbb{C})$-action and more generally with the diagonal action of $\mathsf{GL}(2n,\mathbb{C})$. In particular, $\mathsf{SL}(2,\mathbb{C})$ preserves the quadratic form q, as $q(ax+by,cx+dy)=\omega(ax+by,cx+dy)=(ad-bc)\omega(x,y)$.

Consider now the open simply connected subset $\mathcal{D}=\mathcal{D}_{\mathsf{Sp}(2n,\mathbb{C})}$ of the quadric $\mathrm{Q}_{4n-2}(\mathbb{C})$ corresponding to the projectivization of the open subset of the q-light-cone, $\{(x,y)\mid q(x,y)=0,\mathbb{C}x\neq\mathbb{C}y\}$. The group $\mathsf{PSp}(2n,\mathbb{C})$ acts transitively and faithfully on it, and we aim to understand its isotropy group, say Q.

Let $\mathcal{X}$ be the space of ω isotropic 2-planes of $\mathbb{C}^{2n}$. We have a well defined $\mathsf{PSp}(2n,\mathbb{C})$-equivariant map $\pi:\mathcal{D}\to\mathcal{X}$, associating to $(x,\ y)$ the 2-plane $\mathbb{C}x\oplus\mathbb{C}y$. The π-fiber of an ω-isotropic 2-plane p is the set of all its bases (b_1,b_2), that is $\mathbb{C}b_1\oplus\mathbb{C}b_2=p$. By its true definition, the $\mathsf{PGL}(2,\mathbb{C})$-action preserves the π -fibres. In fact π is a $\mathsf{PGL}(2,\mathbb{C})$-principal fibration. In particular, $\mathsf{PGL}(2,\mathbb{C})$ acts properly and freely on $\mathcal{D}$.

Let $p=\mathbb{C}e_1\oplus\mathbb{C}e_{n+1}\in\mathcal{X}$ where (e_i) is the canonical basis of $\mathbb{C}^{2n}$. Its stabilizer Q' in $\mathsf{PSp}(2n,\mathbb{C})$ preserves the fiber $\mathcal{Y}=\pi^{-1}(p)$ and acts transitively on it, since the $\mathsf{PSp}(2n,\mathbb{C})$-action on $\mathcal{D}$ is transitive and commutes with π. So, on $\mathcal{Y}$, we have two commuting transitive actions of Q' and $\mathsf{PGL}(2,\mathbb{C})$. But, $\mathcal{Y}$ itself is identified with $\mathsf{PGL}(2,\mathbb{C})$, acting on itself on the left (since this action is free and transitive). It follows that Q' acts on the right on $\mathcal{Y}$ via a homomorphism $Q'\to\mathsf{PGL}(2,\mathbb{C})$. Since $\mathsf{PGL}(2,\mathbb{C})$ is semisimple, this homomorphism splits, up to finite index, and thus, up to finite index $Q'=\mathsf{PGL}(2,\mathbb{C})\ltimes Q$, where Q is the kernel of $Q'\to\mathsf{PGL}(2,\mathbb{C})$.

Clearly, Q acts trivially on $\mathcal{Y}$. In fact Q is the stabilizer for the $\mathsf{PSp}(2n,\mathbb{C})$-action on $\mathcal{D}$ of any point of the fiber $\mathcal{Y}$. Therefore, $\mathcal{D}$ as a homogeneous space can be identified with $\mathsf{PSp}(2n,\mathbb{C})/Q$.

Since $\mathcal{X}$ is compact, Q' is a parabolic subgroup of $\mathsf{PSp}(2n,\mathbb{C})$, and in particular the normalizer of Q is parabolic. To finish, take H to be a semi-direct product $\Gamma\ltimes Q$, where Γ is a co-compact lattice in $\mathsf{PGL}(2,\mathbb{C})$. Then $H\subset Q'$ with identity component $H^0=Q$, $M_1=\mathsf{PSp}(2n,\mathbb{C})/H$ is compact and covered by $\mathcal{D}=\mathsf{PSp}(2n,\mathbb{C})/Q$.

1.1.3 The $\mathsf{SL}(n, \mathbb{C})$-case

Given an n-dimensional complex vector space E. The diagonal action of $\mathsf{GL}(E)$ on $E \times E^*$ preserves the quadratic form $q(x, f) = f(x)$. In addition, the $\mathsf{PSL}(E)$-action is transitive and faithful on $\mathrm{Q}(E \times E^*)$, the projectivization of $\{(x, f) \mid f(x) = 0, (x, f) \neq (0, 0)\}$.

Let Q be the stabilizer of a point in the open simply connected subset $\mathcal{D}_{\mathsf{SL}(n,\mathbb{C})}$ of the quadric $\mathrm{Q}(E \times E^*)$ corresponding to the projectivization of the open subset of the q-light-cone, $\{(x, f) \mid f(x) = 0, x \neq 0, f \neq 0\}$. It has codimension 1 in its normalizer P. To see this, let $e_1, \ldots, e_n$ be a basis of E and $e_1^*, \ldots, e_n^*$ its dual bases. Consider p the point in the projective space corresponding to $(e_1, e_n^*) \in \mathcal{D}$. Its stabilizer Q consists of matrices of the form $\begin{pmatrix} \lambda & u^t & v \\ 0 & D & C \\ 0 & 0 & \frac{1}{\lambda} \end{pmatrix}$, where u is a vector of dimension $n-2$, λ, v are scalars, D is a $(n-2) \times (n-2)$-matrix, and C is a vector of dimension $n-2$, such that $\det D = 1$. Its normalizer Q' consists of matrices of the form $\begin{pmatrix} \lambda & u^t & v \\ 0 & D & C \\ 0 & 0 & \lambda' \end{pmatrix}$, with $\lambda(\det D)\lambda' = 1$. This is the stabilizer of the flag $(\mathbb{C}e_1, \mathbb{C}e_1 \oplus \ldots \oplus \mathbb{C}e_{n-1})$ and hence is parabolic. The quotient group Q'/Q has dimension 1. More precisely, up to a finite index, Q' is a semi-direct product $L \ltimes Q$, where $L \cong \mathbb{C}^*$ is represented as matrices of the form $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-2} & 0 \\ 0 & 0 & \alpha \end{pmatrix}$. If Γ is a lattice in $\mathbb{C}^*$, then, $H = \Gamma \ltimes Q$ yields a compact quotient $M_2 = \mathsf{PSL}(n, \mathbb{C})/H$ covered by $\mathcal{D} = \mathsf{PSL}(n, \mathbb{C})/Q$.

Remark 1.1 (Uniqueness) Although we will not need it, let us observe that in both cases the invariant domains $\mathcal{D}$ are unique. More precisely, there are unique (irreducible) representations $\mathsf{Sp}(2n, \mathbb{C}) \to \mathsf{SO}(4n, \mathbb{C})$ and $\mathsf{SL}(n, \mathbb{C}) \to \mathsf{SO}(2n, \mathbb{C})$. Both have a unique dense invariant domain $\mathcal{D}_{\mathsf{Sp}(2n)}$ (resp. $\mathcal{D}_{\mathsf{SL}_n}$).

1.2 Rigidity, Main Result

D'Ambra and Gromov conjectured in [2] that compact pseudo-Riemannian conformal manifolds with an essential action of the conformal group are conformally flat. This conjecture, often known as the pseudo-Riemannian Lichnerowicz conjecture, was later disproved by Frances in [8]. Additionally, this conjecture has been studied under signature restrictions, in the works of Zimmer, Bader, Nevo, Frances, Zeghib, Melnick and Pecastaing (see [3, 10, 18, 20–22, 24]). The present paper is the second in a series, exploring the Lichnerowicz conjecture in the homogeneous context. In [4], we provided a positive affirmation of the conjecture when the non compact semi-simple component of the conformal group is the M bius group. This article deals with the homogeneous Lichnerowicz conjecture in the complex (or real split) cases. More precisely, we will show that the examples constructed in Section 1.1 are essentially the only ones:

Theorem 1.2 *Let M be a compact connected complex manifold endowed with a faithful conformal holomorphic Riemannian structure, invariant under an essential and transitive action of a complex semisimple Lie group G. Then, M is conformally flat. Furthermore:*

- If M is simply connected, then, we have one of the following situations:
(1) $G = \mathsf{PSO}(n+2, \mathbb{C})$ *and* $M = \mathrm{Eins}_n(\mathbb{C})$ *with* $n \geq 1$ *(in particular, for* $n = 1$*,* $G = \mathsf{PSL}(2, \mathbb{C})$ *and* $M = \mathbb{CP}^1$*, and for* $n = 2$*,* $G = \mathsf{PSL}(2, \mathbb{C}) \times \mathsf{PSL}(2, \mathbb{C})$ *and* $M = \mathbb{CP}^1 \times \mathbb{CP}^1$*) or;*
(2) *G is the exceptional group* G_2 *and* $M = \mathrm{Eins}_5(\mathbb{C})$.
- If *M* is not simply connected, then it fits into one of the examples in Section 1.1. In particular:
(1) $G = \mathsf{PSp}(2n, \mathbb{C})$ *and M is a quotient of a* $\mathsf{PSp}(2n, \mathbb{C})$*-homogeneous open subset in* $\mathrm{Eins}_{2n-2}(\mathbb{C})$ *(*$n \geq 3$*). The fundamental group* $\pi_1(M)$ *is a co-compact lattice in* $\mathsf{PGL}(2, \mathbb{C})$ *(i.e. the fundamental group of a closed hyperbolic 3-manifold).*
(2) $G = \mathsf{PSL}(n, \mathbb{C})$ *and M is a quotient of a* $\mathsf{PSL}(n, \mathbb{C})$*-homogeneous open subset in* $\mathrm{Eins}_{2n-2}(\mathbb{C})$ *(*$n \geq 3$*). The fundamental group* $\pi_1(M)$ *is infinite cyclic.*

Remark 1.3 Let us observe, following a suggestion by the referee, that it is also possible to construct essential conformally flat but non-homogeneous examples. For this, note that in our previous homogeneous examples *M*, the fundamental group $\pi_1(M)$, up to a finite index, is $\mathbb{Z}$ (in the case of $\mathsf{SL}(n, \mathbb{C})$) or is isomorphic to the fundamental group of a compact hyperbolic 3-manifold (in the case of $\mathsf{Sp}(2n, \mathbb{C})$). The identity component of the conformal group of *M* is the centralizer of $\pi_1(M)$ in the general conformal group of the Einstein complex universe. One can then deform $\pi_1(M)$ within the latter group in order to obtain a small centralizer. We hope to provide details elsewhere, showing that one can choose this centralizer to have a small dimension while still acting essentially on *M*.

1.3 Organization of the Article

The paper is organized as follows: in Section 2, we provide an algebraic formulation of our initial problem using Lie algebra terminology. Section 3 delves into a detailed examination of the structure of the isotropy subalgebra. We will specifically distinguish between three different cases based on the size of the isotropy subalgebra. Sections 4, 5, and 6 are dedicated to proving the classification theorem in these distinct cases.

2 Algebraic Formulation

Assume that *M* is endowed with a conformal holomorphic Riemannian structure $\mathcal{G}$, invariant under the action of a complex semisimple Lie group *G*. We will assume, in addition, that *G* acts transitively and essentially on $(M, \mathcal{G})$.

Let x_0 be a fixed point of M, and denote by H its stabilizer in G, so that $T_{x_0}M$ is identified with $\mathfrak{g}/\mathfrak{h}$. The conformal structure $\mathcal{G}$ defines a conformal class of a non-degenerate complex bilinear symmetric form g on $\mathfrak{g}/\mathfrak{h}$ which in turn gives rise to a conformal class of a degenerate complex bilinear symmetric form $\langle .,. \rangle$ on $\mathfrak{g}$ admitting $\mathfrak{h}$ as its kernel. More precisely, the form $\langle .,. \rangle$ is defined by

$$\langle X, Y\rangle = g\left(X^*(x), Y^*(x)\right),$$

where X^*, Y^* are the left-invariant fundamental vector fields associated with X and Y.

Consider $P = \overline{H}^{Zariski}$, the Zariski closure of the isotropy group H. It preserves the conformal class of $\langle .,. \rangle$. More precisely, there is a morphism $\delta : P \longrightarrow \mathbb{C}^*$, such that for every $p \in P$ and every $u, v \in \mathfrak{g}$,

$$\langle \mathrm{Ad}_p(u), \mathrm{Ad}_p(v)\rangle = \delta(p)\langle u, v\rangle = \left(\det\left(\mathrm{Ad}_p\right)_{|\mathfrak{g}/\mathfrak{h}}\right)^{\frac{2}{n}}\langle u, v\rangle. \qquad (1)$$

In particular, the group P normalizes H.

Differentiating Eq. 1, we obtain a linear function, that we continue to denote δ, from $\mathfrak{p}$, the Lie algebra of P, to $\mathbb{C}$, such that for every $p \in \mathfrak{p}$ and every $u, v \in \mathfrak{g}$

$$\langle \mathrm{ad}_p(u), v\rangle + \langle u, \mathrm{ad}_p(v)\rangle = \delta(p)\langle u, v\rangle. \qquad (2)$$

In particular, if $p \in \mathfrak{p}$ preserves the metric, then $\delta(p) = 0$, and

$$\langle \mathrm{ad}_p(u), v\rangle + \langle u, \mathrm{ad}_p(v)\rangle = 0. \qquad (3)$$

Since $\mathfrak{p}$ is a complex uniform algebraic subalgebra of the semisimple algebra $\mathfrak{g}$, there exists a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, together with an ordered root system $\Delta = \Delta^- \sqcup \Delta^+$ and a root space decomposition $\mathfrak{g} = \bigoplus_{\alpha\in\Delta^-}\mathfrak{g}_\alpha \oplus \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Delta^+}\mathfrak{g}_\alpha = \mathfrak{g}_- \oplus \mathfrak{a} \oplus \mathfrak{g}_+$ such that $\mathfrak{a} \oplus \mathfrak{g}_+ \subset \mathfrak{p}$ [6, Corrolaire 16.13].

Definition 2.1 Two elements α, β of $\Delta \cup 0$ are said to be paired if $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are not $\langle .,. \rangle$ −orthogonal.

Note that, because $\langle .,. \rangle$ is nontrivial, there always exist two paired elements (possibly the same) α, β of $\Delta \cup 0$. Any such elements α and β verify $\alpha + \beta = \delta$. This shows that, for any element α, there is at most one β (depending whether $\mathfrak{g}_\alpha \subset \mathfrak{h}$ or not) paired with it. More precisely,

Fact 2.1 *If α is a root such that $\mathfrak{g}_\alpha \not\subseteq \mathfrak{h}$, then $\delta - \alpha$ is also a root, and $\mathfrak{g}_{\delta-\alpha} \not\subseteq \mathfrak{h}$ as well.*

Moreover:

Proposition 2.2 *We have:*

(1) $\mathfrak{h}$ *is a nontrivial ideal of* $\mathfrak{p}$*;*
(2) $\mathfrak{p} \subsetneq \mathfrak{g}$*;*
(3) *The restriction of* δ *to* $\mathfrak{a}$ *is a nontrivial linear form.*

Proof 1) By [4, Proposition 2.6], $\mathfrak{h}$ is non trivial.

2) Suppose the contrary. Since $\mathfrak{h}$ is a nontrivial ideal of $\mathfrak{p}$, Eq. 3 is verified for every $u, v \in \mathfrak{g}$, and every $p \in \mathfrak{h}$, which contradicts the essentiality of the action.

3) Now as $\mathfrak{g}_-$ and $\mathfrak{g}_+$ are nilpotent subalgebras, we have that δ is trivial on $(\mathfrak{p} \cap \mathfrak{g}_-) \oplus \mathfrak{g}_+$. If δ were trivial on $\mathfrak{a}$, then δ would be trivial on $\mathfrak{p} = (\mathfrak{p} \cap \mathfrak{g}_-) \oplus \mathfrak{a} \oplus \mathfrak{g}_+$ which clearly contradicts the essentiality hypothesis. □

Definition 2.2 The restriction of δ to $\mathfrak{a}$ is called **distortion**.

In the rest of this paper we will abandon our original group formulation and instead adopt the following Lie algebra one:

- There is a root space decomposition as above,
- There is a distortion $\delta : \mathfrak{a} \longrightarrow \mathbb{C}$,
- The pairing condition of two weight spaces implies their sum is δ,
- The essentiality is translated into the fact that $\delta \neq 0$, and the compactness of *G*/*H* is replaced by the fact that $\mathfrak{a} \oplus \mathfrak{g}_+$ normalizes $\mathfrak{h}$.

Remark 2.3 Although we are working with complex semisimple Lie algebras, what matters to us primarily is the associated root system and the properties it satisfies. This is more than sufficient for our purposes. In this sense, our focus is essentially on the underlying real Lie algebra structure, and our results apply perfectly in the real split case.

We finish this section by the following useful definition:

Definition 2.3 We say that a subalgebra $\mathfrak{g}'$ is a modification of $\mathfrak{g}$, if $\mathfrak{g}'$ projects surjectively onto $\mathfrak{g}/\mathfrak{h}$. Equivalently, $M = G'/(G' \cap H)$, where G' is the connected subgroup of G associated to $\mathfrak{g}'$.

3 Structure of the Isotropy Subalgebra: A Synthetic Study

In this part, we will study in detail the structure of the subalgebra $\mathfrak{h}$. Let us start with the following proposition:

Proposition 3.1 *We have:*

(1) *If* $\mathfrak{a} \subset \mathfrak{h}$*, then the Borel subalgebra* $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{g}_+$ *is contained in* $\mathfrak{h}$*;*
(2) *If* $\mathfrak{a} \not\subseteq \mathfrak{h}$*, then* δ *is a root paired with 0. In particular,* $\mathfrak{g}_\delta$ *is not contained in* $\mathfrak{p}$*. Moreover, the subalgebra* $\mathfrak{a} \cap \mathfrak{h}$ *has codimension one in* $\mathfrak{a}$*.*

Proof Suppose first that $\mathfrak{a} \subset \mathfrak{h}$. Then $\mathfrak{g}_+ = [\mathfrak{a}, \mathfrak{g}_+] \subset [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{h}$. This implies that the Borel subalgebra $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{g}_+ \subset \mathfrak{h}$.

On the contrary, if $\mathfrak{a}$ is not contained in $\mathfrak{h}$, then 0 is paired with δ and hence δ is a root. Let $p \in \mathfrak{g}_\delta \cap \mathfrak{p}$ and $u = v$ in $\mathfrak{a}$. Substituting this into Eq. 2, we obtain $\delta(u)\,\langle p, u\rangle = 0$ for every $p \in \mathfrak{g}_\delta$ and $u \in \mathfrak{a}$. Thus by density we get $\langle \mathfrak{a}, \mathfrak{g}_\delta\rangle = 0$ which contradicts the fact that δ is paired with 0. So, $\mathfrak{g}_\delta \cap \mathfrak{p} = \emptyset$

As $\mathfrak{g}_\delta$ is of dimension one and $\mathfrak{h}$ is the kernel of $\langle\cdot,\cdot\rangle$, we obtain that $\mathfrak{a} \cap \mathfrak{h}$ is of codimension one in $\mathfrak{a}$. □

3.1 Case One: $\mathfrak{a} \not\subseteq \mathfrak{h}$

Proposition 3.2 *Up to modification,* $\mathfrak{g}$ *is simple.*

Proof Assume that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the direct sum of a simple Lie algebra $\mathfrak{g}_1 \not\subseteq \mathfrak{h}$ and a semisimple Lie algebra $\mathfrak{g}_2 \not\subseteq \mathfrak{h}$. Thus there exist a root α of $\mathfrak{g}_1$ and a root β of $\mathfrak{g}_2$ such that $\mathfrak{g}_\alpha \not\subseteq \mathfrak{h}$ and $\mathfrak{g}_\beta \not\subseteq \mathfrak{h}$. Therefore, $\delta - \alpha$, $\delta - \beta$ are also roots of $\mathfrak{g}$. But the roots of $\mathfrak{g}$ are the disjoint union of the those of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. This implies that δ would be a root of both $\mathfrak{g}_1$ and $\mathfrak{g}_2$, which is a contradiction. □

By [16, Proposition 2.17], for every root α, there exists an element $H_\alpha \in \mathfrak{a}$ such that $\mathrm{B}(H_\alpha, .) = \alpha$, where here B is the non-degenerate Killing form of $\mathfrak{a}$.

Let $p \in \mathfrak{g}_{-\delta}$, and choose $0 \neq u \in \mathfrak{g}_\delta$ such that $[p, u] = H_\delta$. Applying Eq. 3 with $p, u = v$ we obtain $\langle H_\delta, u\rangle = 0$ and hence $\langle H_\delta, \mathfrak{g}_\delta\rangle = 0$. However, by Proposition 3.1, δ is a root paired with 0. Therefore:

Fact 3.3 $H_\delta \in \mathfrak{a} \cap \mathfrak{h}$.

Now we have the following important Lemma:

Lemma 3.4 *Let* α *be a root, which we will assume to be positive. Then:*

(1) *If* $\delta(H_\alpha) \neq 0$, $\mathfrak{g}_\alpha$ *is contained in* $\mathfrak{h}$;
(2) *If* $\delta(H_\alpha) = 0$ *and* $\delta - \alpha$ *is a root,* $\mathfrak{a} \cap \mathfrak{h} = H_\alpha^\perp$, *where the orthogonality is with respect to the Killing form* B. *In particular, such an* α *is unique.*
(3) *If* $\delta(H_\alpha) = 0$, $\mathfrak{g}_{-\alpha} \oplus \mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha$ *preserves* $\langle ., .\rangle$. *Moreover, if* $\mathfrak{g}_\alpha \subset \mathfrak{h}$, *then* $\mathfrak{g}_{-\alpha} \oplus \mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha \subset \mathfrak{h}$.

Proof First, assume that $\delta(H_\alpha) \neq 0$. Thus $\mathfrak{g}_\alpha = \delta(H_\alpha)\mathfrak{g}_\alpha = [H_\delta, \mathfrak{g}_\alpha]$. But $\mathfrak{h}$ is an ideal of $\mathfrak{p}$, $H_\delta \in \mathfrak{h}$ and $\mathfrak{g}_\alpha \subset \mathfrak{p}$. Therefore, $\mathfrak{g}_\alpha \subset \mathfrak{h}$.

Assume, on the contrary, that $\delta(H_\alpha) = 0$ and $\delta - \alpha$ is a root. Take $H \in H_\alpha^\perp$ so that $\alpha(H) = 0$. On the one hand, using Eq. 3, with $p \in \mathfrak{g}_\alpha$, $u = H$ and $v \in \mathfrak{g}_{\delta-\alpha}$ gives us $\langle H, [p, v]\rangle = 0$. However, according to [16, Corollary 2.35], $[\mathfrak{g}_\alpha, \mathfrak{g}_{\delta-\alpha}] = \mathfrak{g}_\delta$, implying $H \in \mathfrak{a} \cap \mathfrak{h}$. On the other hand, Proposition 3.1 tells us that $H_\alpha^\perp$ and $\mathfrak{a} \cap \mathfrak{h}$ have the same dimension. Thus, $H_\alpha^\perp = \mathfrak{a} \cap \mathfrak{h}$.

To finish, assume just that $\delta(H_\alpha) = 0$. Then $\mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha$ preserves $\langle .,. \rangle$. Thus its orbit under the action of $\mathfrak{g}_{-\alpha} \oplus \mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha \cong \mathfrak{sl}(2,\mathbb{C})$ is compact and hence trivial by [4, Lemma 2.7].

If $\mathfrak{g}_\alpha \subset \mathfrak{h}$, then, since $\mathfrak{h}$ is an ideal of the subalgebra preserving the conformal class of $\langle .,. \rangle$, we have $\mathfrak{g}_{-\alpha} \oplus \mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha = [\mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha} \oplus \mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha]$ $= [[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha} \oplus \mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha], \mathfrak{g}_{-\alpha} \oplus \mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha] \subset [[\mathfrak{h}, \mathfrak{p}], \mathfrak{p}] \subset [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{h}$. □

For every root α, let us fix two elements $u_\alpha \in \mathfrak{g}_\alpha$ and $u_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $[u_\alpha, u_{-\alpha}] = H_\alpha$. Let α, β be two roots such that $\alpha + \beta$ is a root. By [16, Corollary 2.35], we have that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. Therefore, there is a non zero complex number $k_{\alpha,\beta}$ such that $[u_\alpha, u_\beta] = k_{\alpha,\beta} u_{\alpha+\beta}$. Now, if α is a root such that $\mathfrak{g}_\alpha \not\subset \mathfrak{h}$, then $\delta - \alpha$ is also root. By assuming α negative if necessary, we use Eq. 3, with $p = u_{-\alpha}$, $u = u_\alpha$ and $v = u_\delta$ to obtain: $\langle u_\alpha, u_{\delta-\alpha} \rangle = \frac{1}{k_{\alpha,\delta}} \langle H_\alpha, u_\delta \rangle$. As a consequence we obtain the following uniqueness property:

Proposition 3.5 *If $\mathfrak{a} \not\subset \mathfrak{h}$, then the conformal class of $\langle .,. \rangle$ depends only on $\mathfrak{a} \cap \mathfrak{h}$ and $\mathfrak{g}_\delta$.*

3.2 Case Two: $\mathfrak{a} \subset \mathfrak{h}$

In this case, δ is no longer a root, but rather a sum of two roots. We have:

Proposition 3.6 *Up to modification, $\mathfrak{g}$ is:*

- Simple, or
- The direct sum of two rank-one complex simple Lie algebras.

Proof Assume we are not in the first case. So, one can write $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as the direct sum of simple Lie algebra $\mathfrak{g}_1 \not\subset \mathfrak{h}$ and a semisimple one $\mathfrak{g}_2 \not\subset \mathfrak{h}$. Hence, there is a root α of $\mathfrak{g}_1$ and a root β of $\mathfrak{g}_2$ such that $\mathfrak{g}_\alpha \not\subset \mathfrak{h}$ and $\mathfrak{g}_\beta \not\subset \mathfrak{h}$. Now, if we were not in the second case, there would exist a third root $\gamma \neq \alpha, \beta$ such that $\mathfrak{g}_\gamma \not\subset \mathfrak{h}$. Consequently, $\delta - \alpha$, $\delta - \beta$, and $\delta - \gamma$ are also roots of $\mathfrak{g}$. However, this is impossible since the roots of $\mathfrak{g}$ are the union of the roots of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. □

4 The $\mathrm{Sp}(n, \mathbb{C})$ Case

In this part, we will prove Theorem 1.2 when $\mathfrak{a} \not\subset \mathfrak{h}$ and $\mathfrak{g}_+ \not\subset \mathfrak{h}$. By Proposition 3.2, up to modification, the Lie algebra $\mathfrak{g}$ is simple. The root systems associated to a simple complex Lie algebra are well known and classified. They are of A_n, B_n, C_n, and D_n types, as well as the exceptional ones E_6, E_7, E_8, F_4, and G_2. Up to isomorphism, they are described by means of the canonical basis of $\mathbb{R}^n$. Detailed descriptions of these root systems, along with their associated canonical simple roots, can be found in [16]. From now until the end of the paper, we will assume, up to isomorphism, that the root system Δ is a canonical root system endowed with its canonical order. The notations and terminology used here follow [16, Appendix C].

In this case δ is a root, and there exists a positive root α such that $\mathfrak{g}_\alpha \not\subset \mathfrak{h}$. Hence, by Fact 2.1, $\delta - \alpha$ is also a root. By Lemma 3.4, we have that $\delta(H_\alpha) = 0$, $\mathfrak{a} \cap \mathfrak{h} = H_\alpha^\perp$ and α is unique. We have:

Proposition 4.1 *Let β be a positive root different from α. If β is not orthogonal to α, then $\mathfrak{g}_{-\beta} \not\subset \mathfrak{h}$.*

Proof Assume that $\mathfrak{g}_{-\beta} \subset \mathfrak{h}$. As $\beta \neq \alpha$, then by Lemma 3.4, $\mathfrak{g}_\beta \subset \mathfrak{h}$ and hence $\mathbb{C}H_\beta = [\mathfrak{g}_{-\beta}, \mathfrak{g}_\beta] \subset \mathfrak{a} \cap \mathfrak{h} = H_\alpha^\perp$. This means that β is orthogonal to α. □

Now, we have the following proposition:

Proposition 4.2 *Up to the action of the Weyl group, the pairs of roots $(-\delta, \alpha)$ such that: α is orthogonal to δ and $\delta - \alpha$ is a root are:*

- B_n: $(-\delta, \alpha) = (e_1, e_2)$;
- C_n: $(-\delta, \alpha) = (e_1 + e_2, e_1 - e_2)$;
- F_4: $(-\delta, \alpha) = (e_1, e_2)$ or $(-\delta, \alpha) = \left(\frac{1}{2}(e_1 + e_2 - e_3 - e_4), \frac{1}{2}(e_1 + e_2 + e_3 + e_4)\right)$.

Proof As δ is orthogonal to α we have:

$$|\delta - \alpha|^2 = |\delta|^2 + |\alpha|^2 . \tag{4}$$

First, assume that our root system is of type A_n, D_n, E_6, E_7, or E_8. In these cases, all the roots have the same length. Putting this in Eq. 4 gives us a contradiction.

Now, if we are in the G_2 type, then we have 12 roots: six of them have length 2 and the other six have length 6. Again, these do not verify Eq. 4.

Finally, in all the remaining types (B_n, C_n and F_4), we can easily verify that such pairs exist. We then use the action of the Weyl group to conclude. □

We are left with three types of root systems, namely B_n, C_n and F_4.

Proposition 4.3 *The pair $(-\delta, \alpha)$ exists only in the root systems of type C_n.*

Proof We first prove that the B_n case is impossible. Assume $n > 2$, $(-\delta, \alpha) = (e_i, e_j)$, and let $\beta = e_j + e_k$ with $i \neq j \neq k$. As $\delta - \beta = -e_i - e_j - e_k$ is not a root, we have that $\mathfrak{g}_\beta \subset \mathfrak{h}$. On the other hand $-e_k$ is orthogonal to δ and $\mathfrak{g}_{e_k} \subset \mathfrak{h}$, so by Lemma 3.4, $\mathfrak{g}_{-e_k} \subset \mathfrak{h}$. Thus $[\mathfrak{g}_\beta, \mathfrak{g}_{-e_k}] = \mathfrak{g}_\alpha \subset \mathfrak{h}$ which is a contradiction. So *n* must be equal to 2, and $B_2 = C_2$ [23, Pages 26-27].

As for the F_4 case, the same proof works. □

The only remaining case is the C_n type. In this case, we have only one possibility for the pairs $(-\delta, \alpha)$, namely:

Proposition 4.4 $(-\delta, \alpha) = (e_1 + e_2, e_1 - e_2)$.

Proof By contradiction, assume that $(-\delta, \alpha) = (e_i + e_j, e_i - e_j)$ for some $1 \leq i < j \leq n$ such that $i \neq 1$ or $j \neq 2$. If $i \neq 1$, then $\beta = e_1 - e_j \neq \alpha$ is a positive root which is not orthogonal to α. Thus, by Proposition 4.1, $\mathfrak{g}_{-\beta} \not\subseteq \mathfrak{h}$, and hence $\delta + \beta = -e_i - 2e_j + e_1$ is also a negative root, which is clearly false. If, in contrast, $j \neq 2$, then take $\beta = e_2 - e_j$, and the same proof works. □

The fact that we already have an example of such type (Example 1.1.2), together with the uniqueness property in Proposition 3.5 gives us:

Corollary 4.5 *If $\mathfrak{a} \not\subseteq \mathfrak{h}$ and $\mathfrak{g}_+ \not\subseteq \mathfrak{h}$, then $G = \mathsf{Sp}(n, \mathbb{C})$ and $M = M_1$. In particular, M is conformally flat.*

5 The $\mathrm{SL}(n, \mathbb{C})$ Case

In this part, we will prove Theorem 1.2 when $\mathfrak{a} \not\subseteq \mathfrak{h}$ and $\mathfrak{g}_+ \subset \mathfrak{h}$. In this case, δ is a negative root. Let α be a positive root such that $\delta - \alpha$ is also a root. Consequently, $\mathfrak{g}_{\delta-\alpha} \subset \mathfrak{h}$. If this were not the case, then $\delta - \alpha$ would be paired with α, leading to a contradiction. Now, on the one hand $\mathfrak{g}_\delta = [\mathfrak{g}_{\delta-\alpha}, \mathfrak{g}_\alpha] \subset \mathfrak{h} \subset \mathfrak{p}$. On the other hand, according to Proposition 3.1 (2), $\mathfrak{g}_\delta \cap \mathfrak{p} = \{0\}$. This leads to a contradiction. Thus:

Proposition 5.1 *The negative root δ is the minimal root.*

As a consequence, we get:

Proposition 5.2 *The only possible type is A_n. In particular, $-\delta = e_1 - e_{n+1}$.*

Proof First, assume that we are in the B_n type. In this case, we have $\delta = -e_1 - e_2$. Here $e_1 - e_2$, e_i for $i \geq 3$ are all orthogonal to δ. Using Lemma 3.4, this implies that $H_{e_1-e_2}$ and H_{e_i} for $i \geq 3$ belong to $\mathfrak{a} \cap \mathfrak{h}$. As $H_\delta \in \mathfrak{a} \cap \mathfrak{h}$ (see Fact 3.3), we get that $\mathfrak{a} \cap \mathfrak{h} = \mathfrak{a}$, which contradicts Proposition 3.1.

The same proof works for the C_n and D_n types.

In the exceptional case E_6, $-\delta = \frac{1}{2}(e_8 - e_7 - e_6 + e_5 + e_4 + e_3 + e_2 + e_1)$. On the one hand $\delta + \alpha_1$, $\delta + \alpha_3$, $\delta + \alpha_4$, $\delta + \alpha_5$ and $\delta + \alpha_6$ are not roots. So $\mathfrak{g}_{-\alpha_1}$, $\mathfrak{g}_{-\alpha_3}$, $\mathfrak{g}_{-\alpha_4}$, $\mathfrak{g}_{-\alpha_5}$, $\mathfrak{g}_{-\alpha_6}$ are all in $\mathfrak{h}$. This shows that H_{α_1}, H_{α_3}, H_{α_4}, H_{α_5}, H_{α_6} are all in $\mathfrak{a} \cap \mathfrak{h}$. On the other hand, $H_\delta \in \mathfrak{a} \cap \mathfrak{h}$. But δ, α_1, α_3, α_4, α_5, α_6 are linearly independent. Thus $\mathfrak{a} \cap \mathfrak{h} = \mathfrak{a}$, which contradicts Proposition 3.1.

In the exceptional case E_7, $-\delta = e_8 - e_7$. In this case, for every $1 \leq i \leq 7$, $\delta + \alpha_i$ is not a root. This means that all the $\mathfrak{g}_{-\alpha_i}$ are in $\mathfrak{h}$. Hence $\mathfrak{g} = \mathfrak{h}$, which is a contradiction.

In the exceptional case E_8, $-\delta = \frac{1}{2}(e_8 + e_7 + e_6 + e_5 + e_4 + e_3 + e_2 + e_1)$, and the same proof as in exceptional case E_6 works here too.

Now let us consider the exceptional case G_2. Here $-\delta = 2e_3 - e_2 - e_1$. Consequently, $\delta + \alpha_1$ is not a root, and hence $\mathfrak{g}_{-\alpha_1} \subset \mathfrak{h}$. Thus $H_{\alpha_1} \in \mathfrak{a} \cap \mathfrak{h}$. Together with the fact that $H_\delta \in \mathfrak{a} \cap \mathfrak{h}$, we conclude that $\mathfrak{a} \cap \mathfrak{h} = \mathfrak{a}$, which is in contradiction with Proposition 3.1.

To conclude, let's consider the exceptional case F_4. Here, we also have $-\delta = e_1 + e_2$. Consequently, $\delta + \alpha_1$, $\delta + \alpha_2$, and $\delta + \alpha_3$ are not roots. This implies that $\mathfrak{g}_{-\alpha_1}$, $\mathfrak{g}_{-\alpha_2}$, $\mathfrak{g}_{-\alpha_3}$ are all in $\mathfrak{h}$, and therefore H_{α_1}, H_{α_2}, and H_{α_3} are in $\mathfrak{a} \cap \mathfrak{h}$. Together with the fact that $H_\delta \in \mathfrak{a} \cap \mathfrak{h}$, we deduce that $\mathfrak{a} \cap \mathfrak{h} = \mathfrak{a}$, which once more contradicts Proposition 3.1. □

In the remaining A_n case, the subalgebra $\mathfrak{a} \cap \mathfrak{h}$ is completely determined by the root δ. Indeed, $-\delta = e_1 - e_{n+1}$, and so $\mathfrak{a} \cap \mathfrak{h}$ is generated by the vector $H_{e_1 - e_{n+1}}$ and all vectors $H_{e_i - e_j}$, where $i < j \in \{1, ..., n+1\} \backslash \{1, n+1\}$. The uniqueness property in Proposition 3.5, along with the existence of such Example (as in Example 1.1.3) gives us:

Corollary 5.3 *If* $\mathfrak{a} \not\subseteq \mathfrak{h}$ *and* $\mathfrak{g}_+ \subseteq \mathfrak{h}$, *then* $G = \mathsf{SL}(n, \mathbb{C})$ *and* $M = M_2$. *In particular, M is conformally flat.*

6 The Case of Parabolic Isotropy

In this part, we assume that the Borel subalgebra $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{g}_+$ is contained in $\mathfrak{h}$. In this case, by [9, Theorem 1.4], *M* is conformally flat (see [4, Proposition 3.3]). Then, by simple connectivity, one shows that *M* is conformally equivalent to $\mathrm{Eins}_n(\mathbb{C})$. However, it is not easy to see what *G* is in this case. In other words, it is not an obvious task to see which semisimple Lie groups act transitively and essentially on $\mathrm{Eins}_n(\mathbb{C})$. In the sequel, we will not use the conformal flatness result from [9, Theorem 1.4]. Instead, we will identify *G* and *H* and observe, in particular, that *G*/*H* is $\mathrm{Eins}_n(\mathbb{C})$.

There is a subalgebra of $\mathfrak{g}_-$ such that $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{g}_+$. One can describe more precisely the subalgebra . Indeed, since the root spaces are 1-dimensional, $\mathfrak{g}_+ \subset \mathfrak{h}$, there is a subset Δ' of positive roots of Δ such that $\mathfrak{l} = \bigoplus_{\beta \in -\Delta'} \mathfrak{g}_\beta$ (see [16, Section 5.7]). Let Π be the standard basis of the canonical root system Δ. By [16, Proposition 5.90]), there is a subset Π' of Π such that $\Delta' = \mathrm{span}(\Pi')$.

6.1 Maximality of the Isotropy Subalgebra

Definition 6.1 The parabolic subalgebra $\mathfrak{h}$ is said to be maximal if $|\Pi'| = |\Pi| - 1$.
Let α be a simple root in Π such that $\mathfrak{g}_{-\alpha} \not\subseteq \mathfrak{h}$ (note that this always exists since *M* is non-trivial). Then $\delta + \alpha$ is also a negative root such that $\mathfrak{g}_{\delta+\alpha} \not\subseteq \mathfrak{h}$. Actually we have more:

Proposition 6.1 *The negative root* $\delta + \alpha$ *is the minimal root.*

Proof Assume, by contradiction, that there is a positive root β such that $\delta+\alpha-\beta$ is a negative root. Thus $\mathfrak{g}_{\delta+\alpha-\beta} \nsubseteq \mathfrak{h}$, and hence $\delta-(\delta+\alpha-\beta)=\beta-\alpha$ is also a negative root, which is impossible. □

As a consequence, we get:

Corollary 6.2 *The parabolic subalgebra* $\mathfrak{h}$ *is maximal.*

Proof Assume that there are two simple roots $\alpha_1, \alpha_2 \in \Pi\backslash\Pi'$. By Proposition 6.1 both $\delta+\alpha_1$ and $\delta+\alpha_2$ are minimal roots of Δ. By uniqueness $\delta+\alpha_1=\delta+\alpha_2$, and hence $\alpha_1=\alpha_2$. □

Remark 6.3 Note that, so far, we have not imposed any restriction on the rank of $\mathfrak{g}$, and thus Corollary 6.2 remains valid for lower rank semisimple algebras.

6.2 Higher Rank Parabolic Case

We assume, in light of Proposition 3.6, that after modification, the Lie algebra $\mathfrak{g}$ is simple of $\mathrm{rank}(\mathfrak{g}) \geq 3$. Thus, by Proposition 3.6, it is simple.

6.2.1 Elimination of Cases: First Step Toward Classification

Let α be the unique simple root in $\Pi\backslash\Pi'$. Then using Proposition 6.1, we obtain:

Proposition 6.4 *The simple Lie algebra* $\mathfrak{g}$ *is of non exceptional type.*

Proof Assume the converse. We now distinguish several cases depending on the type of $\mathfrak{g}$:

(1) **If $\mathfrak{g}$ is of type E_6.** Here, $\delta+\alpha=-\frac{1}{2}\left(e_8-e_7-e_6+e_5+e_4+e_3+e_2+e_1\right)$. Therefore:

   (a) If $\alpha=\alpha_1$, then $\delta=-\frac{1}{2}\left(e_8-e_7-e_6+e_5+e_4+e_3+e_2+e_1\right)-\alpha_1$. We have $\mathfrak{g}_{-(\alpha_1+e_2-e_1)} \nsubseteq \mathfrak{h}$. However, $\delta+\alpha_1+e_2-e_1$ is not a root, leading to a contradiction.

   (b) If $\alpha=\alpha_2$, then $\delta=-\frac{1}{2}\left(e_8-e_7-e_6+e_5+e_4+e_3+e_2+e_1\right)-\alpha_2$. We have $\mathfrak{g}_{-(\alpha_2+e_3-e_2)} \nsubseteq \mathfrak{h}$. But $\delta+\alpha_2+e_3-e_2$ is not a root, leading to a contradiction.

   (c) If $\alpha=e_{k+1}-e_k$, then $\delta=-\frac{1}{2}\left(e_8-e_7-e_6+e_5+e_4+e_3+e_2+e_1\right)-\left(e_{k+1}-e_k\right)$. For $1<k\leq 4$, we have $\mathfrak{g}_{-(e_{k+1}-e_{k-1})} \nsubseteq \mathfrak{h}$. However, $\delta+\left(e_{k+1}-e_{k-1}\right)$ leading to a contradiction. For $k=1$, we have $\mathfrak{g}_{-(e_3-e_1)} \nsubseteq \mathfrak{h}$. But $\delta+\left(e_3-e_1\right)$ is also not a root,since the coefficient of e_2 is $-\frac{3}{2}$, so we obtain a contradiction.

(2) **If $\mathfrak{g}$ is of type E_7.** Here, $\delta+\alpha=-\left(e_8-e_7\right)$. Thus:

(a) If $\alpha = \alpha_1$, then $\delta = -(e_8 - e_7) - \alpha_1$. We have $\mathfrak{g}_{-(\alpha_1+e_3+e_2)} \not\subseteq \mathfrak{h}$. But $\delta + (\alpha_1 + e_3 + e_2) = (e_3 + e_2) - (e_8 - e_7)$ is not a root leading to a contradiction.
(b) If $\alpha = \alpha_2$, then $\delta = -(e_8 - e_7) - \alpha_2$. We have $\mathfrak{g}_{-(\alpha_2+e_3-e_2)} \not\subseteq \mathfrak{h}$. But $\delta + (\alpha_2 + e_3 - e_2) = (e_3 - e_2) - (e_8 - e_7)$ is not a root. So we get a contradiction.
(c) If $\alpha = \alpha_i$ with $i > 3$, then $\delta = -(e_8 - e_7) - \alpha_i$. We have $\mathfrak{g}_{-(\alpha_i+e_{i-2}+e_1)} \not\subseteq \mathfrak{h}$. However, $\delta + (\alpha_i + e_{i-2} + e_1) = (e_{i-2} + e_1) - (e_8 - e_7)$ is not a root, leading again to a contradiction.
(d) If $\alpha = \alpha_3$, then $\delta = -(e_8 - e_7) - \alpha_3$. But $\delta + (\alpha_3 + e_3 - e_2) = (e_3 - e_2) - (e_8 - e_7)$ is not a root, which contradicts the fact that $\mathfrak{g}_{-(\alpha_3+e_3-e_2)} \not\subseteq \mathfrak{h}$.

(3) **If $\mathfrak{g}$ is of type E_8.** Here, $\delta + \alpha = -\frac{1}{2}(e_8 + e_7 + e_6 + e_5 + e_4 + e_3 + e_2 + e_1)$ and exactly the same proof as for the E_6 type works.

(4) **If $\mathfrak{g}$ is of type F_4.** Here $\delta + \alpha = -(e_1 + e_2)$. Thus

(a) If $\alpha = \alpha_1$, then $\delta = -\alpha_1 - (e_1 + e_2)$. We have $\mathfrak{g}_{-(\alpha_1+e_2+e_3)} \not\subseteq \mathfrak{h}$. But $\delta + (\alpha_1 + e_2 + e_3)$ is not a root, leading to a contradiction.
(b) If $\alpha = \alpha_2$, then $\delta = -\alpha_2 - (e_1 + e_2)$. We have $\mathfrak{g}_{-(\alpha_2+e_3-e_4)} \not\subseteq \mathfrak{h}$. But $\delta + (\alpha_2 + e_3 - e_4)$ is not a root, leading to a contradiction.
(c) If $\alpha = \alpha_3$, then $\delta = -\alpha_3 - (e_1 + e_2)$. We have $\mathfrak{g}_{-(\alpha_3+e_4)} \not\subseteq \mathfrak{h}$. However, $\delta + (\alpha_3 + e_4)$ is not a root, leading to a contradiction.
(d) If $\alpha = \alpha_4$ then $\delta = -\alpha_4 - (e_1 + e_2)$. We have $\mathfrak{g}_{-(\alpha_4+e_1-e_2)} \not\subseteq \mathfrak{h}$. However, $\delta + (\alpha_4 + e_1 - e_2)$ is not a root, leading to a contradiction.□

This leads us to the following initial classification of $\mathfrak{g}$:

Proposition 6.5 *The simple Lie algebra $\mathfrak{g}$ is of type:*

(1) *B_3 with $\alpha = e_3$ and $\delta = -(e_1 + e_2 + e_3)$ or;*
(2) *D_4 with $\alpha = e_3 + e_4$ and $\delta = -(e_1 + e_2 + e_3 + e_4)$ or;*
(3) *D_4 with $\alpha = e_3 - e_4$ and $\delta = -(e_1 + e_2 + e_3 - e_4)$ or;*
(4) *B_n with $n \geq 3$ and $\alpha = e_1 - e_2$ and $\delta = -2e_1$ or;*
(5) *D_n with $n \geq 3$ and $\alpha = e_1 - e_2$ and $\delta = -2e_1$.*

Proof For this, we distinguish several cases depending on the type of $\mathfrak{g}$. By Proposition 6.4, it is sufficient to consider the non exceptional types:

(1) **If $\mathfrak{g}$ is of type B_n.** Here $\delta + \alpha = -(e_1 + e_2)$. Thus:

(a) If $\alpha = e_k - e_{k+1}$ with $k \geq 2$. Since e_k is a positive root such that $\mathfrak{g}_{-e_k} \not\subseteq \mathfrak{h}$, we would then have $\delta + e_k = -(e_1 + e_2 - e_{k+1})$ is a negative root, which is clearly not true;
(b) If $\alpha = e_1 - e_2$. In this case $\delta = -2e_1$.

(c) If $\alpha = e_n$ with $n > 3$. Since $e_n + e_3$ is a positive root such that $\mathfrak{g}_{-(e_n+e_3)} \not\subseteq \mathfrak{h}$, we would then have $\delta + e_n + e_3 = -(e_1 + e_2 - e_3)$ is a negative root, which is clearly not true;
(d) If $n = 3$ and $\alpha = e_3$. In this case $\delta = -(e_1 + e_2 + e_3)$.

(2) **If $\mathfrak{g}$ is of type C_n.** Here $\delta + \alpha = -2e_1$. Thus:

(a) If $\alpha = e_k - e_{k+1}$, then $\mathfrak{g}_{-(e_k+e_n)} \not\subseteq \mathfrak{h}$. This implies that $\delta + e_k + e_n = -(2e_1 - e_{k+1} - e_n)$ is a negative root, which is clearly not true;
(b) If $\alpha = 2e_n$, then $\mathfrak{g}_{-(e_{n-1}+e_n)} \not\subseteq \mathfrak{h}$. This implies that $\delta + e_{n-1} + e_n = -(2e_1 + e_n - e_{n-1})$ is a negative root, which is clearly not true.

(3) **If $\mathfrak{g}$ is of type D_n.** Here again $\delta + \alpha = -(e_1 + e_2)$. Thus:

(a) If $\alpha = e_k - e_{k+1}$ with $2 \le k \le n-2$. Then $\mathfrak{g}_{-(e_k+e_{n-1})} \not\subseteq \mathfrak{h}$. This implies that $\delta + e_k + e_{n-1} = -(e_1 + e_2 - e_{k+1} - e_{n-1})$ is a negative root, which is clearly not true;
(b) If $\alpha = e_{n-1} - e_n$ and $n \neq 4$. Then $\mathfrak{g}_{-(e_{n-2}-e_n)} \not\subseteq \mathfrak{h}$. This implies that $\delta + e_{n-2} - e_n = -(e_1 + e_2 + e_{n-1} - e_{n-2})$ is a negative root which is clearly not true;
(c) If $n = 4$ and $\alpha = e_3 - e_4$, then $\delta = -(e_1 + e_2 + e_3 - e_4)$.
(d) If $\alpha = e_1 - e_2$, then in this case, $\delta = -2e_1$.
(e) If $\alpha = e_{n-1} + e_n$ with $n \neq 4$. Then $\delta = -(e_1 + e_2 + e_{n-1} + e_n)$. But $\delta + (e_3 + e_n) = -e_1 - e_2 - e_{n-1} + e_3$ is not a negative root
(f) If $n = 4$ and $\alpha = e_3 + e_4$, then $\delta = -(e_1 + e_2 + e_3 + e_4)$.

(4) **If $\mathfrak{g}$ is of type A_n.** Here $\delta + \alpha = -(e_1 - e_{n+1})$ and $\alpha = e_k - e_{k+1}$. If $n \neq 3$ or $k = 1, n$, then either $\mathfrak{g}_{-(e_{k-1}-e_{k+1})} \not\subseteq \mathfrak{h}$ or $\mathfrak{g}_{-(e_k-e_{k+2})} \not\subseteq \mathfrak{h}$.However,neither $\delta + (e_{k-1} - e_{k+1}) = -(e_1 - e_{n+1} + e_k - e_{k+1})$ nor $\delta + (e_{k-1} - e_{k+1}) = -(e_1 - e_{n+1} + e_{k+1} - e_{k+2})$ are negative roots. If $n = 3$ and $k = 2$, then $\alpha = e_2 - e_3$ so that $\delta = -(e_2 - e_3) - (e_1 - e_4)$. In this case, $\mathfrak{g}_{e_1-e_2} \subset \mathfrak{h}$ and $\mathfrak{g}_{e_3-e_4} \subset \mathfrak{h}$. But $A_3 = D_3$, so we are in the last case.□

6.2.2 Recovering the Einstein Space

Using the fact that the nilpotent part of $\mathfrak{h}$ acts isometrically we show:

Proposition 6.6 *The simple Lie algebra $\mathfrak{g}$ is of type:*

(1) *B_n with $n \geq 3$, $\alpha = e_1 - e_2$ and $\delta = -2e_1$ or;*
(2) *D_n with $n \geq 3$, $\alpha = e_1 - e_2$ and $\delta = -2e_1$.*

Proof Following Proposition 6.5, all we need to prove is that cases (1), (2), and (3) are impossible.

The $\mathfrak{so}(7,\mathbb{C})$ case. We assume that $\mathfrak{g}$ is $\mathfrak{so}(7,\mathbb{C})$. It is a complex simple Lie algebra of type B_3. Its standard root decomposition is described in [16, Pages 127-128]. In particular, the root spaces are given by, $\mathfrak{g}_\alpha = \mathbb{C}E_\alpha$.

We assume that the subalgebra $\mathfrak{h}$ is generated by $\mathfrak{a}$, $\mathfrak{g}_+$, $\mathfrak{g}_{e_2-e_1}$ and $\mathfrak{g}_{e_3-e_2}$. In this case, $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{g}_{-e_1} \oplus \mathfrak{g}_{-e_2} \oplus \mathfrak{g}_{-e_3} \oplus \mathfrak{g}_{-e_2-e_3} \oplus \mathfrak{g}_{-e_1-e_3} \oplus \mathfrak{g}_{-e_1-e_2}$ and $\delta = -e_1 - e_2 - e_3$.

On the one hand, using Eq. 3 with:

(1) $p_1 = E_{(e_1-e_2)}, u_1 = E_{-e_1}, v_1 = E_{-(e_1+e_3)}$
(2) $p_2 = E_{(e_2-e_3)}, u_2 = E_{-e_2}, v_2 = E_{-(e_1+e_2)}$
(3) $p_3 = E_{-(e_1-e_3)}, u_3 = E_{-e_3}, v_3 = E_{-(e_2+e_3)}$

gives us:

(1) $\langle u_1, \mathrm{ad}_{p_1} v_1\rangle + \langle \mathrm{ad}_{p_1} u_1, v_1\rangle = 0$
(2) $\langle u_2, \mathrm{ad}_{p_2} v_2\rangle + \langle \mathrm{ad}_{p_2} u_2, v_2\rangle = 0$
(3) $\langle u_3, \mathrm{ad}_{p_3} v_3\rangle + \langle \mathrm{ad}_{p_3} u_3, v_3\rangle = 0$

On the other hand, we have: $\mathrm{ad}_{p_1} u_1 = -2u_2$, $\mathrm{ad}_{p_1} v_1 = -2v_3$, $\mathrm{ad}_{p_2} u_2 = -2u_3$, $\mathrm{ad}_{p_2} v_2 = -2v_1$, $\mathrm{ad}_{p_3} u_3 = -2u_1$, and $\mathrm{ad}_{p_3} v_3 = -2v_2$. This leads to

$$\langle u_1, v_3\rangle = -\langle u_2, v_1\rangle = \langle u_3, v_2\rangle = -\langle u_1, v_3\rangle,$$

and hence $\langle u_1, v_3\rangle = 0$, which contradicts the fact that $\mathfrak{g}_{-e_1}$ is paired with $\mathfrak{g}_{-e_2-e_3}$.

The $\mathfrak{so}(8,\mathbb{C})$ case. We assume that $\mathfrak{g}$ is $\mathfrak{so}(8,\mathbb{C})$. It is a complex simple Lie algebra of type D_4. Its standard root decomposition is described in [16, Pages 128]. In particular, the root spaces are given by, $\mathfrak{g}_\alpha = \mathbb{C}E_\alpha$.

We assume that the subalgebra $\mathfrak{h}$ is generated by $\mathfrak{a}$, $\mathfrak{g}_+$, $\mathfrak{g}_{e_2-e_1}$, $\mathfrak{g}_{e_3-e_2}$, and $\mathfrak{g}_{e_4-e_3}$. In this case, $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{g}_{-(e_1+e_2)} \oplus \mathfrak{g}_{-(e_2+e_4)} \oplus \mathfrak{g}_{-(e_1+e_4)} \oplus \mathfrak{g}_{-(e_2+e_3)} \oplus \mathfrak{g}_{-(e_1+e_3)} \oplus \mathfrak{g}_{-(e_3+e_4)}$, and $\delta = -e_1 - e_2 - e_3 - e_4$.

Using Eq. 3 with:

(1) $p_1 = E_{-(e_2-e_3)}, u_1 = E_{-(e_1+e_3)}, v_1 = E_{-(e_3+e_4)}$
(2) $p_2 = E_{-(e_1-e_2)}, u_2 = E_{-(e_2+e_3)}, v_2 = E_{-(e_2+e_4)}$
(3) $p_3 = E_{(e_1-e_3)}, u_3 = E_{-(e_1+e_2)}, v_3 = E_{-(e_1+e_4)}$

along with the commutation relations: $\mathrm{ad}_{p_1} u_1 = 2u_3$, $\mathrm{ad}_{p_1} v_1 = 2v_2$, $\mathrm{ad}_{p_2} u_2 = 2u_1$, $\mathrm{ad}_{p_2} v_2 = 2v_3$, $\mathrm{ad}_{p_3} u_3 = -2u_2$, and $\mathrm{ad}_{p_3} v_3 = -2v_1$, give us

$$\langle u_1, v_2\rangle = -\langle u_3, v_1\rangle = \langle u_2, v_3\rangle = -\langle u_1, v_2\rangle,$$

and hence $\langle u_1, v_2\rangle = 0$, which contradicts the fact that $\mathfrak{g}_{-(e_1+e_3)}$ is paired with $\mathfrak{g}_{-(e_2+e_4)}$.

To finish, assume that the subalgebra $\mathfrak{h}$ is generated by $\mathfrak{a}$, $\mathfrak{g}_+$, $\mathfrak{g}_{e_2-e_1}$, $\mathfrak{g}_{e_3-e_2}$, and $\mathfrak{g}_{-e_3-e_4}$. In this case,

$\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{g}_{-(e_1+e_2)} \oplus \mathfrak{g}_{-(e_1+e_3)} \oplus \mathfrak{g}_{-(e_2+e_3)} \oplus \mathfrak{g}_{-(e_1-e_4)} \oplus \mathfrak{g}_{-(e_2-e_4)} \oplus \mathfrak{g}_{-(e_3-e_4)}$, and $\delta = -e_1 - e_2 - e_3 + e_4$.

Again, we use Eq. 3 with:

(1) $p_1 = E_{-(e_2-e_3)}$, $u_1 = E_{-(e_1+e_3)}$, $v_1 = E_{-(e_3-e_4)}$
(2) $p_2 = E_{-(e_1-e_2)}$, $u_2 = E_{-(e_2+e_3)}$, $v_2 = E_{-(e_2-e_4)}$
(3) $p_3 = E_{(e_1-e_3)}$, $u_3 = E_{-(e_1+e_2)}$, $v_3 = E_{-(e_1-e_4)}$,

together with the commutation relations: $\mathrm{ad}_{p_1} u_1 = 2u_3$, $\mathrm{ad}_{p_1} v_1 = 2v_2$, $\mathrm{ad}_{p_2} u_2 = 2u_1$, $\mathrm{ad}_{p_2} v_2 = 2v_3$, $\mathrm{ad}_{p_3} u_3 = -2u_2$, and $\mathrm{ad}_{p_3} v_3 = -2v_1$, to get

$$\langle u_1, v_2 \rangle = -\langle u_3, v_1 \rangle = \langle u_2, v_3 \rangle = -\langle u_1, v_2 \rangle,$$

and hence $\langle u_1, v_2 \rangle = 0$, which contradicts the fact that $\mathfrak{g}_{-(e_1+e_3)}$ is paired with $\mathfrak{g}_{-(e_2-e_4)}$. □

Now, this last Proposition, together with the fact that we already have examples of such types (see Example 1.1.1) gives us:

Corollary 6.7 *If $\mathfrak{a} \oplus \mathfrak{g}_+ \subseteq \mathfrak{h}$ and rank$(\mathfrak{g}) \geq 3$, then M is conformally flat. Moreover, $G = \mathsf{SO}(n+2, \mathbb{C})$ and $M = Eins_n(\mathbb{C})$.*

6.3 Classification Theorem: Lower Rank Parabolic Case

In this part we need to deal with the parabolic case where, after modification, the Lie algebra $\mathfrak{g}$ is of $\mathrm{rank}(\mathfrak{g}) \leq 2$.

If $\mathrm{rank}(\mathfrak{g}) = 1$, then M is conformally equivalent to $\mathbb{CP}^1$. If $\mathfrak{g}$ is of type $A_1 \times A_1$, then, up to finite cover, G is $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ and $H = P_1 \times P_2$ where P_1, P_2 are Borel subgroups of G. Hence M is conformally equivalent to $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Now we are left with A_2, B_2 or G_2 types. We have:

Proposition 6.8 *The Lie algebra $\mathfrak{g}$ is of type:*

(1) *B_2 with $\alpha = e_1 - e_2$ and $\delta = -2e_1$; or*
(2) *G_2 with $\alpha = e_1 - e_2$ and $\delta = -2(e_3 - e_2)$.*

Proof Assume first that $\mathfrak{g}$ is of type A_2. In this case $\delta + \alpha = -(e_1 - e_3)$, and without loss of generality we can suppose that $\alpha = e_1 - e_2$. As $\mathfrak{g}_{-(e_2-e_3)}$ acts isometrically, we use Eq. 3 with $0 \neq p \in \mathfrak{g}_{-(e_2-e_3)}$, $0 \neq u = v \in \mathfrak{g}_{-(e_1-e_2)}$ to get $\langle [p,u], u \rangle = 0$. But this contradicts the fact that $\mathfrak{g}_{-(e_1-e_2)}$ is paired with $\mathfrak{g}_{-(e_1-e_3)}$.

In the case where $\mathfrak{g}$ is of type B_2, we have $\delta + \alpha = -(e_1 + e_2)$. If $\alpha = e_2$, then $\delta = -e_1 - 2e_2$. But $\mathfrak{g}_{e_1} \nsubseteq \mathfrak{h}$ thus $\delta + e_1 = -2e_2$ is a negative root which is a clearly false. Thus $\alpha = e_1 - e_2$ and $\delta = -2e_1$.

Finally, if $\mathfrak{g}$ is of type G_2, then $\delta + \alpha = 2e_3 - e_1 - e_2$. Assume that $\alpha = -2e_1 + e_2 + e_3$, thus $\delta = -3(e_3 - e_1)$. As $\mathfrak{g}_{-(e_3-e_2)} \nsubseteq \mathfrak{h}$, we have $\delta + (e_3 - e_2) = 3e_1 - e_2 - 2e_3$ is a negative root, which is not true. □

End of proof of Theorem 1.2 *Assume first that $\mathfrak{g}$ is of type B_2. Then by Proposition 6.8, $\alpha = e_1 - e_2$ and $\delta = -2e_1$. As we already have an example of such situation, we get that $G = \mathrm{SO}(5, \mathbb{C})$ and $M = Eins_3(\mathbb{C})$.*

To finish, we assume that $\mathfrak{g}$ is of type G_2 with $\alpha = e_1 - e_2$ and $\delta = -2(e_3 - e_2)$. In this case the subalgebra $\mathfrak{h}$ is generated by $\mathfrak{a}$, $\mathfrak{g}_+$, and $\mathfrak{g}_{-(-2e_1+e_2+e_3)}$, so that $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{g}_{-(e_1-e_2)} \oplus \mathfrak{g}_{-(e_3-e_1)} \oplus \mathfrak{g}_{-(e_3-e_2)} \oplus \mathfrak{g}_{-(-2e_2+e_1+e_3)} \oplus \mathfrak{g}_{-(2e_3-e_1-e_2)}$. Recall that the root space decomposition of $\mathfrak{g}$ is given by $\mathfrak{g}_\alpha = \mathbb{C}E_\alpha$, with in particular the following commutation relations, among others:

(1) $\left[E_{-(-2e_1+e_2+e_3)}, E_{-(e_1-e_2)}\right] = -E_{-(e_3-e_1)}$;
(2) $\left[E_{-(-2e_1+e_2+e_3)}, E_{-(-2e_2+e_1+e_3)}\right] = -E_{-(2e_3-e_1-e_2)}$;
(3) $\left[E_{e_3-e_1}, E_{-(e_3-e_2)}\right] = -2E_{-(e_1-e_2)}$;
(4) $\left[E_{e_3-e_1}, E_{-(2e_3-e_1-e_2)}\right] = E_{-(e_3-e_2)}$;
(5) $\left[E_{e_1-e_2}, E_{-(e_3-e_2)}\right] = -2E_{-(e_3-e_1)}$;
(6) $\left[E_{e_1-e_2}, E_{-(-2e_2+e_1+e_3)}\right] = -E_{-(e_3-e_2)}$.

On the one hand, *M* is identified, as a homogeneous space, to the complex Einstein space.

On the other hand, let $\langle ., . \rangle$ be the complex bilinear form defined on $\mathfrak{g}/\mathfrak{h}$ by:

(1) $\mathfrak{g}_{-(e_1-e_2)}$ *is paired with* $\mathfrak{g}_{-(2e_3-e_1-e_2)}$, $\mathfrak{g}_{-(e_3-e_1)}$ *with* $\mathfrak{g}_{-(-2e_2+e_1+e_2)}$, *and* $\mathfrak{g}_{-(e_3-e_2)}$ *with itself;*
(2) $\langle E_{-(e_1-e_2)}, E_{-(2e_3-e_1-e_2)} \rangle = 1$;
(3) $\langle E_{-(e_3-e_1)}, E_{-(-2e_2+e_1+e_2)} \rangle = -1$;
(4) $\langle E_{-(e_3-e_2)}, E_{-(e_3-e_2)} \rangle = 2$;

Then it is straightforward to verify that the conformal class of $\langle ., . \rangle$ is uniquely preserved by $\mathfrak{h}$. Thus *M* admits a unique conformal holomorphic Riemannian structure invariant under the action of the simple Lie group G_2. In addition, this conformal structure is flat. Hence *M* is the Einstein space and G_2 admits a representation in $SO(7, \mathbb{C})$ (see also [1]).

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflicts of Interest The authors have no conflict of interest to declare that are relevant to this article.

References

1. Agricola, I.: Old and new on the exceptional group G_2. Notices Amer. Math. Soc. **55**(8), 922–929 (2008)
2. Ambra, G.D., Gromov, M.: Lectures on transformation groups: Geometry and dynamics. Surv. Differ. Geom. **1**, 19–111 (1990)
3. Bader, U., Nevo, A.: Conformal actions of simple Lie groups on compact pseudo-Riemannian manifolds. J. Differential Geom. **60**(3), 355–387 (2002)

4. Belraouti, M., Deffaf, M., Raffed, Y., Zeghib, A.: Pseudo-conformal actions of the mobius group. Differ. Geom. Appl. **91**, 102070 (2023)
5. Biswas, I., Dumitrescu, S.: Holomorphic Riemannian metric and the fundamental group. Bull. Soc. Math. France **147**(3), 455–468 (2019)
6. Borel, A.: Groupes lineaires algebriques. Ann. Math. **64**(1), 20–82 (1956)
7. Dumitrescu, S., Zeghib, A.: Global rigidity of holomorphic Riemannian metrics on compact complex 3-manifolds. Math. Ann. **345**(1), 53–81 (2009)
8. Frances, C.: About pseudo-Riemannian Lichnerowicz conjecture. Transformation Groups **20**(4), 1015–1022 (2015)
9. Frances, C., Melnick, K.: Formes normales pour les champs conformes pseudo-riemanniens. Bull. Soc. Math. France **141**(3), 377–421 (2013)
10. Frances, C., Zeghib, A.: Some remarks on conformal pseudo-Riemannian actions of simple Lie groups. Math. Res. Lett. **12**(1), 49–56 (2005)
11. Gasqui, J., Goldschmidt, H.: On the geometry of the complex quadric. Hokkaido Math. J. **20**(2), 279–312 (1991)
12. Gasqui, J., Goldschmidt, H.: The infinitesimal rigidity of the complex quadric of dimension four. American Journal of Mathematics **116**(3), 501–539 (1994)
13. Inoue, M., Kobayashi, S., Ochiai, T.: Holomorphic affine connections on compact complex surfaces. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27**(2), 247–264 (1980)
14. Klein S.: The complex quadric from the standpoint of Riemannian geometry. PhD thesis, Universität zu Köln (2005)
15. Klein, S.: Totally geodesic submanifolds of the complex quadric. Differential Geometry and its Applications **26**(1), 79–96 (2008)
16. Knapp, A.W.: Lie Groups Beyond an Introduction. Progress in Mathematics, vol. 140. Birkhäuser Boston Inc, Boston, MA (1996)
17. Kobayashi S., Nomizu K.: Foundations of Differential Geometry. Vol. II, volume Vol. II of Interscience Tracts in Pure and Applied Mathematics, No. 15. Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney (1969)
18. Melnick, K., Pecastaing, V.: The conformal group of a compact simply connected Lorentzian manifold. J. Amer. Math. Soc. **35**(1), 81–122 (2022)
19. Musso, E., Nicolodi, L.: Conformal geometry of isotropic curves in the complex quadric. Internat. J. Math. **33**(8), 2250054 (2022)
20. Pecastaing, V.: Essential conformal actions of PSL(2, **R**) on real-analytic compact Lorentz manifolds. Geom. Dedicata **188**, 171–194 (2017)
21. Pecastaing, V.: Conformal actions of real-rank 1 simple Lie groups on pseudo-Riemannian manifolds. Transformation Groups **24**(4), 1213–1239 (2019)
22. Pecastaing, V.: Conformal actions of higher rank lattices on compact pseudo-Riemannian manifolds. Geom. Funct. Anal. **30**(3), 955–987 (2020)
23. Serre J.-P.: Complex Semisimple Lie Algebras. Springer-Verlag, New York . Translated from the French by G. A. Jones (1987)
24. Zimmer, R.J.: Split rank and semisimple automorphism groups of *G*-structures. J. Differential Geom. **26**(1), 169–173 (1987)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.