SIMPLY CONNECTED INDEFINITE HOMOGENEOUS SPACES OF FINITE VOLUME

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Abstract. Let M be a simply connected pseudo-Riemannian homogeneous space of finite volume with isometry group G. We show that M is compact and that the solvable radical of G is abelian and the Levi factor is a compact semisimple Lie group acting transitively on M. For metric index less than three, we find that the isometry group of M is compact itself. Examples demonstrate that G is not necessarily compact for higher indices. To prepare these results, we study Lie algebras with abelian solvable radical and a nil-invariant symmetric bilinear form. For these, we derive an orthogonal decomposition into three distinct types of metric Lie algebras.

Contents

1. Introduction and main results

In this article we are interested in the isometry groups of simply connected homogeneous pseudo-Riemannian manifolds of finite volume. D'Ambra [3, Theorem 1.1] showed that a simply connected compact analytic Lorentzian manifold (not necessarily homogeneous) has compact isometry group, and she also gave an example of a simply connected compact analytic manifold of metric signature $(7,2)$ that has a non-compact isometry group.

Here we study homogeneous spaces for arbitrary metric signature. Our main tool is the structure theory of the isometry Lie algebras developed by the authors in [2]. The metric on the homogeneous space induces a symmetric bilinear form on the isometry Lie algebra, and as shown in [1, 2], the existence of a finite invariant measure then implies that this bilinear form is nil-invariant. The first main result is the following theorem:

Theorem A. *Let M be a connected and simply connected pseudo-Riemannian homogeneous space of finite volume, $G = \mathrm{Iso}(M)^\circ$, and let H be the stabilizer subgroup in G of a point in M. Let $G = KR$ be a Levi decomposition, where R is the solvable radical of G. Then:*

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(1) *M is compact.*
(2) *K is compact and acts transitively on M.*
(3) *R is abelian. Let A be the maximal compact subgroup of R. Then $A = \mathrm{Z}(G)^\circ$. More explicitely, $R = A \times V$ where $V \cong \mathbb{R}^n$ and $V^K = \mathbf{0}$.*
(4) *H is connected. If $\dim R > 0$, then $H = (H \cap K)E$, where E and $H \cap K$ are normal subgroups in H, $(H \cap K) \cap E$ is finite, and E is the graph of a non-trivial homomorphism $\varphi : R \to K$, where the restriction $\varphi|_A$ is injective.*

In Section 4 we give examples of isometry groups of compact simply connected homogeneous M with non-compact radical. However, for metric index 1 or 2 the isometry group of a simply connected M is always compact:

Theorem B. *The isometry group of any simply connected pseudo-Riemannian homogeneous manifold of finite volume with metric index $\ell \leq 2$ is compact.*

As follows from Theorem A, the isometry Lie algebra of a simply connected pseudo-Riemannian homogeneous space of finite volume has abelian radical. This motivates a closer investigation of Lie algebras with abelian radical that admit nil-invariant symmetric bilinear forms in Section 3. Our main result is the following algebraic theorem:

Theorem C. *Let $\mathcal{G}$ be a Lie algebra whose solvable radical $\mathcal{R}$ is abelian. Suppose $\mathcal{G}$ is equipped with a nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ such that the kernel $\mathcal{G}^\perp$ of $\langle\cdot,\cdot\rangle$ does not contain a non-trivial ideal of $\mathcal{G}$. Let $\mathcal{K} \times \mathcal{S}$ be a Levi subalgebra of $\mathcal{G}$, where $\mathcal{K}$ is of compact type and $\mathcal{S}$ has no simple factors of compact type. Then $\mathcal{G}$ is an orthogonal direct product of ideals*

$$\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3,$$

with

$$\mathcal{G}_1 = \mathcal{K} \ltimes \mathcal{A}, \quad \mathcal{G}_2 = \mathcal{S}_0, \quad \mathcal{G}_3 = \mathcal{S}_1 \ltimes \mathcal{S}_1^*,$$

where $\mathcal{R} = \mathcal{A} \times \mathcal{S}_1^*$ and $\mathcal{S} = \mathcal{S}_0 \times \mathcal{S}_1$ are orthogonal direct products, and $\mathcal{G}_3$ is a metric cotangent algebra. The restrictions of $\langle\cdot,\cdot\rangle$ to $\mathcal{G}_2$ and $\mathcal{G}_3$ are invariant and non-degenerate. In particular, $\mathcal{G}^\perp \subseteq \mathcal{G}_1$.

For the definition of metric cotangent algebra, see Section 2. We call an algebra $\mathcal{G}_1 = \mathcal{K} \ltimes \mathcal{A}$ with $\mathcal{K}$ semisimple of compact type and $\mathcal{A}$ abelian a Lie algebra of *Euclidean type.* By Theorem A, isometry Lie algebras of compact simply connected pseudo-Riemannian homogeneous spaces are of Euclidean type. However, not every Lie algebra of Euclidean type appears as the isometry Lie algebra of a compact pseudo-Riemannian homogeneous space. In fact, this is the case for the Euclidean Lie algebras $\mathcal{E}_n = \mathcal{SO}_n \ltimes \mathbb{R}^n$ with $n \neq 3$.

Theorem D. *The Euclidean group $\mathrm{E}_n = \mathrm{O}_n \ltimes \mathbb{R}^n$, $n \neq 1, 3$, does not have compact quotients with a pseudo-Riemannian metric such that E_n acts isometrically and almost effectively.*

Note that E_n acts transitively and effectively on compact manifolds with finite fundamental group, as we remark at the end of Section 3.

Notations and conventions. For a Lie group G, we let G° denote the connected component of the identity. For a subgroup H of G, we write $\mathrm{Ad}_{\mathcal{G}}(H)$ for the adjoint representation of H on the Lie algebra $\mathcal{G}$ of G, to distinguish it from the adjoint representation $\mathrm{Ad}(H)$ on its own Lie algebra $\mathcal{H}$.

The *solvable radical* R of G is the maximal connected solvable normal subgroup of G. The *solvable radical* $\mathfrak{R}$ of $\mathfrak{G}$ is the maximal solvable ideal of $\mathfrak{G}$. The semisimple Lie algebra $\mathfrak{F} = \mathfrak{G}/\mathfrak{R}$ is a direct product $\mathfrak{F} = \mathfrak{K} \times \mathfrak{S}$, where $\mathfrak{K}$ is a semisimple Lie algebra of *compact type*, meaning its Killing form is definite, and $\mathfrak{S}$ is semisimple without factors of compact type.

The center of a group G, or a Lie algebra $\mathfrak{G}$, is denoted by $\mathrm{Z}(G)$, or $\mathfrak{Z}(\mathfrak{G})$, respectively. Similarly, the centralizer of a subgroup H in G (or a subalgebra $\mathfrak{H}$ in $\mathfrak{G}$) is denoted by $\mathrm{Z}_G(H)$ (or $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{H})$).

The action of a Lie group G on a homogeneous space M is *(almost) effective* if the stabilizer of any point in M does not contain a non-trivial (connected) normal subgroup of G.

If V is a G-module, then we write $V^G = \{v \in V \mid gv = v \text{ for all } g \in G\}$ for the module of *G-invariants*. Similary, $V^{\mathfrak{G}} = \{v \in V \mid xv = 0 \text{ for all } x \in \mathfrak{G}\}$ for a $\mathfrak{G}$-module.

For direct products of Lie algebras $\mathfrak{G}_1$, $\mathfrak{G}_2$ we write $\mathfrak{G}_1 \times \mathfrak{G}_2$, whereas $\mathfrak{G}_1 + \mathfrak{G}_2$ or $\mathfrak{G}_1 \oplus \mathfrak{G}_2$ refers to sums as vector spaces.

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2. Nil-invariant bilinear forms

Let $\mathfrak{G}$ be a finite-dimensional real Lie algebra, let $\mathrm{Inn}(\mathfrak{G})$ denote the inner automorphism group of $\mathfrak{G}$ and $\overline{\mathrm{Inn}(\mathfrak{G})}^{\mathrm{z}}$ its Zariski closure in $\mathrm{Aut}(\mathfrak{G})$. A symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{G}$ is called *nil-invariant* if for all $x_1, x_2 \in \mathfrak{G}$,

$$\langle \varphi x_1, x_2\rangle = -\langle x_1, \varphi x_2\rangle \tag{2.1}$$

for all nilpotent elements φ of the Lie algebra of $\overline{\mathrm{Inn}(\mathfrak{G})}^{\mathrm{z}}$. For a subalgebra $\mathfrak{H}$ of $\mathfrak{G}$, we say $\langle\cdot,\cdot\rangle$ is *$\mathfrak{H}$-invariant* if for all $x \in \mathfrak{H}$, $\mathrm{ad}_{\mathfrak{G}}(x)$ is skew-symmetric for $\langle\cdot,\cdot\rangle$.

The *kernel* of $\langle\cdot,\cdot\rangle$ is the subspace

$$\mathfrak{G}^\perp = \{x \in \mathfrak{G} \mid \langle x, y\rangle = 0 \text{ for all } y \in \mathfrak{G}\}.$$

We use a Levi decomposition of $\mathfrak{G}$,

$$\mathfrak{G} = (\mathfrak{K} \times \mathfrak{S}) \ltimes \mathfrak{R},$$

where $\mathfrak{K}$ is semisimple of compact type, $\mathfrak{S}$ is semisimple without factors of compact type, and $\mathfrak{R}$ is the solvable radical of $\mathfrak{G}$. Let further $\mathfrak{G}_{\mathrm{s}} = \mathfrak{S} \ltimes \mathfrak{R}$.

Theorem 2.1 ([2, Theorem A]). *Let $\mathfrak{G}$ be a finite-dimensional real Lie algebra with nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$. Let $\langle\cdot,\cdot\rangle_{\mathfrak{G}_{\mathrm{s}}}$ denote the restriction of $\langle\cdot,\cdot\rangle$ to $\mathfrak{G}_{\mathrm{s}}$. Then:*

1. *$\langle\cdot,\cdot\rangle_{\mathfrak{G}_{\mathrm{s}}}$ is invariant by the adjoint action of $\mathfrak{G}$ on $\mathfrak{G}_{\mathrm{s}}$.*
2. *$\langle\cdot,\cdot\rangle$ is invariant by the adjoint action of $\mathfrak{G}_{\mathrm{s}}$.*

This implies some orthogonality relations that will be useful later on:

$$\mathfrak{S} \perp [\mathfrak{K}, \mathfrak{G}], \quad \mathfrak{K} \perp [\mathfrak{S}, \mathfrak{G}]. \tag{2.2}$$

Theorem 2.2 ([2, Corollary C]). *Let $\mathcal{G}$ be a finite-dimensional real Lie algebra with nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$, where we further assume that $\mathcal{G}^\perp$ does not contain any non-zero ideal of $\mathcal{G}$. Let $\mathcal{Z}(\mathcal{G}_{\mathrm{s}})$ denote the center of $\mathcal{G}_{\mathrm{s}}$. Then*

$$\mathcal{G}^\perp \subseteq \mathcal{K} \ltimes \mathcal{Z}(\mathcal{G}_{\mathrm{s}}) \quad \text{and} \quad [\mathcal{G}^\perp, \mathcal{G}_{\mathrm{s}}] \subseteq \mathcal{Z}(\mathcal{G}_{\mathrm{s}}) \cap \mathcal{G}^\perp.$$

We say that $\langle\cdot,\cdot\rangle$ has *relative index* ℓ if the induced scalar product on $\mathcal{G}/\mathcal{G}^\perp$ has index ℓ. For relative index $\ell \leq 2$, we have a general structure theorem for $\mathcal{G}$.

Theorem 2.3 ([2, Theorem D]). *Let $\mathcal{G}$ be a finite-dimensional real Lie algebra with nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ of relative index $\ell \leq 2$, and assume that $\mathcal{G}^\perp$ does not contain any non-zero ideal of $\mathcal{G}$. Then:*

1. *The Levi decomposition of $\mathcal{G}$ is a direct sum of ideals $\mathcal{G} = \mathcal{K} \times \mathcal{S} \times \mathcal{R}$.*
2. *$\mathcal{G}^\perp$ is contained in $\mathcal{K} \times \mathcal{Z}(\mathcal{R})$ and $\mathcal{G}^\perp \cap \mathcal{R} = \mathbf{0}$.*
3. *$\mathcal{S} \perp (\mathcal{K} \times \mathcal{R})$ and $\mathcal{K} \perp [\mathcal{R}, \mathcal{R}]$.*

2.1. **Cotangent algebras.** Let $\mathcal{L}$ be a Lie algebra. A *cotangent algebra* constructed from $\mathcal{L}$ is a Lie algebra $\mathcal{G} = \mathcal{L} \ltimes \mathcal{L}^*$ where $\mathcal{L}$ acts on its dual space $\mathcal{L}^*$ by its coadjoint representation. We call $\mathcal{G}$ a *metric cotangent algebra* if it has a non-degenerate invariant scalar product $\langle\cdot,\cdot\rangle$ such that $\mathcal{L}^*$ is totally isotropic.

2.2. **Invariance by $\mathcal{G}^\perp$.** We are mainly interested in nil-invariant bilinear forms $\langle\cdot,\cdot\rangle$ on $\mathcal{G}$ induced by pseudo-Riemannian metrics on homogeneous spaces. In this case, $\langle\cdot,\cdot\rangle$ is invariant by the stabilizer subalgebra $\mathcal{G}^\perp$. We can then further sharpen the statement of Theorem 2.2.

Proposition 2.4. *Let $\mathcal{G}$ and $\langle\cdot,\cdot\rangle$ be as in Theorem 2.2. If in addition $\langle\cdot,\cdot\rangle$ is $\mathcal{G}^\perp$-invariant, then*

$$[\mathcal{G}^\perp, \mathcal{G}_{\mathrm{s}}] = \mathbf{0}.$$

The proof is based on the following immediate observations:

Lemma 2.5. *Suppose $\langle\cdot,\cdot\rangle$ is $\mathcal{G}^\perp$-invariant. Then $[[\mathcal{K}, \mathcal{G}^\perp], \mathcal{G}_{\mathrm{s}}] \subseteq \mathcal{G}^\perp \cap \mathcal{G}_{\mathrm{s}}$.*

and

Lemma 2.6. *Let $\mathcal{H}$ be any Lie algebra and V a module for $\mathcal{H}$. Suppose that the subalgebra $\mathcal{Q}$ of $\mathcal{H}$ is generated by the subspace $\mathcal{M}$ of $\mathcal{H}$. Then $\mathcal{Q} \cdot V = \mathcal{M} \cdot V$.*

Together with

Lemma 2.7. *Let $\mathcal{K}$ be semisimple of compact type and $\mathcal{K}_0$ a subalgebra of $\mathcal{K}$. Then the subalgebra $\mathcal{Q}$ generated $\mathcal{M} = \mathcal{K}_0 + [\mathcal{K}, \mathcal{K}_0]$ is an ideal of $\mathcal{K}$.*

Proof. Put $\mathcal{Z} = \mathcal{Z}_{\mathcal{K}}(\mathcal{K}_0)$. Then $[\mathcal{Z}, \mathcal{M}] \subseteq \mathcal{M}$ and $[[\mathcal{K}, \mathcal{K}_0], \mathcal{M}] \subseteq \mathcal{M} + [\mathcal{M}, \mathcal{M}]$. Since $\mathcal{K} = [\mathcal{K}, \mathcal{K}_0] + \mathcal{Z}$, this shows $[\mathcal{K}, \mathcal{M}] \subseteq \mathcal{Q}$. Since $\mathcal{Q}$ is linearly spanned by the iterated commutators of elements of $\mathcal{M}$, $[\mathcal{K}, \mathcal{Q}] \subseteq \mathcal{Q}$. □

Proof of Proposition 2.4. Let $\mathcal{K}_0$ be the image of $\mathcal{G}^\perp$ under the projection homomorphism $\mathcal{G} \to \mathcal{K}$. Note that by Theorem 2.2 above, $[\mathcal{G}^\perp, \mathcal{G}_{\mathrm{s}}] = [\mathcal{K}_0, \mathcal{G}_{\mathrm{s}}]$. Let $\mathcal{Q} \subseteq \mathcal{K}$ be the subalgebra generated by $\mathcal{M} = \mathcal{K}_0 + [\mathcal{K}, \mathcal{K}_0]$ and consider $V = \mathcal{G}_{\mathrm{s}}$ as a module for $\mathcal{Q}$. Since $\mathcal{Q}$ is an ideal of $\mathcal{K}$, $[\mathcal{Q}, V]$ is a submodule for $\mathcal{K}$, that is, $[\mathcal{K}, [\mathcal{Q}, V]] \subseteq [\mathcal{Q}, V]$. By Lemmas 2.5, 2.6 and Theorem 2.2 we have $[\mathcal{Q}, V] = [\mathcal{M}, V] \subseteq \mathcal{G}^\perp \cap \mathcal{Z}(\mathcal{G}_{\mathrm{s}})$. Hence, $\mathcal{J} = [\mathcal{M}, V] \subseteq \mathcal{G}^\perp$ is an ideal in $\mathcal{G}$, with $\mathcal{J} \supseteq [\mathcal{G}^\perp, \mathcal{G}_{\mathrm{s}}] = [\mathcal{K}_0, \mathcal{G}_{\mathrm{s}}]$. Since $\mathcal{G}^\perp$ contains no non-trivial ideals of $\mathcal{G}$ by assumption, we conclude that $\mathcal{J} = \mathbf{0}$. □

3. Metric Lie algebras with abelian radical

In this section we study finite-dimensional real Lie algebras $\mathcal{G}$ whose solvable radical $\mathcal{R}$ is abelian and which are equipped with a nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$.

3.1. An algebraic theorem.

The Lie algebras with abelian radical and a nil-invariant symmetric bilinear form decompose into three distinct types of metric Lie algebras.

Theorem C. *Let $\mathcal{G}$ be a Lie algebra whose solvable radical $\mathcal{R}$ is abelian. Suppose $\mathcal{G}$ is equipped with a nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ such that the kernel $\mathcal{G}^\perp$ of $\langle\cdot,\cdot\rangle$ does not contain a non-trivial ideal of $\mathcal{G}$. Let $\mathcal{K}\times\mathcal{S}$ be a Levi subalgebra of $\mathcal{G}$, where $\mathcal{K}$ is of compact type and $\mathcal{S}$ has no simple factors of compact type. Then $\mathcal{G}$ is an orthogonal direct product of ideals*

$$\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3,$$

with

$$\mathcal{G}_1 = \mathcal{K} \ltimes \mathcal{A}, \quad \mathcal{G}_2 = \mathcal{S}_0, \quad \mathcal{G}_3 = \mathcal{S}_1 \ltimes \mathcal{S}_1^*,$$

where $\mathcal{R} = \mathcal{A}\times\mathcal{S}_1^*$ and $\mathcal{S} = \mathcal{S}_0\times\mathcal{S}_1$ are orthogonal direct products, and $\mathcal{G}_3$ is a metric cotangent algebra. The restrictions of $\langle\cdot,\cdot\rangle$ to $\mathcal{G}_2$ and $\mathcal{G}_3$ are invariant and non-degenerate. In particular, $\mathcal{G}^\perp\subseteq\mathcal{G}_1$.

We split the proof into several lemmas. Consider the submodules of invariants $\mathcal{R}^{\mathcal{S}},\mathcal{R}^{\mathcal{K}}\subseteq\mathcal{R}$. Since $\mathcal{S}$, $\mathcal{K}$ act reductively, we have

$$[\mathcal{S},\mathcal{R}]\oplus\mathcal{R}^{\mathcal{S}} = \mathcal{R} = [\mathcal{K},\mathcal{R}]\oplus\mathcal{R}^{\mathcal{K}}.$$

Then $\mathcal{A}=\mathcal{R}^{\mathcal{S}}$, $\mathcal{B}=[\mathcal{S},\mathcal{R}^{\mathcal{K}}]$ and $\mathcal{C}=[\mathcal{S},\mathcal{R}]\cap[\mathcal{K},\mathcal{R}]$ are ideals in $\mathcal{G}$ and $\mathcal{R}=\mathcal{A}\oplus\mathcal{B}\oplus\mathcal{C}$. Recall from Theorem 2.1 that $\langle\cdot,\cdot\rangle$ is in particular $\mathcal{S}$- and $\mathcal{R}$-invariant.

Lemma 3.1. *$\mathcal{C}=\mathbf{0}$ and $\mathcal{R}$ is an orthogonal direct sum of ideals in $\mathcal{G}$*

$$\mathcal{R}=\mathcal{A}\oplus\mathcal{B}$$

where $[\mathcal{K},\mathcal{R}]\subseteq\mathcal{A}$ and $[\mathcal{S},\mathcal{R}]=\mathcal{B}$.

Proof. The $\mathcal{S}$-invariance of $\langle\cdot,\cdot\rangle$ immediately implies $\mathcal{A}\perp\mathcal{B}$. Since $\mathcal{R}$ is abelian, $\mathcal{R}$-invariance implies $\mathcal{C}\perp\mathcal{R}$. Since $\mathcal{C}\perp(\mathcal{S}\times\mathcal{K})$ by (2.2), this shows $\mathcal{C}$ is an ideal contained in $\mathcal{G}^\perp$, hence $\mathcal{C}=\mathbf{0}$. Now $[\mathcal{K},\mathcal{R}]\subseteq\mathcal{A}$ and $[\mathcal{S},\mathcal{R}]=\mathcal{B}$ by definition of $\mathcal{A}$ and $\mathcal{B}$. □

Lemma 3.2. *$\mathcal{G}$ is a direct product of ideals*

$$\mathcal{G}=(\mathcal{K}\ltimes\mathcal{A})\times(\mathcal{S}\ltimes\mathcal{B}),$$

where $(\mathcal{K}\ltimes\mathcal{A})\perp(\mathcal{S}\ltimes\mathcal{B})$.

Proof. The splitting as a direct product of ideals follows from Lemma 3.1. The orthogonality follows together with (2.2) and the fact that the $\mathcal{S}$-invariance of $\langle\cdot,\cdot\rangle$ implies $\mathcal{S}\perp\mathcal{A}$ and $\mathcal{K}\perp\mathcal{B}$. □

Lemma 3.3. *$\mathcal{G}^\perp\subseteq\mathcal{K}\ltimes\mathcal{A}$ and $\mathcal{S}\ltimes\mathcal{B}$ is a non-degenerate ideal of $\mathcal{G}$.*

Proof. $\mathcal{Z}(\mathcal{G}_{\mathrm{s}})=\mathcal{A}$, therefore $\mathcal{G}^\perp\subseteq\mathcal{K}\ltimes\mathcal{A}$ by Theorem 2.2. Since also $(\mathcal{S}\ltimes\mathcal{B})\perp(\mathcal{K}\ltimes\mathcal{A})$, we have $(\mathcal{S}\ltimes\mathcal{B})\cap(\mathcal{S}\ltimes\mathcal{B})^\perp\subseteq\mathcal{G}^\perp\subseteq\mathcal{K}\ltimes\mathcal{A}$. It follows that $(\mathcal{S}\ltimes\mathcal{B})\cap(\mathcal{S}\ltimes\mathcal{B})^\perp=\mathbf{0}$. □

To complete the proof of Theorem C, it remains to understand the structure of the ideal $\mathcal{S} \ltimes \mathcal{B}$, which by Theorem 2.1 and the preceding lemmas is a Lie algebra with an invariant non-degenerate scalar product given by the restriction of $\langle\cdot,\cdot\rangle$.

Lemma 3.4. *$\mathcal{B}$ is totally isotropic. Let $\mathcal{S}_0$ be the kernel of the $\mathcal{S}$-action on $\mathcal{B}$. Then $\mathcal{S}_0 = \mathcal{B}^\perp \cap \mathcal{S}$.*

Proof. Since $\langle\cdot,\cdot\rangle$ is $\mathcal{R}$-invariant and $\mathcal{R}$ is abelian, $\mathcal{B}$ is totally isotropic. For the second claim, use $\mathcal{B} \cap \mathcal{S}^\perp = \mathbf{0}$ and the invariance of $\langle\cdot,\cdot\rangle$. □

Lemma 3.5. *$\mathcal{S}$ is an orthogonal direct product of ideals $\mathcal{S} = \mathcal{S}_0 \times \mathcal{S}_1$ with the following properties:*

(1) *$\mathcal{S}_1 \ltimes \mathcal{B}$ is a metric cotangent algebra.*
(2) *$[\mathcal{S}_0, \mathcal{B}] = \mathbf{0}$ and $\mathcal{S}_0 = \mathcal{B}^\perp \cap \mathcal{S}$.*

Proof. The kernel $\mathcal{S}_0$ of the $\mathcal{S}$-action on $\mathcal{B}$ is an ideal in $\mathcal{S}$, and by Lemma 3.4 orthogonal to $\mathcal{B}$. Let $\mathcal{S}_1$ be the ideal in $\mathcal{S}$ such that $\mathcal{S} = \mathcal{S}_0 \times \mathcal{S}_1$. Then $\mathcal{S}_0 \perp \mathcal{S}_1$ by invariance of $\langle\cdot,\cdot\rangle$.

$\mathcal{S}_1$ acts faithfully on $\mathcal{B}$ and so $\mathcal{S}_1 \cap \mathcal{B}^\perp = \mathbf{0}$ by Lemma 3.4. Moreover, $\mathcal{S}_1 \ltimes \mathcal{B}$ is non-degenerate since $\mathcal{S} \ltimes \mathcal{B}$ is. But $\mathcal{B}$ is totally isotropic by Lemma 3.4, so non-degeneracy implies $\dim \mathcal{S}_1 = \dim \mathcal{B}$. Therefore $\mathcal{B}$ and $\mathcal{S}_1$ are dually paired by $\langle\cdot,\cdot\rangle$.

Now identify $\mathcal{B}$ with $\mathcal{S}_1^*$ and write $\xi(s) = \langle \xi, s\rangle$ for $\xi \in \mathcal{S}_1^*$, $s \in \mathcal{S}_1$. Then, once more by invariance of $\langle\cdot,\cdot\rangle$,

$$[s,\xi](s') = \langle [s,\xi], s'\rangle = \langle \xi, -[s,s']\rangle = \xi(-\mathrm{ad}(s)s') = (\mathrm{ad}^*(s)\xi)(s')$$

for all $s, s' \in \mathcal{S}_1$. So the action of $\mathcal{S}_1$ on $\mathcal{S}_1^*$ is the coadjoint action. This means $\mathcal{S} \ltimes \mathcal{B}$ is a metric cotangent algebra (cf. Subsection 2.1). □

Proof of Theorem C. The decomposition into the desired orthogonal ideals follows from Lemmas 3.2 to 3.5. The structure of the ideals $\mathcal{G}_2$ and $\mathcal{G}_3$ is Lemma 3.5. □

The algebra $\mathcal{G}_1$ in Theorem C is of Euclidean type. Let $\mathcal{G} = \mathcal{K} \ltimes V$, with $V \cong \mathbb{R}^n$, be an algebra of Euclidean type and let $\mathcal{K}_0$ be the kernel of the $\mathcal{K}$-action on V. Proposition 2.4 and the fact that the solvable radical V is abelian immediately give the following:

Proposition 3.6. *Let $\mathcal{G} = \mathcal{K} \ltimes V$ be a Lie algebra of Euclidean type, and suppose $\mathcal{G}$ is equipped with a symmetric bilinear form that is nil-invariant and $\mathcal{G}^\perp$-invariant, such that $\mathcal{G}^\perp$ does not contain a non-trivial ideal of $\mathcal{G}$. Then*

$$\mathcal{G}^\perp \subseteq \mathcal{K}_0 \times V. \tag{3.1}$$

The following Examples 3.7 and 3.8 show that in general a metric Lie algebra of Euclidean type cannot be further decomposed into orthogonal direct sums of metric Lie algebras. Both examples will play a role in Section 4.

Example 3.7. Let $\mathcal{K}_1 = \mathcal{SO}_3$, $V_1 = \mathbb{R}^3$, $V_0 = \mathbb{R}^3$ and $\mathcal{G} = (\mathcal{SO}_3 \ltimes V_1) \times V_0$ with the natural action of $\mathcal{SO}_3$ on V_1. Let $\varphi : V_1 \to V_0$ be an isomorphism and put

$$\mathcal{H} = \{(0, v, \varphi(v)) \mid v \in V_0\} \subset (\mathcal{K}_0 \ltimes V_1) \times V_0.$$

We can define a nil-invariant symmetric bilinear form on $\mathcal{G}$ by identifying $V_1 \cong \mathcal{SO}_3^*$ and requiring for $k \in \mathcal{K}_1$, $v_0 \in V_0$, $v_1 \in V_1$,

$$\langle k, v_0 + v_1\rangle = v_1(k) - \varphi^{-1}(v_0)(k),$$

and further $\mathcal{K}_1 \perp \mathcal{K}_1$, $(V_0 \oplus V_1) \perp (V_0 \oplus V_1)$. Then $\langle\cdot,\cdot\rangle$ has signature $(3,3,3)$ and kernel $\mathcal{H} = \mathcal{G}^\perp$, which is not an ideal in $\mathcal{G}$. Note that $\langle\cdot,\cdot\rangle$ is not invariant. Moreover, $\mathcal{K}_1 \ltimes V_1$ is not orthogonal to V_0. A direct factor $\mathcal{K}_0$ can be added to this example at liberty.

Example 3.8. We can modify the Lie algebra $\mathcal{G}$ from Example 3.7 by embedding the direct summand $V_0 \cong \mathbb{R}^3$ in a torus subalgebra in a semisimple Lie algebra $\mathcal{K}_0$ of compact type, say $\mathcal{K}_0 = \mathcal{SO}_6$, to obtain a Lie algebra $\mathcal{F} = (\mathcal{K}_1 \ltimes V_1) \times \mathcal{K}_0$. We obtain a nil-invariant symmetric bilinear form of signature $(15,3,3)$ on $\mathcal{F}$ by extending $\langle\cdot,\cdot\rangle$ by a definite form on a vector space complement of V_0 in $\mathcal{K}_0$. The kernel of the new form is still $\mathcal{G}^\perp = \mathcal{H}$.

3.2. **Nil-invariant bilinear forms on Euclidean algebras.** A *Euclidean algebra* is a Lie algebra $\mathcal{E}_n = \mathcal{SO}_n \ltimes \mathbb{R}^n$, where $\mathcal{SO}_n$ acts on $\mathbb{R}^n$ by the natural action.

By a *skew pairing* of a Lie algebra $\mathcal{L}$ and an $\mathcal{L}$-module V we mean a bilinear map $\langle\cdot,\cdot\rangle : \mathcal{L} \times V \to \mathbb{R}$ such that $\langle x, yv\rangle = -\langle y, xv\rangle$ for all $x, y \in \mathcal{L}$, $v \in V$. Note that any nil-invariant symmetric bilinear form on $\mathcal{G} = \mathcal{K} \ltimes \mathbb{R}^n$ yields a skew pairing of $\mathcal{K}$ and $\mathbb{R}^n$.

Proposition 3.9 ([2, Proposition A.5])**.** *Let $\langle\cdot,\cdot\rangle : \mathcal{SO}_3 \times V \to \mathbb{R}$ be a skew pairing for the (non-trivial) irreducible module V. If the skew pairing is non-zero, then V is isomorphic to the adjoint representation of $\mathcal{SO}_3$ and $\langle\cdot,\cdot\rangle$ is proportional to the Killing form.*

Example 3.10. Consider $\mathcal{G} = \mathcal{SO}_3 \ltimes \mathbb{R}^n$ with a nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$, and assume that the action of $\mathcal{SO}_3$ is irreducible. By Proposition 3.9, either $\mathcal{SO}_3 \perp \mathbb{R}^n$, or $n = 3$ and $\mathcal{SO}_3$ acts by its coadjoint representation on $\mathbb{R}^3 \cong \mathcal{SO}_3^*$, and $\langle\cdot,\cdot\rangle$ is the dual pairing. In the first case, $\mathbb{R}^n$ is an ideal in $\mathcal{G}^\perp$, and in the second case, $\langle\cdot,\cdot\rangle$ is invariant and non-degenerate.

Example 3.11. Let $\mathcal{G}$ be the Euclidean Lie algebra $\mathcal{SO}_4 \ltimes \mathbb{R}^4$ with a nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$. Since $\mathcal{SO}_4 \cong \mathcal{SO}_3 \times \mathcal{SO}_3$, and here both factors $\mathcal{SO}_3$ act irreducibly on $\mathbb{R}^4$, it follows from Example 3.10 that in $\mathcal{G}$, $\mathbb{R}^4$ is orthogonal to both factors $\mathcal{SO}_3$, hence to all of $\mathcal{SO}_4$. In particular, $\mathbb{R}^4$ is an ideal contained in $\mathcal{G}^\perp$.

Theorem 3.12. *Let $\langle\cdot,\cdot\rangle$ be a nil-invariant symmetric bilinear form on the Euclidean Lie algebra $\mathcal{SO}_n \ltimes \mathbb{R}^n$ for $n \geq 4$. Then the ideal $\mathbb{R}^n$ is contained in $\mathcal{G}^\perp$.*

Proof. For $n = 4$, this is Example 3.11. So assume $n > 4$. Consider an embedding of $\mathcal{SO}_4$ in $\mathcal{SO}_n$ such that $\mathbb{R}^n = \mathbb{R}^4 \oplus \mathbb{R}^{n-4}$, where $\mathcal{SO}_4$ acts on $\mathbb{R}^4$ in the standard way and trivially on $\mathbb{R}^{n-4}$. By Example 3.11, $\mathcal{SO}_4 \perp \mathbb{R}^4$. Since $\mathbb{R}^{n-4} \subseteq [\mathcal{SO}_n, \mathbb{R}^n]$, the nil-invariance of $\langle\cdot,\cdot\rangle$ implies $\mathcal{SO}_4 \perp \mathbb{R}^{n-4}$. Hence $\mathbb{R}^n \perp \mathcal{SO}_4$.

The same reasoning shows that $\mathrm{Ad}(g)\mathcal{SO}_4 \perp \mathbb{R}^n$, where $g \in \mathrm{SO}_n$. Then $\mathcal{B} = \sum_{g \in \mathrm{SO}_n} \mathrm{Ad}(g)\mathcal{SO}_4$ is orthogonal to $\mathbb{R}^n$. But $\mathcal{B}$ is clearly an ideal in $\mathcal{SO}_n$, so $\mathcal{B} = \mathcal{SO}_n$ by simplicity of $\mathcal{SO}_n$ for $n > 4$. □

Theorem D. *The Euclidean group $\mathrm{E}_n = \mathrm{O}_n \ltimes \mathbb{R}^n$, $n \neq 1, 3$, does not have compact quotients with a pseudo-Riemannian metric such that E_n acts isometrically and almost effectively.*

Proof. For $n > 3$, such an action of E_n would induce a nil-invariant symmetric bilinear form on the Lie algebra $\mathcal{SO}_n \ltimes \mathbb{R}^n$ without non-trivial ideals in its kernel, contradicting Theorem 3.12.

For $n = 2$, the Lie algebra $\mathcal{E}_2$ is solvable, and hence by Baues and Globke [1], any nil-invariant symmetric bilinear form must be invariant. For such a form, the ideal $\mathbb{R}^2$ of $\mathcal{E}_2$ must be contained in $\mathcal{E}_2^{\perp}$, and therefore action cannot be effective.

Note that $\mathcal{E}_3$ is an exception, as it is the metric cotangent algebra of $\mathcal{SO}_3$. □

Remark. Clearly the Lie group E_n admits compact quotient manifolds on which E_n acts almost effectively. For example take the quotient by a subgroup $F \ltimes \mathbb{Z}^n$, where $F \subset \mathrm{O}_n$ is a finite subgroup preserving $\mathbb{Z}^n$. Compact quotients with finite fundamental group also exist. For example, for any non-trivial homomorphism $\varphi : \mathbb{R}^n \to \mathrm{O}_n$, the graph H of φ is a closed subgroup of E_n isomorphic to $\mathbb{R}^n$, and the quotient $M = \mathrm{E}_n/H$ is compact (and diffeomorphic to O_n). Since H contains no non-trivial normal subgroup of E_n, the E_n-action on M is effective. Theorem D tells us that none of these quotients admit E_n-invariant pseudo-Riemannian metrics.

4. Simply connected compact homogeneous spaces with indefinite metric

Let M be a connected and simply connected pseudo-Riemannian homogeneous space of finite volume. Then we can write

$$M = G/H \tag{4.1}$$

for a connected Lie group G acting almost effectively and by isometries on M, and H is a closed subgroup of G that contains no non-trivial connected normal subgroup of G (for example, $G = \mathrm{Iso}(M)^{\circ}$). Note that H is connected since M is simply connected.

Let $\mathcal{G}$, $\mathcal{H}$ denote the Lie algebras of G, H, respectively. Recall that the pseudo-Riemannian metric on M induces an $\mathcal{H}$-invariant and nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathcal{G}$, and the kernel of $\langle\cdot,\cdot\rangle$ is precisely $\mathcal{G}^{\perp} = \mathcal{H}$ and contains no non-trivial ideal of $\mathcal{G}$.

We decompose $G = KSR$, where K is a compact semisimple subgroup, S is a semisimple subgroup without compact factors, R the solvable radical of G

Proposition 4.1. *The subgroup S is trivial and M is compact.*

Proof. As M is simply connected, $H = H^{\circ}$. Now $H \subseteq KR$ by Theorem 2.2. On the other hand, since M has finite invariant volume, the Zariski closure of $\mathrm{Ad}_{\mathcal{G}}(H)$ contains $\mathrm{Ad}_{\mathcal{G}}(S)$, see Mostow [7, Lemma 3.1]. Therefore S must be trivial. It follows from Mostow's result [6, Theorem 6.2] on quotients of solvable Lie groups that $M = (KR)/H$ is compact. □

We can therefore restrict ourselves in (4.1) to groups $G = KR$ and connected uniform subgroups H of G.

The structure of a general compact homogeneous manifold with finite fundamental group is surveyed in Onishchik and Vinberg [8, II.5.§2]. In our context it follows that

$$G = L \ltimes V \tag{4.2}$$

where V is a normal subgroup isomorphic to $\mathbb{R}^n$ and $L = KA$ is a maximal compact subgroup of G. The solvable radical is $R = A \ltimes V$. Moreover, $V^L = \mathbf{0}$. By a theorem of Montgomery [5] (also [8, p. 137]), K acts transitively on M.

The existence of a G-invariant metric on M further restricts the structure of G.

Proposition 4.2. *The solvable radical R of G is abelian. In particular, $R = A \times V$, $V^K = \mathbf{0}$ and $A = \mathrm{Z}(G)^\circ$.*

Proof. Let $\mathrm{Z}(R)$ denote the center of R and N its nilradical. Since H is connected, $H \subseteq K\mathrm{Z}(R)^\circ$ by Theorem 2.2. Hence there is a surjection $G/H \to G/(K\mathrm{Z}(R)^\circ) = R/\mathrm{Z}(R)^\circ$. It follows that $\mathrm{Z}(R)^\circ$ is a connected uniform subgroup. Therefore the nilradical N of R is $N = T\mathrm{Z}(R)^\circ$ for some compact torus T. But a compact subgroup of N must be central in R, so $T \subseteq \mathrm{Z}(R)$. Hence $N \subseteq \mathrm{Z}(R)$, which means $R = N$ is abelian. □

Combined with (4.2), we obtain

$$G = KR = (K_0 A) \times (K_1 \ltimes V), \tag{4.3}$$

with $K = K_0 \times K_1$, $R = A \times V$, where K_0 is the kernel of the K-action on V.

For any subgroup Q of G we write $H_Q = H \cap Q$.

Lemma 4.3. *$[H, H] \subseteq H_K$. In particular, H_K is a normal subgroup of H.*

Proof. By Proposition 3.6 and the connectedness of H, we have $H \subseteq K_0 R$. Since R is abelian, $[H, H] \subseteq H_{K_0}$. □

If G is simply connected, we have $K \cap R = \{e\}$. Then let p_K, p_R denote the projection maps from G to K, R.

Lemma 4.4. *Suppose G is simply connected. Then $\mathsf{p}_R(H) = R$.*

Proof. Since K acts transitively on M, we have $G = KH$. Then $R = \mathsf{p}_R(G) = \mathsf{p}_R(H)$. □

Proposition 4.5. *Suppose G is simply connected. Then the stabilizer H is a semidirect product $H = H_K \times E$, where E is the graph of a homomorphism $\varphi : R \to K$ that is non-trivial if $\dim R > 0$. Moreover, $\varphi(R \cap H) = \{e\}$.*

Proof. The subgroup H_K is a maximal compact subgroup of the stabilizer H. By Lemma 4.3, $H = H_K \times E$ for some normal subgroup E diffeomorphic to a vector space. By Lemma 4.4, H projects onto R with kernel H_K, so that $E \cong R$. Then E is necessarily the graph of a homomorphism $\varphi : R \to K$. If $\dim R > 0$, then φ is non-trivial, for otherwise $R \subseteq H$, in contradiction to the almost effectivity of the action. Observe that $R \cap H \subseteq E$. Therefore $\varphi(R \cap H) \subseteq H_K \cap E = \{e\}$. □

Now we can state our main result:

Theorem A. *Let M be a connected and simply connected pseudo-Riemannian homogeneous space of finite volume, $G = \mathrm{Iso}(M)^\circ$, and let H be the stabilizer subgroup in G of a point in M. Let $G = KR$ be a Levi decomposition, where R is the solvable radical of G. Then:*

1. *M is compact.*
2. *K is compact and acts transitively on M.*
3. *R is abelian. Let A be the maximal compact subgroup of R. Then $A = \mathrm{Z}(G)^\circ$. More explicitely, $R = A \times V$ where $V \cong \mathbb{R}^n$ and $V^K = \mathbf{0}$.*
4. *H is connected. If $\dim R > 0$, then $H = (H \cap K)E$, where E and $H \cap K$ are normal subgroups in H, $(H \cap K) \cap E$ is finite, and E is the graph of a non-trivial homomorphism $\varphi : R \to K$, where the restriction $\varphi|_A$ is injective.*

Proof. Claims (1), (2) and (3) follow from Proposition 4.1, Proposition 4.2 and (4.2), applied to $G = \mathrm{Iso}(M)^\circ$.

For claim (4), let $\widetilde{G}$ be the universal cover of G. Since G acts effectively on M, $\widetilde{G}$ acts almost effectively on M with stabilizer $\widetilde{H}$, the preimage of H in $\widetilde{G}$. Let $\widetilde{\varphi} : \widetilde{R} \to \widetilde{K}$ be the homomorphism given by Proposition 4.5 for $\widetilde{G}$. Then $\widetilde{R} = \widetilde{A} \oplus V$, with $\widetilde{A} \cong \mathbb{R}^k$ for some k, and $R = \widetilde{R}/Z$ for some central discrete subgroup $Z \subset \widetilde{A} \cap \widetilde{H}$. Since $Z \subset \widetilde{R} \cap \widetilde{H}$ we have $Z \subseteq \ker \widetilde{\varphi}$. Therefore $\widetilde{\varphi}$ descends to a homomorphism $R \to \widetilde{K}$, and by composing with the canonical projection $\widetilde{K} \to K$, we obtain a homomorphism $\varphi : R \to K$ with the desired properties. Observe that $\ker \varphi|_A \subset A \cap H$ is a normal subgroup in G. Hence it is trivial, as G acts effectively. □

Now assume further that the index of the metric on M is $\ell \leq 2$. Theorem 2.3 has strong consequences in the simply connected case.

Theorem B. *The isometry group of any simply connected pseudo-Riemannian homogeneous manifold of finite volume and metric index $\ell \leq 2$ is compact.*

Proof. We know from Theorem A that M is compact. Let $G = \mathrm{Iso}(M)^\circ$, with $G = KR$ as before. By Theorem 2.3, R commutes with K and thus $R = A$ by part 3 of Theorem A. It follows that $G = KA$ is compact.

Then K is a characteristic subgroup of G which also acts transitively on M. Therefore we may identify $\mathrm{T}_x M$ at $x \in M$ with $\mathcal{K}/(\mathcal{H} \cap \mathcal{K})$, where $\mathcal{K}$ is the Lie algebra of K. Hence the isotropy representation of the stabilizer $\mathrm{Iso}(M)_x$ factorizes over a closed subgroup of the automorphism group of $\mathcal{K}$. As this latter group is compact, the isotropy representation has compact closure in $\mathrm{GL}(\mathrm{T}_x M)$. If follows that there exists a Riemannian metric on M that is preserved by $\mathrm{Iso}(M)$. Hence $\mathrm{Iso}(M)$ is compact. □

Remark. Note that in fact the isometry group of every compact analytic simply connected pseudo-Riemannian manifold has finitely many connected components (Gromov [4, Theorem 3.5.C]).

For indices higher than two, the identity component of the isometry group of a simply connected M can be non-compact. This is demonstrated by the examples below in which we construct some M on which a non-compact group acts isometrically. The following Lemma 4.6 then ensures that these groups cannot be contained in any compact Lie group.

Lemma 4.6. *Assume that the action of K on V in the semidirect product $G = K \ltimes V$ is non-trivial. Then G cannot be immersed in a compact Lie group.*

Proof. Suppose there is a compact Lie group C that contains G as a subgroup. As the action of K on V is non-trivial, there exists an element $v \in V \subseteq C$ such that $\mathrm{Ad}_C(v)$ has non-trivial unipotent Jordan part. But by compactness of C, every $\mathrm{Ad}_C(g)$, $g \in C$, is semisimple, a contradiction. □

Example 4.7. Start with $G_1 = (\widetilde{\mathrm{SO}}_3 \ltimes \mathbb{R}^3) \times \mathrm{T}^3$, where $\widetilde{\mathrm{SO}}_3$ acts on $\mathbb{R}^3$ by the coadjoint action, and let $\varphi : \mathbb{R}^3 \to \mathrm{T}^3$ be a homomorphism with discrete kernel. Put

$$H = \{(\mathrm{I}_3, v, \varphi(v)) \mid v \in \mathbb{R}^3\}.$$

The Lie algebras $\mathcal{G}_1$ of G_1 and $\mathcal{H}$ of H are the corresponding Lie algebras from Example 3.7. We can extend the nil-invariant scalar product $\langle\cdot,\cdot\rangle$ on $\mathcal{G}_1$ from

Example 3.7 to a left-invariant tensor on G_1, and thus obtain a G_1-invariant pseudo-Riemannian metric of signature $(3,3)$ on the quotient $M_1 = G_1/H = \widetilde{\mathrm{SO}}_3 \times \mathrm{T}^3$. Here, M_1 is a non-simply connected manifold with a non-compact connected stabilizer.

In order to obtain a simply connected space, embed T^3 in a simply connected compact semisimple group K_0, for example $K_0 = \widetilde{\mathrm{SO}}_6$, so that G_1 is embedded in $G = (\widetilde{\mathrm{SO}}_3 \ltimes \mathbb{R}^3) \times K_0$. As in Example 3.8, we can extend $\langle \cdot,\cdot \rangle$ from $\mathcal{G}_1$ to $\mathcal{G}$, and thus obtain a compact simply connected pseudo-Riemannian homogeneous manifold $M = G/H = \widetilde{\mathrm{SO}}_3 \times K_0$.

Example 4.8. Example 4.7 can be generalized by replacing $\widetilde{\mathrm{SO}}_3$ by any simply connected compact semisimple group K, acting by the coadjoint representation on $\mathbb{R}^d$, where $d = \dim K$, and trivially on T^d. Define H similarly as a graph in $\mathbb{R}^d \times \mathrm{T}^d$, and embed T^d in a simply connected compact semisimple Lie group K_0.

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