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On homogeneous 3-dimensional spacetimes: focus on plane waves

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Abstract
We revisit the classification of Lorentz homogeneous spaces of dimension 3, and relax usual completeness assumptions. In particular, non-unimodular elliptic plane waves, and only them, are neither locally symmetric nor locally isometric to a left-invariant Lorentz metric on a 3-dimensional Lie group. We characterize homogeneous plane waves in dimension 3, and prove they are non-extendable, and geodesically complete only if they are symmetric. Finally, only one non-flat plane wave has a compact model.

Keywords Homogeneous · Plane wave · Heisenberg extension

Contents

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1 Introduction

In dimension two, homogeneous pseudo-Riemannian spaces have constant curvature, and so they have the maximal possible symmetries. This is no longer true in higher dimensions, some homogeneous spaces are more homogeneous than others! This applies to any pseudo-Riemannian type, but let us focus on the Lorentzian case in dimension three. We have two (non-disjoint) distinguished classes:

- On the one hand, the symmetric spaces. Among them, the most symmetric are those of constant sectional curvature, they are three, up to scaling, and they have isometry groups of maximal dimension equal to six. We then have three decomposable symmetric spaces, and two reducible non-decomposable (Cahen-Wallach) spaces having solvable isometry groups of dimension four.
- On the other hand, there is the wide world of *Lorentz groups*, i.e. Lie groups of dimension three endowed with a left-invariant Lorentz metric. They, generically, have an isometry group equal to the supporting group itself, but can sometimes have extra isometries.

One may ask whether any homogeneous Lorentz space of dimension 3 fits into one of these two classes, that is, whether any homogeneous Lorentz space of dimension three is locally symmetric or locally isometric to a left-invariant metric on a Lie group of dimension three. This was proven in the Riemannian case in [20] and in the Lorentzian case, assuming completeness, in [4].

The initial motivation of the present work was, on the one hand, to provide a simpler proof of this fact in the Lorentzian case, say a Lie theoretical proof instead of using tools from the homogeneous structures theory, which involves complicated computations. On the other hand, we aimed to fix some completeness as well as local vs. global issues in the literature. Trying this, we realized that there is a one-parameter family of examples of (incomplete) homogeneous Lorentz spaces that are neither locally symmetric nor locally isometric to Lorentz groups. In [10] the same family was ruled out incorrectly in the proofs of the classification of homogeneous Lorentz geometries having compact models, which we fix here without altering the statements (see 1.2.2).

1.1 Results on plane waves

Our first results concern the global structure of 3-dimensional *plane waves*. Most studies on the subject have a mathematical-physical origin and usually consider infinitesimal and local aspects, e.g. Killing fields, ... [1, 3, 18], instead of global isometry groups as in our present work. Actually, [3] contains classification results on homogeneous plane waves of arbitrary dimension, but from the point of view of theoretical physics.

1.1.1 Heisenberg extensions

Let Heis denote the 3-dimensional Heisenberg group and $\mathfrak{heis}$ denote its Lie algebra. Consider a semi-direct product $G_\rho = \mathbb{R} \ltimes \text{Heis}$, where $\mathbb{R}$ acts on Heis via a representation $t \to \rho(t) \in \text{Aut(Heis)}$.

Theorem 1.1 *Let I be any non-central one parameter subgroup of* Heis *generated by a vector $W \in \mathfrak{heis}$ and put $\mathrm{P}_\rho = G_\rho / I$. Then we have*

(1) The G_ρ-action on P_ρ preserves a Lorentz metric if and only if the projection of W to $\mathfrak{heis}/\mathrm{Z}$ is not an eigenvector of the action of $\rho(t)$ on $\mathfrak{heis}/\mathrm{Z}$ where Z is the center of $\mathfrak{heis}$. In addition, this Lorentz metric is unique up to scaling and automorphisms fixing W.

(2) There is no 3-dimensional Lie group acting on P_ρ isometrically with an open orbit if and only if the action of $\rho(t)$ on $\mathfrak{heis}/\mathrm{Z}$ is by similarities with non-real eigenvalues.

(3) The space P_ρ is symmetric exactly for those representations $t \to \rho(t) \in \text{Aut(Heis)}$ fixing the center Z, or equivalently, G_ρ is unimodular. In this case

- *P_ρ is the Minkowski space if ρ is unipotent.*
- *P_ρ is a hyperbolic or an elliptic Cahen-Wallach space.*

And besides the globally symmetric cases, P_ρ is locally symmetric exactly for the non-unimodular case for which G_ρ is isomorphic to $\text{Aff} \ltimes \mathbb{R}^2$, where Aff is a copy in $\mathrm{O}(1,2)$ of the affine group $\text{Aff}^+(\mathbb{R})$, preserving a degenerate plane $\mathbb{R}^2 \subset \mathbb{R}^3$. In this case, P_ρ is globally isometric to half Minkowski.

(4) P_ρ is isometric to the Minkowski space exactly when ρ is unipotent. In this case, The group G_ρ is isomorphic to $N \ltimes \mathbb{R}^3 \subset \mathrm{O}(1,2) \ltimes \mathbb{R}^3$ where N is a unipotent one-parameter subgroup of $\mathrm{O}(1,2)$.

(5) *There is only one flat non-complete case, isometric to half Minkowski, for which the group* G_ρ *is isomorphic to* $\mathrm{Aff} \ltimes \mathbb{R}^2 \subset \mathrm{O}(1,2) \ltimes \mathbb{R}^3$. *This corresponds to any non-unimodular representation* $t \to \rho(t) \in \mathrm{Aut}(\mathrm{Heis})$ *having a fixed vector in* $\mathfrak{heis}$.
(6) *Except the Minkowski case,* $\mathrm{Isom}_0(\mathrm{P}_\rho) = G_\rho$.
(7) *Two spaces* P_ρ *and* $\mathrm{P}_{\rho'}$ *are isometric if and only if* G_ρ *and* $G_{\rho'}$ *are isomorphic.*
(8) *Besides the two flat cases, two spaces* P_ρ *and* $\mathrm{P}_{\rho'}$ *are locally isometric if and only if* G_ρ *and* $G_{\rho'}$ *are isomorphic.*

Remark 1.2 Some of the results in Theorem 1.1 are known and we added them here for completeness. In particular, flat non-complete homogeneous connected Lorentz manifolds are classified in [8]. They are quotients of half Minkowski, whose boundary is a lightlike hyperplane, by discrete translation subgroups.

1.1.2 Plane waves

Plane waves are special pp-waves which in turn are special Brinkmann spacetimes. All these classes play a central role in General Relativity both from the mathematical and applied sides, especially since the discovery of gravitational waves.

A plane wave spacetime is defined by having a null parallel vector field V such that there are local *Brinkmann coordinates* in which the metric has the form

$$2dvdu + \sum H_{ij}(u)x^i x^j du^2 + \sum_{i=1}^{n-2}(dx^i)^2$$

where V corresponds to $\frac{\partial}{\partial v}$. They can also be defined by having *Rosen* coordinates in which the metric has the form $2dvdu + g_u$, where g_u is a flat metric depending on u i.e. of the form $g_u = g_{ij}(u)dx^i dx^j$. In an intrinsic way, a plane wave is characterized by having a null parallel vector field V such that its orthogonal distribution $V^\perp$ has flat leaves and, in addition, the space is "almost symmetric" i.e. $\nabla_W R = 0$ for any W tangent to $V^\perp$, where R is the Riemannian tensor.

The following classification result is due to [3] (at the Lie algebra level). We yield here a self-contained proof which we think is more limpid, especially regarding globality questions.

Theorem 1.3 *Let* (M, g) *be a simply connected homogeneous Lorentzian three-dimensional manifold. Then* (M, g) *is a plane wave if and only if it is globally isometric to some* P_ρ. *If a homogeneous plane wave* M *is not simply connected, then it is either flat or isometric to a cyclic central quotient of one of the Cahen-Wallach spaces.*

In fact, it is the center of the Heisenberg algebra which plays the role of the null parallel vector field. Observe, however, that the Heisenberg center is central in G_ρ exactly when P_ρ is a symmetric space (see Lemma 7.1).

1.1.3 Literature

There are lots of works on homogeneous plane waves. The most important is surely [3] by M. Blau and M. O'Loughlin, in 2003. It has very interesting mathematical content although it is intended for a mathematical physics audience. A relatively recent work in a pure mathematical context is [12], where the authors W. Globke and T. Leistner, aimed to know when a homogeneous pp-wave is in fact a plane wave.

We are inspired by these two references, and especially by [3] with which we have natural overlaps at many places. Our approach is however different. The authors in [3] start with a plane wave in local coordinates (of Brinkmann or Rosen type) and ask when it is (infinitesimally) homogeneous. By a direct analysis of the Killing equations, they determine the isometry algebra as generated by Killing fields satisfying some bracket relations, and prove that it contains a Heisenberg algebra. It appears then that the Killing algebra contains an $\mathbb{R}$-extension of the Heisenberg algebra (although the action of the one-parameter group of (infinitesimal) automorphisms on the Heisenberg algebra is not visible).

To exhibit simply connected homogeneous plane waves explicitly as Lorentz homogeneous spaces, which is the aim of Theorem 1.3, one has to determine the isometry group of such structures. This could be achieved by identifying the extended Heisenberg algebra in [3], and studying the completeness of the Killing fields (which in general is difficult to check). However, the approach we adopt here to the classification result is different and (mostly) Lie theoretical, dealing directly with groups. Our starting point is to consider a Lorentz homogeneous space of dimension 3, assume (after reduction) that it has a 4-dimensional isometry group, observe that (generally) this 4-dimensional group is a Heisenberg extension, and then bring out its plane wave structure. This approach has the advantage of providing the homogeneous space directly, overcoming the (highly non-trivial) analysis of the Killing equation on the one hand, and the completeness problem of the Killing fields on the other hand (to see the difference in a toy example, observe that the Euclidean plane and any of its open subsets share the same Killing algebra, but this open set has generically a trivial isometry group).

We hope that our present article plays a role of complement and companion of Blau and O'Loughlin's paper in this subject of homogeneous plane waves. Of course, contrary to [3], we do not deal here with physical aspects like supergravity and quantization... and conversely, [3] does not cover geometric (and global) aspects like completeness, extendibility and existence of compact quotients.

1.1.4 Some terminology

It appears worthwhile for us to give names to the plane waves classes in the following way

- The flat complete case corresponding to ρ unipotent.
- The Cahen-Wallach, hyperbolic or elliptic, case corresponding to ρ unimodular semi-simple on $\mathfrak{heis}/\mathrm{Z}$.
- In the remaining non-unimodular cases, let us call P_ρ a *hyperbolic* (resp. *elliptic*, resp. *parabolic*) non-unimodular plane wave, according to the action of G_ρ on $\mathfrak{heis}/\mathrm{Z}$ being diagonalizable with real eigenvalues (resp. non-real eigenvalues, resp. real non-diagonalizable).

1.1.5 Global coordinates system

It is known that a Cahen-Wallach symmetric space has a global chart where the metric has the form $g = 2dudv \pm x^2du^2 + dx^2$ (the minus sign corresponds to the elliptic case and the plus to the hyperbolic one, see for instance [13]). The global coordinate system, in the next theorem, already appears in [3].

Theorem 1.4 *Assume ρ is non-unimodular. Then P_ρ has a global coordinate system $(u, v, x) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ where the metric has the form*

$$g = 2dudv + \frac{b(\rho)x^2}{u^2}du^2 + dx^2.$$

The invariant $b(\rho)$ is defined by $\rho(t) = e^{tA}$ where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & b & 1 \end{pmatrix}$ acting on the Lie algebra $\mathfrak{heis}$ *(endowed with the basis $\{Z, X, Y\}$, with $[X, Y] = Z$). Moreover, G_ρ and $G_{\rho'}$ are isomorphic if and only if $b(\rho) = b(\rho')$.*

Observe that one can modify any derivation A of $\mathfrak{heis}$ with $A(Z) \neq 0$ by $\alpha A + \mathrm{ad}_u$, $u \in \mathfrak{heis}$ and gets a matrix of the form above (the two derivations generate isomorphic semi-direct products).

- The case $b(\rho) = 0$ corresponds to $G_\rho = \mathrm{Aff} \ltimes \mathbb{R}^2$, where $\mathrm{Aff} \subset \mathrm{SL}(2, \mathbb{R})$ is the affine group acting by its usual representation in $\mathbb{R}^2$. The space P_ρ in this case is isometric to the half Minkowski space $\{(u, v, x), u > 0\}$ endowed with the metric $2dudv + dx^2$.
- For $b < -\frac{1}{4}$, the space P_ρ is elliptic i.e. the A_ρ-action on $\mathfrak{heis}/Z$ has non-real eigenvalues and for $b > -1/4$ the space P_ρ is hyperbolic i.e. A_ρ has two different real eigenvalues.
- For $b = -\frac{1}{4}$, P_ρ is the parabolic non-unimodular plane wave i.e. A_ρ is conjugate to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

1.1.6 Compact models of homogeneous plane waves

Theorem 1.5 *There is exactly one non-flat plane wave that admits a compact model. This is given in Brinkmann coordinates defined on the half space $\{u > 0\}$ as*

$$g = 2dudv + \frac{2x^2}{u^2}du^2 + dx^2.$$

In Rosen coordinates, the metric takes the form $g = 2dudv + u^{-2}dx^2$.

Its isometry group contains a copy of the group $\mathrm{Sol} = \mathrm{SO}_0(1, 1) \ltimes \mathbb{R}^2$ acting properly. Any lattice $\Gamma \subset \mathrm{Sol}$ gives a compact quotient $\Gamma \backslash \mathrm{P}$. All compact models arise in this way.

Remark 1.6 Non-existence of compact quotients of 3-dimensional non-flat Cahen-Wallach spaces is already known in [13] and [10] (See Remarks 12.3 and 12.4 for further discussion).

1.1.7 Completeness and extendibility

Theorem 1.7 *The non-unimodular homogeneous plane waves are not complete. More precisely*

a) Any null geodesic, which is not an orbit of the parallel field $\frac{\partial}{\partial v}$, is incomplete.
b) Any timelike geodesic is incomplete.

In addition, a homogeneous plane wave P_ρ is non-extendable.

1.2 Classification of homogeneous Lorentz 3-spaces

Coming back to the issue in the classification of homogeneous Lorentz 3-manifolds, we have

Theorem 1.8 *Any simply connected homogeneous Lorentz 3-manifold is globally isometric to*

- *A Lorentz group i.e. a* 3*-dimensional Lie group endowed with a left-invariant Lorentz metric,*
- *Or, a globally symmetric space,*
- *Or, a non-unimodular elliptic plane wave i.e.* P_ρ *with* ρ *acting on* $\mathfrak{heis}/\mathrm{Z}$ *by similarities with non-real eigenvalues. In this case,* P_ρ *is neither locally symmetric nor locally isometric to a Lorentz group.*

In other words, we add elliptic non-unimodular plane waves to the existing classifications in the literature (e.g. in [4]). More importantly, we drop the completeness assumption usually made to get such a classification.

1.2.1 Homogeneous locally symmetric case

It is natural to ask what exactly happens in the locally symmetric case.

Theorem 1.9 *Let* X *be a Lorentz symmetric space of dimension* 3*. Homogeneous (simply connected) spaces* M *modeled on* X *correspond to open orbits of connected closed subgroups* $L \subset \mathrm{Isom}(X)$*. Such* L *may have dimension* 3, 4, 5 *or* 6*.*

- *Let* L *be a subgroup of* $\mathrm{Isom}(X)$ *of dimension* 4*. Then* L *has at least one open orbit. It can have a discrete quantity of* 2*-dimensional orbits, which are furthermore (complete) totally geodesic lightlike surfaces in* X*.*
- *If* $\dim L \geq 5$ *then it acts transitively and* X *has constant sectional curvature.*

1.2.2 Locally homogeneous compact Lorentz manifolds

In the vein of Thurston's eight Riemannian homogeneous geometries in dimension 3, the authors of [10] study Lorentz geometries in dimension 3 possessing compact forms, that is, pairs (G, X) with X a Lorentzian 3-manifold on which G acts isometrically transitively and there exists some compact manifold that can be modeled on this (G, X)-structure. If the isotropy of G acting on X is compact then G preserves a Riemannian metric and we find ourselves with Thurston's geometries. It is then natural to assume the Lorentz geometry to be of non-Riemannian type, that is, it has non-compact isotropy.

Theorem 1.10 *([10]) Let* (M, g) *be a locally homogeneous compact Lorentz* 3*-manifold of non-Riemannian type. Then,* M *admits a Lorentz metric, probably different from* g*, of (non-positive) constant sectional curvature. More precisely, up to finite cover,* (M, g) *is isometric to*

- *An anti de Sitter manifold* $\Gamma\backslash\mathrm{AdS}_3$*, where* AdS_3 *is* $\mathrm{PSL}(2, \mathbb{R})$ *endowed with its Killing form, and* $\Gamma \subset \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ *acts properly co-compactly on it.*
- *Or* M *is a quotient* $\Gamma\backslash L$ *where* L *a 3-dimensional Lie group and* $\Gamma \subset L$ *is a co-compact lattice, and metrically* L *is endowed with a left-invariant Lorentz metric which fits into one of the following cases*

  (1) $L = \mathbb{R}^3$ *endowed with a flat metric.*
  (2) $L = \mathrm{Heis}$ *or* $L = \mathrm{Sol}$*. In both cases,* L *has (up to isomorphism) two left-invariant metrics with non-compact isotropy, one of which is flat.*
  (3) $L = \mathrm{PSL}(2, \mathbb{R})$*. It has, up to isomorphism, two left-invariant metrics of non-compact isotropy, in addition to the* AdS*-one which is given by its Killing form.*

In the proofs of [10], some cases of Heisenberg extensions (actually corresponding to elliptic plane waves) were ruled out incorrectly because they were treated as algebraic groups (while they are not)! The present proof (Theorem 1.5) of the non-existence of compact models of elliptic plane waves fills this gap.

Let us finally observe that the algebraically most difficult case is the anti de Sitter one. It is indeed delicate to describe discrete subgroups $\Gamma \subset \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ acting properly co-compactly on $\mathrm{PSL}(2,\mathbb{R})$ (see [7, 19] for recent results).

1.3 Organization of the article

We start Sect. 2 by classifying homogeneous 3-dimensional Riemannian manifolds, essentially for two purposes: to provide a simpler proof (compared to the ones in the literature, for instance in [20]), and to compare with the Lorentzian situation to see what is new.

In Sect. 3 we define our family of Lorentz homogeneous spaces P_ρ, as quotients of Heisenberg extensions. These spaces will play the major role in the two main parts of this paper, on the one hand, they are the spaces of discussion in the part about plane waves and, on the other hand, a sub-family of them will constitute the new examples that are neither locally symmetric nor locally isometric to Lorentz groups.

Sections 4 and 5 will be devoted to the proof of the classification theorem 1.8. For easiness, the proof is divided into two parts: Sect. 4 treats the case where the isometry group is solvable and Sect. 5 treats the non-solvable case.

Proof of Theorem 1.9, about the classification of homogeneous locally symmetric Lorentz 3-manifolds, is presented in Sect. 6.

Theorem 1.1 concerning the algebraic structure of the spaces P_ρ is proven in Sect. 7.

Proofs of the statements about plane waves start, essentially, from Sect. 8. Sect. 8 is devoted to the proof of Theorem 1.3 that characterizes all homogeneous 3-dimensional plane waves. Theorem 1.4, about the global coordinates for non-unimodular plane-waves, is proven in Sect. 9. Proof of Theorem 1.7 is divided into two parts: the completeness statement is proven in Sect. 10, and the non-extendibility statement is discusses in Sect. 11. Finally, proof of the statement of the (non-)existence of compact models (Theorem 1.5) is detailed in Sect. 12.

2 What's new in the non-Riemannian case.

As we mentioned before, it is known that any homogeneous simply connected Riemannian manifold of dimension three is either symmetric or isometric to a left-invariant metric on a Lie group of dimension three, see [20]. Comparing this to the Lorentz situation led us to new examples. This section is devoted to this comparison.

2.1 The Riemannian case

We give here a proof of the classification of three-dimensional homogeneous simply connected Riemannian manifolds.

Theorem 2.1 *A simply connected homogeneous Riemannian 3-manifold (M, g) is globally symmetric or isometric to a Lie group with a left-invariant Riemannian metric.*

Proof Denote by G the identity component of the full isometry group of (M, g) and $\mathfrak{g}$ its Lie algebra. Let m be a point in M whose stabilizer in G is denoted by I (which is a connected

compact subgroup of G). The isotropy representation of I in $\mathrm{SO}(g_m) \simeq \mathrm{SO}(3)$ is faithful. Since $\mathrm{SO}(3)$ has no two-dimensional subgroups then I is either three-dimensional, one-dimensional, or trivial. If the dimension is zero then we are done (i.e. (M, g) is a Riemannian Lie group). If I is three-dimensional then the sectional curvature is constant which means that M is globally isometric to one of the Riemannian models with constant sectional curvature. □

Suppose now that I is one dimensional i.e. G is four-dimensional.

Fact 2.2 *The solvable radical of* $\mathfrak{g}$*, denoted by* $\mathrm{R}(\mathfrak{g})$*, is non-trivial.*

Proof If not, then G is by definition a semi-simple Lie group of dimension 4 which is impossible. □

If $\dim(\mathrm{R}(\mathfrak{g})) = 1$ then by Levi's decomposition either $\mathfrak{g} = \mathfrak{su}(2) \ltimes \mathbb{R}$ or $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \ltimes \mathbb{R}$. But, semi-simplicity of $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{su}(2)$ implies that the above semi-direct product is a direct product. In both cases if $\mathfrak{i} = \mathrm{Lie}(I) \subset \mathfrak{su}(2)$ or $\mathrm{Lie}(I) \subset \mathfrak{sl}(2, \mathbb{R})$ then $I = \mathrm{SO}(2)$ up to conjugacy, and the tangent space of G/I at I is $\mathbb{R} \oplus (\mathfrak{s}/\mathfrak{i})$ with $\mathfrak{s}$ equals $\mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{su}(2)$. Hence $\mathbb{R}$ is exactly the factor on which I acts trivially and $\mathfrak{s}/\mathfrak{i}$ is orthogonal to $\mathbb{R}$ since this is the unique I-invariant subspace transverse to $\mathfrak{i}$. This implies that the metric on $\mathbb{R} \oplus (\mathfrak{s}/\mathfrak{i})$ is a product metric. Therefore, up to scaling, (M, g) is isometric to the symmetric spaces

$$\mathbb{R} \times (\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)) = \mathbb{R} \times \mathbb{H}^2 \text{ or } \mathbb{R} \times (\mathrm{SU}(2)/\mathrm{SO}(2)) = \mathbb{R} \times \mathbb{S}^2.$$

And, if $\mathrm{Lie}(I)$ is transverse to the semi-simple factor, then (M, g) is isometric to a left-invariant metric on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ or $\mathrm{SU}(2)$.

If $\dim(\mathrm{R}(\mathfrak{g})) \geq 2$ then $\mathfrak{g}$ is solvable and $\mathfrak{i}$ cannot be contained in $[\mathfrak{g}, \mathfrak{g}]$ because, if not, we would have that ad_h is nilpotent for every $h \in \mathfrak{i}$ and it cannot be the infinitesimal isometry of a Riemannian inner product (unless ad_h is identically zero, but this is not the case since the derivative representation of I is faithful). Using this along with the fact that $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, one can find an ideal L supplementary to $\mathfrak{i}$ by pulling-back a hyperplane in $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ transverse to the projection of $\mathfrak{i}$. Since the normal subgroup corresponding to L is transverse to I, it follows that its action on $M = G/I$ is transitive and locally free. But, since M is simply connected then the action is in fact free and (M, g) is globally isometric to a left-invariant metric on this Lie group.

□

2.2 The Lorentzian case

One may observe that many ideas in the Riemannian case can be generalized immediately to the case of homogeneous Lorentzian three-manifolds except, a priori, two of them:

(1) The one concerning the fact that I cannot be two-dimensional. Here, one observes that $\mathrm{SO}(1, 2)$ contains two-dimensional subgroups (the affine group of $\mathbb{R}$ which is unique, up to conjugacy, inside $\mathrm{SO}(1, 2)$). However, one still manages to show that it cannot represent the full isotropy group of a point in a Lorentzian 3-manifold.
(2) The fact that non-zero nilpotent endomorphisms cannot be the infinitesimal isometry of a Riemannian inner product, which is possible in the Lorentzian setting. This is the main difference which leads to new examples. We will investigate all of this in what follows.

Let us first give some general considerations. Suppose that (M, g) is a locally homogeneous Lorentz three-manifold, meaning that any $p, q \in M$ have neighborhoods U and V such that

there exists $f : U \to V$ isometric with $f(p) = q$. In particular, such (M, g) is analytic and it is known that if it is simply connected, then any locally defined Killing field extends (uniquely) globally [17]. In this case, let $\mathfrak{g}$ be the Lie algebra of (all) Killing fields and G its associated simply connected group. Choose a base point p_0 and let $\mathfrak{i}$ be its (full) isotropy subalgebra, i.e. those Killing fields vanishing at p_0. We have the following

Proposition 2.1 *The isotropy subalgebra* $\mathfrak{i}$ *cannot have dimension two.*

Proof The idea relies on the fact that if we have a representation of $\mathrm{SL}(2, \mathbb{R})$ on some finite dimensional vector space V such that there is $v \in V$ fixed by a two-dimensional (connected) subgroup $H \subset \mathrm{SL}(2, \mathbb{R})$ (which is unique up to conjugacy) then v is fixed by $\mathrm{SL}(2, \mathbb{R})$ (this is due to the fact that $\mathrm{SL}(2, \mathbb{R})/H \sim \mathbb{S}^1$ and $\mathrm{SL}(2, \mathbb{R})$ cannot have non-trivial compact orbits when acting linearly on V). We have in our situation that the faithful representation, of local isometries fixing p, contains a 2-dimensional subgroup inside $\mathrm{SO}_0(1, 2)$ that fixes the Ricci tensor Ric_p. One deduces from the previous discussion that Ric_p is fixed by all $\mathrm{SO}_0(1, 2)$. But any quadratic form invariant by $\mathrm{SO}_0(1, 2)$ is proportional to the metric itself. So our Lorentz manifold is Einstein and it is known that, in dimension three, this is equivalent to having constant sectional curvature which implies that $\mathfrak{i}$ is not the full isotropy subalgebra. □

So, if (M, g) does not have constant sectional curvature then $\dim(I)$ can be 0 or 1 and, hence $\dim(G) = 3$ or 4. It is known that in a simply connected Lie group G of dimension ≤ 5, any connected Lie subgroup is closed. Indeed, if G is solvable then this is true in every dimension (see [16] page 187, last paragraph). If $\dim(G) = 4$ and G is not solvable, then its Lie algebra is either $\mathfrak{su}(2) \oplus \mathbb{R}$ or $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$ and in both cases we know that any connected Lie subgroup is closed. If $\dim(G) = 5$ and G is not solvable, then its Lie algebra is either $\mathfrak{su}(2) \oplus \mathbb{R}^2$, or $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}^2$, or $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ and, again, the same holds in this case. But, observe that this is no longer true in higher dimensions as can be shown in dimension 6, with I a dense subgroup in $\mathrm{SO}(2) \times \mathrm{SO}(2) \subset \mathrm{SU}(2) \times \mathrm{SU}(2)$, which is simply connected. Therefore, in our situation, the quotient G/I exists and M is locally isometric to it.

Observe that one can also find simply connected Lorentz homogeneous spaces $M = G/I$ with $\dim(G) = 5$ and $\dim(I) = 2$. But in this case, the Lie algebra $\mathfrak{g}$ of G is not the full Killing algebra of M. The full Killing algebra $\mathfrak{g}'$ has dimension 6 and M has constant curvature, but its associated group G', a priori, acts only locally on M. This is because M could be incomplete. However, one can prove that a constant curvature manifold admitting an isometric action of a 5-dimensional group is complete (see Theorem 1.9).

3 The Heisenberg extensions.

Definition 3.1 We say that a semi-direct product $G = \mathbb{R} \ltimes \mathrm{Heis}$ is non-real if the $\mathbb{R}$-action on $\mathbb{R}^2 = \mathrm{Heis}/\mathrm{Z}$ (where Z denotes the center of Heis) has complex non-real spectrum. Equivalently, this action is conjugate to a non-trivial similarity action, i.e. a one-parameter group $R \subset \mathbb{R}^+ \times \mathrm{SO}(2) \subset \mathrm{GL}(2, \mathbb{R})$ with a non-trivial projection on $\mathrm{SO}(2)$. For example, if R is $\mathrm{SO}(2)$, then G is the oscillator group [2, 21]. In the opposite case, we say that G is real.

Proposition 3.1 *Let* I *be any non-central one parameter group of* Heis *with* $\mathrm{Lie}(I) = \mathfrak{i}$. *Let* G *be any semi-direct product* $G = \mathbb{R} \ltimes \mathrm{Heis}$. *Then*

(1) G/I *admits a* G*-invariant Lorentz metric if and only if the* $\mathbb{R}$*-action does not preserve the abelian group of rank* 2 *generated by* I *and* Z.

(2) Assume G/I has a G-invariant Lorentz metric. Then there is a subalgebra $\mathfrak{l}$ of dimension 3 transverse to $\mathfrak{i}$ if and only if G is real.

Lemma 3.2 *A nilpotent (non-zero) endomorphism A of a 3-dimensional linear space is an infinitesimal isometry of some Lorentz scalar product if and only if its nilpotency order equals 3, that is, $A^3 = 0$ but $A^2 \neq 0$.*

Proof If A has nilpotency order equals 3 then, for u generic, the system $\left\{A^2(u), A(u), u\right\}$ is a basis in which A has a matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. This is skew symmetric with respect to the Lorentz form $-2dxdz + dy^2$. Suppose now that A has nilpotency order two and g is an invariant Lorentz metric. Then $\mathrm{Im}(A)$ is one-dimensional. But, for arbitrary X we have

$$g(A(X), X) + g(X, A(X)) = 2g(A(X), X) = 0.$$

So every X, not in $\ker(A)$, is orthogonal to $\mathrm{Im}(A)$, which is impossible since g is non-degenerate.

□

Proof of (1) G preserves a Lorentz metric on G/I if and only if the $\mathrm{Ad}(I)$ action on the quotient vector space $\mathfrak{g}/\mathfrak{i}$ (identified to the tangent space at the base point I) preserves a Lorentz scalar product. Equivalently, the infinitesimal action ad_a is skew symmetric with respect to a Lorentz scalar product on $\mathfrak{g}/\mathfrak{i}$, where a generates $\mathfrak{i}$. Consider a standard generating system $\{Z, X, Y\}$ of the Heisenberg algebra $\mathfrak{heis}$ with bracket relation $Z = [X, Y]$. The Lie algebra $\mathfrak{g}$ is generated by adding a fourth element T (with brackets $[T, W] \in \mathfrak{heis}$, for any $W \in \mathfrak{heis}$). Since all one-dimensional subspaces in Heis different from $\mathbb{R}Z$ are equivalent by automorphisms, we can assume $\mathfrak{i} = \mathbb{R}X$. The action of ad_X on $\mathfrak{g}/\mathbb{R}X$ has nilpotency degree 3 if and only if $\mathrm{ad}_X(T) \notin \mathbb{R}X \oplus \mathbb{R}Z$. This proves the first part of the proposition. □

Proof of (2) Assume G is real. Then ad_T has a non-vanishing eigenvector u when acting on $\mathfrak{heis}/\mathbb{R}Z \cong \mathbb{R}X \oplus \mathbb{R}Y$. We have that $u \notin \mathfrak{i} \oplus \mathbb{R}Z$ since we assume the existence of an $\mathrm{ad}_{\mathfrak{i}}$-invariant scalar product. Thus, $\mathfrak{l} = \mathbb{R}T \oplus \mathbb{R}u \oplus \mathbb{R}Z$ is a Lie algebra transverse to $\mathfrak{i}$. Conversely, suppose that such $\mathfrak{l}$ exists and consider its intersection $\mathfrak{l}' = \mathfrak{l} \cap \mathfrak{heis}$. Then $\mathfrak{l}'$ is a 2-dimensional ideal in $\mathfrak{l}$. Thus, $\mathfrak{l}'$ contains Z and another element $u \in \mathbb{R}X \oplus \mathbb{R}Y$. The Lie algebra $\mathfrak{l}$ also contains an element $T' = T + v$, $v \in \mathfrak{heis}$. Now, $[T, u] + [v, u] = [T', u] \in \mathfrak{l}' = \mathbb{R}Z \oplus \mathbb{R}u$. Since $[v, u] \in \mathbb{R}Z$ then the class of u is an eigenvector of ad_T acting on $\mathfrak{heis}/\mathbb{R}Z$. Therefore, this last endomorphism has a real spectrum. □

3.1 Description of the Heisenberg extensions

Let $\rho : \mathbb{R} \to \mathrm{Aut}(\mathfrak{heis})$ be a representation into the automorphism group of the Heisenberg algebra. This one-parameter group is of the form $\exp(tA_\rho)$ where A_ρ is a derivation of the Heisenberg algebra. Put $G_\rho = \mathrm{Heis} \rtimes_\rho \mathbb{R}$ the associated Lie group with Lie algebra, denoted by $\mathrm{Lie}(G_\rho)$, generated by Z, X, Y and T such that

$$[X, Y] = Z \text{ and } [T, W] = A_\rho(W) \text{ for every } W \in \mathfrak{heis}.$$

In particular, $[\mathrm{Lie}(G_\rho), \mathrm{Lie}(G_\rho)] \subset \mathfrak{heis}$ (more precisely, the derived subalgebra is generated by the span of the image of A_ρ and Z). Let $V \in \mathfrak{heis}$ be a non-central element. Hence $\mathrm{ad}_V : \mathrm{Lie}(G_\rho) \to \mathrm{Lie}(G_\rho)$ is nilpotent ($\mathfrak{heis}$ is contained in the nil-radical and we have

equality if A_ρ is not nilpotent). Suppose that ad_V has nilpotency degree 3 and let $I \subset G_\rho$ be the one-parameter subgroup (necessarily closed) tangent to V and consider the homogeneous space $\mathrm{P}_\rho = G_\rho/I$. It follows from Proposition 3.1 that P_ρ possesses a G_ρ-invariant Lorentz metric. In what follows, we will show the uniqueness, up to isometry and scaling, of P_ρ, i.e. independence on the choice of the non-central element $V \in \mathfrak{heis}$.

Observe that we always have $A_\rho(Z) = \alpha Z$ and A_ρ induces a linear transformation

$$\widetilde{A_\rho} : \mathfrak{heis}/Z = \mathbb{R}^2 \to \mathbb{R}^2$$

where $\mathrm{tr}(\widetilde{A_\rho}) = \alpha$.

Remark 3.3 Notice that multiplying A_ρ by a non-zero real number α or adding an inner derivation of the form ad_u, $u \in \mathfrak{heis}$, does not change the semi-direct product. More precisely, if $A_{\rho'} = \alpha A_\rho + \mathrm{ad}_u$ then the extensions G_ρ and $G_{\rho'}$ are isomorphic by an isomorphism preserving Heis. In particular, we can always assume that A_ρ preserves a plane transverse to the center of $\mathfrak{heis}$.

3.1.1 The real diagonalizable case

Suppose that $\widetilde{A_\rho}$ is diagonalizable with real eigenvalues. Then, up to conjugacy by automorphisms, we have

$$A_\rho = \begin{pmatrix} a+b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$$

with respect to the basis $\{Z, X, Y\}$ where $a \neq b$ (since we are in the situation of the existence of elements in $\mathfrak{heis}$ with nilpotency order 3). We can assume $a \neq 0$ (observe that in this case G_ρ is unimodular if and only if $a = -b$). Furthermore, we can assume that $a = 1$ because scaling A_ρ doesn't change the one-parameter group. Thus, we have a one-parameter family of Lie algebras, denoted $\mathrm{Lie}(G_b)$ this time, given by the derivations

$$A_b = \begin{pmatrix} 1+b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b \end{pmatrix}$$

for b different from 1 (observe that b and $1/b$ give isomorphic algebras for $b \neq 0$). The brackets of this Lie algebra are given by

$$[T, Z] = (1+b)Z, \quad [T, X] = X, \quad [T, Y] = bY, \quad \text{and} \quad [X, Y] = Z$$

Let now $V \in \mathfrak{heis} \subset \mathrm{Lie}(G_b)$ be a non-central element such that ad_V has nilpotency order three. Put $V = \alpha X + \beta Y + \gamma Z$. We can assume, up to inner conjugacy in $\mathfrak{heis}$, that $\gamma = 0$ and observe that both α and β must be different from zero otherwise ad_V has order two. In addition, all such V are equivalent by automorphisms of $\mathrm{Lie}(G_b)$, it suffices to consider an automorphism $\varphi : \mathrm{Lie}(G_b) \to \mathrm{Lie}(G_b)$ given by $\varphi(X) = t_1 X$, $\varphi(Y) = t_2 Y$, $\varphi(Z) = t_1 t_2 Z$, and $\varphi(T) = T$.

Suppose now that $V = X + Y$ and put $m = \mathrm{span}(T, Y' = X + bY, Z)$. Then m is ad_V-invariant and we have

$$\mathrm{ad}_V(T) = -Y', \quad \mathrm{ad}_V(Y') = (b-1)Z, \quad \text{and} \quad \mathrm{ad}_V(Z) = 0$$

So giving a G_b-invariant Lorentzian metric on $P_b = G_b/I$, where $I = \exp(tV)$, is equivalent to giving a Lorentzian ad_V-invariant inner product $g(.,.)$ on m. That is

$$g(\mathrm{ad}_V(W), U) + g(W, \mathrm{ad}_V(U)) = 0$$

for every $W, U \in m$. By direct evaluations we find

$$g(T,T) = \hat{\beta}, \quad g(Y',Y') = \hat{\alpha}, \quad g(T,Z) = \frac{\hat{\alpha}}{b-1}$$

and

$$g(Z,Z) = g(Y',Z) = g(T,Y') = 0$$

where $\hat{\alpha}$ is different from zero. But, up to automorphisms fixing V and preserving m, we can assume $\hat{\beta} = 0$. It suffices to consider the automorphism $\varphi : \mathrm{Lie}(G_b) \to \mathrm{Lie}(G_b)$ defined by $\varphi(X) = X$, $\varphi(Y) = Y$, $\varphi(T) = T + \delta Z$, and $\varphi(Z) = Z$ with a suitable choice of δ. This shows that the Lorentzian homogeneous space P_b is unique up to isometry and scaling.

3.1.2 The real parabolic case

Suppose now that $\widetilde{A_\rho}$ has real non-zero eigenvalues but not diagonalizable. In this case, up to conjugacy by automorphisms, we have

$$A_\rho = \begin{pmatrix} 2t & 0 & 0 \\ 0 & t & s \\ 0 & 0 & t \end{pmatrix}$$

with respect to the basis $\{Z, X, Y\}$ where t and s are non-zero. We can assume that $t = s = 1$ up to scaling and automorphisms of the Heisenberg algebra. Thus, we have a Lie algebra $\mathrm{Lie}(G_\rho)$ given by the derivation

$$A_\rho = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The brackets of this Lie algebra are given by

$$[T,Z] = 2Z, \quad [T,X] = X, \quad [T,Y] = X + Y, \quad \text{and} \quad [X,Y] = Z$$

Let $V = \alpha X + \beta Y + \gamma Z$ be a non-central element whose nilpotency order is 3. We can assume, up to inner conjugacy in $\mathfrak{heis}$, that $\gamma = 0$. Observe that $\beta \neq 0$ otherwise ad_V has order two. In addition, all such V are equivalent by automorphisms of $\mathrm{Lie}(G_\rho)$, it suffices to consider $\varphi : \mathrm{Lie}(G_\rho) \to \mathrm{Lie}(G_\rho)$ given by $\varphi(X) = X$, $\varphi(Y) = Y + tX$, $\varphi(Z) = Z$, and $\varphi(T) = T$ with a suitable value of t.

Suppose now that $V = Y$ and put $m = \mathrm{span}(T, Y' = X + Y, Z)$. Then m is ad_V-invariant and we have

$$\mathrm{ad}_V(T) = -Y', \quad \mathrm{ad}_V(Y') = -Z, \quad \text{and} \quad \mathrm{ad}_V(Z) = 0$$

Suppose $g(.,.)$ is a Lorentzian ad_V-invariant inner product on m. That is

$$g(\mathrm{ad}_V(W), U) + g(W, \mathrm{ad}_V(U)) = 0$$

for every $W, U \in m$. By evaluations we find

$$g(T,T) = \hat{\beta}, \quad g(Y',Y') = \hat{\alpha}, \quad g(T,Z) = -\hat{\alpha}$$

and

$$g(Z,Z) = g(Y',Z) = g(T,Y') = 0$$

where $\hat{\alpha}$ is different from zero. We can also assume $\hat{\beta} = 0$ by considering the automorphism $\varphi : \mathrm{Lie}(G_\rho) \to \mathrm{Lie}(G_\rho)$ defined by $\varphi(X) = X, \varphi(Y) = Y, \varphi(T) = T + \delta Z$, and $\varphi(Z) = Z$ with a suitable choice of δ. This shows, similar to the previous cases, that the Lorentzian homogeneous space $\mathrm{P}_\rho = G_\rho / \exp(tV)$ is unique up to isometry and scaling.

3.1.3 The non-real case

Consider the case where $\widetilde{A_\rho}$ has complex eigenvalues. In this case A_ρ preserves a plane transverse to the central direction and acting on it by a similarity. Since all transverse planes are equivalent by Heisenberg inner automorphisms, we can assume that, up to scaling,

$$A_\rho = \begin{pmatrix} 2\cos(\theta) & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

with respect to the basis $\{Z, X, Y\}$ where $\theta \in (0, \pi)$. Here, one thinks of A_ρ acting on the plane $\mathrm{span}(X, Y)$ as a multiplication by a complex number in the upper half of the unit circle. But, in order to simplify the brackets, we can scale and assume that A_ρ acts on $\mathrm{span}(X, Y)$ by multiplying by a complex number of the form $i + c$ with $c \in \mathbb{R}$. Thus, we have a one-parameter family of Lie algebras, which we denote by $\mathrm{Lie}(G_c)$, given by the derivation (observe also that one can take only $c \geq 0$)

$$A_c = \begin{pmatrix} 2c & 0 & 0 \\ 0 & c & -1 \\ 0 & 1 & c \end{pmatrix}$$

The brackets of this Lie algebra are given by

$$[T,Z] = 2cZ, \quad [T,X] = Y + cX, \quad [T,Y] = -X + cY, \quad \text{and} \quad [X,Y] = Z$$

As before, if $V = \alpha X + \beta Y + \gamma Z$ is a non-central element then we can always assume that $\gamma = 0$. All such V are equivalent to $V = X$ by automorphisms $\varphi : \mathrm{Lie}(G_c) \to \mathrm{Lie}(G_c)$ of the form $\varphi(X) = \alpha X + \beta Y$, $\varphi(Y) = -\beta X + \alpha Y$, $\varphi(Z) = (\alpha^2 + \beta^2)Z$, and $\varphi(T) = T$ which send X to V.

Suppose now that $V = X$ and $m = \mathrm{span}(T, Y' = Y + cX, Z)$. Then m is ad_V-invariant and we have

$$\mathrm{ad}_V(T) = -Y', \quad \mathrm{ad}_V(Y') = Z, \quad \text{and} \quad \mathrm{ad}_V(Z) = 0$$

Suppose $g(.,.)$ is a Lorentzian ad_V-invariant inner product on m. By evaluations, as before, we find

$$g(T,T) = \hat{\beta}, \quad g(Y',Y') = \hat{\alpha}, \quad g(T,Z) = \hat{\alpha}$$

and

$$g(Z,Z) = g(Y',Z) = g(T,Y') = 0$$

where $\hat{\alpha}$ is different from zero. By considering an automorphism $\varphi : \mathrm{Lie}(G_c) \to \mathrm{Lie}(G_c)$ of the form $\varphi(X) = X, \varphi(Y) = Y, \varphi(T) = T + \delta Z$, and $\varphi(Z) = Z$ we can assume $\hat{\beta} = 0$. This shows the uniqueness of the Lorentzian homogeneous space $P_c = G_c / \exp(tV)$ up to isometry and scaling.

3.1.4 The nilpotent case

Lastly, suppose that $\widetilde{A_\rho}$ has only zero eigenvalues but not diagonalizable. We have in this case that, up to scaling, conjugacy by automorphisms, and adding a derivation of the form ad_u with $u \in \mathfrak{heis}$

$$A_\rho = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to a basis $\{Z, X, Y\}$. The brackets of this Lie algebra are given by

$$[T, Y] = X, \quad \text{and} \quad [X, Y] = Z$$

Let $V = \alpha X + \beta Y + \gamma Z$ be a non-central element whose nilpotency order is 3. We can always assume, up to inner conjugacy in $\mathfrak{heis}$, that $\gamma = 0$. Observe that $\beta \neq 0$ otherwise ad_V has order two. In addition, all such V are equivalent by automorphisms of $\mathrm{Lie}(G_\rho)$, it suffices to consider $\varphi : \mathrm{Lie}(G_\rho) \to \mathrm{Lie}(G_\rho)$ given by $\varphi(X) = \beta X$, $\varphi(Y) = \beta Y + \alpha X$, $\varphi(Z) = \beta^2 Z$, and $\varphi(T) = T$ which sends Y to V.

Suppose now that $V = Y$ and put $m = \mathrm{span}(T, X, Z)$. Then m is ad_V-invariant and we have

$$\mathrm{ad}_V(T) = -X, \quad \mathrm{ad}_V(X) = -Z, \quad \text{and} \quad \mathrm{ad}_V(Z) = 0$$

Suppose $g(.,.)$ is a Lorentzian ad_V-invariant inner product on m. That is

$$g(\mathrm{ad}_V(W), U) + g(W, \mathrm{ad}_V(U)) = 0$$

for every $W, U \in m$. By evaluations we find

$$g(T, T) = \hat{\beta}, \quad g(X, X) = \hat{\alpha}, \quad g(T, Z) = -\hat{\alpha}$$

and

$$g(Z, Z) = g(X, Z) = g(T, X) = 0$$

where $\hat{\alpha}$ is different from zero. We can also assume $\hat{\beta} = 0$ by considering the automorphism $\varphi : \mathrm{Lie}(G_\rho) \to \mathrm{Lie}(G_\rho)$ defined by $\varphi(X) = X$, $\varphi(Y) = Y$, $\varphi(T) = T + \delta Z$, and $\varphi(Z) = Z$ with a suitable choice of δ. This shows that the Lorentzian homogeneous space $\mathrm{P}_\rho = G_\rho / \exp(tV)$ is unique up to isometry and scaling.

4 Proof of Theorem 1.8 in the solvable case.

Recall that we have shown in Proposition 2.1 that if (M, g) is a homogeneous simply connected Lorentzian manifold with $\dim(\mathrm{Isom}(M)) \geq 5$, then M has constant sectional curvature. In fact, one shows in this case that M is complete (see Theorem 1.9). Now , since the case $\dim(\mathrm{Isom}(M)) = 3$ corresponds to the fact that M is a left-invariant metric on $\mathrm{Isom}_0(M)$, then all what remains to consider is the case $\dim(\mathrm{Isom}(M)) = 4$.

Proposition 4.1 *Let G/I be a Lorentz space with G solvable and has dimension 4. Then there exists a Lie subalgebra $\mathfrak{l}$ transverse to $\mathfrak{i}$ if and only if G is not a non-real semi-direct product $\mathbb{R} \ltimes \mathrm{Heis}$ with $I \subset \mathrm{Heis}$.*

Proof Observe first that $\mathfrak{i}$ is not central since we always assume that the action of G on G/I is faithful. Let $[\mathfrak{g},\mathfrak{g}]$ be the derived subalgebra. Since $\mathfrak{g}$ is solvable we have that $[\mathfrak{g},\mathfrak{g}]$ is a proper subalgebra of $\mathfrak{g}$, and we have the natural projection $\pi : \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] = \mathbb{R}^k$ for $k \geq 1$. Suppose that $\mathfrak{i}$ is not contained in $[\mathfrak{g},\mathfrak{g}]$, then $\mathfrak{i}$ projects injectively to a line $\pi(\mathfrak{i})$ inside $\mathbb{R}^k$. Let $H \subset \mathbb{R}^k$ be any hyperplane transverse to $\pi(\mathfrak{i})$ (if $k = 1$ then H is reduced to $\{0\}$). Hence, $\pi^{-1}(H)$ is an ideal transverse to $\mathfrak{i}$ in $\mathfrak{g}$ which proves the proposition in this case.

So, it remains to consider the case where $\mathfrak{i} \subset [\mathfrak{g},\mathfrak{g}]$. For $a \in \mathfrak{i} - \{0\}$ we have that ad_a is nilpotent since $[\mathfrak{g},\mathfrak{g}]$ is nilpotent (recall that $\mathrm{ad}_a \neq 0$ since $\mathfrak{i}$ is not central). If $[\mathfrak{g},\mathfrak{g}]$ is abelian then $\mathrm{ad}_a^2 = 0$ which implies that ad_a can not infinitesimally preserve a Lorentz product (by Lemma 3.2). So $[\mathfrak{g},\mathfrak{g}]$ is non-abelian. Being nilpotent and non-abelian, $[\mathfrak{g},\mathfrak{g}]$ is therefore the Heisenberg algebra (see [15]) and the group G is then a semi-direct product $\mathbb{R} \ltimes \mathrm{Heis}$ with $I \subset G$ is a non-central one-parameter subgroup of Heis, which is already treated in Proposition 3.1. This completes the proof of Proposition 4.1. □

Remark 4.1 Observe from the proof that in all cases where G is not a semi-direct product of Heis, we found $\mathfrak{l}$ transverse to $\mathfrak{i}$ which is an ideal.

4.1 Global identification

So far, we proved the existence of a sub-algebra $\mathfrak{l}$ transverse to $\mathfrak{i}$ if G is not a non-real semi-direct product of Heis. But, we want to prove that globally G/I is identified to L, where L is the group determined by $\mathfrak{l}$. This essentially means that L acts transitively on G/I, whereas the transversality means that the L-orbit of the point I in G/I is open. It is likely, in our setting here (e.g. G solvable and G/I of Lorentz type), that any open orbit necessarily equals the full space. We will prove this for the L we found in the proofs of Propositions 4.1 and 3.1.

First, let us notice that for simply connected groups of dimension 4, any connected subgroup is closed (see the discussion after the proof of Proposition 2.1). Let L be the subgroup determined by the subalgebra $\mathfrak{l}$. If L is normal, then G/L is isomorphic to $\mathbb{R}$ as a group, and I projects to it isomorphically. That is, $G/L = I$ or equivalently $G = LI$, and $I \cap L = \{1\}$. In other words, L acts freely transitively on G/I.

From Remark 4.1, we can find L normal unless G is a semi-direct product $\mathbb{R} \ltimes \mathrm{Heis}$. In this case, let I be the one-parameter group $\{\exp(tX), t \in \mathbb{R}\} \subset \mathrm{Heis}$ and the group L transverse to I as constructed in the proof of Proposition 3.1. The orbit of a point $gI \in G/I$ under the action of L is open if and only if L is transverse to $\mathrm{Stab}(gI) = gIg^{-1}$ where $\mathrm{Stab}(gI)$ is the stabilizer of gI. We have, as in the proof of Proposition 3.1, that $\mathfrak{l} = \mathbb{R}T \oplus \mathbb{R}u \oplus \mathbb{R}Z$ and $\mathbb{R}u \oplus \mathbb{R}Z$ is an ideal in $\mathfrak{g}$. So the conjugates of X are never inside $\mathfrak{l}$ since they are never inside the ideal $\mathbb{R}u \oplus \mathbb{R}Z = \mathfrak{l} \cap \mathfrak{heis}$. This shows that L is always transverse to the stabilizer of any point in G/I which implies that the L-orbit of every point $gI \in G/I$ is open. Hence L acts transitively locally freely on G/I because G/I is connected and the orbits are open. By simple-connectedness, we get that the L-action is in fact free. □

Theorem 4.2 *Let G be a simply connected solvable group of dimension 4 and G/I a Lorentz 3-dimensional G homogeneous space. Then there exists a simply connected 3-dimensional group L acting isometrically, freely, and transitively on G/I if and only if G is not a non-real semi-direct product $\mathbb{R} \ltimes \mathrm{Heis}$ with $I \subset \mathrm{Heis}$.*

5 Proof of Theorem 1.8 in the non-solvable case.

Let G/I be a 3-dimensional homogeneous Lorentz space with G of dimension 4, non-solvable, and simply connected. Necessarily, for the same reason that we mentioned in the Riemannian case, G is isomorphic to $\mathbb{R} \times S$, with $S = \widetilde{\mathrm{SL}}(2, \mathbb{R})$ or $\mathrm{SU}(2)$. The isotropy group I cannot be the factor $\mathbb{R}$ since it is central (and the G-action would be non-faithful). If I is not contained in S, then S acts freely transitively and isometrically on G/I since S is normal and transverse to I. In other words, the Lorentz space G/I is identified with S endowed with a left-invariant metric.

It remains to consider the case $I \subset S$ i.e. $G/I = \mathbb{R} \times (S/I)$. In this case, $\mathfrak{g}/\mathfrak{i} = \mathbb{R} \oplus (\mathfrak{s}/\mathfrak{i})$. Consider the case $S = \widetilde{\mathrm{SL}}(2, \mathbb{R})$. If $\mathfrak{i} \subset \mathfrak{sl}(2, \mathbb{R})$ is nilpotent, then it has nilpotency of order 2 when acting on $\mathfrak{s}/\mathfrak{i}$ (since the dimension of this space is 2) and, hence, has the same order when acting on $\mathbb{R} \times (\mathfrak{s}/\mathfrak{i})$. Therefore, it cannot infinitesimally preserve a Lorentz product. It follows that I must be hyperbolic or elliptic. In both cases, the factor $\mathbb{R}$ is exactly the trivial factor of the I-action. Thus, it is orthogonal to $\mathfrak{s}/\mathfrak{i}$ (observe that $\mathbb{R}$ is not null).

One concludes that G/I is a direct product $\mathbb{R} \times (S/I)$. The metric on $\mathbb{R}$ can be positive or negative. As for S/I, it can be the sphere $\mathbb{S}^2$ if $S = \mathrm{SU}(2)$, the hyperbolic plane $\mathbb{H}^2$ or the (anti) de Sitter plane $\mathbb{DS}^2 = \widetilde{\mathrm{SO}}_0(1, 2)/\mathrm{SO}_0(1, 1)$ if $S = \widetilde{\mathrm{SL}}(2, \mathbb{R})$. In particular, the space G/I is symmetric.

Remark 5.1 In the case of $G = \mathbb{R} \times \mathrm{SU}(2)$, the unique 3-dimensional subgroup of G is $\mathrm{SU}(2)$. Hence, if the isotropy group I is included in $\mathrm{SU}(2)$ then there is no Lie subgroup of G transverse to it, that is, there is no 3-dimensional Lie subgroup of G having an open orbit. Whereas in the case of $G = \mathbb{R} \times \widetilde{\mathrm{SL}}(2, \mathbb{R})$ and $I \subset \widetilde{\mathrm{SL}}(2, \mathbb{R})$ hyperbolic, the subgroup $L = \mathbb{R} \times \mathrm{Aff}$, where Aff is the group of upper triangular matrices, has an open orbit in G/I but it does not act transitively.

6 Homogeneous locally symmetric spaces, proof of Theorem 1.9.

6.1 Let M be modeled on X and homogeneous. Let us begin with assuming that M is simply connected. Then it has a developing map $d : M \to X$, equivariant with respect to a local isomorphism $\psi : G \to L \subset \mathrm{Isom}(X)$ (that is, $d \circ g = \psi(g) \circ d$ for every $g \in G$). The d-image is an open L-orbit. Conversely, any open orbit of a subgroup of $\mathrm{Isom}(X)$ is a homogeneous space modeled on X.

Therefore, the proof of Theorem 1.9 reduces to the study of L-orbits, for L a subgroup of dimension ≥ 4, essentially $\dim L = 4$, since the 5-dimensional case follows. Observe first that in the case where X is symmetric, but not of constant curvature, $\dim(\mathrm{Isom}(X)) = 4$ and there is nothing to prove, and hence, we will assume that X has constant curvature.

6.2 The stabilizer of a line $\mathbb{R}v \in T_xX$ in $\mathrm{SO}(T_xX, g_x) = \mathrm{SO}(1, 2)$ has dimension 2 if v is isotropic, and dimension 1 otherwise. It follows that a curve can be preserved by a subgroup $\subset \mathrm{Isom}(X)$ of dimension at most 3. Hence, if $\dim(L) = 4$ then any L-orbit is either a surface or is open.

Similarly, the stabilizer of a plane $F \subset T_xX$ has dimension 1 unless F is lightlike. It follows that if a connected surface Σ is preserved by a subgroup $L \subset \mathrm{Isom}(X)$ of dimension 4, then it is lightlike and homogeneous.

6.1 Action of Aff $\subset$ SO(1, 2) near 0 in $\mathbb{R}^{1,2}$

Write the metric of $\mathbb{R}^{1,2}$ as $2dudv + dx^2$. The subgroup $H \subset \mathrm{SO}(1,2)$ preserving the direction $\frac{\partial}{\partial u} = (1,0,0)$ is isomorphic to the affine group Aff. In fact any subgroup in SO(1, 2) of dimension 2 is conjugate to H and equals the stabilizer of an isotropic direction.

Let Σ_0 be the lightlike affine 2-plane tangent to $\frac{\partial}{\partial u}^{\perp} = \mathbb{R}\frac{\partial}{\partial u} \oplus \mathbb{R}\frac{\partial}{\partial x}$ at 0. So, $\Sigma_0 = \{v = 0\}$. The H-orbits of the Σ_0-points are $\{0\}, \mathbb{R}(\frac{\partial}{\partial u})\setminus\{0\}$, and (null) affine lines parallel to $\frac{\partial}{\partial u}$ defined explicitly by $\{v, = 0, x = a\}$ (for $a \in \mathbb{R}$). A level E_r^- defined by

$$2uv + x^2 = -r^2, \quad r \neq 0$$

is homothetic to a hyperbolic plane, and H acts on it transitively. Similarly, a level E_r^+ given by

$$2uv + x^2 = +r^2, \quad r \neq 0$$

corresponds to a de Sitter plane. It intersects Σ_0 on the two lines $\{v = 0, x = \pm r\}$. Then E_r^+ minus these two lines is an H-orbit. Finally $E_0 - \Sigma_0$ is a H-orbit. This is the light cone with the affine line tangent to $\frac{\partial}{\partial u}$ removed.

From all this, we infer that if a lightlike surface is H-invariant, then it is either $E_0 - \Sigma_0$ or an open subset of Σ_0.

6.2 2-dimensional *L*-orbits are lightlike and geodesic

Let Σ be a L-orbit of dimension 2. It is, thus, lightlike and has isotropy of dimension 2. In the Minkowski case, we can assume $0 \in \Sigma$ and its isotropy is H. From our previous analysis, since Σ is lightlike and H-invariant, Σ is $E_0 - \Sigma_0$ or contained in Σ_0.

Now, the subgroup of isometries preserving the cone E_0 is exactly SO(1, 2), and in particular $E_0 - \Sigma_0$ can not be preserved by a 4-dimensional group. It follows that Σ is open in Σ_0 and hence geodesic. In fact, the subgroup of elements preserving Σ_0 is $H \ltimes T$, where $T \cong \mathbb{R}^2$ acts by translation on Σ_0. It has dimension 4 and, thus, coincides with L. We then see that L has Σ_0 as a unique 2-dimensional orbit. It has two half-Minkowski spaces as open orbits.

6.5 In the non-flat case, one uses the exponential map at a point p to get a similar picture as that in Minkowski. One also uses Gauss lemma to see that the metric levels at p are orthogonal to the geodesic rays through p and hence are not lightlike, unless in the light cone. Similarly, one concludes that L is the stabilizer of a lightlike geodesic surface Σ in X.

The de Sitter space dS_3 has a model $2uv+x^2+y^2 = 1$ in $\mathbb{R}^4$. A lightlike geodesic surface is given by $\{(u, 0, x, y)/x^2 + y^2 = +1\}$. Its stabilizer L is the stabilizer of the direction $\mathbb{R}.(1, 0, 0, 0)$ in SO(1, 3). Thus, the exterior of Σ consists of two open orbits.

Consider now the anti de Sitter space AdS_3. It is represented as the level $2uv + x^2 - y^2 = -1$. Its intersection with a lightlike hyperplane in $\mathbb{R}^{2,2}$ is up to isometry $\{(u, 0, x, y)/x^2 - y^2 = -1\}$. It has two connected components, and Σ is one of them. In other words, the stabilizer of Σ has exactly another 2-dimensional orbit, and has 2 open orbits.

6.3 Case where *M* is not simply connected

In the flat case, the group L preserving a lightlike geodesic surface is, up to conjugacy, $L = \mathrm{Aff} \ltimes \mathbb{R}^2$. If M, a flat homogeneous manifold with $\dim \mathrm{Isom}(M) = 4$, is not simply

connected, then, its isometry group is a non-trivial central quotient of L. But L has trivial center, and thus M has to be simply connected.

In the case of dS_3, the corresponding L is a maximal parabolic subgroup of $SO(1,3)$, which is isomorphic to $(\mathbb{R} \times SO(2)) \ltimes \mathbb{R}^2$. Here, too, L has a trivial center. The same conclusion applies to AdS_3 as L in this case is isomorphic to $\mathrm{Aff} \times \mathrm{Aff}$. □

7 Proof of Theorem 1.1.

Lemma 7.1 *Suppose that the representation $\rho : \mathbb{R} \to \mathrm{Aut}(\mathrm{Heis})$ is unimodular. Then P_ρ is globally symmetric. Furthermore, if ρ is unipotent then P_ρ is isometric to Minkowski, and if ρ is hyperbolic (resp. elliptic) then P_ρ is a Cahen-Wallach space.*

Proof First, suppose that ρ is unipotent, that is (3.1.4) ρ is given by the derivation

$$\mathrm{ad}_T = A_\rho = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

in the basis $\{Z, X, Y\}$. The isotropy in this case can be chosen to correspond to the subalgebra $\mathfrak{i} = \mathbb{R}Y$. One sees that $\mathrm{span}(T, X, Z)$ is an abelian subalgebra transverse to all conjugates of $\mathfrak{i}$. This shows (see Sect. 4.1) that P_ρ is isometric to a translation invariant metric on $\mathbb{R}^3$ *i.e* to Minkowski.

Suppose now that ρ is hyperbolic, given by

$$A_b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The brackets of $\mathrm{Lie}(G_\rho)$ are given by

$$[T, X] = X, \quad [T, Y] = -Y, \quad \text{and} \quad [X, Y] = Z$$

And an inner Lorentz product (3.1.1) on $m = \mathrm{span}(T, Y' = X - Y, Z)$ given by

$$g(Y', Y') = 1, \quad g(T, Z) = -\frac{1}{2}$$

and

$$g(T, T) = g(Z, Z) = g(Y', Z) = g(T, Y') = 0$$

Consider the automorphism of $\mathrm{Lie}(G_\rho)$ given by

$$\varphi(T) = -T, \quad \varphi(X) = Y, \quad \varphi(Y) = X, \quad \text{and} \quad \varphi(Z) = -Z.$$

So, φ fixes $\mathfrak{i} = \mathbb{R}(X + Y)$ corresponding to the isotropy and acts on $\mathrm{Lie}(G_\rho/\mathfrak{i})$ by $-Id$. Furthermore, φ preserves g (on m) and the associated automorphism $\widetilde{\varphi} : G_\rho \to G_\rho$ (i.e. $\widetilde{\varphi}$ induces the automorphism φ on $\mathrm{Lie}(G_\rho)$) preserves I and acts on $\mathrm{P}_\rho = G_\rho/I$ by a global isometry fixing the point I with derivative $-Id$. Since P_ρ is homogeneous, then there exists such a global symmetry around every point in P_ρ. This shows that P_ρ is globally symmetric.

Similarly, if ρ is elliptic (3.1.3) given by

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

The brackets given by

$$[T, X] = Y, \quad [T, Y] = -X, \quad \text{and} \quad [X, Y] = Z$$

As before if $m = \text{span}(T, Y, Z)$ (3.1.3). Then m is ad_X-invariant and we have

$$\text{ad}_V(T) = -Y, \quad \text{ad}_V(Y) = Z, \quad \text{and} \quad \text{ad}_V(Z) = 0$$

with Lorentzian inner product

$$g(Y, Y) = 1, \quad g(T, Z) = 1$$

and

$$g(T, T) = g(Z, Z) = g(Y, Z) = g(T, Y) = 0$$

And if we consider the automorphism

$$\varphi(T) = -T, \quad \varphi(X) = X, \quad \varphi(Y) = -Y, \quad \text{and} \quad \varphi(Z) = -Z.$$

we obtain, the same as before, that P_ρ is symmetric. The last two cases are known to be the only Cahen-Wallach spaces in dimension three (see for instance [6, 13]). □

Lemma 7.2 *P_ρ has constant curvature if and only if it is flat. In addition, if P_ρ is flat then it is either isometric to the Minkowski space with G_ρ isomorphic to $\mathbb{R} \ltimes \mathbb{R}^3$ where $\mathbb{R}$ acts via a unipotent one-parameter subgroup of $\mathrm{SO}(1, 2)$, or P_ρ is isometric to half Minkowski with G_ρ non-unimodular isomorphic to $\mathrm{Aff} \ltimes \mathbb{R}^2$.*

Proof The first statement follows from the following fact: the (local) holonomy of a non-flat three dimensional Lorentz manifold with constant curvature (i.e. locally isometric, up to scaling, to de Sitter or anti-de Sitter space) is maximal (i.e. equal to $\mathrm{SO}_0(1, 2)$). In particular, there cannot be a parallel null vector field if the curvature is constant non-zero (since $\mathrm{SO}_0(1, 2)$ has no non-zero fixed vector on $\mathbb{R}^3$). On the other hand, P_ρ are plane waves (Theorem 1.3) and, by definition, admit a non-trivial null parallel vector field. So if P_ρ has constant curvature then it must be flat.
Suppose now that P_ρ is flat. Then $\text{Lie}(G_\rho)$ has a faithful representation in $\mathfrak{so}(1, 2) \ltimes \mathbb{R}^3$. Consider the projection homomorphism

$$\pi : \mathfrak{so}(1, 2) \ltimes \mathbb{R}^3 \longrightarrow \mathfrak{so}(1, 2)$$

We know that $\text{Lie}(G_\rho) \cap \mathbb{R}^3$ is non-trivial and equals the kernel of π restricted to $\text{Lie}(G_\rho)$ and also $\text{Lie}(G_\rho) \cap \mathfrak{so}(1, 2)$ has dimension at least 1. The case $\dim(\pi(\text{Lie}(G_\rho))) = 3$ is not possible since G_ρ is solvable. If $\dim(\pi(\text{Lie}(G_\rho))) = 1$ then necessarily $\pi(\text{Lie}(G_\rho)) = \text{Lie}(G_\rho) \cap \mathfrak{so}(1, 2)$. In this case, $\mathbb{R}^3 \subset \text{Lie}(G_\rho)$ and G_ρ is isomorphic to $\mathbb{R} \ltimes \mathbb{R}^3$ where $\mathbb{R}$ must act via a unipotent one parameter group (otherwise G_ρ doesn't contain a copy of Heisenberg which is not the case). In this case, P_ρ is isometric to Minkowski (see Lemma 7.1).

Suppose now that $\dim(\pi(\text{Lie}(G_\rho))) = 2$, then $\dim(\text{Lie}(G_\rho) \cap \mathbb{R}^3) = 2$. So $\text{Lie}(G_\rho)$ is contained in the stabilizer of a (degenerate) plane $\mathbb{R}^2$ inside $\mathfrak{so}(1, 2) \ltimes \mathbb{R}^3$ (acting by conjugacy), which is isomorphic to $\text{aff} \ltimes \mathbb{R}^3$. Denote the basis of the Lie algebra $\text{aff} \ltimes \mathbb{R}^3$ by $\{h, f, e_1, e_2, e_3\}$ such that the stabilized plane is $\text{span}(e_1, e_2)$ and the brackets are given by

$$[h, f] = f \quad \text{and} \quad [e_i, e_j] = 0$$

and the matrices of ad_h, ad_f acting on $\mathbb{R}^3$ with respect to the basis $\{e_1, e_2, e_3\}$ are given by

$$\mathrm{ad}_f = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{ad}_h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

So we can suppose that $\mathrm{Lie}(G_\rho) = \mathrm{span}(h + \beta e_3, f + \alpha e_3, e_1, e_2)$. We have that $[h + \beta e_3, f + \alpha e_3] = f - \alpha e_3 - \beta e_2$. Since the derived subalgebra must be contained in $\mathrm{span}(f + \alpha e_3, e_1, e_2)$, we deduce that $\alpha = 0$. So

$$\mathrm{Lie}(G_\rho) = \mathrm{span}(h + \beta e_3, f, e_1, e_2)$$

It is easy to see that $\mathrm{span}(h + \beta e_3, f - \beta e_2)$ is a subalgebra transverse to the subalgebra $\mathrm{span}(e_1, e_2)$. So G_ρ is isomorphic to $\mathrm{Aff} \ltimes \mathbb{R}^2$ and it preserves half Minkowski (given by the stabilized plane for $\beta = 0$) which must be isometric to P_ρ since this latter is unique (3.1.1). □

This proves (4) and (5) in Theorem 1.1. □

Lemma 7.3 *Except the two flat cases (see Lemma 7.2), the following statements are equivalent*

1) *Up to scaling,* P_ρ *is locally isometric to* $\mathrm{P}_{\rho'}$.
2) G_ρ *is isomorphic* $G_{\rho'}$.
3) *Up to scaling,* P_ρ *is globally isometric to* $\mathrm{P}_{\rho'}$.

Proof Let G_ρ and $G_{\rho'}$ be the associated groups. We know that their derived subgroups are exactly Heis (the only cases where this is not true are the flat ones). Suppose that P_ρ is locally isometric to $\mathrm{P}_{\rho'}$. Then their Killing algebras $\mathrm{Kil}(\mathrm{P}_\rho)$, $\mathrm{Kil}(\mathrm{P}_{\rho'})$ are isomorphic. Taking into account that $\mathrm{Lie}(G_\rho) \subset \mathrm{Kil}(\mathrm{P}_\rho)$ and $\mathrm{Lie}(G_{\rho'}) \subset \mathrm{Kil}(\mathrm{P}_{\rho'})$ and that the dimension of the Killing algebras is equal to 4 (since we excluded the flat cases 7.2), we deduce that $\mathrm{Lie}(G_\rho) = \mathrm{Kil}(\mathrm{P}_\rho) = \mathrm{Kil}(\mathrm{P}_{\rho'}) = \mathrm{Lie}(G_{\rho'})$ which means that G_ρ and $G_{\rho'}$ are isomorphic. Suppose now that G_ρ and $G_{\rho'}$ are isomorphic and put $\mathrm{P}_\rho = G_\rho/I$, $\mathrm{P}_{\rho'} = G_{\rho'}/I'$. Let $\phi : G_\rho \longrightarrow G_{\rho'}$ be a Lie group isomorphism. Since ϕ sends Heis to Heis (the derived subgroups), we have $\phi(I) \subset$ Heis and $G_{\rho'}/\phi(I)$ admits a $G_{\rho'}$-invariant Lorentz metric isometric to P_ρ. But, the uniqueness of the Lorentz metric on $\mathrm{P}_{\rho'}$ (up to scaling and automorphisms) shows that $G_{\rho'}/\phi(I)$, up to scaling, is isometric to $\mathrm{P}_{\rho'}$. Hence P_ρ and $\mathrm{P}_{\rho'}$ are isometric up to scaling. □

Hence, Lemma 7.3 together with Lemma 7.2 imply (6), (7), and (8) in Theorem 1.1.

Proof of part (3) in Theorem 1.1. Suppose that P_ρ is locally symmetric. Then P_ρ is locally isomorphic to a (simply connected) symmetric space X. Hence $\mathrm{Kil}(\mathrm{P}_\rho) = \mathrm{Kil}(X)$, that is, P_ρ and X have isomorphic Killing algebras. Assume that P_ρ is not flat, then (by Lemma 7.2) we have $\mathrm{Kil}(\mathrm{P}_\rho) = \mathrm{Lie}(G_\rho) = \mathrm{Kil}(X)$. Let $\mathfrak{i}_x \subset \mathrm{Kil}(X)$ be the (infinitesimal) isotropy subalgebra of a point $x \in X$. Then we have, since X is symmetric, $\mathrm{Lie}(\mathrm{Hol}_x) \subset \mathfrak{i}_x$ where Hol_x denotes the holonomy group at x. But, since X is not flat, one has $\mathrm{Lie}(\mathrm{Hol}_x) = \mathfrak{i}_x$. This shows that the Hol_x-action on $T_x X$ is indecomposable because the $\mathrm{ad}(\mathfrak{i}_x)$-action is indecomposable since it corresponds to the ad-action of the isotropy $\mathfrak{i} \subset \mathrm{Lie}(G_\rho)$ which is nilpotent with nilpotency order equals 3 (Lemma 3.2). Therefore, X is indecomposable. Furthermore, X admits, locally, a parallel null vector field (coming from P_ρ) which extends, uniquely, to a global null parallel vector field. Thus, X is in fact a Cahen-Wallach space. But in dimension three, we have only two such spaces corresponding to the elliptic and hyperbolic plane waves with unimodular isometry groups G_ρ [13]. So, from Lemma 7.3, P_ρ is a symmetric Cahen-Wallach space with G_ρ unimodular elliptic or hyperbolic. Finally, all this shows that if P_ρ is

locally symmetric not isometric to a Cahen-Wallach, then it is necessarily flat, and in this case (Lemma 7.2) we know that we have only two possibilities corresponding to G_ρ unimodular nilpotent which gives the Minkowski space and G_ρ non-unimodular isomorphic to Aff $\ltimes \mathbb{R}^2$ which gives half Minkowski.

Lastly, let us mention that parts (1) and (2) in Theorem 1.1 are proven in Proposition 3.1. □

8 Plane waves, proof of Theorem 1.3.

8.1 P_ρ is a plane wave.

Remember the definition of the Lorentz metric on P_ρ in Proposition 3.1. One sees that the Killing field Z is null, and that its orthogonal distribution is generated by the Heisenberg algebra, i.e. the three vector fields Z, X, Y.

Let $\mathfrak{l} = \mathbb{R}Y + \mathbb{R}Z$. This is an abelian 2-dimensional Lie sub-algebra of $\mathfrak{g}$. In a neighbourhood of the basis point $1I$, Z and Y are linearly independent (we assume that I is the one parameter group generated by X). It follows that $\mathfrak{l}$ generates the orthogonal distribution $Z^\perp$ (locally).

It is proven in [12] (Theorem 3) that if a Lorentz 3-manifold admits a degenerate distribution generated by two commuting Killing fields, then the Lorentz metric is a plane wave. This applies to our situation and yields that P_ρ is a plane wave in a neighbourhood of the basis point $1I$. This obviously implies that P_ρ is everywhere a plane wave, by homogeneity. For easiness, let us give a proof of the Brinkmann property, that is, Z is a parallel field on P_ρ.

Lemma 8.1 *Let (M, g) be a Lorentzian manifold and let A, B, and C be three Killing vector fields on M. Suppose that A is null, $g(A, B) = 0$, and*

$$[B, A] = 0, \quad [C, A] = \alpha A, \quad \text{and} \quad [C, B] = \beta A + \gamma B$$

Then we have $\nabla_W A = 0$ for every $W = A, B,$ or C.

Proof The Koszul formula arbitrary three Killing vector fields X', Y', Z' on M we have

$$2g(\nabla_{X'} Y', Z') = g([X', Y'], Z') + g([Y', Z'], X') - g([Z', X'], Y')$$

By evaluations we find

$$g(\nabla_A A, A) = g(\nabla_A A, B) = g(\nabla_B A, A) = g(\nabla_C A, A) = g(\nabla_B A, B) = 0$$

We have also

$$2g(\nabla_B A, C) = g(-\beta A - \gamma B, A) - g(\alpha A, B) = 0$$

$$2g(\nabla_C A, B) = g(\alpha A, B) + g(\beta A + \gamma B, A) = 0$$

This proves that $\nabla_W A = 0$ for every $W = A, B$, or C. In particular we have

$$\nabla_{(fA+gB+hC)} A = 0 \text{ for every } f, g, h \in C^\infty(M).$$

□

In our case, we apply the lemma for the basis $A = Z$, $B = Y$ and $C = T$, and get that Z is parallel. □

The flatness of the $Z^\perp$-leaves follows straightforwardly from the fact that Z and Y commute. The proof of the last property of plane waves, that is $\nabla_U R = 0$, for any $U \in Z^\perp$ is detailed in [12]. □

8.2 A homogeneous 3-plane wave is isometric to some P_ρ

We are now going to prove that a simply connected homogeneous 3-plane wave M is isometric to some P_ρ. The Heisenberg group Heis preserves the flat metric $g_0 = 2dudv + dx^2$. It acts by translations on x and v, and by a linear one parameter group of unipotent transformations

$$(v, x, u) \mapsto (v + tx - \frac{t^2}{2}u, x - tu, u)$$

Consider now more generally a metric of the form $g_\delta = 2dudv + \delta(u)dx^2$ where δ is a function depending on u. It is obviously invariant by translation on x and v. It is also invariant under a modification of the previous unipotent one parameter group as follows

$$(v, x, u) \mapsto (v + tx - \frac{t^2}{2}F_\delta(u), x - tF_\delta(u), u)$$

where $F_\delta(u)$ is an anti-derivative of $\frac{1}{\delta(u)}$. The obtained 3-dimensional isometry group of g_δ, is in fact isomorphic to Heis (and acts trivially on u). To summarize, we have

Fact 8.2 *A metric $g_\delta = 2dudv + \delta(u)dx^2$ where δ is a function depending on u admits an isometric action of the* 3*-dimensional Heisenberg group* Heis.

Remember (1.1.2) that a plane wave can be defined by having a local Rosen coordinates where the metric has the form $g_\delta = 2dudv + \delta(u)dx^2$. It follows that it admits an isometric infinitesimal action of the Heisenberg algebra. This action preserves individually the leaves of the orthogonal distribution $\frac{\partial}{\partial v}^\perp$.

In fact, any Killing field U tangent to the $\frac{\partial}{\partial v}^\perp$-distribution belongs to $\mathfrak{heis}$. Indeed, let F_0 be a leaf of $\frac{\partial}{\partial v}^\perp$, then it is (locally) affinely isomorphic to the flat $\mathbb{R}^2$ endowed with a degenerate Riemannian metric dy^2. Transformations preserving such a structure have the form $(z, y) \to (\lambda z + ty + a, y + b)$. If $\lambda \neq 1$, this can not be the restriction of an isometry preserving individually the leaves of $\frac{\partial}{\partial v}^\perp$, since such a transformation has a transverse linear distortion λ^{-1}. But $\lambda = 1$ exactly means that the transformation belongs to Heis. Therefore, the Killing field U coincides with a Killing field $W \in \mathfrak{heis}$ on F_0. The Killing field $U - W$ vanishes on the hypersurface F_0 and hence equals 0 (at a singularity, a non-zero Killing form has a derivative, element of $\mathfrak{so}(1, 2)$ and one easily sees that its 0-set has a most dimension 1). From all this, one infers that $\mathfrak{heis}$ is an ideal in the Killing algebra of the plane wave.

It follows that if the plane wave is locally homogeneous, its Killing algebra contains at least one Killing field T allowing one to pass from a $\frac{\partial}{\partial v}^\perp$-leaf to another. Thus $\mathfrak{g} = \mathbb{R}T \oplus \mathfrak{heis}$ is an extension of the Heisenberg algebra. The associated simply connected Lie group has the form G_ρ, for some ρ, and furthermore the homogeneous plane wave M is locally isometric to P_ρ.

We will assume P_ρ is not flat, since this particular case is treated in Theorem 1.9. So we have a developing map $d : M \to \mathrm{P}_\rho$. The developing map d is equivariant with respect to a homomorphism $h : G \to G_\rho$ where G is the Lie group acting (transitively) on M. Since M and P_ρ have same Killing algebra, and G_ρ is simply connected, it follows that h is an isomorphism. By equivariance, d is a diffeomorphism. In other words, M is identified with P_ρ.

8.3 The non-simply connected case

If M is homogeneous non-simply connected, then the previous discussion shows that $\widetilde{M} = \mathrm{P}_\rho$ for some ρ and $M = \Gamma \backslash \mathrm{P}_\rho$ where $\Gamma \subset G_\rho$ is a normal subgroup (hence central). But the center of G_ρ is trivial unless ρ is unimodular where the center is exactly the direction Z. And ρ unimodular corresponds to either the Minkowski case, or to the two Cahen-Wallach cases (see Lemma 7.1).

9 Global coordinates, proof of Theorem 1.4.

Consider a plane wave metric given in Brinkmann coordinates as

$$g_b = 2dudv + \frac{bx^2}{u^2}du^2 + dx^2$$

on $\{(u, v, x), u > 0\}$. Let us say that most computations are actually done in [3], but only at a Killing fields level. The metric g_b obviously admits a one parameter group of isometries (boosts) $\phi : (u, v, x) \to (e^t u, e^{-t} v, x)$. These isometries act transitively on the set of u-levels.

Now, as any plane wave, the Heisenberg algebra acts (locally) by preserving u-levels and transitively on each of them. It follows that g_b is locally homogeneous and, if it is non-flat, its Killing algebra equals some extension $\mathfrak{g}_\rho$ of the Heisenberg algebra (this will be detailed).

9.1 Global action of G_ρ

We are now going to prove that the $\mathfrak{g}_\rho$ infinitesimal action integrates to a global G_ρ-action. For this, we have to consider the $\mathfrak{heis}$-action in Brinkmann coordinates. But understanding such an action involves analyzing the transform Brinkmann to Rosen, which sounds highly "transcendental"!

Fact 9.1 *Let a metric g in Brinkmann form $g = 2dudv + H(u)x^2du^2 + dx^2$, defined on $(v, x), \in \mathbb{R}^2$, and $u \in J$, J an interval. Then* Heis *acts (globally, not merely locally) isometrically by preserving the u-levels.*

Proof Equivalence of Brinkmann and Rosen coordinates is done in Appendix 13 (see [1] for more details).

Then, we describe the Heis-action in the Rosen form $g_\delta = 2dudv + \delta(u)dx^2$ as given in Fact 8.2. Let us focus on the $\mathbb{R}^2$ ($\subset$ Heis)-action given by $(v, x, u) \mapsto (v + a, x + b, u)$ which is isometric and generates the horizontal foliation $\mathscr{F}$, whose leaves are the u-levels.

All these leaves are thus $\mathbb{R}^2$-affine homogeneous spaces. In particular, the geodesics of g_δ tangent to $\mathscr{F}$ do not depend of δ, that is, they are usual straight lines as in the case of the Minkowski space corresponding to $g_1 = 2dudv + dx^2$.

The Killing field W corresponding to the translation action on the x-coordinate, $(v, x, u) \mapsto (v, x + b, u)$ is thus parallel on each $\mathscr{F}$-leaf. This Killing field is however not parallel for a general g_δ (on the whole space). Let us say, there exists a map $u \mapsto W_u$ (depending on δ), such that, on each $\mathscr{F}_u$, W is parallel to W_u *i.e* directed by W_u.

As for the Brinkmann form $g = 2dudv + H(u)x^2du^2 + dx^2$, the same Killing field W is parallel on each u-level. In order to prove that W has a (complete) global flow, it is enough to show that the u-levels are complete. It turns out that geodesics of these levels are still straight

lines in the (Brinkmann) (v, x)-coordinates (see [5] where geodesic equations are explained and shown to have affine solutions for horizontal initial conditions) and since $(v, x) \in \mathbb{R}^2$, they are complete.

As for the action $(v, x, u) \mapsto (v + tx - \frac{t^2}{2} F_\delta(u), x - t F_\delta(u), u)$ of the other one parameter subgroup of Heis in Rosen coordinates, it does not have geodesic orbits, but it is affine on the u-levels endowed with the Rosen (v, x)-coordinates. This is still affine in the Brinkmann (v, x)-coordinates since the geodesics are straight lines. By completeness of these u-levels, the flow acts globally. □

9.1.1 Identification of ρ as a function of b

Let us now show that any P_ρ (for ρ non-unimodular) is isometric to a metric g_b, for some $b = b(\rho)$. For this, it is enough to check that the Killing algebra of $g_{b(\rho)}$ equals $\mathfrak{g}_\rho$. Now, the computation of the Killing algebra of g_b is done in [3] (Formula 3.33). It is stated there that the Heisenberg algebra $\mathfrak{heis} = \mathrm{span}(Z, X, Y)$, is extended by an element T, such that $[T, X] = bY$, $[T, Y] = X + Y$ and $[T, Z] = Z$, in other words ad_T equals $A_b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & b & 1 \end{pmatrix}$ in the basis $\{Z, X, Y\}$. The associated semi-direct product is thus defined by $\rho(t) = e^{tA}$. Conversely, let A be a derivation of $\mathfrak{heis}$ with $A(Z) \neq 0$, that is, $A(Z) = aZ$ with $a \neq 0$. Up to scaling (of A), we can assume $a = 1$, and up to adding an interior derivation ad_u, $u \in \mathfrak{heis}$, we can assume that A preserves a supplementary of Z, say the plane $\mathbb{R}X \oplus \mathbb{R}Y$. And up to conjugacy by automorphisms, A has the form A_b. This completes the proof of Theorem 1.4 □

9.1.2 Hyperbolic case

We also have an explicit global Rosen formula for hyperbolic plane waves 1.1.4

$$g_\alpha = 2dudv + u^{2\alpha} dx^2$$

defined on $(v, x) \in \mathbb{R}^2$, $u > 0$ (considered, in particular, in [18]). So Heis acts isometrically affinely in these coordinates. Observe that the linear one parameter group $(v, x, u) \mapsto (e^t v, e^{\alpha t} x, e^{-t} u)$, also preserves g_α. We know that Heis is normal in $\mathrm{Isom}(g_\alpha)$, and thus an extension G_ρ, acts isometrically in an affine manner. So G_ρ is in fact a subgroup of $\mathrm{Aff}(\mathbb{R}^3)$. We can show that the representation associated to α is $\rho(t) = e^{tA}$, where A is diagonal with entries $(1, 1 - \alpha, \alpha)$. We deduce that any hyperbolic non-unimodular plane wave P_ρ has a global Rosen model g_α.

9.1.3 Parabolic (non-unimodular) case

$$g_b = 2dudv + \tfrac{x^2}{4u^2} du^2 + dx^2$$

Proof that G_ρ is isomorphic to $G_{\rho'}$ if and only if $b(\rho) = b(\rho')$.

In what follows, we denote $\mathfrak{g}_\rho$ by $\mathfrak{g}$. Since the center $\mathfrak{z}$ of $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is equal to $[\mathfrak{g}', \mathfrak{g}']$, and $[\mathfrak{g}, \mathfrak{z}] = \mathfrak{z}$, the adjoint representation ad of $\mathfrak{g}$ induces an action of $\mathfrak{g}/\mathfrak{g}'$ on $\mathfrak{z}$, and since this action is not trivial ($\mathfrak{z}$ is not central), there is a unique $[T] \in \mathfrak{g}/\mathfrak{g}'$ that acts as the identity.

Moreover, ad induces an action $\overline{\mathrm{ad}}$ of $\mathfrak{g}/\mathfrak{g}'$ on $\mathfrak{g}'/\mathfrak{z}$ (this is well-defined, thanks to the two properties mentioned at the beginning of the proof), and $\det(\overline{\mathrm{ad}}_{[T]}) = -b$, this shows that b is determined by the algebra structure of $\mathfrak{g}$. □

10 Proof of the completeness statement in Theorem 1.7.

Fix a metric $g = 2dudv + \frac{bx^2}{u^2}du^2 + dx^2$, $b \neq 0$ and let P_ρ the so defined plane wave and G_ρ its isometry group (or sometimes simply P and G).

10.1 Space of null geodesics

Let $\mathscr{G}^0$ be the space of all inextendible geometric (i.e. non-parametric) isotropic geodesics of P_ρ. Let $\mathscr{H}^0$ be the space of horizontal ones, i.e. those for which u is constant and are in fact orbits of the parallel field $\frac{\partial}{\partial v}$. The set of vertical i.e. non-horizontal ones is denoted $\mathscr{V}^0$, so that $\mathscr{G}^0 = \mathscr{V}^0 \cup \mathscr{H}^0$.

10.1.1 Example of an incomplete (vertical) null geodesic

Observe that $(v, x, u) \mapsto (v, -x, u)$ is isometric and hence $\{x = 0\}$ is geodesic. The metric on the latter plane is $dudv$. Therefore, $t \mapsto (0, 0, t)$ is a null geodesic. Its maximal existence domain is $]0, +\infty[$. Let us denote this geodesic γ_0.

10.1.2 Vertical null geodesics are non-complete

Indeed, for such a geodesic $\gamma(t) = (v(t), x(t), u(t))$, it is known that, up to a scaling and a shift of time, one can choose $u(t) = t$, that is the coordinate t plays the role of (an affine) time (see [5, Prop. 3.1]). The (second order) equation on $x(t)$ is linear, and $v(t)$ is an integral of it. More precisely, as detailed in [5], $x(t)$ satisfies, up to a constant, an equation $\ddot{x}(t) = \frac{x}{t^2}$ (after identifying u with a multiple ct). Solutions are, thus, defined on $]0, +\infty[$. Observe in fact, that this applies to all non-horizontal geodesics, null or not, in particular to all timelike geodesics.

10.1.3 Action of $\mathrm{Isom}(\mathrm{P}_\rho)$ on $\mathscr{G}^0$

Proposition 10.1 *The action of G_ρ on $\mathscr{G}^0$ has exactly two orbits $\mathscr{V}^0$ and $\mathscr{H}^0$. In particular, any non-horizontal null geodesic is congruent, mod G_ρ, to the vertical geodesic γ_0. Also, any such geodesic is the orbit of a one parameter group of G_ρ.*

Proof Let T^0 be the space of null tangent directions of P_ρ. This is circle bundle over

$$\left\{(v, x, u) \in \mathbb{R}^2 \times \mathbb{R}^+\right\}.$$

The set H^0 of horizontal null directions gives a section, and so is connected. The set V^0 of non horizontal directions forms a bundle with fiber $\mathbb{S}^1 - 1\,\mathrm{pt} \cong \mathbb{R}$, and thus is connected, too. Spaces of geodesics $\mathscr{H}^0$ and $\mathscr{V}^0$ are just quotients of H^0 and V^0 by the geodesic flow, and hence are connected. The isotropy in G_ρ is unipotent (of dimension 1) fixing only horizontal directions. It follows that G_ρ acts freely on V^0 and has 1-dimensional isotropy on H^0. Since

dim $V^0 = 4$, and dim $G_\rho = 4$, connectedness of G_ρ implies that it acts transitively (and freely) on V^0. In particular $\mathscr{V}^0$ is a homogeneous space G_ρ/L, where L is a one parameter group. From this follows that any vertical null geodesic in an isometric image of γ_0, and that any such geodesic is preserved by a one parameter group of isometries acting freely (and hence transitively since the dimension is one) on it. As for $\mathscr{H}^0$, it is a G_ρ homogeneous surface homeomorphic to $\mathbb{R} \times \mathbb{R}^+$. □

11 Proof of the non-extendibility statement in Theorem 1.7.

In this section, ρ is fixed once for all, and hence P_ρ will be simply denoted P. A non-trivial extension of (P, g_b) consists of a Lorentz manifold (N, g) together with an embedding $\phi : \mathrm{P} \to N$, such that $\phi^* g = g_b$, but ϕ is not onto, so $\mathrm{P}' = \phi(\mathrm{P})$ is an open proper subset of N. We say that P is inextendible if such ϕ does not exist, say alternatively if any isometric embedding is bijective. It is important to note that this depends on the regularity of the metric g on N.

11.1 Analytic case

We can assume N is simply connected, and then use that any locally defined Killing field extends everywhere. The Lie algebra $\mathfrak{g}_\rho$ then acts on N. The leaves can not be all of dimension 3, since this means they are all open, but then by connectedness $\mathrm{P}' = N$. One can exclude orbits of dimension one (paragraph 6.2), and conclude that some orbit in the closure of P' has dimension 2. The same argument applies to the $\mathfrak{heis}$-action and yields that all its orbits have dimension 2. Therefore, this 2-dimensional $\mathfrak{g}_\rho$ orbit coincides with a $\mathfrak{heis}$-orbit. Now, these $\mathfrak{heis}$- leaves are affine $\mathbb{R}^2$ endowed with a degenerate form dy^2. The automorphism group of such a structure is Aff $\ltimes \mathbb{R}^2$, where $\mathbb{R}^2$ acts by translation and Aff, is the stabilizer in $\mathrm{GL}(2, \mathbb{R})$ of the parallel direction $\frac{\partial}{\partial v}$, identified to $(1, 0) \in \mathbb{R}^2$ (more explicitly Aff consists of matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, preserving the direction of $(1, 0)$ and acting isometrically transversely). We then deduce that the Lie algebra $\mathfrak{g}_\rho$ embeds in the Lie algebra of Aff $\ltimes \mathbb{R}^2$, but the latter corresponds to the flat metric g_0. Let us mention that the non-extendibility in the analytic case can also be derived from the main result of [9]. It says that if the Killing algebra of an analytic Lorentz metric on a connected 3-manifold N has an open orbit, then this manifold is locally homogeneous, i.e. its (full) Killing algebra has one orbit. In our case, if $\mathrm{P}' \neq N$, then the Lie algebra of N has dimension ≥ 5, which implies (Lemma 7.2) that N has constant curvature, hence flat.

11.1.1 Smooth complete case

It is shown in [11] that there exists a geodesic $\gamma(t)$ in P, $t \in]-\infty, 1[$ and a parallel 2-plane field $Q(t)$ along γ such that the sectional curvature of $Q(t)$ tends to ∞ when $t \to 1$. This implies that P can not be embedded in a geodesically complete space. Here the metric g (on N) is assumed to be C^2 (and **complete**).

11.1.2 A synthetic proof

Let us give a brief outline of a proof that works in the C^2 case, but one may expect to adapt it even to the C^1 (perhaps C^0) case (we hope to give more details elsewhere). The lightlike geodesic foliation $\mathscr{F}$ tangent to $\frac{\partial}{\partial v}^{\perp}$ is defined on P'. It has complete leaves. Let us now recall a Lipschitz regularity property of such foliations. For this, it might be better to give an example brought out in particular from [22, 23]. Let B^3 be the unit open Euclidean ball, and consider $\mathscr{L}_s$ a family of disjoint affine 2-dimensional subsets, which are relatively complete in B^3, to mean that $\mathscr{L}_s$ equals the intersection with B^3 of a complete affine 2-plane of $\mathbb{R}^3$ (so in B^3, $\mathscr{L}_s$ is as complete as possible). Let $\Sigma = \cup_s \mathscr{L}_s$, be the support of our disjoint family. The Lipschitz regularity fact is that the family is (locally) Lipschitz and in particular extends to a similar disjoint geodesic family on the closure $\overline{\Sigma}$. It turns out this generalizes to disjoint families of geodesic hypersurfaces in general pseudo-Riemannian manifolds (in fact manifolds with a connection), see [22, 23]. In our situation, the foliation $\mathscr{F}$ of P' extends to a foliation on $\overline{\mathrm{P}'}$. If τ is a transverse curve to $\mathscr{F}$, then $\mathrm{P} \cap \tau$ is an open set of τ, and because $\dim \tau = 1$, we infer that $\overline{\mathrm{P}'}$ is a regular open domain in N, i.e. a submanifold of codimension 0 with regular (in fact geodesic) boundary.

Let F_0 be a boundary leaf of $\mathscr{F}$. By continuity, as above (11.1), this is an affine $\mathbb{R}^2$ endowed with a degenerate Riemannian metric. In order to get a contradiction as above, we need to prove that G_ρ acts on F_0 (preserving its structures).

For this goal, let p a point of P' close enough to F_0, and $\phi \in G_\rho$ a small isometry of P', say ϕ is in a small neighbourhood of 1 in G_ρ and denote $q = \phi(p)$. Consider a cone of directions in $T_p \mathrm{P}'$ whose geodesics all cut F_0 transversely. These geodesics cut F_0 and also nearby leaves F of $\mathscr{F}$ in an open subset (of these leaves). For γ such a geodesic, F a leaf of $\mathscr{F}$, close to F_0, $\gamma \cap F$ is one point, and $\phi(\gamma \cap F) = \phi(\gamma) \cap \phi(F)$. This formula allows one to define an extension of ϕ on some open subset of F_0 by $\phi(\gamma \cap F_0) = \phi(\gamma) \cap F_0$. Observe here that ϕ preserves F_0.

Now, to check that everything is coherent, that is, these extensions are well defined and they give rise to a group, or at least an algebra action, observe that since we have a true action on P', the construction, (in P') does not depend on the choice of the point p. Therefore, the extensions to the leaf F_0 coincide since they are limits of actions on interior leaves in P'.

From all this, we infer that G_ρ acts on F_0, which implies that it embeds in the automorphism group of its structure, $\mathrm{Aff} \ltimes \mathbb{R}^2$, as discussed in 11.1, which is impossible.

11.1.3 Conformal extensions

Observe that all plane waves (in dimension 3) are conformally flat. Indeed, in Rosen coordinates $g = 2dudv + \delta(u)dx^2 = \delta(u)(2dwdv + dx^2)$, where $dw = \frac{du}{\delta(u)}$. Therefore, their conformal developing in the Einstein universe $\mathrm{Ein}_{1,2}$ is a non-trivial conformal extension.

12 Compact models.

The aim of this section is to prove Theorem 1.5. Suppose that P_ρ is not flat. Then $\mathrm{Isom}_0(\mathrm{P}_\rho) = G_\rho$ (by Theorem 1.1). Let $\mathrm{Isom}(\mathrm{P}_\rho)$ denote the full isometry group and $I_x \subset \mathrm{Isom}(\mathrm{P}_\rho)$ the full isotropy of a point $x \in \mathrm{P}_\rho$ with identity component $I_x^0 = I_x \cap G_\rho$ consisting of a unipotent one-parameter group of $\mathrm{SO}(1,2)$ when acting on $T_x \mathrm{P}_\rho$. Furthermore, I_x / I_x^0 is finite since the image of I_x inside $\mathrm{O}(1,2)$ (by the faithful derivative representation) is an

algebraic group. This shows that $\mathrm{Isom}(\mathrm{P}_\rho)/G_\rho$ is finite, which implies that if we have a compact $(\mathrm{Isom}(\mathrm{P}_\rho), \mathrm{P}_\rho)$-manifold, then it has a finite cover which is a $(G_\rho, \mathrm{P}_\rho)$-manifold. So we restrict our attention only to compact $(G_\rho, \mathrm{P}_\rho)$-manifolds.

Proposition 12.1 *Let M be a closed $(G_\rho, \mathrm{P}_\rho)$-manifold. Then M is complete (as a $(G_\rho, \mathrm{P}_\rho)$-structure), that is, the developing map $D : \widetilde{M} \to \mathrm{P}_\rho$ is a diffeomorphism.*

Proof Let $Z \in \mathrm{Lie}(G_\rho)$ be the vector that generates the central direction in $\mathfrak{heis}$ and V a generator of $\mathrm{Lie}(I)$ where $\mathrm{P}_\rho = G_\rho/I$. We have that Z is ad_V-invariant *i.e* $\mathrm{ad}_V(Z) = 0$. This implies that the vector field $L_*(Z)$ on G_ρ obtained by left translating Z is I-right invariant. So $L_*(Z)$ projects to a G_ρ-invariant vector field on P_ρ which we denote by $\mathscr{Z}$. Thus, $\mathscr{Z}$ can be pulled back to a well defined vector field $\mathscr{Z}_M$ on M. Let $\pi^*(\mathscr{Z}_M)$ be the pull back of $\mathscr{Z}_M$ on $\widetilde{M}$ where $\pi : \widetilde{M} \to M$ is the covering projection. The developing map D sends $\pi^*(\mathscr{Z}_M)$ to $\mathscr{Z}$, so it sends a $\pi^*(\mathscr{Z}_M)$-trajectory to a $\mathscr{Z}$-trajectory where D, restricted to each trajectory, is a local diffeomorphism into the corresponding $\mathscr{Z}$-trajectory. Since the trajectories are one-dimensional, D must be injective on each $\pi^*(\mathscr{Z}_M)$-trajectory. But, since $\pi^*(\mathscr{Z}_M)$ is complete (because $\mathscr{Z}_M$ is complete) then D sends a $\pi^*(\mathscr{Z}_M)$-trajectory bijectively onto a $\mathscr{Z}$-trajectory.

Consider now the action of the Heisenberg group $\mathrm{Heis} \subset G_\rho$ on P_ρ. Since the stabilizer of every point $x \in \mathrm{P}_\rho$ must be contained in Heis (because Heis is normal), then the orbits of this action define a G_ρ-invariant foliation by (totally geodesics degenerate) surfaces of P_ρ. Thus, this foliation, denoted by $\mathscr{F}$, can be pulled back to a well defined foliation $\mathscr{F}_M$ on M. Let $\widetilde{\mathscr{F}}_M$ be the lift of the foliation $\mathscr{F}_M$ to the universal cover $\widetilde{M}$. The vector field $\pi^*(\mathscr{Z}_M)$ is everywhere tangent to $\widetilde{\mathscr{F}}_M$, so each leaf $\widetilde{\mathscr{F}}_M(p)$ is foliated by the $\pi^*(\mathscr{Z}_M)$-trajectories and the developing D map sends each leaf of $\widetilde{\mathscr{F}}_M$ equivariantly to the $\mathscr{F}$-leaves. Let now $Y \in \mathfrak{heis} \subset \mathrm{Lie}(G_\rho)$ transverse to $\mathbb{R}Z \oplus \mathbb{R}V$ inside $\mathfrak{heis}$, and V, as indicated above, a generator of $\mathrm{Lie}(I)$. We have that $\mathrm{ad}_V(Y)$ is proportional to Z. This implies that the flow of the left-invariant vector field $L_*(Y)$ on G_ρ preserves the surfaces gK where K is the subgroup of G_ρ associated to the abelian subalgebra $\mathrm{span}(V, Z)$. Hence, since G_ρ/K is nothing but the space of all $\mathscr{Z}$-leaves on P_ρ, we have that Y generates a G_ρ-equivariant flow ϕ^t on the space of $\mathscr{Z}$-leaves and preserving each Heisenberg leaf. This flow, thus, can be pulled back to a complete flow (an $\mathbb{R}$-action), ϕ^t_M, on the space of the $\mathscr{Z}_M$-trajectories (completeness is due to the compactness of M, see Remark 12.1). Moreover, ϕ^t_M can be lifted to a complete flow, $\widetilde{\phi}^t_M$, on the space of all $\pi^*(\mathscr{Z}_M)$-trajectories on $\widetilde{M}$. Now, since D sends a leaf $\widetilde{\mathscr{F}}_M(p)$ to the leaf $\mathscr{F}(D(p))$ by sending a $\pi^*(\mathscr{Z}_M)$-trajectory bijectively to a $\mathscr{Z}$-trajectory, we have then, for similar reasons, that D sends a $\widetilde{\phi}^t_M$-orbit to a ϕ^t-orbit injectively (because the flow ϕ^t is injective since the space of $\mathscr{Z}$-trajectories inside each Heisenberg leaf is identified with $\mathbb{R}$ and ϕ_t acts on it properly freely) and surjectively (because $\widetilde{\phi}^t_M$ is complete). So, D sends each $\widetilde{\mathscr{F}}_M(p)$ bijectively onto $\mathscr{F}(D(p))$.

Finally, let $T \in \mathrm{Lie}(G_\rho)$ not inside $\mathfrak{heis}$. We have that $\mathrm{ad}_V(T) \in \mathfrak{heis}$. this implies, with the same ideas, that the flow of the left-invariant vector field $L_*(T)$ preserves the cosets gL where $L = \mathrm{Heis}$. Thus, this flow is well defined on G_ρ/L which is the space of all $\mathscr{F}$-leaves. By similar arguments, this defines a flow on the space of all $\widetilde{\mathscr{F}}_M$-leaves on $\widetilde{M}$ which corresponds bijectively to the flow on the space of the $\mathscr{F}$-leaves on P_ρ. This proves that D is bijective, hence, a diffeomorphism. □

Remark 12.1 Observe that the space of $\mathscr{Z}_M$-trajectories, introduced in the proof, is pathological in general yet the $\mathbb{R}$-action of ϕ^t_M is well defined on it and it can also have a wild behaviour (like a transverse $\mathbb{R}$-action, on the leaves of a dense linear foliation on the 2-torus,

which has a dense stabilizer). In fact, one can even construct a vector field on M, tangent to the Heisenberg leaves and transverse to the $\mathscr{Z}_M$-orbits, and whose flow sends orbit to orbit realizing our flow ϕ_M^t on the space of $\mathscr{Z}_M$-orbits.

Corollary 12.2 *If M is a closed $(G_\rho, \mathrm{P}_\rho)$-manifold. Then $M = \Gamma\backslash \mathrm{P}_\rho$ where $\Gamma \subset G_\rho$ is a discrete subgroup acting freely and properly discontinuously on P_ρ*

Remark 12.3 The completeness result of the previous Proposition 12.1 and Corollary 12.2 is already known in the case where P_ρ is symmetric. It follows, in particular, from Theorem 2.1 in [10] or from Corollary 2 in [14].

12.1 (Non-)existence of compact models

We have seen so far that if P_ρ admits compact models, that is, closed manifolds locally isometric to P_ρ, then these compact models are, up to finite covers, of the form $\Gamma\backslash \mathrm{P}_\rho$, in the sense that they are quotients of P_ρ by a discrete subgroup acting freely and properly.

Remark 12.4 As said in Remark 1.6, the symmetric case in Theorem 1.5 is known. Indeed, it is shown in [13] that non-flat Cahen-Wallach spaces do not admit compact quotients in dimension 3. So the non-existence of compact models follows form the completeness result (as mentioned in Remark 12.3).

We will give here a direct proof of this fact.

Theorem 12.5 *All Lorentz spaces P_ρ, except two, do not support compact quotients (i.e of the form $\Gamma\backslash \mathrm{P}_\rho$). The two exceptional cases correspond to left-invariant metrics on* SOL *and* Heis*, where the latter is globally isometric to the Minkowski space.*

Proof We will prove that there is no discrete subgroup Γ of G_ρ acting freely and cocompactly on P_ρ, except for two cases. this is done by considering all the possible representations $\rho : \mathbb{R} \to \mathrm{Aut}(\mathrm{Heis})$ (up to equivalence) then giving separate proofs of the (non-)existence in each case. The basis for $\mathrm{Lie}(G_\rho)$ will be $\{Z, X, Y, T\}$ as usual where Z, X, and Y generate $\mathfrak{heis}$ and $A_\rho = \mathrm{ad}_T$ acting on $\mathfrak{heis}$.

• Let $\mathbb{R}$ act on Heis via the representation $\rho : \mathbb{R} \to \mathrm{Aut}(\mathrm{Heis})$ generated by the derivation (3.1.1)

$$A_\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and let $G_\rho = \mathbb{R} \ltimes_\rho \mathrm{Heis}$. We have seen (3.1.1) that G_ρ/I where $\mathrm{Lie}(I) = \mathfrak{i}$ is not included in $\mathrm{span}(Z, X)$ or in $\mathrm{span}(Z, Y)$, is a Lorentz space. Let $\Gamma \subset G_\rho$ be a discrete subgroup acting freely and co-compactly on $\mathrm{P}_\rho = G_\rho/I$. We have that the connected Lie subgroup $[G_\rho, G_\rho]$ is generated by $\mathrm{span}(Z, X)$. Indeed $[G_\rho, G_\rho]$ is generated by the image of A_ρ and Z. Hence $[\Gamma, \Gamma]$ lies inside the abelian subgroup H given by Z and X. Clearly Γ is not contained in $\mathrm{Heis}\backslash \mathrm{P}_\rho \sim \mathbb{R}$ is not compact. So there is $\gamma \in \Gamma$ of the form $\gamma = h \exp(tT)$ where $t \neq 0$ and $h \in \mathrm{Heis}$. The adjoint action of γ on G_ρ (the action by conjugacy) preserves H and acts on it via the automorphism $\mathrm{Ad}(\gamma) = \mathrm{Ad}(h) \circ \exp(tA_\rho)$. Since $\exp(tA_\rho)$ restricted to H is of the form $\begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}$, then the action of $\mathrm{Ad}(\gamma)$ on H is a contraction (up to forward or backward iterations). But $\mathrm{Ad}(\gamma)$ preserves $\Gamma \cap H$, so we necessarily have $\Gamma \cap H = \{1\}$ since $\mathrm{Ad}(\gamma)$ is a contraction on H and $\Gamma \cap H$ is discrete. We conclude that $[\Gamma, \Gamma] = \{1\}$ which

means that Γ is abelian. Consider now the quotient $\pi : G_\rho \to G_\rho/Z$ where Z refers to the center of Heis. We have that $G_\rho/Z = \mathbb{R} \ltimes \mathbb{R}^2$ where $\mathbb{R}$ acts on $\mathbb{R}^2$ via the representation $t \to A_t = \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}$ with $\mathbb{R}^2$ endowed the basis $\{\widetilde{X}, \widetilde{Y}\}$ obtained by projecting X and Y. The restriction of π to Γ is injective since $\Gamma \cap H = \{1\}$. Thus $\pi(\Gamma)$ is an abelian subgroup of $\mathbb{R} \ltimes \mathbb{R}^2$. Furthermore, $\pi(\Gamma)$ is not contained in the translation factor, $\mathbb{R}^2$, of $\mathbb{R} \ltimes \mathbb{R}^2$ because $\pi^{-1}(\mathbb{R}^2) = \text{Heis}$ and Γ is not contained in Heis. So, there is an element $\alpha = A_t + v$ in $\pi(\Gamma)$ with $t \neq 0$. Suppose that α has a fixed point when acting on $\mathbb{R}^2$, then, up to conjugacy, we can assume that $A_t \in \pi(\Gamma)$. Since the line $\text{span}(\widetilde{Y})$ is exactly the fixed subspace of A_t and $\pi(\Gamma)$ is abelian, then all elements of $\pi(\Gamma)$ preserve $\text{span}(\widetilde{Y})$ and, hence, they are all of the form $A_t + v$ with $v \in \text{span}(\widetilde{Y})$. That is $\pi(\Gamma) \subset \mathbb{R} \oplus \text{span}(\widetilde{Y})$ which implies that $\Gamma \subset \pi^{-1}(\mathbb{R} \oplus \text{span}(\widetilde{Y})) = L$ where L is the Lie subgroup generated by Z, Y, and T. But, we have seen that (4.1) P_ρ as a homogeneous Lorentz space, is identified with a left-invariant metric on L. This shows that $\Gamma\backslash \text{P}_\rho = \Gamma\backslash L$ and $\Gamma \subset L$ is a co-compact lattice which is impossible since L is not unimodular. It remains now to treat the case where all elements of $\pi(\Gamma)$ are without fixed points when acting on $\mathbb{R}^2$. Suppose that we are in this situation and consider the quotient $p : \mathbb{R} \ltimes \mathbb{R}^2 \to (\mathbb{R} \ltimes \mathbb{R}^2)/\text{span}(\widetilde{Y})$ (observe that $\text{span}(\widetilde{Y})$ is central in $\mathbb{R} \ltimes \mathbb{R}^2$). Then

$$(\mathbb{R} \ltimes \mathbb{R}^2)/\text{span}(\widetilde{Y}) = \mathbb{R} \ltimes \mathbb{R} = \text{Aff}$$

and the projection $p(\pi(\Gamma))$ is an abelian subgroup which is not contained in the normal translation factor $\mathbb{R}$ (because $\pi^{-1}(p^{-1}(\mathbb{R})) = \text{Heis}$). Since every element of Aff, which is not in the normal factor, has exactly one fixed point when acting on $\mathbb{R}$. We can assume then, up to conjugacy, that all elements of $p(\pi(\Gamma))$ fix 0. This implies that $p(\pi(\Gamma)) \subset \mathbb{R}$ (where $\mathbb{R}$ here refers to the "linear" factor). This gives that, as before, that $\Gamma \subset L$ (where L is as above) and, for the same reason, this is impossible. Observe, in this case, that P_ρ supports compact models (*i.e* compact manifolds locally isometric to P_ρ) since it is flat (Lemma 7.2), but the previous discussion shows that P_ρ does not support (G_ρ, P_ρ)-compact models.

- Let $\mathbb{R}$ act on Heis via the representation $\rho : \mathbb{R} \to \text{Aut}(\text{Heis})$ generated by the derivation (3.1.1)

$$A_\rho = \begin{pmatrix} 1+b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b \end{pmatrix}$$

where we suppose $b > 0$. In this case, $\mathbb{R}$ acts on Heis via

$$t \mapsto \begin{pmatrix} e^{t(1+b)} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{tb} \end{pmatrix}$$

in the usual basis $\{Z, X, Y\}$. So the $\mathbb{R}$-action on Heis is a contraction (for $t \to -\infty$), which implies that $\Gamma \cap \text{Heis} = \{1\}$ since $\Gamma \cap \text{Heis}$ is preserved by the conjugacy action (necessarily contracting up to forward or backward iteration) of an element of Γ which is not in Heis. But $\Gamma \cap \text{Heis}$ is discrete, which implies $\Gamma \cap \text{Heis} = \{1\}$. In particular Γ is abelian. Consider now the quotient $\pi : G_\rho \to G_\rho/Z$. Remark that $G_\rho/Z = \mathbb{R} \ltimes \mathbb{R}^2 \subset \text{Aff}(\mathbb{R}^2)$ where $\mathbb{R}$ acts on $\mathbb{R}^2$ via the representation $t \mapsto A_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{bt} \end{pmatrix}$. We know that, except translations, every element of $\mathbb{R} \ltimes \mathbb{R}^2$ has exactly one fixed point. Moreover, we have that $\pi(\Gamma) \cap \mathbb{R}^2 = \{1\}$, because $\pi^{-1}(\mathbb{R}^2) = \text{Heis}$ and $\Gamma \cap \text{Heis} = \{1\}$. So (up to conjugacy), all elements of $\pi(\Gamma)$ fix $0 \in \mathbb{R}^2$, i.e $\pi(\Gamma) \subset \mathbb{R}$, where $\mathbb{R}$ denotes the linear factor, which

implies that $\Gamma \subset \pi^{-1}(\mathbb{R}) = H$ with H generated by $\mathrm{span}(Z, T)$. But $H \backslash \mathrm{P}_\rho \cong \mathbb{R}$ where H acts freely and properly on P_ρ since it doesn't intersect any conjugate of I. This can be seen easily, since $I \subset \mathrm{Heis}$ and Heis is a normal subgroup, moreover $H \cap \mathrm{Heis} = Z$. We conclude that P_ρ doesn't admit any compact quotient.

- Let $\mathbb{R}$ act on Heis via the representation $\rho : \mathbb{R} \to \mathrm{Aut}(\mathrm{Heis})$ generated by the derivation (3.1.3)

$$A_\rho = \begin{pmatrix} 2c & 0 & 0 \\ 0 & c & -1 \\ 0 & 1 & c \end{pmatrix}$$

with $c \neq 0$, where $\mathbb{R}$ acts on Z by $t \mapsto e^{2tc}$ and acts by similarities on $\mathbb{R}X \oplus \mathbb{R}Y$ with non-trivial homothety factor. A similar reasoning to the preceding case (using the same notations) shows that Γ is abelian and that, up to conjugacy, $\Gamma \subset H$ with H given by $\mathrm{span}(Z, T)$ which is impossible. Hence there can't be any compact quotient of P_ρ.

- Let $\mathbb{R}$ act on Heis via the representation $\rho : \mathbb{R} \to \mathrm{Aut}(\mathrm{Heis})$ generated by the derivation (3.1.1)

$$A^b_\rho = \begin{pmatrix} 1+b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b \end{pmatrix}$$

with $b < 0$. We can suppose that $b \in [-1, 0[$, because A^b_ρ is similar to $A^{\frac{1}{b}}_\rho$. Observe that

$$G_\rho \text{ is unimodular if, and only if } b = -1$$

We know that (3.1.1) in order for $\mathrm{P}_\rho = G_\rho / I$ to carry a Lorentz metric, it is necessary and sufficient that I avoids $\mathrm{span}(X, Z) \cup \mathrm{span}(Y, Z)$. Let us assume, up to conjugacy, that I lives inside $\mathrm{span}(X, Y)$ and generated by $X + Y$. The orbits of $\mathbb{R}$ action on $\mathrm{span}(X, Y)$ are hyperbolas, so the orbit of I under this action covers two opposite quarters among the four quarters of $\mathrm{span}(X, Y) \setminus (\mathbb{R}X \cup \mathbb{R}Y)$. This shows that all the conjugates of I cover two opposite quarters of the complimentary of $\mathrm{span}(X, Z) \cup \mathrm{span}(Y, Z)$ inside Heis. Now in order for Γ to act freely, it must not intersect any conjugate of I, So $\Gamma \cap \mathrm{Heis}$ must be contained in the other two quarters (with their boundaries included). Consider the quotient $\pi : G_\rho \to G_\rho / Z = \mathbb{R} \ltimes \mathbb{R}^2$, so $\pi(\Gamma) \cap \mathbb{R}^2$ must be inside some line $l \subset \mathbb{R}^2$ because, otherwise, the intersection contains a lattice and this would contradict that it is inside two quarters. Since $\pi(\Gamma)$ is not contained in $\mathbb{R}^2$, we can pick an element of $\alpha \in \pi(\Gamma) \setminus \mathbb{R}^2$, of the form $\alpha = \begin{pmatrix} e^t & 0 \\ 0 & e^{bt} \end{pmatrix} + v$. So the induced action of α on $\mathbb{R}^2$ (by conjugacy) is equivalent to the action of $\begin{pmatrix} e^t & 0 \\ 0 & e^{bt} \end{pmatrix}$. Hence either $\pi(\Gamma) \cap \mathbb{R}^2 \subset \mathbb{R}X$ or $\pi(\Gamma) \cap \mathbb{R}^2 \subset \mathbb{R}Y$. Suppose that $\pi(\Gamma) \cap \mathbb{R}^2 \subset \mathbb{R}X$ (the case where $\pi(\Gamma) \cap \mathbb{R}^2 \subset \mathbb{R}Y$ can be treated similarly) and consider the projection

$$P : \mathbb{R} \ltimes \mathbb{R}^2 \to \mathbb{R} \ltimes \mathbb{R}^2 / \mathbb{R}X \cong \mathrm{Aff}$$

Since $P(\pi(\Gamma))$ is abelian, and it is not contained in the normal factor, then, up to conjugacy, it is contained in the linear factor $\mathbb{R} \subset \mathrm{Aff}$. This means that up to conjugacy we have $\Gamma \subset L_X$ is a cocompact lattice with L_X generated by $\mathrm{span}(Z, X, T)$, but this is

impossible since L_X is not unimodular. Thus it remains (similarly) that $\Gamma \subset L_Y$ with L_Y generated by $\mathrm{span}(Z, Y, T)$. For $b = -1$, we have G_ρ is unimodular but L_Y is not. In fact, L_Y is unimodular, if and only if $b = -1/2$ (remember $b \in [-1, 0[$), and in this case, $L_Y = \mathrm{SOL}$ which admits cocompact lattices.

- Suppose now that (3.1.2) $A_\rho = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. The $\mathbb{R}$-action on Heis is given by

$$t \mapsto \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^t & f(t) \\ 0 & 0 & e^t \end{pmatrix}$$

  Consider the projection $\pi : G_\rho \to G_\rho/Z \cong \mathbb{R} \ltimes \mathbb{R}^2 \subset \mathrm{Aff}(\mathbb{R}^2)$. For similar reasons as before we have that $\Gamma \cap \mathrm{Heis} = \{1\}$ since the $\mathbb{R}$-action is expanding on Heis. So $\pi(\Gamma)$ is abelian, since $\pi(\Gamma)$ cannot be contained in $\mathbb{R}^2$. We conclude that $\pi(\Gamma)$ is (up to conjugacy) contained in the linear factor, which is impossible since in this case Γ is inside H given by $\mathrm{span}(Z, T)$.

- Suppose now (3.1.3) $A_\rho = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. The $\mathbb{R}$-action fixes Z and acts by rotations on $\mathrm{span}(X, Y)$. We have $I \subset \mathrm{Heis} \setminus Z$ and by hypothesis, Γ acts freely. So $\Gamma \cap \mathrm{Heis} \subset Z$ because the orbit of I under the conjugacy action covers $\mathrm{Heis} \setminus Z$. Consider Now the projection $\pi : G_\rho \to G_\rho/Z \cong \widetilde{\mathrm{Euc}}(\mathbb{R}^2) = \mathbb{R} \ltimes \mathbb{R}^2$. Then $\pi(\Gamma) \subset \widetilde{\mathrm{Euc}}(\mathbb{R}^2)$ is abelian (because $[\Gamma, \Gamma] \subset Z$) and it is not contained in the normal factor $\mathbb{R}^2$. Since $\Gamma \cap \mathrm{Heis} \subset Z$, we deduce that $\pi(\Gamma) \cap \mathbb{R}^2 = 0$. Now, we know that, except the center (which is isomorphic to $\mathbb{Z}$), every element of $\widetilde{\mathrm{Euc}}(\mathbb{R}^2)$ either has exactly one fixed point or has no fixed point if it belongs $\mathbb{R}^2$ (the normal) which is not our case since $\pi(\Gamma) \cap \mathbb{R}^2 = \{0\}$. Suppose that there is $\alpha \in \pi(\Gamma)$ with exactly one fixed point, then, up to conjugacy, one can suppose that all elements fix the origin, that is, $\pi(\Gamma) \subset \mathrm{Stab}(0) = \mathbb{R}$. So $\Gamma \subset H$ given by $\mathrm{span}(Z, T)$ which is clearly impossible. Now if all elements of $\pi(\Gamma)$ act trivially, then $\pi(\Gamma)$ is contained in the center of $\widetilde{\mathrm{Euc}}(\mathbb{R}^2)$, this means that Γ is contained in the subgroup generated by Z and a discrete subgroup of $\mathbb{R}$ generated by T, which is impossible.
- Lastly, let $\mathbb{R}$ act on Heis via the representation $\rho : \mathbb{R} \to \mathrm{Aut}(\mathrm{Heis})$ generated by the derivation (3.1.4)

$$A_\rho = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

  In this case, the $\mathbb{R}$-action on Heis is unipotent. Since all vectors in the complement of $\mathrm{Im}(A_\rho) \oplus \mathbb{R}Z = \mathbb{R}X \oplus \mathbb{R}Z$ are equivalent, we can fix I to be the one parameter subgroup generated by Y, in this case, the Lie subalgebra, isomorphic to $\mathfrak{heis}$, generated by Z, X, T is transverse to all the conjugates of $\mathbb{R}Y$. So P_ρ is isometric to a left-invariant metric on Heis which is globally isometric to the Minkowski space (Lemma 7.1) and admits compact quotients.

□

Remark 12.6 Non-existence of compact models in the unimodular case (that is on spaces that are locally isometric to Cahen-Wallach spaces) follows from [14] Corollary 2, and the non-existence of compact quotients of 3-dimensional Cahen-Wallach spaces follows from [13].

13 Appendix: Rosen and Brinkmann coordinates

Consider a metric g written in Rosen coordinates $(v, x, u),\ u > 0$ as $g = 2dudv + u^{2a}dx^2$. Consider the following coordinate change

$$v = \overline{v} + \frac{a}{2}\overline{u}^{-1}\overline{x}^2,\ x = \overline{u}^{-a}\overline{x},\ u = \overline{u}$$

We have $\frac{\partial}{\partial \overline{x}} = (au^{a-1}x, u^{-a}, 0)$, $\frac{\partial}{\partial \overline{u}} = (-\frac{a}{2}u^{2a-2}x^2, -au^{-1}x, 1)$, and $\frac{\partial}{\partial \overline{v}} = (1, 0, 0)$. One verifies that

$$g(\frac{\partial}{\partial \overline{x}}, \frac{\partial}{\partial \overline{x}}) = 1,\ \ g(\frac{\partial}{\partial \overline{u}}, \frac{\partial}{\partial \overline{u}}) = (a^2 - a)\overline{u}^{-2}\overline{x}^2,\ \ g(\frac{\partial}{\partial \overline{v}}, \frac{\partial}{\partial \overline{v}}) = 0$$

And,

$$g(\frac{\partial}{\partial \overline{x}}, \frac{\partial}{\partial \overline{u}}) = 0,\ \ g(\frac{\partial}{\partial \overline{u}}, \frac{\partial}{\partial \overline{v}}) = 1,\ \ g(\frac{\partial}{\partial \overline{x}}, \frac{\partial}{\partial \overline{v}}) = 0$$

Therefore, we obtain that the metric g in the coordinates $(\overline{v}, \overline{x}, \overline{u})$ has the form

$$g = 2d\overline{u}d\overline{v} + (a^2 - a)\overline{u}^{-2}\overline{x}^2 d\overline{u}^2 + d\overline{x}^2$$

This correspondence Rosen-Brinkmann covers cases of metrics of the form $g = 2dudv + bu^{-2}x^2du^2 + dx^2$, with $b \geq -\frac{1}{4}$.

In general, for a metric $g = 2dudv + \delta(u)dx^2$, one uses a coordinate change of the form $v = \overline{v} + c(u)\overline{x}^2,\ x = \delta(u)^{-1/2}\overline{x},\ u = \overline{u}.$, where c is a function to be determined.

In order to get a Brinkmann form for the metric, we must have $g(\frac{\partial}{\partial \overline{x}}, \frac{\partial}{\partial \overline{u}}) = 0$. This is equivalent to $4c(u) + \delta(u)^{1/2}a(u) = 0$, where $a(u)$ is the derivative of $\delta(u)^{-1/2}$. One then verifies that indeed, the new form of the metric is of Brinkmann type $g = 2d\overline{u}d\overline{v} + d(\overline{u})\overline{x}^2 d\overline{u}^2 + d\overline{x}^2$, where $d(u) = 2c'(u) + \delta(u)a(u)^2$.

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Declarations

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