ISOMETRY LIE ALGEBRAS OF INDEFINITE HOMOGENEOUS SPACES OF FINITE VOLUME

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Abstract. Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra equipped with a symmetric bilinear form $\langle\cdot,\cdot\rangle$. We assume that $\langle\cdot,\cdot\rangle$ is nil-invariant. This means that every nilpotent operator in the smallest algebraic Lie subalgebra of endomomorphims containing the adjoint representation of $\mathfrak{g}$ is an infinitesimal isometry for $\langle\cdot,\cdot\rangle$. Among these Lie algebras are the isometry Lie algebras of pseudo-Riemannian manifolds of finite volume. We prove a strong invariance property for nil-invariant symmetric bilinear forms, which states that the adjoint representations of the solvable radical and all simple subalgebras of non-compact type of $\mathfrak{g}$ act by infinitesimal isometries for $\langle\cdot,\cdot\rangle$. Moreover, we study properties of the kernel of $\langle\cdot,\cdot\rangle$ and the totally isotropic ideals in $\mathfrak{g}$ in relation to the index of $\langle\cdot,\cdot\rangle$. Based on this, we derive a structure theorem and a classification for the isometry algebras of indefinite homogeneous spaces of finite volume with metric index at most two. Examples show that the theory becomes significantly more complicated for index greater than two.

Contents

1. Introduction and main results

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra equipped with a symmetric bilinear form $\langle\cdot,\cdot\rangle$. The pair is called a metric Lie algebra. Traditionally, the bilinear form $\langle\cdot,\cdot\rangle$ is called *invariant* if the adjoint representation of $\mathfrak{g}$ acts by skew linear maps. We will call $\langle\cdot,\cdot\rangle$ *nil-invariant*, if every nilpotent operator in the smallest algebraic Lie subalgebra of endomomorphims containing the adjoint representation of $\mathfrak{g}$ is a

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skew linear map. This nil-invariance condition appears to be significantly weaker than the requirement that $\langle\cdot,\cdot\rangle$ is *invariant*.

Recall that the dimension of a maximal totally isotropic subspace is called the *index* of a symmetric bilinear form, and that the form is called *definite* if its index is zero. Since definite bilinear forms do not admit nilpotent skew maps, the condition of nil-invariance is less restrictive and therefore more interesting for metric Lie algebras with bilinear forms of higher index.

In this paper, we mainly study finite-dimensional real Lie algebras $\mathfrak{g}$ with a nil-invariant symmetric bilinear form. We will discuss the general properties of these metric Lie algebras, compare them with Lie algebras with invariant symmetric bilinear form, and derive elements of a classification theory, which give a complete description for low index, in particular, in the situation of index less than three.

Nil-invariant bilinear forms and isometry Lie algebras. The motivation for this article mainly stems from the theory of geometric transformation groups and automorphism groups of geometric structures.

Namely, consider a Lie group G acting by isometries on a pseudo-Riemannian manifold (M, g) of finite volume. Then at each point $p \in M$, the scalar product g_p naturally induces a symmetric bilinear form $\langle\cdot,\cdot\rangle_p$ on the Lie algebra $\mathfrak{g}$ of G. As we show in Section 2 of this paper, the bilinear form $\langle\cdot,\cdot\rangle_p$ is nil-invariant on $\mathfrak{g}$. Note that, in general, $\langle\cdot,\cdot\rangle_p$ will be degenerate, since the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ tangent to the stabilizer G_p of p is contained in its kernel.

Isometry groups of Lorentzian metrics (where the scalar products g_p are of index one) have been studied intensely. Results obtained by Adams and Stuck [1] in the compact situation, and by Zeghib [11] amount to a classification of the isometry Lie algebras of Lorentzian manifolds of finite volume.

In these works it is used prominently that, for Lorentzian finite volume manifolds, the scalar products $\langle\cdot,\cdot\rangle_p$ are invariant by the elements of the nilpotent radical of $\mathfrak{g}$, cf. [1, §4]. The latter condition is closely related to nil-invariance, but it is also significantly less restrictive. The role played by the stronger nil-invariance condition seems to have gone unnoticed so far.

Aside from Lorentzian manifolds, the classification problem for isometry Lie algebras of finite volume geometric manifolds with metric g of arbitrary index appears to be much more difficult.

Some more specific results have been obtained in the context of *homogeneous* pseudo-Riemannian manifolds. Here, M can be described as a coset space G/H, and any associated metric Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle_p)$ locally determines G and H, as well as the geometry of M. These pseudo-Riemannian manifolds are model spaces of particular interest.

Based on [11], a structure theory for Lorentzian homogeneous spaces of finite volume is given by Zeghib [12].

Pseudo-Riemannian homogeneous spaces of arbitrary index were studied by Baues and Globke [2] for solvable Lie groups G. They found that, for solvable G, the finite volume condition implies that the stabilizer H is a lattice in G and that the metric on M is induced by a bi-invariant metric on G. Also it was observed in [2] that the nil-invariance condition holds for the isometry Lie algebras of finite volume homogeneous spaces, where it appears as a direct consequence of the Borel density theorem. The main result in [2] amounts to showing the surprising fact that any nil-invariant symmetric bilinear form on a solvable Lie algebra $\mathfrak{g}$ is, in fact, an invariant form.

By studying metric Lie algebras with nil-invariant symmetric bilinear form, the present work aims to further understand the isometry Lie algebras of pseudo-Riemannian manifolds of finite volume. We will derive a structure theory which allows to completely describe such algebras in index less than three. In particular, this classification contains all local models for pseudo-Riemannian homogeneous spaces of finite volume of index less than three.

1.1. **Main results and structure of the paper.** In Section 2, we prove that the orbit maps of isometric actions of Lie groups on pseudo-Riemannian manifolds of finite volume give rise to nil-invariant scalar products on their tangent Lie algebras.

Some basic definitions and properties of metric Lie algebras are reviewed in Section 3.

In favourable cases, nil-invariance of $\langle\cdot,\cdot\rangle$ already implies invariance. For solvable Lie algebras $\mathfrak{g}$, this is always the case, as was shown in [2]. These results are briefly summarized in Section 4. In this section, we will also review the classification of solvable Lie algebras with invariant scalar products of index one and two. Their properties will be needed further on.

Strong invariance properties. In Section 5 we begin our investigation of nil-invariant symmetric bilinear forms $\langle\cdot,\cdot\rangle$ on arbitrary Lie algebras. For any Lie algebra $\mathfrak{g}$, we let

$$\mathfrak{g} = (\mathfrak{k} \times \mathfrak{s}) \ltimes \mathfrak{r}$$

denote a Levi decomposition of $\mathfrak{g}$, where $\mathfrak{k}$ is semisimple of compact type, $\mathfrak{s}$ is semisimple of non-compact type and $\mathfrak{r}$ is the solvable radical of $\mathfrak{g}$. For this, recall that $\mathfrak{k}$ is called of compact type if the Killing form of $\mathfrak{k}$ is definite and that $\mathfrak{s}$ is of non-compact type if it has no ideal of compact type. We also write

$$\mathfrak{g}_{\mathrm{s}} = \mathfrak{s} \ltimes \mathfrak{r}.$$

Our first main result is a *strong invariance property* for nil-invariant symmetric bilinear forms:

Theorem A. *Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra, let $\langle\cdot,\cdot\rangle$ be a nil-invariant symmetric bilinear form on $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle_{\mathfrak{g}_{\mathrm{s}}}$ the restriction of $\langle\cdot,\cdot\rangle$ to $\mathfrak{g}_{\mathrm{s}}$. Then:*

(1) *$\langle\cdot,\cdot\rangle_{\mathfrak{g}_{\mathrm{s}}}$ is invariant by the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}_{\mathrm{s}}$.*
(2) *$\langle\cdot,\cdot\rangle$ is invariant by $\mathfrak{g}_{\mathrm{s}}$.*

This result is as good as one can hope for. For example, *any* scalar product on a semisimple Lie algebra $\mathfrak{k}$ of compact type is already nil-invariant, without any further invariance property required. Note that Theorem A does not require any assumption on the index of $\langle\cdot,\cdot\rangle$.

Remark. We would like to point out that the proof of Theorem A works for Lie algebras over any field of characteristic zero if the notion of subalgebra of compact type $\mathfrak{k}$ is replaced by the appropriate notion of maximal anisotropic semisimple subalgebra of $\mathfrak{g}$. The latter condition is equivalent to the requirement that the Cartan subalgebras of $\mathfrak{k}$ do not contain any elements split over the ground field.

We obtain the following striking corollary to Theorem A, or rather to its proof:

Corollary B. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over the field of complex numbers and $\langle\cdot,\cdot\rangle$ a nil-invariant symmetric bilinear form on $\mathfrak{g}$. Then $\langle\cdot,\cdot\rangle$ is invariant.*

For any nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$, it is important to consider its kernel

$$\mathfrak{g}^\perp = \{X \in \mathfrak{g} \mid X \perp \mathfrak{g}\} ,$$

also called the *metric radical* of $\mathfrak{g}$. If $\langle\cdot,\cdot\rangle$ is invariant, then $\mathfrak{g}^\perp$ is an ideal of $\mathfrak{g}$. If $\langle\cdot,\cdot\rangle$ is nil-invariant, then, in general, $\mathfrak{g}^\perp$ is not even a subalgebra of $\mathfrak{g}$. Nevertheless, a considerable simplification of the exposition may be obtained by restricting results to metric Lie algebras whose radical $\mathfrak{g}^\perp$ does not contain any non-trivial ideals of $\mathfrak{g}$. Such metric Lie algebras will be called *effective*. This condition is, of course, natural from the geometric motivation. Moreover, it is not a genuine restriction, since by dividing out the maximal ideal of $\mathfrak{g}$ contained in $\mathfrak{g}^\perp$, one may pass from any metric Lie algebra to a quotient metric Lie algebra that is effective.

Theorem A determines the properties of $\mathfrak{g}^\perp$ significantly as is shown in the following:

Corollary C. *Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra with a nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$. Assume that the metric radical $\mathfrak{g}^\perp$ does not contain any non-trivial ideal of $\mathfrak{g}$. Let $\mathfrak{z}(\mathfrak{g}_{\mathrm{s}})$ denote the center of $\mathfrak{g}_{\mathrm{s}}$. Then*

$$\mathfrak{g}^\perp \subseteq \mathfrak{k} \times \mathfrak{z}(\mathfrak{g}_{\mathrm{s}}) \quad \textit{and} \quad [\mathfrak{g}^\perp, \mathfrak{g}_{\mathrm{s}}] \subseteq \mathfrak{z}(\mathfrak{g}_{\mathrm{s}}) \cap \mathfrak{g}^\perp.$$

The proof of Corollary C can be found in Section 6, which is at the technical heart of our paper.

We start out by studying the totally isotropic ideals in $\mathfrak{g}$, and in particular properties of the metric radical $\mathfrak{g}^\perp$. Our first main result in Section 6 is the proof of Corollary C in Subsection 6.2.

As the form $\langle\cdot,\cdot\rangle$ may be degenerate, it is useful to introduce its *relative index*. By definition, this is the index of the induced scalar product on the vector space $\mathfrak{g}/\mathfrak{g}^\perp$. The relative index mostly determines the geometric and algebraic type of the bilinear form $\langle\cdot,\cdot\rangle$.

For effective metric Lie algebras with relative index $\ell \leq 2$, we further strengthen Corollary C by showing that, with this additional requirement, $\mathfrak{g}^\perp$ does not intersect $\mathfrak{g}_{\mathrm{s}}$. This is formulated in Corollary 6.17.

Classifications for small index. Section 6 culminates in Subsection 6.5, where we give an analysis of the action of semisimple subalgebras on the solvable radical of $\mathfrak{g}$. This imposes strong restrictions on the structure of $\mathfrak{g}$ for small relative index.

The combined results are summarized in Section 7, leading to the following general structure theorem for the case $\ell \leq 2$:

Theorem D. *Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra with nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ of relative index $\ell \leq 2$, and assume that $\mathfrak{g}^\perp$ does not contain a non-trivial ideal of $\mathfrak{g}$. Then:*

(1) *The Levi decomposition* (5.1) *of $\mathfrak{g}$ is a direct sum of ideals: $\mathfrak{g} = \mathfrak{k} \times \mathfrak{s} \times \mathfrak{r}$.*
(2) *$\mathfrak{g}^\perp$ is contained in $\mathfrak{k} \times \mathfrak{z}(\mathfrak{r})$ and $\mathfrak{g}^\perp \cap \mathfrak{r} = \mathbf{0}$.*
(3) *$\mathfrak{s} \perp (\mathfrak{k} \times \mathfrak{r})$ and $\mathfrak{k} \perp [\mathfrak{r}, \mathfrak{r}]$.*

Examples in Section 8 illustrate that the statements in Theorem D may fail for relative index $\ell \geq 3$.

We specialize Theorem D to obtain classifications of the Lie algebras $\mathfrak{g}$ in the cases $\ell = 1$ (Theorem E) and $\ell = 2$ (Theorem F). As follows from the discussion at the beginning, these theorems also describe the structure of isometry Lie algebras of pseudo-Riemannian homogeneous spaces of finite volume with index one or two (real signatures of type $(n-1,1)$ or $(n-2,2)$, respectively).

Our first result concerns the Lorentzian case:

Theorem E. *Let $\mathfrak{g}$ be a Lie algebra with nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ of relative index $\ell = 1$, and assume that $\mathfrak{g}^\perp$ does not contain a non-trivial ideal of $\mathfrak{g}$. Then one of the following cases occurs:*

(I) $\mathfrak{g} = \mathfrak{a} \times \mathfrak{k}$*, where $\mathfrak{a}$ is abelian and either semidefinite or Lorentzian.*
(II) $\mathfrak{g} = \mathfrak{r} \times \mathfrak{k}$*, where $\mathfrak{r}$ is Lorentzian of oscillator type.*
(III) $\mathfrak{g} = \mathfrak{a} \times \mathfrak{k} \times \mathfrak{sl}_2(\mathbb{R})$*, where $\mathfrak{a}$ is abelian and definite, $\mathfrak{sl}_2(\mathbb{R})$ is Lorentzian.*

This classification of isometry Lie algebras for finite volume homogeneous Lorentzian manifolds is contained in Zeghib's [12, Théorème algébrique 1.11], which uses a somewhat different approach in its proof. Moreover, the list in [12] contains two additional cases of metric Lie algebras (Heisenberg algebra and tangent algebra of the affine group, compare Example 3.3 of the present paper) that cannot appear as Lie algebras of transitive Lorentzian isometry groups, since they do not satisfy the effectivity condition. According to [12], models of all three types (I)-(III) actually occur as isometry Lie algebras of homogeneous spaces G/H, in which case $\mathfrak{h} = \mathfrak{g}^\perp$ is a subalgebra tangent to a closed subgroup H of G.

The algebraic methods developed here also lead to a complete understanding in the case of signature $(n-2,2)$:

Theorem F. *Let $\mathfrak{g}$ be a Lie algebra with nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ of relative index $\ell = 2$, and assume that $\mathfrak{g}^\perp$ does not contain a non-trivial ideal of $\mathfrak{g}$. Then one of the following cases occurs:*

(I) $\mathfrak{g} = \mathfrak{r} \times \mathfrak{k}$*, where $\mathfrak{r}$ is one of the following:*
  (a) *$\mathfrak{r}$ is abelian.*
  (b) *$\mathfrak{r}$ is Lorentzian of oscillator type.*
  (c) *$\mathfrak{r}$ is solvable but non-abelian with invariant scalar product of index 2.*
(II) $\mathfrak{g} = \mathfrak{a} \times \mathfrak{k} \times \mathfrak{s}$*. Here, $\mathfrak{a}$ is abelian, $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})$ with a non-degenerate invariant scalar product of index 2. Moreover, $\mathfrak{a}$ is definite.*
(III) $\mathfrak{g} = \mathfrak{r} \times \mathfrak{k} \times \mathfrak{sl}_2(\mathbb{R})$*, where $\mathfrak{sl}_2(\mathbb{R})$ is Lorentzian, and $\mathfrak{r}$ is one of the following:*
  (a) *$\mathfrak{r}$ is abelian and either semidefinite or Lorentzian.*
  (b) *$\mathfrak{r}$ is Lorentzian of oscillator type.*

For the definition of an oscillator algebra, see Example 3.7. The possibilities for $\mathfrak{r}$ in case (I-c) of Theorem F above are discussed in Section 3.3. Note further that the orthogonality relations of Theorem D part (3) are always satisfied.

Remark. Theorem F contains no information which of the possible algebraic models actually do occur as isometry Lie algebras of homogeneous spaces of index two. This question needs to be considered on another occasion.

Notations and conventions. The identity element of a group G is denoted by e.

Let H be a subgroup of a Lie group G. We write $\mathrm{Ad}_{\mathfrak{g}}(H)$ for the adjoint representation of H on the Lie algebra $\mathfrak{g}$ of G, to distinguish it from the adjoint representation $\mathrm{Ad}(H)$ on its own Lie algebra $\mathfrak{h}$.

The centralizer and the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$ are denoted by $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ and $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$, respectively. The center of $\mathfrak{g}$ is denoted by $\mathfrak{z}(\mathfrak{g})$.

If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two Lie algebras, the notation $\mathfrak{g}_1 \times \mathfrak{g}_2$ denotes the direct product of Lie algebras. The notations $\mathfrak{g}_1 + \mathfrak{g}_2$ and $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ are used to indicate sums and direct sums of vector spaces.

The *solvable radical* $\mathfrak{r}$ of $\mathfrak{g}$ is the maximal solvable ideal of $\mathfrak{g}$. The semisimple Lie algebra $\mathfrak{f} = \mathfrak{g}/\mathfrak{r}$ is a direct product $\mathfrak{f} = \mathfrak{k} \times \mathfrak{s}$ of Lie algebras, where $\mathfrak{k}$ is a semisimple Lie algebra of *compact type*, meaning that the Killing form of $\mathfrak{k}$ is definite, and $\mathfrak{s}$ is semisimple without factors of compact type.

For any linear operator φ, $\varphi = \varphi_{\mathrm{ss}} + \varphi_{\mathrm{n}}$ denotes its Jordan decomposition, where φ_{ss} is semisimple, and φ_{n} is nilpotent. Further notation will be introduced in Section 3.

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2. Isometry Lie algebras

Let (M, g) be a pseudo-Riemannian manifold of finite volume, and let

$$G \subseteq \mathrm{Iso}(M, \mathrm{g})$$

be a Lie group of isometries of M. Identify the Lie algebra $\mathfrak{g}$ of G with a subalgebra of Killing vector fields on (M, g). Let $\mathrm{S}^2\mathfrak{g}^*$ denote the space of symmetric bilinear forms on $\mathfrak{g}$, and let

$$\Phi : M \to \mathrm{S}^2\mathfrak{g}^*, \quad p \mapsto \Phi_p$$

be the Gauß map, where

$$\Phi_p(X, Y) = \mathrm{g}_p(X_p, Y_p).$$

The adjoint representation of G on $\mathfrak{g}$ induces a representation $\varrho : G \to \mathrm{GL}(\mathrm{S}^2\mathfrak{g}^*)$.

Theorem 2.1. *Let A be the real Zariski closure of $\varrho(G)$ in the group $\mathrm{GL}(\mathrm{S}^2\mathfrak{g}^*)$. Let $p \in M$. Then the bilinear form Φ_p is invariant by all unipotent elements in A.*

Proof. Put $V = \mathrm{S}^2\mathfrak{g}^*$. For a subset $W \subseteq V$, let $\overline{W}$ denote its image in the projective space $\mathbb{P}(V)$. Similarly, for subsets in $\mathrm{GL}(V)$ and their image in the projective linear group $\mathrm{PGL}(V)$.

Note that the above Gauß map Φ is equivariant with respect to ϱ, since G acts by isometries on M. Since M carries a finite G-invariant Borel measure, there is a finite G-invariant measure ν on the projective space $\mathbb{P}(V)$ with support $\operatorname{supp}\nu = \overline{\Phi(M)} \subset \mathbb{P}(V)$. Let $\mathrm{PGL}(V)_\nu$ denote the stabiliser of ν in the projective linear group. This is a real algebraic subgroup of $\mathrm{PGL}(V)$, cf. [13, Theorem 3.2.4]. Also, by construction, $\overline{\varrho(G)} \subseteq \mathrm{PGL}(V)_\nu$.

There exist vector subspaces $W_1, \ldots, W_r$ of V such that $\operatorname{supp}\nu \subseteq \overline{W} = \overline{W}_1 \cup \ldots \cup \overline{W}_r$ and the quasi-linear subspace $\overline{W}$ is minimal with this property. Note that the identity component of $\mathrm{PGL}(V)_\nu$ preserves all $\overline{W}_i$, and by Furstenberg's Lemma [13, Corollary 3.2.2], its restriction to $\mathrm{PGL}(W_i)$ has compact closure.

Since $\mathrm{PGL}(V)_\nu$ is real algebraic, the image of A in $\mathrm{PGL}(V)$ is contained in $\mathrm{PGL}(V)_\nu$. Choose W_i such that $\Phi_p \in W_i$. Let $u \in \mathsf{A}$ be a unipotent element. Since the restriction of u to $\mathrm{PGL}(W_i)$ is unipotent and it is contained in a compact subset of $\mathrm{PGL}(W_i)$, it must be the identity of $\overline{W}_i$. This implies $u \cdot \Phi_p = \Phi_p$. □

In terms of Definition 3.1 below, this implies the following:

Corollary 2.2. *For $p \in M$, let $\langle\cdot,\cdot\rangle_p$ denote the symmetric bilinear form induced on the Lie algebra $\mathfrak{g}$ of G by pulling back g_p along the orbit map $g \mapsto g\cdot p$. Then $\langle\cdot,\cdot\rangle_p$ is nil-invariant and its kernel contains the Lie algebra $\mathfrak{g}_p$ of the stabilizer G_p of p in G. If G acts transitively on M, then the kernel of $\langle\cdot,\cdot\rangle_p$ equals $\mathfrak{g}_p$.*

3. Metric Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra with a symmetric bilinear form $\langle\cdot,\cdot\rangle$. The pair $(\mathfrak{g},\langle\cdot,\cdot\rangle)$ is called a *metric Lie algebra*.

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. The restriction of $\langle\cdot,\cdot\rangle$ to $\mathfrak{h}$ will be denoted by $\langle\cdot,\cdot\rangle_{\mathfrak{h}}$. The form $\langle\cdot,\cdot\rangle$ is called $\mathfrak{h}$-*invariant* if

$$\langle \mathrm{ad}(X)Y_1, Y_2\rangle = -\langle Y_1, \mathrm{ad}(X)Y_2\rangle \tag{3.1}$$

for all $X \in \mathfrak{h}$ and $Y_1, Y_2 \in \mathfrak{g}$. We define

$$\mathrm{inv}(\mathfrak{g},\langle\cdot,\cdot\rangle) = \{X \in \mathfrak{g} \mid \langle \mathrm{ad}(X)Y_1, Y_2\rangle = -\langle Y_1, \mathrm{ad}(X)Y_2\rangle \text{ for all } Y_1, Y_2 \in \mathfrak{g}\}.$$

This is the maximal subalgebra of $\mathfrak{g}$ under which $\langle\cdot,\cdot\rangle$ is invariant. If $\langle\cdot,\cdot\rangle$ is $\mathfrak{g}$-invariant, we simply say $\langle\cdot,\cdot\rangle$ is *invariant*.

The kernel of $\langle\cdot,\cdot\rangle$ is the subspace

$$\mathfrak{g}^\perp = \{X \in \mathfrak{g} \mid \langle X, Y\rangle = 0 \text{ for all } Y \in \mathfrak{g}\}.$$

It is also called the *metric radical* for $(\mathfrak{g},\langle\cdot,\cdot\rangle)$. It is an invariant subspace for the Lie brackets with elements of $\mathrm{inv}(\mathfrak{g},\langle\cdot,\cdot\rangle)$, and, if $\langle\cdot,\cdot\rangle$ is invariant, then $\mathfrak{g}^\perp$ is an ideal in $\mathfrak{g}$.

3.1. Nil-invariant bilinear forms.

Let $\overline{\mathrm{Inn}(\mathfrak{g})}^{\mathrm{z}}$ denote the Zariski closure of the adjoint group $\mathrm{Inn}(\mathfrak{g})$ in $\mathrm{Aut}(\mathfrak{g})$.

Definition 3.1. A symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ is called *nil-invariant*, if for all $X_1, X_2 \in \mathfrak{g}$,

$$\langle \varphi X_1, X_2\rangle = -\langle X_1, \varphi X_2\rangle, \tag{3.2}$$

for all nilpotent elements φ of the Lie algebra of $\overline{\mathrm{Inn}(\mathfrak{g})}^{\mathrm{z}}$.

In particular, (3.2) holds for the nilpotent parts $\varphi = \mathrm{ad}(Y)_{\mathrm{n}}$ of the Jordan decomposition of the adjoint representation of any $Y \in \mathfrak{g}$.

3.2. **Index of symmetric bilinear forms.** Let $\langle\cdot,\cdot\rangle$ be a symmetric bilinear form on a finite-dimensional vector space V. An element $x \in V$ is called *isotropic* if $\langle x, x\rangle = 0$. A subspace $W \subseteq V$ is called *isotropic*, if there exists $x \in W$, $x \neq 0$, with $\langle x, x\rangle = 0$. W is called *totally isotropic* if $W \subseteq W^\perp$.

The dimension of a maximal totally isotropic subspace of V is called the *index* $\mu(V)$ of V. Set

$$\ell(V) = \mu(V) - \dim V^\perp,$$

so that $\ell(V)$ is the index of the non-degenerate bilinear form induced by $\langle\cdot,\cdot\rangle$ on $V/V^\perp$. We call ℓ the *relative index* of V (or $\langle\cdot,\cdot\rangle$).

When there is no ambiguity about the space V, we simply write $\mu = \mu(V)$ and $\ell = \ell(V)$. We then say that V is of *index ℓ type*. In particular, for $\ell = 1$, we say V is of *Lorentzian type*. We call V *Lorentzian* if $\mu = \ell = 1$.

If $\langle\cdot,\cdot\rangle$ is non-degenerate, that is, if $V^\perp = \mathbf{0}$, then we call $\langle\cdot,\cdot\rangle$ a *scalar product* on V. We say that the scalar product $\langle\cdot,\cdot\rangle$ is *definite* if $\mu = 0$.

Let $W \subseteq V$ be a vector subspace. We say W is *definite*, *Lorentzian*, of relative index $\ell(W)$ or of index $\mu(W)$, respectively, if the restriction $\langle\cdot,\cdot\rangle_W$ is. Observe further that $\mu(W) \leq \mu(V)$ and $\ell(W) \leq \ell(V)$.

3.3. **Examples of metric Lie algebras.**

Example 3.2. Consider $\mathbb{R}^n$ with a scalar product $\langle\cdot,\cdot\rangle$ represented by the matrix $\left(\begin{smallmatrix} \mathrm{I}_{n-s} & 0 \\ 0 & -\mathrm{I}_s \end{smallmatrix}\right)$, where $s \leq n-s$. Then $\langle\cdot,\cdot\rangle$ has index s, and we write $\mathbb{R}^n_s$ for $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$. If we take $\mathbb{R}^n$ to be an abelian Lie algebra, together with $\langle\cdot,\cdot\rangle$ it becomes a metric Lie algebra denoted by $\mathfrak{ab}^n_s$.

The Heisenberg algebra occurs naturally in the construction of Lie algebras with invariant scalar products.

Example 3.3. Let $(\cdot,\cdot)$ be a Hermitian form on $\mathbb{C}^n$. Define the *Heisenberg algebra* $\mathfrak{h}_{2n+1}$ as the vector space $\mathbb{C}^n \oplus \mathfrak{z}$, where $\mathfrak{z} = \operatorname{span}\{Z\}$, with Lie brackets defined by

$$[X, Y] = \operatorname{Im}(X, Y)Z,$$

for any $X, Y \in \mathbb{C}^n$. Thus $\mathfrak{h}_{2n+1}$ is a real $2n+1$-dimensional two-step nilpotent Lie algebra with one-dimensional center (as such it is unique up to isomorphism of Lie algebras). Equip $\mathbb{C}^n$ with the bilinear product $\langle\cdot,\cdot\rangle = \operatorname{Re}(\cdot,\cdot)$. Declaring $\mathfrak{z}$ to be perpendicular to $\mathfrak{h}_{2n+1}$ turns $\mathfrak{h}_{2n+1}$ into a metric Lie algebra, whose relative index $\ell(\mathfrak{h}_{2n+1})$ is determined by the index of the Hermitian form.

Example 3.4. Put $\mathfrak{d} = \operatorname{span}\{J\}$. Define the $2n+2$-dimensional *oscillator algebra* $\mathfrak{osc}$ as the semidirect product

$$\mathfrak{osc} = \mathfrak{d} \ltimes \mathfrak{h}_{2n+1},$$

where J acts by multiplication with the imaginary unit on $\mathbb{C}^n$. Given any metric on $\mathfrak{h}_{2n+1}$ as in Example 3.3, an invariant scalar product $\langle\cdot,\cdot\rangle$ of index $\ell(\mathfrak{h}_{2n+1}) + 1$ on $\mathfrak{osc}$ is obtained by requiring $\langle J, Z\rangle = 1$ and $\mathbb{C}^n \perp J$.

Example 3.4 is an important special case of the following construction:

Example 3.5. Given $\psi \in \mathfrak{so}_{n-s,s}$ define the *oscillator algebra*

$$\mathfrak{g} = \mathfrak{osc}(\psi)$$

as follows. On the vector space $\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{ab}_s^n \oplus \mathfrak{j}$ with $\mathfrak{d} = \mathrm{span}\{D\}$, $\mathfrak{j} = \mathrm{span}\{Z\}$ define a Lie product by declaring:

$$[D, X] = \psi(X), \quad [X, Y] = \langle [D, X], Y\rangle Z \ .$$

where, $X, Y \in \mathfrak{ab}_s^n$. Next extend the indefinite scalar product on $\mathfrak{ab}_s^n$ to $\mathfrak{g}$ by

$$\langle D, D\rangle = \langle Z, Z\rangle = 0, \quad \langle D, Z\rangle = 1, \quad (\mathfrak{a} \oplus \mathfrak{j}) \perp \mathfrak{ab}_s^n.$$

Then $\langle\cdot,\cdot\rangle$ is an invariant scalar product of index $s+1$ on $\mathfrak{g}$. The Lie algebra $\mathfrak{osc}(\psi)$ is solvable. It is nilpotent if and only if ψ is nilpotent. If ψ is a k-step nilpotent operator, then $\mathfrak{g}$ is a k-step nilpotent algebra. If ψ is not zero, then the ideal $\mathfrak{h} = \mathfrak{ab}_s^n \oplus \mathfrak{j}$ is of Heisenberg type with $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{j}$.

3.3.1. *Invariant Lorentzian scalar products.* The main building blocks for metric Lie algebras with invariant Lorentzian scalar products are obtained by:

Example 3.6. The Killing form on $\mathfrak{sl}_2(\mathbb{R})$ is an invariant Lorentzian scalar product. In fact, all semisimple Lie algebras with an invariant Lorentzian scalar product are products of $\mathfrak{sl}_2(\mathbb{R})$ by simple factors of compact type.

Example 3.7. For $\psi \in \mathfrak{so}_n$, the oscillator algebra $\mathfrak{osc}(\psi)$ is Lorentzian. We say that such a metric Lie algebra is Lorentzian of *oscillator type*.

Remark. Classification of Lie algebras with invariant Lorentzian scalar products were derived by Medina [9] and by Hilgert and Hofmann [6]. It can be deduced that algebras of oscillator type are the only non-abelian solvable Lie algebras which admit an invariant Lorentzian scalar product. This is also a direct consequence of the reduction theory of solvable metric Lie algebras, see Section 4.

4. Review of the solvable case

The first two authors studied nil-invariant symmetric bilinear forms on solvable Lie algebras in [2]. The main result [2, Theorem 1.2] is:

Theorem 4.1. *Let $\mathfrak{g}$ be a solvable Lie algebra and $\langle\cdot,\cdot\rangle$ a nil-invariant symmetric bilinear form on $\mathfrak{g}$. Then $\langle\cdot,\cdot\rangle$ is invariant. In particular, $\mathfrak{g}^\perp$ is an ideal in $\mathfrak{g}$.*

An important tool in the study of (nil-)invariant products $\langle\cdot,\cdot\rangle$ on solvable $\mathfrak{g}$ is the reduction by a totally isotropic ideal $\mathfrak{j}$ in $\mathfrak{g}$. Since $\langle\cdot,\cdot\rangle$ is invariant, $\mathfrak{j}^\perp$ is a subalgebra. Therefore, we can consider the quotient Lie algebra

$$\overline{\mathfrak{g}} = \mathfrak{j}^\perp/\mathfrak{j}. \tag{4.1}$$

Since $\mathfrak{j}$ is totally isotropic, $\overline{\mathfrak{g}}$ inherits a non-degenerate symmetric bilinear form from $\mathfrak{j}^\perp$ that is (nil-)invariant as well. The metric Lie algebra $(\overline{\mathfrak{g}}, \langle\cdot,\cdot\rangle)$ is called the *reduction* of $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ by $\mathfrak{j}$. Reduction by $\mathfrak{j}$ decreases the index of $\langle\cdot,\cdot\rangle$.

The ideal

$$\mathfrak{j}_0 = \mathfrak{z}(\mathfrak{n}) \cap [\mathfrak{g}, \mathfrak{n}] \tag{4.2}$$

is a characteristic totally isotropic ideal in $\mathfrak{g}$, whose orthogonal space $\mathfrak{j}_0^\perp$ is also an ideal in $\mathfrak{g}$ and contains $\mathfrak{j}_0$ in its center. Then $\mathfrak{j}_0 = \mathbf{0}$ if and only if $\mathfrak{g}$ is abelian. In particular, $\mathfrak{g}$ is abelian if $\langle\cdot,\cdot\rangle$ is definite. This implies [2, Proposition 5.4]:

Proposition 4.2. *Let $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ be a solvable metric Lie algebra with nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$. After a finite sequence of reductions with respect to totally isotropic and central ideals, $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ reduces to an abelian metric Lie algebra.*

The proposition is useful in particular to derive properties of solvable metric Lie algebras of low index.

4.1. **Invariant scalar products of index 2.**

Example 4.3. Let $\psi \in \mathfrak{so}_{n,1}$. Then the oscillator algebra $\mathfrak{osc}(\psi)$ as defined in Example 3.5 is of index $\ell = 2$.

Let $\alpha^j = (\alpha^j_1, \dots, \alpha^j_n)$, $j = 1, 2$, denote tuples of real numbers, and let us put $\mathfrak{d} = \mathrm{span}\{D_1, D_2\}$, $\mathfrak{j} = \mathrm{span}\{Z_1, Z_2\}$. Let $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ be an orthonormal basis of $\mathfrak{a} = \mathfrak{ab}_0^{2n}$,

Example 4.4. We define a metric Lie algebra $\mathfrak{g} = \mathfrak{osc}(\alpha^1, \alpha^2)$ as follows. The Lie product on

$$\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{ab}_0^{2n} \oplus \mathfrak{j}$$

is given by the relations

$$[X_i, Y_j] = \delta_{ij}(\alpha^1_i Z_1 + \alpha^2_i Z_2), \quad [D_k, X_j] = \alpha^k_j Y_j, \quad [D_k, Y_j] = -\alpha^k_j X_j. \tag{4.3}$$

Define a scalar product $\langle \cdot, \cdot \rangle$ of index 2 on $\mathfrak{g}$ by

$$\langle D_1, D_2 \rangle = \langle Z_1, Z_2 \rangle = 0, \quad \langle D_i, Z_j \rangle = \delta_{ij}, \quad D_i, Z_i \perp \mathfrak{a}\,. \tag{4.4}$$

Then $\mathfrak{g}$ is a solvable Lie algebra with invariant scalar product $\langle \cdot, \cdot \rangle$. Observe that $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{n}] \subseteq \mathfrak{a} \oplus \mathfrak{j}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. Then $\mathfrak{n}$ is at most two-step nilpotent, since $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{j}$.

Example 4.5. We define a metric Lie algebra $\mathfrak{g} = \mathfrak{osc}_1(\alpha^1, \alpha^2)$ as follows. Consider $\mathfrak{a} = \mathfrak{ab}_0^{2n+1} = \mathrm{span}\{W\} + \mathfrak{ab}_0^{2n}$, where $W \perp \mathfrak{ab}_0^{2n}$, and $\langle W, W \rangle = 1$. A Lie product on

$$\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{ab}_0^{2n+1} \oplus \mathfrak{j}$$

is given by the relations (4.3) and

$$[D_1, D_2] = W, \quad [D_1, W] = -Z_2, \quad [D_2, W] = Z_1.$$

Define a scalar product $\langle \cdot, \cdot \rangle$ of index 2 on $\mathfrak{g}$ using (4.4). Then $\mathfrak{g}$ is a solvable Lie algebra with invariant scalar product $\langle \cdot, \cdot \rangle$. Note if $n = 0$, or $\alpha^1 = \alpha^2 = 0$ then $\mathfrak{g}$ is three-step nilpotent. Otherwise, $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{j} \subseteq \mathfrak{z}(\mathfrak{g})$.

The three families of Lie algebras in Examples 4.3, 4.4, 4.5 were found by Kath and Olbrich [7] to contain all indecomposable non-simple metric Lie algebras with invariant scalar product of index 2. Thus we note:

Proposition 4.6. *Any solvable metric Lie algebra with invariant scalar product of index* 2 *is obtained by taking direct products of metric Lie algebras in Examples 3.7, 4.3 to 4.5 or abelian metric Lie algebras.*

We use this to derive the following particular observation, which will play an important role in Section 6.5. An ideal in a metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is called *characteristic* if it is preserved by every skew derivation of $\mathfrak{g}$.

Proposition 4.7. *Let* $\mathfrak{g}$ *be a solvable Lie algebra with invariant bilinear form* $\langle \cdot, \cdot \rangle$ *of index* $\mu \leq 2$. *Then* $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ *has a characteristic ideal* $\mathfrak{q}$ *that satisfies:*

(1) $\dim[\mathfrak{q}, \mathfrak{q}] \leq 2$.
(2) $\mathrm{codim}_{\mathfrak{g}}\, \mathfrak{q} \leq 2$.

Proof. It is easily checked that the characteristic ideal $\mathfrak{q} = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ satisfies (1) and (2) for the Examples 3.7, 4.3 to 4.5, and for products of oscillators as in Example 3.7. Hence, the proposition is satisfied for all invariant scalar products of index $\ell = \mu \leq 2$.

Suppose now that $\langle \cdot, \cdot \rangle$ is degenerate and $\ell = 0$. Then $\mathfrak{g}/\mathfrak{g}^{\perp}$ inherits a definite invariant scalar product. Hence, $\mathfrak{g}/\mathfrak{g}^{\perp}$ is abelian, and $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}^{\perp}$. Since $\dim \mathfrak{g}^{\perp} \leq \mu \leq 2$, it follows that $\mathfrak{q} = \mathfrak{g}$ has the required properties.

Finally, suppose $\langle \cdot, \cdot \rangle$ is degenerate, $\ell = 1$. Then $\mathfrak{g}_0 = \mathfrak{g}/\mathfrak{g}^{\perp}$ is Lorentzian. It follows that $\mathfrak{g}_0$ admits a codimension one characteristic ideal $\mathfrak{q}_0 = [\mathfrak{g}_0, \mathfrak{g}_0] + \mathfrak{z}(\mathfrak{g}_0)$, where $\dim[\mathfrak{q}_0, \mathfrak{q}_0] \leq 1$. Thus the preimage $\mathfrak{q}$ of $\mathfrak{q}_0$ in $\mathfrak{g}$ has the required properties. □

In the following $\mathfrak{n}$ denotes the nilradical of the Lie algebra $\mathfrak{g}$.

Corollary 4.8. *Let $\mathfrak{g}$ be a solvable metric Lie algebra which admits an invariant scalar product of index ≤ 2. If $\mathfrak{g}$ is not nilpotent then*

$$\mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{n}, \mathfrak{n}] = \mathfrak{z}(\mathfrak{n}) \cap [\mathfrak{n}, \mathfrak{n}] = [\mathfrak{n}, \mathfrak{n}] \,.$$

5. Nil-invariant symmetric bilinear forms

Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra with solvable radical $\mathfrak{r}$. Let

$$\mathfrak{g} = (\mathfrak{k} \times \mathfrak{s}) \ltimes \mathfrak{r} \tag{5.1}$$

be a Levi decomposition , where $\mathfrak{k}$ is semisimple of compact type and $\mathfrak{s}$ is semisimple without factors of compact type. Furthermore, we put

$$\mathfrak{g}_{\mathrm{s}} = \mathfrak{s} \ltimes \mathfrak{r} \,.$$

Note that $\mathfrak{g}_{\mathrm{s}}$ is a characteristic ideal of $\mathfrak{g}$.

The purpose of this section is to show:

Theorem A. *Let $\langle \cdot, \cdot \rangle$ be a nil-invariant symmetric bilinear form on $\mathfrak{g}$, and let $\langle \cdot, \cdot \rangle_{\mathfrak{g}_{\mathrm{s}}}$ denote the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{g}_{\mathrm{s}}$. Then:*

1. *$\langle \cdot, \cdot \rangle_{\mathfrak{g}_{\mathrm{s}}}$ is invariant by the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}_{\mathrm{s}}$.*
2. *$\langle \cdot, \cdot \rangle$ is invariant by $\mathfrak{g}_{\mathrm{s}}$.*

The proof of Theorem A begins with a few auxiliary results.

Lemma 5.1. *Let $\mathfrak{s} \subseteq \mathfrak{g}$ be a semisimple subalgebra of non-compact type. Then the subalgebra generated by all $X \in \mathfrak{s}$, such that $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent, is $\mathfrak{s}$.*

Proof. Call $X \in \mathfrak{s}$ nilpotent if $\mathrm{ad}(X) : \mathfrak{s} \to \mathfrak{s}$ is nilpotent. Since, for every representation of $\mathfrak{s}$, nilpotent elements are mapped to nilpotent operators, it is sufficient to prove the statement for $\mathfrak{s} = \mathfrak{g}$. So let $\mathfrak{s}_0$ be the subalgebra of $\mathfrak{s}$ generated by all nilpotent elements. Clearly, $\mathfrak{s}_0$ is an ideal and $\mathfrak{s}_1 = \mathfrak{s}/\mathfrak{s}_0$ does not contain any nilpotent elements. Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{s}_1$, and $\mathfrak{a}_{\mathrm{s}}$ the subspace consisting of elements $X \in \mathfrak{a}$, where $\mathrm{ad}(X)$ is split semisimple (that is, diagonalizable over $\mathbb{R}$). The weight spaces for the non-trivial roots of $\mathfrak{a}_{\mathrm{s}}$ consist of nilpotent elements of $\mathfrak{s}_1$. Since, by construction, $\mathfrak{s}_1$ has no nilpotent elements, this implies that $\mathfrak{a}$ has no elements split over $\mathbb{R}$. This in turn implies that $\mathfrak{s}_1$ is of compact type (cf. Borel [3, §24.6(c)]). By assumption, $\mathfrak{s}$ is of non-compact type, so $\mathfrak{s}_1$ must be trivial. □

Lemma 5.2. *Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$. Then $\langle \cdot, \cdot \rangle$ is invariant by $\mathfrak{s} \ltimes \mathfrak{n}$.*

Proof. Since $\langle\cdot,\cdot\rangle$ is nil-invariant, $\mathrm{inv}(\mathfrak{g},\langle\cdot,\cdot\rangle)$ contains all X such that the operator $\mathrm{ad}(X):\mathfrak{g}\to\mathfrak{g}$ is nilpotent. In particular, $\mathfrak{n}$ is contained in $\mathrm{inv}(\mathfrak{g},\langle\cdot,\cdot\rangle)$. Since $\mathfrak{s}$ is of non-compact type, the subalgebra generated by all $X\in\mathfrak{s}$ with $\mathrm{ad}(X)$ nilpotent is $\mathfrak{s}$, see Lemma 5.1 Therefore, also $\mathfrak{s}\subseteq\mathrm{inv}(\mathfrak{g},\langle\cdot,\cdot\rangle)$. □

Recall that any derivation φ of the solvable Lie algebra $\mathfrak{r}$ satisfies $\varphi(\mathfrak{r})\subseteq\mathfrak{n}$ (Jacobson [8, Theorem III.7]). In particular, if φ is semisimple, there exists a decomposition $\mathfrak{r}=\mathfrak{a}+\mathfrak{n}$ into vector subspaces, where $\varphi(\mathfrak{a})=0$. Similarly, for any subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ acting reductively on $\mathfrak{r}$, we have $\mathfrak{r}=\mathfrak{r}^{\mathfrak{h}}+\mathfrak{n}$, where $[\mathfrak{h},\mathfrak{r}^{\mathfrak{h}}]=\mathbf{0}$.

Lemma 5.3. *Let $\mathfrak{h}$ be a subalgebra of $\mathrm{inv}(\mathfrak{g},\langle\cdot,\cdot\rangle)$, and let $\mathfrak{g}^{\mathfrak{h}}$ be the maximal trivial submodule for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. Then $\mathfrak{g}^{\mathfrak{h}}\perp[\mathfrak{h},\mathfrak{g}]$. Moreover, if $\mathfrak{h}$ is a semisimple subalgebra contained in $\mathfrak{s}$ then $[\mathfrak{g}^{\mathfrak{h}},\mathfrak{g}]\perp\mathfrak{h}$.*

Proof. Let $V\in\mathfrak{g}^{\mathfrak{h}}$ and $X\in\mathfrak{h}$, $Y\in\mathfrak{g}$. Then $\langle V,[X,Y]\rangle=\langle[V,X],Y\rangle=0$. Hence, $\mathfrak{g}^{\mathfrak{h}}\perp[\mathfrak{h},\mathfrak{g}]$.

Now assume $\mathfrak{h}$ is a semisimple subalgebra of $\mathfrak{s}$. We may write $Y=Y_1+Y_2$, where $Y_1\in\mathfrak{g}^{\mathfrak{s}}$ and $Y_2\in\mathfrak{s}\ltimes\mathfrak{n}$. Thus $[V,Y_1]\in\mathfrak{g}^{\mathfrak{h}}$. Since $\mathfrak{h}$ is also semisimple, $\mathfrak{h}=[\mathfrak{h},\mathfrak{h}]$. Therefore, the first part of this lemma shows that $[V,Y_1]\perp\mathfrak{h}$. By Lemma 5.2, $Y_2\in\mathrm{inv}(\mathfrak{g},\langle\cdot,\cdot\rangle)$. Therefore,

$$\langle[V,Y_2],X\rangle=\langle V,[Y_2,X]\rangle=\langle[X,V],Y_2\rangle=0.$$

That is, $[V,Y_2]\perp\mathfrak{h}$ as well. □

Proof of Theorem A, part (1). Since $\mathfrak{s}$ acts reductively on $\mathfrak{g}$, we have $\mathfrak{g}=\mathfrak{g}_{\mathrm{s}}+\mathfrak{g}^{\mathfrak{s}}$. Therefore, by Lemma 5.2, it is enough to prove invariance of $\langle\cdot,\cdot\rangle_{\mathfrak{g}_{\mathrm{s}}}$ under $\mathfrak{g}^{\mathfrak{s}}$.

Let $X\in\mathfrak{g}^{\mathfrak{s}}$, $Y,Z\in\mathfrak{g}_{\mathrm{s}}$. Decompose $Y=Y_{\mathfrak{s}}+Y_{\mathfrak{r}}$, $Z=Z_{\mathfrak{s}}+Z_{\mathfrak{r}}$, according to the direct sum $\mathfrak{g}_{\mathrm{s}}=\mathfrak{s}\oplus\mathfrak{r}$. Using Lemma 5.3, we get $\langle[X,Y],Z\rangle=\langle[X,Y_{\mathfrak{r}}],Z_{\mathfrak{r}}\rangle$. By Theorem 4.1, the restriction of $\langle\cdot,\cdot\rangle$ to the solvable Lie algebra generated by $\mathfrak{r}$ and X is invariant on that subalgebra. Hence, $\langle[X,Y_{\mathfrak{r}}],Z_{\mathfrak{r}}\rangle=-\langle Y_{\mathfrak{r}},[X,Z_{\mathfrak{r}}]\rangle=-\langle Y,[X,Z]\rangle$. □

Lemma 5.4. *Let $\mathfrak{f}$ be a subalgebra of $\mathfrak{g}$ and $\mathfrak{g}^{\mathfrak{f}}$ the maximal trivial submodule for the adjoint action of $\mathfrak{f}$ on $\mathfrak{g}$. Then $[\mathfrak{g}^{\mathfrak{f}},\mathfrak{g}_{\mathrm{s}}]\perp\mathfrak{f}$. In particular, $[\mathfrak{g}^{\mathfrak{k}},\mathfrak{g}]\perp\mathfrak{k}$.*

Proof. Let $X\in\mathfrak{g}^{\mathfrak{f}}$, $Y\in\mathfrak{g}_{\mathrm{s}}$ and $K\in\mathfrak{f}$. Since $\mathfrak{g}_{\mathrm{s}}$ is an ideal in $\mathfrak{g}$, we may write

$$\mathfrak{g}_{\mathrm{s}}=(\mathfrak{g}_{\mathrm{s}}\cap\ker\mathrm{ad}(X)_{\mathrm{ss}})+\mathrm{ad}(X)\mathfrak{g}_{\mathrm{s}}\,.$$

Suppose first that $Y\in\ker\mathrm{ad}(X)_{\mathrm{ss}}$. Then we get

$$\langle[X,Y],K\rangle=\langle\mathrm{ad}(X)_{\mathrm{n}}Y,K\rangle=-\langle Y,\mathrm{ad}(X)_{\mathrm{n}}K\rangle=0\,.$$

The latter term vanishes, since $K\in\ker\mathrm{ad}(X)\subseteq\ker\mathrm{ad}(X)_{\mathrm{n}}$.

Next suppose $Y=\mathrm{ad}(X)Y'$, for some $Y'\in\mathfrak{g}_{\mathrm{s}}$. Since $Y\in[\mathfrak{g},\mathfrak{g}_{\mathrm{s}}]\subseteq\mathfrak{s}\ltimes\mathfrak{n}$, Lemma 5.2 implies that $Y\in\mathrm{inv}(\mathfrak{g},\langle\cdot,\cdot\rangle)$. Thus

$$\langle[X,Y],K\rangle=\langle X,[Y,K]\rangle=\langle X,[[X,Y'],K]\rangle=-\langle X,[[Y',K],X]\rangle=0\,.$$

The latter term is zero, since $[Y',K]\in[\mathfrak{g}_{\mathrm{s}},\mathfrak{g}]\subseteq\mathrm{inv}(\mathfrak{g},\langle\cdot,\cdot\rangle)$, and therefore $\mathrm{ad}([Y',K])$ is a skew-symmetric linear map. This shows $[\mathfrak{g}^{\mathfrak{f}},\mathfrak{g}_{\mathrm{s}}]\perp\mathfrak{f}$. Finally, for the last statement observe that $[\mathfrak{g}^{\mathfrak{k}},\mathfrak{g}]=[\mathfrak{g}^{\mathfrak{k}},\mathfrak{g}_{\mathrm{s}}]$. □

Proof of Theorem A, part (2). By Lemma 5.2, $\langle\cdot,\cdot\rangle$ is invariant by $\mathfrak{s}+\mathfrak{n}$. Since $\mathfrak{g}_{\mathrm{s}}=\mathfrak{s}+\mathfrak{r}^{\mathfrak{k}}+\mathfrak{n}$, to prove that $\langle\cdot,\cdot\rangle$ is $\mathfrak{g}_{\mathrm{s}}$-invariant, it suffices to show that $\mathrm{ad}(X)$ is skew for all $X\in\mathfrak{r}^{\mathfrak{k}}$. By part (1), the restriction of $\mathrm{ad}(X)$ to the ideal $\mathfrak{g}_{\mathrm{s}}$ is skew.

Hence, it remains to show that $\langle [X,Y],Z\rangle = -\langle Y,[X,Z]\rangle$, where at least one of Y,Z, say Y, is in $\mathfrak{k}$. This is satisfied, since

$$0 = \langle [X,Y],Z\rangle = -\langle Y,[X,Z]\rangle \,.$$

Note that the right term is zero because of Lemma 5.4. □

Lemma 5.3 and Lemma 5.4 also imply:

Corollary 5.5. *Let* $\langle\cdot,\cdot\rangle$ *be a nil-invariant symmetric bilinear form on* $\mathfrak{g} = (\mathfrak{k}\times\mathfrak{s})\ltimes\mathfrak{r}$. *Then*

(1) $\mathfrak{s}\perp[\mathfrak{k},\mathfrak{g}]$ *and* $\mathfrak{k}\perp[\mathfrak{s},\mathfrak{g}]$.
(2) *The simple factors of* $\mathfrak{s}$ *are pairwise orthogonal.*

Example 5.6 (Nil-invariant products on semisimple Lie algebras)**.** Let

$$\mathfrak{g} = \mathfrak{k}\times\mathfrak{s}$$

be semisimple, where $\mathfrak{k}$ is an ideal of compact type and $\mathfrak{s}$ is of non-compact type. For any nil-invariant bilinear form $\langle\cdot,\cdot\rangle$,

$$(\mathfrak{g},\langle\cdot,\cdot\rangle) = (\mathfrak{k},\langle\cdot,\cdot\rangle_{\mathfrak{k}})\times(\mathfrak{s},\langle\cdot,\cdot\rangle_{\mathfrak{s}})$$

decomposes as a direct product of metric Lie algebras, where $\langle\cdot,\cdot\rangle_{\mathfrak{s}}$ is invariant.

6. Totally isotropic ideals and metric radicals

Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra with nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ and subalgebras $\mathfrak{k}$, $\mathfrak{s}$, $\mathfrak{r}$, $\mathfrak{g}_{\mathrm{s}}$ as in Section 5. We let ℓ denote the *relative* index of $\langle\cdot,\cdot\rangle$ (which is the index of the non-degenerate bilinear form induced by $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}/\mathfrak{g}^\perp$).

6.1. **Transporter algebras.** Suppose $\mathfrak{b}\subseteq\mathfrak{g}_{\mathrm{s}}$ is a totally isotropic ideal of $\mathfrak{g}$ contained in $\mathfrak{g}_{\mathrm{s}}$. Then consider

$$\mathfrak{b}_0 = \mathfrak{b}\cap\mathfrak{g}^\perp\subseteq\mathfrak{b}.$$

By Theorem A part (2), $\langle\cdot,\cdot\rangle$ is invariant by $\mathfrak{g}_{\mathrm{s}}$. Therefore, $\mathfrak{b}_0$ is an ideal of $\mathfrak{g}_{\mathrm{s}}$.

For any subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ define the *transporter subalgebra* of $\mathfrak{b}$ in $\mathfrak{q}$ as

$$\mathfrak{n}_{\mathfrak{q}}(\mathfrak{b},\mathfrak{b}_0) = \{X\in\mathfrak{q}\mid [X,\mathfrak{b}]\subseteq\mathfrak{b}_0\}. \tag{6.1}$$

Clearly, $\mathfrak{n}_{\mathfrak{q}}(\mathfrak{b},\mathfrak{b}_0)$ is a subalgebra of $\mathfrak{q}$. Also, $[\mathfrak{q},\mathfrak{b}]\subseteq\mathfrak{b}_0$ if and only if $\mathfrak{n}_{\mathfrak{q}}(\mathfrak{b},\mathfrak{b}_0)=\mathfrak{q}$.

Lemma 6.1 (Transporter Lemma)**.** *For* $\mathfrak{q}$, $\mathfrak{b}$, $\mathfrak{b}_0$ *as above, we have*

$$\mathfrak{n}_{\mathfrak{q}}(\mathfrak{b},\mathfrak{b}_0) = \mathfrak{q}\cap[\mathfrak{g},\mathfrak{b}]^\perp, \tag{6.2}$$

$$\operatorname{codim}_{\mathfrak{q}}\mathfrak{n}_{\mathfrak{q}}(\mathfrak{b},\mathfrak{b}_0) \;\le\; \operatorname{codim}_{\mathfrak{q}}\mathfrak{q}\cap\mathfrak{b}^\perp \;\le\; \dim\mathfrak{b}-\dim\mathfrak{b}_0 \;\le\; \ell. \tag{6.3}$$

Proof. Let $Z\in\mathfrak{b}$ and $X\in\mathfrak{q}$ and $Y\in\mathfrak{g}$. Since $Z\in\mathfrak{g}_{\mathrm{s}}$, we have $\langle [Y,Z],X\rangle = -\langle Y,[X,Z]\rangle$. This shows the equivalence of $X\perp[\mathfrak{g},\mathfrak{b}]$ and $X\in\mathfrak{n}_{\mathfrak{q}}(\mathfrak{b},\mathfrak{b}_0)$. Hence, the equation (6.2) holds.

As $[\mathfrak{g},\mathfrak{b}]\subseteq\mathfrak{b}$ and thus $[\mathfrak{g},\mathfrak{b}]^\perp\supseteq\mathfrak{b}^\perp$, we clearly have $\operatorname{codim}_{\mathfrak{q}}\mathfrak{q}\cap[\mathfrak{g},\mathfrak{b}]^\perp\le\operatorname{codim}_{\mathfrak{q}}\mathfrak{q}\cap\mathfrak{b}^\perp$. Now $\operatorname{codim}_{\mathfrak{g}}\mathfrak{b}^\perp = \dim\mathfrak{b}-\dim\mathfrak{b}\cap\mathfrak{g}^\perp$. Since $\mathfrak{b}$ is totally isotropic, this means $\operatorname{codim}_{\mathfrak{g}}\mathfrak{b}^\perp\le\ell$. Since $\operatorname{codim}_{\mathfrak{q}}\mathfrak{q}\cap\mathfrak{b}^\perp\le\operatorname{codim}_{\mathfrak{g}}\mathfrak{b}^\perp$, the inequalities (6.3) follow. □

Remark. Equality holds in (6.3) if and only if $\dim\mathfrak{q}+\mathfrak{b}^\perp=\dim\mathfrak{g}$.

The following relations between transporters are satisfied:

Lemma 6.2.

(1) $[\mathfrak{g}_{\rm s}, \mathfrak{n}_{\mathfrak{g}}(\mathfrak{b},\mathfrak{b}_0)] \subseteq \mathfrak{b}^\perp \cap \mathfrak{g}_{\rm s} \subseteq \mathfrak{n}_{\mathfrak{g}_{\rm s}}(\mathfrak{b},\mathfrak{b}_0)$

(2) $\mathfrak{n}_{\mathfrak{g}_{\rm s}}(\mathfrak{b},\mathfrak{b}_0)$ *is an ideal of* $\mathfrak{g}$.

(3) $\mathfrak{b}^\perp \subseteq \mathfrak{n}_{\mathfrak{g}}(\mathfrak{b},\mathfrak{b}_0)$.

Proof. Let $Y \in \mathfrak{g}_{\rm s}$ and $X \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{b},\mathfrak{b}_0)$, $Z \in \mathfrak{b}$. Since $[X,Z] \in \mathfrak{b}_0$, we get $\langle [Y,X],Z]\rangle = \langle X,[Z,Y]\rangle = \langle [X,Z],Y\rangle = 0$. This shows $[Y,X] \perp \mathfrak{b}$. By (6.2), $\mathfrak{b}^\perp \cap \mathfrak{g}_{\rm s} \subseteq \mathfrak{n}_{\mathfrak{g}_{\rm s}}(\mathfrak{b},\mathfrak{b}_0)$. This shows (1).

Let $Y \in \mathfrak{g}$ and $X \in \mathfrak{n}_{\mathfrak{g}_{\rm s}}(\mathfrak{b},\mathfrak{b}_0)$, $Z \in [\mathfrak{g},\mathfrak{b}]$. We get $\langle [Y,X],Z\rangle = \langle Y,[Z,X]\rangle = \langle [Y,Z],X\rangle$. Since $[Y,Z] \in [\mathfrak{g},\mathfrak{b}]$, (6.2) shows that $\langle [Y,X],Z\rangle = 0$. Thus, $[Y,X] \in \mathfrak{n}_{\mathfrak{g}_{\rm s}}(\mathfrak{b},\mathfrak{b}_0)$. This shows (2).

Finally, $\mathfrak{b}^\perp \subseteq [\mathfrak{g},\mathfrak{b}]^\perp$, which, in light of (6.2), shows (3). □

6.1.1. *Totally isotropic ideals.*

Proposition 6.3. *Let* $\mathfrak{i}$ *be an ideal of* $\mathfrak{k}$ *contained in* $\mathfrak{n}_{\mathfrak{k}}(\mathfrak{b},\mathfrak{b}_0)$. *Then* $\mathfrak{i} + \mathfrak{n}_{\mathfrak{g}_{\rm s}}(\mathfrak{b},\mathfrak{b}_0)$ *is an ideal of* $\mathfrak{g}$ *contained in* $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{b},\mathfrak{b}_0)$. *In particular,* $[\mathfrak{i} + \mathfrak{n}_{\mathfrak{g}_{\rm s}}(\mathfrak{b},\mathfrak{b}_0),\mathfrak{b}]$ *is an ideal of* $\mathfrak{g}$ *contained in* $\mathfrak{g}^\perp$.

Proof. Clearly, $\mathfrak{j} = \mathfrak{i} + \mathfrak{n}_{\mathfrak{g}_{\rm s}}(\mathfrak{b},\mathfrak{b}_0)$ is contained in $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{b},\mathfrak{b}_0)$. Recall that $\mathfrak{g} = \mathfrak{k} + \mathfrak{g}_{\rm s}$. Since $\mathfrak{i}$ is an ideal in $\mathfrak{k}$, $[\mathfrak{k},\mathfrak{j}] \subseteq \mathfrak{i} + [\mathfrak{k},\mathfrak{n}_{\mathfrak{g}_{\rm s}}(\mathfrak{b},\mathfrak{b}_0)]$. Using (2) of Lemma 6.2, we conclude $[\mathfrak{k},\mathfrak{j}] \subseteq \mathfrak{j}$. By (1) of Lemma 6.2, $[\mathfrak{g}_{\rm s},\mathfrak{j}] \subseteq \mathfrak{n}_{\mathfrak{g}_{\rm s}}(\mathfrak{b},\mathfrak{b}_0) \subseteq \mathfrak{j}$. □

Corollary 6.4. *Assume that* $\mathfrak{g}^\perp$ *does not contain any non-trivial ideal of* $\mathfrak{g}$. *Then every totally isotropic ideal* $\mathfrak{b}$ *of* $\mathfrak{g}$ *contained in* $\mathfrak{g}_{\rm s}$ *is abelian.*

Proof. By (3) of Lemma 6.2, $\mathfrak{b} \subseteq \mathfrak{n}_{\mathfrak{g}_{\rm s}}(\mathfrak{b},\mathfrak{b}_0)$. Therefore, $[\mathfrak{b},\mathfrak{b}]$ is an ideal of $\mathfrak{g}$ and contained in $\mathfrak{g}^\perp$. Hence, $[\mathfrak{b},\mathfrak{b}] = \mathbf{0}$. □

The case of a large transporter in $\mathfrak{k}$ has particularly strong consequences:

Proposition 6.5. *Assume that* $\mathfrak{n}_{\mathfrak{k}}(\mathfrak{b},\mathfrak{b}_0) = \mathfrak{k}$. *Then:*

(1) $\mathfrak{b} \cap \mathfrak{g}^\perp$ *is an ideal in* $\mathfrak{g}$.

If furthermore $\mathfrak{g}^\perp$ *does not contain any non-trivial ideal of* $\mathfrak{g}$*, then:*

(2) $\mathfrak{b} \cap \mathfrak{g}^\perp = \mathbf{0}$*,* $\dim \mathfrak{b} \leq \ell$*,* $[\mathfrak{k},\mathfrak{b}] = \mathbf{0}$.

Proof. Since, by Theorem A, $\langle\cdot,\cdot\rangle$ is invariant by $\mathfrak{g}_{\rm s}$, $[\mathfrak{g}_{\rm s},\mathfrak{g}^\perp] \subseteq \mathfrak{g}^\perp$. Hence, $[\mathfrak{g}_{\rm s},\mathfrak{b}\cap\mathfrak{g}^\perp] \subseteq \mathfrak{b} \cap \mathfrak{g}^\perp$. Since $\mathfrak{n}_{\mathfrak{k}}(\mathfrak{b},\mathfrak{b}_0) = \mathfrak{k}$ this implies that $\mathfrak{b}_0 = \mathfrak{b} \cap \mathfrak{g}^\perp$ is an ideal in $\mathfrak{g}$. □

6.1.2. *Metric radical of* $\mathfrak{g}_{\rm s}$. In the following consider the special case

$$\mathfrak{b} = \mathfrak{g}_{\rm s}^\perp \cap \mathfrak{g}_{\rm s}\,.$$

Thus $\mathfrak{b}$ is totally isotropic and it is the metric radical of $\mathfrak{g}_{\rm s}$ (with respect to the induced metric $\langle\cdot,\cdot\rangle_{\mathfrak{g}_{\rm s}}$). By Theorem A, $\langle\cdot,\cdot\rangle_{\mathfrak{g}_{\rm s}}$ is invariant by $\mathfrak{g}$. Therefore, $\mathfrak{b}$ is an ideal in $\mathfrak{g}$. Moreover,

$$\mathfrak{b}_0 = \mathfrak{g}^\perp \cap \mathfrak{g}_{\rm s}$$

is an ideal in $\mathfrak{g}_{\rm s}$.

Lemma 6.6. $[\mathfrak{g}_{\rm s},\mathfrak{g}_{\rm s}^\perp] \subseteq \mathfrak{b}_0$. *In particular,* $[\mathfrak{g}_{\rm s},\mathfrak{b}] \subseteq \mathfrak{b}_0 \subseteq \mathfrak{g}^\perp$.

Proof. Let $Y \in \mathfrak{g}$, $X \in \mathfrak{g}_{\rm s}$, $B \in \mathfrak{g}_{\rm s}^\perp$. Since $\mathfrak{g}_{\rm s}$ is an ideal, $[Y,X] \in \mathfrak{g}_{\rm s}$. Since $\langle\cdot,\cdot\rangle$ is $\mathfrak{g}_{\rm s}$-invariant, we obtain $\langle Y,[X,B]\rangle = \langle [Y,X],B\rangle = 0$. This shows $\mathfrak{g} \perp [\mathfrak{g}_{\rm s},\mathfrak{g}_{\rm s}^\perp]$. □

Since $[\mathfrak{g}_{\rm s},\mathfrak{b}]$ is an ideal in $\mathfrak{g}$, we deduce:

Corollary 6.7. *If $\mathfrak{g}^\perp$ does not contain a non-trivial ideal of $\mathfrak{g}$, then $\mathfrak{g}_{\mathrm{s}}^\perp \cap \mathfrak{g}_{\mathrm{s}} \subseteq \mathfrak{z}(\mathfrak{g}_{\mathrm{s}})$. In particular, $\mathfrak{g}^\perp \cap \mathfrak{g}_{\mathrm{s}} \subseteq \mathfrak{z}(\mathfrak{g}_{\mathrm{s}})$.*

The following strengthens Proposition 6.5 for $\mathfrak{b} = \mathfrak{g}_{\mathrm{s}}^\perp \cap \mathfrak{g}_{\mathrm{s}}$ and $\mathfrak{b}_0 = \mathfrak{g}^\perp \cap \mathfrak{g}_{\mathrm{s}}$:

Proposition 6.8. *Assume that $\mathfrak{n}_{\mathfrak{k}}(\mathfrak{b},\mathfrak{b}_0) = \mathfrak{k}$. Then:*

(1) $[\mathfrak{g}, \mathfrak{g}_{\mathrm{s}}^\perp \cap \mathfrak{g}_{\mathrm{s}}] \subseteq \mathfrak{g}^\perp \cap \mathfrak{g}_{\mathrm{s}}$. *In particular, $\mathfrak{g}^\perp \cap \mathfrak{g}_{\mathrm{s}}$ is an ideal in $\mathfrak{g}$.*

If furthermore $\mathfrak{g}^\perp$ contains no non-trivial ideal of $\mathfrak{g}$, then:

(2) $\mathfrak{g}^\perp \cap \mathfrak{g}_{\mathrm{s}} = \mathbf{0}$ *and* $[\mathfrak{g}, \mathfrak{g}_{\mathrm{s}}^\perp \cap \mathfrak{g}_{\mathrm{s}}] = \mathbf{0}$.
(3) $[\mathfrak{g}_{\mathrm{s}}^\perp, \mathfrak{g}_{\mathrm{s}}] = [\mathfrak{g}^\perp, \mathfrak{g}_{\mathrm{s}}] = \mathbf{0}$.

Proof. By assumption, $[\mathfrak{k},\mathfrak{b}] \subseteq \mathfrak{b}_0$. By Lemma 6.6, $[\mathfrak{g}_{\mathrm{s}},\mathfrak{b}] \subseteq \mathfrak{b}_0 \subseteq \mathfrak{g}^\perp$, so that $[\mathfrak{g},\mathfrak{b}] \subseteq \mathfrak{b}_0$. In particular, $\mathfrak{b}_0$ is an ideal in $\mathfrak{g}$. Thus (1), (2) follow, and also (3), since $[\mathfrak{g}_{\mathrm{s}}^\perp, \mathfrak{g}_{\mathrm{s}}] \subseteq \mathfrak{b}_0$, by Lemma 6.6. □

Remark. It is not difficult to see (compare Lemma 6.9 below) that the centralizer of $\mathfrak{g}_{\mathrm{s}}$ in $\mathfrak{g}$ is $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_{\mathrm{s}}) = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{g}_{\mathrm{s}}) \times \mathfrak{z}(\mathfrak{g}_{\mathrm{s}})$.

6.2. **Metric radical of $\mathfrak{g}$.**

Lemma 6.9. *Let W be a $\mathfrak{g}$-module. Suppose that $\mathfrak{c} = \{Z \in \mathfrak{g} \mid Z \cdot W = \mathbf{0}\}$, the centralizer of W, is contained in $\mathfrak{k} + \mathfrak{r}$. Then $\mathfrak{c} = (\mathfrak{c} \cap \mathfrak{k}) + (\mathfrak{c} \cap \mathfrak{r})$.*

Proof. Let $\mathfrak{g} = \mathfrak{f} \ltimes \mathfrak{r}$ (where $\mathfrak{f} \supseteq \mathfrak{k}$ is a semisimple subalgebra, and $\mathfrak{r}$ the maximal solvable ideal of $\mathfrak{g}$) be a Levi decomposition of $\mathfrak{g}$. Assume first that W is an irreducible $\mathfrak{g}$-module. Then the action of $\mathfrak{r}$ on W is reductive and commutes with $\mathfrak{f}$. Since the image of $\mathfrak{f}$ in $\mathfrak{gl}(W)$ has trivial center, the claim of the lemma follows in this case. For the general case, consider a Hölder sequence of submodules $W \supseteq W_1 \supseteq \ldots \supseteq W_k = \mathbf{0}$ such that the $\mathfrak{g}$-module W_i/W_{i+1} is irreducible. The above implies that, for any $Z = K + X \in \mathfrak{c}$, where $K \in \mathfrak{f}$ and $X \in \mathfrak{r}$, K (and X) act trivially on W_i/W_{i+1}. Since $K \in \mathfrak{k}$ is semisimple on W, this implies that K acts trivially on W. That is, $K \in \mathfrak{c} \cap \mathfrak{k}$ and therefore also $X \in \mathfrak{c} \cap \mathfrak{r}$. □

Proposition 6.10. *If $\mathfrak{g}^\perp$ does not contain a non-trivial ideal of $\mathfrak{g}$, then*

(1) $\mathfrak{g}_{\mathrm{s}}^\perp \subseteq \mathfrak{n}_{\mathfrak{k}}(\mathfrak{g}_{\mathrm{s}}, \mathfrak{z}(\mathfrak{g}_{\mathrm{s}}) \cap \mathfrak{g}_{\mathrm{s}}^\perp) + \mathfrak{n}_{\mathfrak{n}}(\mathfrak{g}_{\mathrm{s}}, \mathfrak{z}(\mathfrak{g}_{\mathrm{s}}) \cap \mathfrak{g}_{\mathrm{s}}^\perp)$.
(2) $\mathfrak{g}^\perp \subseteq \mathfrak{n}_{\mathfrak{k}}(\mathfrak{g}_{\mathrm{s}}, \mathfrak{z}(\mathfrak{g}_{\mathrm{s}}) \cap \mathfrak{g}^\perp) + \mathfrak{n}_{\mathfrak{n}}(\mathfrak{g}_{\mathrm{s}}, \mathfrak{z}(\mathfrak{g}_{\mathrm{s}}) \cap \mathfrak{g}^\perp)$.

Proof. By Lemma 6.6, $[\mathfrak{g}_{\mathrm{s}}^\perp, \mathfrak{g}_{\mathrm{s}}] \subseteq \mathfrak{b}_0 = \mathfrak{g}_{\mathrm{s}} \cap \mathfrak{g}^\perp \subseteq \mathfrak{g}_{\mathrm{s}} \cap \mathfrak{g}_{\mathrm{s}}^\perp = \mathfrak{b}$. Since $\mathfrak{b}$ is an ideal of $\mathfrak{g}$, $W = \mathfrak{g}_{\mathrm{s}}/\mathfrak{b}$ is a $\mathfrak{g}$-module. Now $\mathfrak{g}_{\mathrm{s}}^\perp$ is contained in $\mathfrak{c} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{g}_{\mathrm{s}}, \mathfrak{g}_{\mathrm{s}} \cap \mathfrak{g}_{\mathrm{s}}^\perp)$, which is the centralizer of W. In view of our assumption on ideals in $\mathfrak{g}^\perp$, observe that $[\mathfrak{c}, \mathfrak{g}_{\mathrm{s}}] \subseteq \mathfrak{b} \subseteq \mathfrak{z}(\mathfrak{g}_{\mathrm{s}}) \subseteq \mathfrak{r}$ by Corollary 6.7. Now $[\mathfrak{c}, \mathfrak{g}_{\mathrm{s}}] \subseteq \mathfrak{r}$ implies that $\mathfrak{c}$ is contained in $\mathfrak{k} + \mathfrak{r}$. Therefore, Lemma 6.9 applies, showing $\mathfrak{g}_{\mathrm{s}}^\perp \subseteq \mathfrak{n}_{\mathfrak{k}}(\mathfrak{g}_{\mathrm{s}}, \mathfrak{g}_{\mathrm{s}} \cap \mathfrak{g}_{\mathrm{s}}^\perp) + \mathfrak{n}_{\mathfrak{r}}(\mathfrak{g}_{\mathrm{s}}, \mathfrak{g}_{\mathrm{s}} \cap \mathfrak{g}_{\mathrm{s}}^\perp)$. Since $[\mathfrak{r}, \mathfrak{z}(\mathfrak{g}_{\mathrm{s}})] = \mathbf{0}$, it follows that $\mathfrak{n}_{\mathfrak{r}}(\mathfrak{g}_{\mathrm{s}}, \mathfrak{z}(\mathfrak{g}_{\mathrm{s}})) \subseteq \mathfrak{n}_{\mathfrak{n}}(\mathfrak{g}_{\mathrm{s}}, \mathfrak{z}(\mathfrak{g}_{\mathrm{s}}))$. Hence, (1) holds.

To prove (2), suppose $Z = K + X \in \mathfrak{n}(\mathfrak{g}_{\mathrm{s}}, \mathfrak{g}_{\mathrm{s}} \cap \mathfrak{g}^\perp)$, where $K \in \mathfrak{k}$, $X \in \mathfrak{r}$. By (1), $K \in \mathfrak{c} = \mathfrak{n}(\mathfrak{g}_{\mathrm{s}}, \mathfrak{g}_{\mathrm{s}} \cap \mathfrak{g}_{\mathrm{s}}^\perp)$. Since K acts as a semisimple derivation on $\mathfrak{g}_{\mathrm{s}}$, we can decompose $\mathfrak{g}_{\mathrm{s}} = W_1 + (\mathfrak{g}_{\mathrm{s}} \cap \mathfrak{g}_{\mathrm{s}}^\perp)$, where $[K, W_1] = \mathbf{0}$. Now, for $w \in \mathfrak{g}_{\mathrm{s}}$, write $w = w_1 + v$, where $w_1 \in W_1$, $v \in \mathfrak{g}_{\mathrm{s}} \cap \mathfrak{g}^\perp$. Note that $0 = \langle [Z,v], Y\rangle = \langle [K,v], Y\rangle + \langle [X,v], Y\rangle$ for all $Y \in \mathfrak{g}$. By Lemma 6.6, $[X,v] \in \mathfrak{g}^\perp$. This implies $[K,w] = [K,v] \in \mathfrak{g}^\perp$. It follows also that $[X,w] \in \mathfrak{g}^\perp$. □

Lemma 6.11. *Let $\mathfrak{j} = \mathfrak{n}_{\mathfrak{n}}(\mathfrak{g}_{\mathrm{s}}, \mathfrak{z}(\mathfrak{g}_{\mathrm{s}}) \cap \mathfrak{g}^\perp)$. Then $\mathfrak{j}$ is an ideal in $\mathfrak{g}$.*

Proof. Let $N \in \mathfrak{j}$, and $K, Y \in \mathfrak{g}$, $v \in \mathfrak{g}_{\mathrm{s}}$. Then $\langle [[K,N],v],Y\rangle = \langle [K,N],[v,Y]\rangle = \langle K,[N,[v,Y]]\rangle = 0$, since $v' = [v,Y] \in \mathfrak{g}_{\mathrm{s}}$ and therefore, $[N,v'] \in \mathfrak{g}^{\perp}$. This shows $[[K,N],\mathfrak{g}_{\mathrm{s}}] \subseteq \mathfrak{g}^{\perp}$.

Similarly, $[[K,N],v] = [[K,v],N] + [[v,N],K]$, where $v' = [K,v] \in \mathfrak{g}_{\mathrm{s}}$ and $[v',N] \in \mathfrak{z}(\mathfrak{g}_{\mathrm{s}})$, as well as $[v,N] \in \mathfrak{z}(\mathfrak{g}_{\mathrm{s}})$ and therefore, $[[v,N],K] \in \mathfrak{z}(\mathfrak{g}_{\mathrm{s}})$. It follows $[[K,N],\mathfrak{g}_{\mathrm{s}}] \subseteq \mathfrak{z}(\mathfrak{g}_{\mathrm{s}})$. We conclude that $[K,N] \in \mathfrak{j}$. Hence, $\mathfrak{j}$ is an ideal of $\mathfrak{g}$. □

These considerations yield the following important property of $\mathfrak{g}^{\perp}$:

Theorem 6.12. *Suppose that $\mathfrak{g}^{\perp}$ does not contain a non-trivial ideal of $\mathfrak{g}$. Then*

$$\mathfrak{g}^{\perp} \subseteq \mathfrak{n}_{\mathfrak{k}}(\mathfrak{g}_{\mathrm{s}}, \mathfrak{z}(\mathfrak{g}_{\mathrm{s}}) \cap \mathfrak{g}^{\perp}) + \mathfrak{z}(\mathfrak{g}_{\mathrm{s}}).$$

Proof. Consider the ideal $\mathfrak{j}$, as defined in Lemma 6.11. By (2) of Proposition 6.10, we have $\mathfrak{g}^{\perp} \subseteq \mathfrak{n}_{\mathfrak{k}}(\mathfrak{g}_{\mathrm{s}}, \mathfrak{z}(\mathfrak{g}_{\mathrm{s}}) \cap \mathfrak{g}^{\perp}) + \mathfrak{j}$. Since $\mathfrak{j}$ is an ideal in $\mathfrak{g}$, so is $[\mathfrak{j},\mathfrak{g}_{\mathrm{s}}]$. Since $[\mathfrak{j},\mathfrak{g}_{\mathrm{s}}] \subseteq \mathfrak{g}^{\perp}$, the assumption on ideals in $\mathfrak{g}^{\perp}$ implies that $[\mathfrak{j},\mathfrak{g}_{\mathrm{s}}] = \mathbf{0}$. It follows that $\mathfrak{j}$ is contained in $\mathfrak{z}(\mathfrak{g}_{\mathrm{s}})$. □

6.3. Transporter in $\mathfrak{k}$ and low relative index.

Lemma 6.13. *Let $\mathfrak{k}$ be a semisimple Lie algebra of compact type and $\mathfrak{l}$ a subalgebra of $\mathfrak{k}$. Then either $\mathfrak{l} = \mathfrak{k}$ or $m = \operatorname{codim}_{\mathfrak{k}} \mathfrak{l} > 1$. Assume further that $\mathfrak{l}$ does not contain any non-trivial ideal of $\mathfrak{k}$. Then, up to conjugation by an automorphism of $\mathfrak{k}$:*

1. *if $m = 2$, then $\mathfrak{k} = \mathfrak{so}_3$ and $\mathfrak{l} = \mathfrak{so}_2$,*
2. *if $m = 3$, one of the following holds:*
   - (a) *$\mathfrak{k} = \mathfrak{so}_3$ and $\mathfrak{l} = \mathbf{0}$,*
   - (b) *$\mathfrak{k} = \mathfrak{so}_3 \times \mathfrak{so}_3$, $\mathfrak{l}$ is the image of a diagonal embedding $\mathfrak{so}_3 \to \mathfrak{so}_3 \times \mathfrak{so}_3$.*

Proof. As an $\mathrm{ad}_{\mathfrak{k}}(\mathfrak{l})$-module, $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{w}$ for a submodule $\mathfrak{w}$. For this, note that any Lie subalgebra of $\mathfrak{k}$ acts reductively, since $\mathfrak{k}$ is of compact type.

Suppose $\operatorname{codim}_{\mathfrak{k}} \mathfrak{l} = 1$, that is, $\mathfrak{w}$ is one-dimensional. Then $[\mathfrak{w},\mathfrak{w}] = \mathbf{0}$ and it follows that $\mathfrak{w}$ is also an ideal of $\mathfrak{k}$. A one-dimensional ideal cannot exist, since $\mathfrak{k}$ is semisimple. It follows that $\operatorname{codim}_{\mathfrak{k}} \mathfrak{l} > 1$.

Since $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{w}$, the kernel of the adjoint action of $\mathfrak{l}$ on $\mathfrak{w}$ is an ideal in $\mathfrak{k}$. Assume further that $\mathfrak{l}$ contains no non-trivial ideals of $\mathfrak{k}$. Then $\mathfrak{l}$ acts faithfully on $\mathfrak{w}$.

For $m = 2$, this means $\mathfrak{l} = \mathfrak{so}_2$ and $\dim \mathfrak{k} = m + \dim \mathfrak{l} = 3$. Hence, $\mathfrak{k} = \mathfrak{so}_3$.

For $m = 3$, $\mathfrak{l}$ embeds into $\mathfrak{so}_3$. If $\mathfrak{l} = \mathbf{0}$, we have $\dim \mathfrak{k} = 3$ and thus $\mathfrak{k} = \mathfrak{so}_3$. Otherwise, either $\mathfrak{l} = \mathfrak{so}_2$ or $\mathfrak{l} = \mathfrak{so}_3$. In the first case, $\dim \mathfrak{k} = 4$. Since there is no four-dimensional simple Lie algebra, this is not possible. In the latter case, $\dim \mathfrak{k} = 6$. This leaves $\mathfrak{k} = \mathfrak{so}_3 \times \mathfrak{so}_3$ (being isomorphic to $\mathfrak{so}_4$) as the only possibility. Since $\mathfrak{l}$ is not an ideal of $\mathfrak{k}$, $\mathfrak{l}$ projects injectively onto both factors of $\mathfrak{k}$. It follows that, up to automorphism of $\mathfrak{k}$, $\mathfrak{l}$ is the image of an embedding $\mathfrak{so}_3 \to \mathfrak{so}_3 \times \mathfrak{so}_3$, $X \mapsto (X,X)$. □

6.3.1. *Totally isotropic ideals and low relative index.* Let $\mathfrak{b}$ be any totally isotropic ideal of $\mathfrak{g}$ contained in $\mathfrak{g}_{\mathrm{s}}$ and put $\mathfrak{b}_0 = \mathfrak{b} \cap \mathfrak{g}^{\perp}$.

Proposition 6.14. *If $\ell \leq 2$, then $\mathfrak{n}_{\mathfrak{k}}(\mathfrak{b},\mathfrak{b}_0) = \mathfrak{k}$.*

Proof. Put $\mathfrak{l} = \mathfrak{n}_{\mathfrak{k}}(\mathfrak{b},\mathfrak{b}_0)$ and $m = \operatorname{codim}_{\mathfrak{k}} \mathfrak{l}$. By Lemma 6.1, $\mathfrak{l} = \mathfrak{k} \cap [\mathfrak{g},\mathfrak{b}]^{\perp}$ and $m \leq \ell$.

Assume now that $m \geq 1$. According to Lemma 6.13, the case $m = 1$ never occurs. Hence, in this case, we have $m = 2$.

Let $\mathfrak{i} \subseteq \mathfrak{l}$ be the maximal ideal of $\mathfrak{k}$ contained in $\mathfrak{l}$. Using Proposition 6.3, we see that there exists an ideal $\mathfrak{b}_1$ of $\mathfrak{g}$, such that $[\mathfrak{i},\mathfrak{b}] \subseteq \mathfrak{b}_1 \subseteq [\mathfrak{g},\mathfrak{b}] \cap \mathfrak{g}^{\perp}$. Since $[\mathfrak{g},\mathfrak{b}]$ and

$\mathfrak{b}_1$ are ideals, $U = [\mathfrak{g},\mathfrak{b}]/\mathfrak{b}_1$ is a module for $\mathfrak{k}$. In fact, since $[\mathfrak{i},\mathfrak{b}] \subseteq \mathfrak{b}_1$, U is a module for $\mathfrak{k}/\mathfrak{i}$. Also, since $\mathfrak{b}_1 \subseteq \mathfrak{g}^\perp$, $\langle\cdot,\cdot\rangle$ restricted to $\mathfrak{k}\times[\mathfrak{g},\mathfrak{b}]$ induces a skew pairing on $\mathfrak{k}\times U$, such that $U^\perp = \mathfrak{l}$. Since we have $\mathfrak{i} \perp [\mathfrak{g},\mathfrak{b}]$, this shows that $\langle\cdot,\cdot\rangle$ restricted to $\mathfrak{k}\times[\mathfrak{g},\mathfrak{b}]$ descends to a skew pairing

$$\langle\cdot,\cdot\rangle : (\mathfrak{k}/\mathfrak{i})\times U \to \mathbb{R}\,,\ \text{where } U^\perp = \mathfrak{l}/\mathfrak{i}. \tag{6.4}$$

If $m = 2$, then by Lemma 6.13, $\mathfrak{k}/\mathfrak{i} = \mathfrak{so}_3$ and $\mathfrak{l}/\mathfrak{i} = \mathfrak{so}_2$. By Corollary A.6, either the skew pairing $\langle\cdot,\cdot\rangle$ in (6.4) is zero (that is, $U^\perp = \mathfrak{k}/\mathfrak{i}$) or $\mathfrak{k}\cap[\mathfrak{g},\mathfrak{b}]^\perp = \mathfrak{i}$. In the first case, $\mathfrak{l} = \mathfrak{k}\cap[\mathfrak{g},\mathfrak{b}]^\perp = \mathfrak{k}$. In the second case, $\mathfrak{l} = \mathfrak{i}$, a contradiction to $\mathfrak{l}/\mathfrak{i} = \mathfrak{so}_2$. Therefore, $m = 0$. □

Combining with Proposition 6.5 (1) we arrive at:

Corollary 6.15. *If $\ell \le 2$ then, for any totally isotropic ideal $\mathfrak{b}$ of $\mathfrak{g}$ contained in $\mathfrak{g}_{\mathrm{s}}$, $\mathfrak{g}^\perp\cap\mathfrak{b}$ is an ideal in $\mathfrak{g}$. In particular, $\mathfrak{g}^\perp\cap\mathfrak{g}_{\mathrm{s}}$ is an ideal in $\mathfrak{g}$.*

The following now summarizes our results on totally isotropic ideals in case $\ell \le 2$:

Corollary 6.16. *Assume that $\mathfrak{g}^\perp$ does not contain any non-trivial ideal of $\mathfrak{g}$ and that $\ell \le 2$. Then, for any totally isotropic ideal $\mathfrak{b}$ of $\mathfrak{g}$ contained in $\mathfrak{g}_{\mathrm{s}}$,*

(1) $\mathfrak{b}\cap\mathfrak{g}^\perp = \mathbf{0}$, $\dim\mathfrak{b} \le \ell$, $[\mathfrak{k},\mathfrak{b}] = \mathbf{0}$.

Furthermore, the following hold:

(2) $\mathfrak{g}^\perp\cap\mathfrak{g}_{\mathrm{s}} = \mathbf{0}$.
(3) $[\mathfrak{g}_{\mathrm{s}}^\perp,\mathfrak{g}_{\mathrm{s}}] = \mathbf{0}$.
(4) $[\mathfrak{g},\mathfrak{g}_{\mathrm{s}}\cap\mathfrak{g}_{\mathrm{s}}^\perp] = \mathbf{0}$.

Proof. Since $\ell \le 2$, according to Proposition 6.14 $\mathfrak{n}_{\mathfrak{k}}(\mathfrak{b},\mathfrak{b}_0) = \mathfrak{k}$. Thus (1) holds due to part (2) of Proposition 6.5.

Now (2), (3), (4) are consequence of Proposition 6.8. □

Combining with Theorem 6.12, we also obtain:

Corollary 6.17. *Assume that $\mathfrak{g}^\perp$ does not contain any non-trivial ideal of $\mathfrak{g}$ and that $\ell \le 2$. Then $\mathfrak{g}^\perp$ is contained in $\mathfrak{z}(\mathfrak{g}_{\mathrm{s}})\times\mathfrak{k}$ and $\mathfrak{g}^\perp\cap\mathfrak{g}_{\mathrm{s}} = \mathbf{0}$.*

Remark. As Example 8.2 shows, these conclusions do not necessarily hold if $\ell \ge 3$.

6.4. **Metric radicals of the characteristic ideals.** This section serves to clarify the relations between the metric radicals of $\mathfrak{g}_{\mathrm{s}}$, $\mathfrak{r}$ and $\mathfrak{n}$, where $\mathfrak{n}$ denotes the nilradical of $\mathfrak{r}$.

Lemma 6.18.

(1) $[\mathfrak{r},[\mathfrak{g},\mathfrak{r}]^\perp] \perp \mathfrak{g}$.
(2) $[\mathfrak{r},\mathfrak{n}^\perp] \perp \mathfrak{g}$ *and* $[\mathfrak{g},\mathfrak{n}^\perp\cap\mathfrak{g}_{\mathrm{s}}] \perp \mathfrak{r}$.
(3) $[\mathfrak{g}_{\mathrm{s}},(\mathfrak{s}+\mathfrak{n})^\perp] \perp \mathfrak{g}$ *and* $[\mathfrak{g},(\mathfrak{s}+\mathfrak{n})^\perp\cap\mathfrak{g}_{\mathrm{s}}] \perp \mathfrak{g}_{\mathrm{s}}$.
(4) $[\mathfrak{g}_{\mathrm{s}},\mathfrak{n}^\perp] \perp (\mathfrak{k}+\mathfrak{r})$ *and* $[\mathfrak{k}+\mathfrak{r},\mathfrak{n}^\perp\cap\mathfrak{g}_{\mathrm{s}}] \perp \mathfrak{g}_{\mathrm{s}}$.

The lemma is clearly implied by:

Remark. Let $\mathfrak{a} \subseteq \operatorname{inv}(\mathfrak{g},\langle\cdot,\cdot\rangle)$, $\mathfrak{b},\mathfrak{c} \subseteq \mathfrak{g}$ be subspaces such that $[\mathfrak{a},\mathfrak{c}] \subseteq \mathfrak{b}$. Then $[\mathfrak{a},\mathfrak{b}^\perp] \perp \mathfrak{c}$. Furthermore, this implies $\mathfrak{a} \perp [\mathfrak{b}^\perp\cap\operatorname{inv}(\mathfrak{g},\langle\cdot,\cdot\rangle),\mathfrak{c}]$.

Lemma 6.19. *Let $\mathfrak{j} \subseteq \mathfrak{g}_{\mathrm{s}}$ be an ideal in $\mathfrak{g}$. Then the following hold:*

(1) $\mathfrak{j}^\perp\cap\mathfrak{g}_{\mathrm{s}}$ *and* $\mathfrak{j}^\perp\cap\mathfrak{j}$ *are ideals of* $\mathfrak{g}$.
(2) $[\mathfrak{j},\mathfrak{j}^\perp] \subseteq \mathfrak{g}^\perp$.

Proof. Since $\langle\cdot,\cdot\rangle$ restricted to $\mathfrak{g}_{\mathrm{s}}$ is $\mathfrak{g}$-invariant by Theorem A, $\mathfrak{j}^\perp \cap \mathfrak{g}_{\mathrm{s}}$ is an ideal in $\mathfrak{g}$. It follows that $\mathfrak{j}^\perp \cap \mathfrak{j}$ is an ideal. Hence (1) holds. Now (2) follows using the above remark with $\mathfrak{a} = \mathfrak{j}$, $\mathfrak{c} = \mathfrak{g}$ and $\mathfrak{b} = \mathfrak{j}$. □

6.4.1. *Radicals in effective metric Lie algebras.* For all following results, we shall also require that the metric Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is effective. That is, we assume for now that $\mathfrak{g}^\perp$ *does not contain any non-trivial ideal of* $\mathfrak{g}$.

Lemma 6.20. *Let* $\mathfrak{i},\mathfrak{j} \subseteq \mathfrak{g}_{\mathrm{s}}$ *be ideals in* $\mathfrak{g}$. *Then:*

1. $[\mathfrak{j}, \mathfrak{j}^\perp \cap \mathfrak{g}_{\mathrm{s}}] = \mathbf{0}$ *and* $\mathfrak{j}^\perp \cap \mathfrak{j} = \mathfrak{j}^\perp \cap \mathfrak{z}(\mathfrak{j})$.
2. *If* $\mathfrak{z}(\mathfrak{i}) \subseteq \mathfrak{j} \subseteq \mathfrak{i}$ *then* $\mathfrak{i}^\perp \cap \mathfrak{i} \;\subseteq\; \mathfrak{j}^\perp \cap \mathfrak{j}$.

Proof. By Lemma 6.19 (1), $[\mathfrak{j}, \mathfrak{j}^\perp \cap \mathfrak{g}_{\mathrm{s}}]$ is an ideal of $\mathfrak{g}$ and contained in $\mathfrak{g}^\perp$. Since $\mathfrak{g}^\perp$ does not contain any non-trivial ideal of $\mathfrak{g}$, $[\mathfrak{j}, \mathfrak{j}^\perp \cap \mathfrak{g}_{\mathrm{s}}] = \mathbf{0}$. Hence, (1) holds. Under the assumption of (2), this means $\mathfrak{i}^\perp \cap \mathfrak{i} \subseteq \mathfrak{z}(\mathfrak{i}) \subseteq \mathfrak{j}$. Since also $\mathfrak{i}^\perp \subseteq \mathfrak{j}^\perp$, (2) follows. □

The next result somewhat strengthens Corollary 6.7.

Proposition 6.21. *The following hold:*

1. $[\mathfrak{r}, [\mathfrak{g},\mathfrak{r}]^\perp \cap \mathfrak{g}_{\mathrm{s}}] = \mathbf{0}$.
2. $[\mathfrak{r}, \mathfrak{n}^\perp \cap \mathfrak{g}_{\mathrm{s}}] = \mathbf{0}$. *In particular,* $\mathfrak{n}^\perp \cap \mathfrak{r} \subseteq \mathfrak{z}(\mathfrak{r})$.
3. $[\mathfrak{g}_{\mathrm{s}}, (\mathfrak{s}+\mathfrak{n})^\perp \cap \mathfrak{g}_{\mathrm{s}}] = \mathbf{0}$. *In particular,* $(\mathfrak{s}+\mathfrak{n})^\perp \cap \mathfrak{g}_{\mathrm{s}} \subseteq \mathfrak{z}(\mathfrak{g}_{\mathrm{s}})$.

Proof. By Lemma 6.19 (1), $\mathfrak{j}^\perp \cap \mathfrak{g}_{\mathrm{s}}$ is an ideal of $\mathfrak{g}$ for any ideal $\mathfrak{j}$ of $\mathfrak{g}$ contained in $\mathfrak{g}_{\mathrm{s}}$. Then for any ideal $\mathfrak{i}$ of $\mathfrak{g}$, $[\mathfrak{i}, \mathfrak{j}^\perp \cap \mathfrak{g}_{\mathrm{s}}]$ is also an ideal in $\mathfrak{g}$. Therefore, if $[\mathfrak{i}, \mathfrak{j}^\perp \cap \mathfrak{g}_{\mathrm{s}}] \subseteq \mathfrak{g}^\perp$, then $[\mathfrak{i}, \mathfrak{j}^\perp \cap \mathfrak{g}_{\mathrm{s}}] = \mathbf{0}$. In the view of Lemma 6.18, (1), (2), (3) follow. □

We can deduce from (2) of Proposition 6.21 the equalities

$$\mathfrak{n}^\perp \cap \mathfrak{r} = \mathfrak{n}^\perp \cap \mathfrak{n} = \mathfrak{n}^\perp \cap \mathfrak{z}(\mathfrak{n}) = \mathfrak{n}^\perp \cap \mathfrak{z}(\mathfrak{r}), \tag{6.5}$$

$$\mathfrak{r}^\perp \cap \mathfrak{r} = \mathfrak{r}^\perp \cap \mathfrak{n} = \mathfrak{r}^\perp \cap \mathfrak{z}(\mathfrak{n}) = \mathfrak{r}^\perp \cap \mathfrak{z}(\mathfrak{r}). \tag{6.6}$$

Also (3) of Proposition 6.21 shows that

$$\mathfrak{g}_{\mathrm{s}}^\perp \cap \mathfrak{g}_{\mathrm{s}} \subseteq \mathfrak{z}(\mathfrak{g}_{\mathrm{s}}) \subseteq \mathfrak{z}(\mathfrak{r}), \tag{6.7}$$

Moreover, using nil-invariance of $\langle\cdot,\cdot\rangle$ and Corollary 5.5 (1), the above yield

$$[\mathfrak{g}, \mathfrak{n}^\perp \cap \mathfrak{n}] \subseteq \mathfrak{r}^\perp \cap \mathfrak{r},\ \ [\mathfrak{s}, \mathfrak{n}^\perp \cap \mathfrak{n}] \subseteq \mathfrak{r}^\perp \cap \mathfrak{r} \cap \mathfrak{k}^\perp \ \text{ and }\ [\mathfrak{k}+\mathfrak{r}, \mathfrak{n}^\perp \cap \mathfrak{n}] \subseteq \mathfrak{g}_{\mathrm{s}}^\perp \cap \mathfrak{r}. \tag{6.8}$$

Thus there is a tower of totally isotropic ideals of $\mathfrak{g}$ contained in $\mathfrak{z}(\mathfrak{r})$:

$$\mathfrak{g}_{\mathrm{s}}^\perp \cap \mathfrak{g}_{\mathrm{s}} \;\subseteq\; \mathfrak{r}^\perp \cap \mathfrak{r} \;\subseteq\; \mathfrak{n}^\perp \cap \mathfrak{n}. \tag{6.9}$$

6.5. **Actions of semisimple subalgebras on the solvable radical.** Let $\mathfrak{q}$ be a subalgebra of $\mathfrak{g}$. We call the subspace $W \subseteq \mathfrak{g}$ a submodule for $\mathfrak{q}$ if $[\mathfrak{q}, W] \subseteq W$. In the following we let $\mathfrak{f} \subseteq \mathfrak{g}$ denote a semisimple subalgebra of $\mathfrak{g}$. As usual, we decompose $\mathfrak{f} = \mathfrak{k} \times \mathfrak{s}$, where $\mathfrak{k}$ is an ideal of compact type and $\mathfrak{s}$ has no factor of compact type.

Lemma 6.22. *Let* $W \subseteq \mathfrak{g}$ *be a submodule for* $\mathfrak{f}$, *with* $\dim[\mathfrak{f}, W] \leq 2$. *Then:*

$$\mathfrak{f} \perp [\mathfrak{f}, W],\ \ [\mathfrak{k}, W] = \mathbf{0},\ \ \mathfrak{s} \perp W.$$

Proof. Assume first that W is not a trivial module. Thus $\dim[\mathfrak{f}, W] = 2$ and $\mathfrak{f} = \mathfrak{f}_0 \times \mathfrak{sl}_2(\mathbb{R})$, where $\mathfrak{f}_0$ is the kernel of the representation of $\mathfrak{f}$ on W. As $W \subseteq \mathfrak{g}^{\mathfrak{f}_0}$, Lemma 5.4 states that $[W, \mathfrak{g}_{\mathrm{s}}] \perp \mathfrak{f}_0$. Clearly,

$$[W, \mathfrak{f}] = [W, \mathfrak{sl}_2(\mathbb{R})] \subseteq [W, \mathfrak{g}_{\mathrm{s}}].$$

Therefore, $\mathfrak{f}_0 \perp [W, \mathfrak{f}]$ and $[W, \mathfrak{f}] \subseteq \mathfrak{g}_{\mathrm{s}}$.

Since $\mathfrak{g}_{\mathrm{s}} \subseteq \mathrm{inv}(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ (part (2) of Theorem A), $\langle\cdot,\cdot\rangle$ induces a skew pairing $\mathfrak{sl}_2(\mathbb{R}) \times [W, \mathfrak{f}] \to \mathbb{R}$ for the module $[W, \mathfrak{f}]$. Since $[W, \mathfrak{f}]$ is of dimension two and non-trivial, Proposition A.4 shows that $\mathfrak{sl}_2(\mathbb{R}) \perp [W, \mathfrak{f}]$. This now implies $\mathfrak{f} \perp [\mathfrak{f}, W]$ and $[\mathfrak{k}, W] = \mathbf{0}$, as $\mathfrak{k} \subseteq \mathfrak{f}_0$. Since $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}] \subseteq \mathrm{inv}(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, $[\mathfrak{s}, W] \perp \mathfrak{s}$ implies that $W \perp \mathfrak{s}$. □

Lemma 6.23. *Let $\mathfrak{q}$ be a subalgebra of $\mathfrak{g}$, and let $W \subseteq \mathfrak{r}$ be a submodule for $\mathfrak{q}$. Then the following hold for $\mathfrak{l} = \mathfrak{q} \cap [W, W]^{\perp}$:*

(1) *$\mathfrak{l}$ is a subalgebra, and $[\mathfrak{l}, W] \subseteq W^{\perp}$.*

Assume further that $\mathfrak{q}$ acts reductively on W. Then:

(2) *$\mathfrak{l} = \mathfrak{q} \cap [W_1, W_1]^{\perp}$, where $W_1 = [\mathfrak{q}, W]$.*
(3) *$\mathfrak{l}$ is an ideal in $\mathfrak{q}$.*

If $\mathfrak{q} = \mathfrak{f}$ is semisimple and $\dim[W_1, W_1] \leq 2$ *then:*

(4) *$\mathfrak{l} = \mathfrak{f}$ and $\mathfrak{f} \perp [W, W]$.*
(5) *$[\mathfrak{f}, W]$ is totally isotropic.*

Proof. Observe that for any $u, v \in W$, $K \in \mathfrak{g}$, $\langle K, [u, v]\rangle = \langle [K, u], v\rangle$. In particular, $K \perp [W, W]$ is equivalent to $[K, W] \perp W$. To finish the proof of (1), assume that $K_1, K_2 \perp [W, W]$, where $K_1, K_2 \in \mathfrak{q}$. Then also $\langle [[K_1, K_2], u], v\rangle = \langle [[K_1, u], K_2], v\rangle + \langle [[u, K_2], K_1], v\rangle = 0$. Hence, $[K_1, K_2] \perp [W, W]$. This shows that $\mathfrak{l}$ is a subalgebra.

Next we show (2). Since $\mathfrak{q}$ acts reductively on W, $W = W_0 \oplus W_1$, with $W_1 = [\mathfrak{q}, W]$ and $[\mathfrak{q}, W_0] = \mathbf{0}$. For any $u, v \in W$, decompose $u = u_0 + u_1$, $v = v_0 + v_1$, where $u_i, v_i \in W_i$. Then compute $\langle K, [u, v]\rangle = \langle K, [u_1, v_1]\rangle$.

Finally, if $\mathfrak{q}$ acts reductively, there is a decomposition into submodules $W = (W \cap W^{\perp}) \oplus W'$. Correspondingly, $K \in \mathfrak{l}$ if and only if $[K, W'] = \mathbf{0}$. This shows that $\mathfrak{l}$ is an ideal in $\mathfrak{q}$. Hence, (3) holds.

If $\mathfrak{f}$ is semisimple, then $\mathfrak{f}$ acts reductively on W. By part (3), $\mathfrak{l} = \mathfrak{f} \cap [W, W]^{\perp}$ is an ideal of $\mathfrak{f}$. Since $\dim[W, W] \leq 2$ it is an ideal of codimension at most two. Since $\mathfrak{f}$ is semisimple this implies $\mathfrak{l} = \mathfrak{f}$. Hence, (4) holds. Now (4) together with (1) implies that $[\mathfrak{f}, W] \subseteq W^{\perp}$ is totally isotropic. □

For any subspace W of $\mathfrak{g}$, recall that $\mu(W)$ denotes the index of W.

Lemma 6.24. *Let $W \subseteq \mathfrak{r}$ be a submodule for $\mathfrak{f}$, such that* $\dim[W, W] \leq 2$. *Then:*

(1) *$[\mathfrak{f}, W] \subseteq W^{\perp}$ is totally isotropic.*
(2) *If $\mu(\mathfrak{g}_{\mathrm{s}}) \leq 2$ then $[\mathfrak{f}, W] = \mathbf{0}$.*

Proof. By Lemma 6.23 part (5), $[\mathfrak{f}, W] \subseteq W^{\perp}$ is totally isotropic. In particular, assuming $\mu(\mathfrak{g}_{\mathrm{s}}) \leq 2$, $\dim[\mathfrak{f}, W] \leq 2$. Then Lemma 6.22 implies $[\mathfrak{k}, W] = \mathbf{0}$, $\mathfrak{s} \perp W$. Assuming $[\mathfrak{f}, W] \neq \mathbf{0}$, $\dim[\mathfrak{f}, W] = 2$ and $\mathfrak{s}$ contains $\mathfrak{sl}_2(\mathbb{R})$, so that $\mu(\mathfrak{s}) \geq \mu(\mathfrak{sl}_2(\mathbb{R})) \geq 1$. We get $2 = \mu([\mathfrak{f}, W]) \leq \mu(\mathfrak{g}_{\mathrm{s}}) - 1 \leq 1$. Thus (2) follows. □

We are ready to give the main result of this subsection.

Proposition 6.25. *If* $\mu(\mathfrak{g}_{\mathrm{s}}) \leq 2$ *then* $[\mathfrak{k} \times \mathfrak{s}, \mathfrak{r}] = \mathbf{0}$.

Proof. We have $\mu(\mathfrak{r}) \leq \mu(\mathfrak{g}_{\mathrm{s}}) \leq 2$. Thus Proposition 4.7 implies that there exists an ideal $\mathfrak{q}$ of $\mathfrak{g}$ with $\dim[\mathfrak{q}, \mathfrak{q}] \leq 2$, and the codimension of $\mathfrak{q}$ in $\mathfrak{r}$ is at most two. Since $\mu(\mathfrak{g}_{\mathrm{s}}) \leq 2$, $[\mathfrak{k} \times \mathfrak{s}, \mathfrak{r}] = \mathbf{0}$, by Lemma 6.24. This also implies $\mathfrak{s} \perp \mathfrak{r}$ (compare Lemma 6.22). □

As a consequence we further get:

Lemma 6.26. *Suppose* $\mu(\mathfrak{g}_{\mathrm{s}}) \leq 2$. *Then the following hold:*

(1) $\mathfrak{s}$ *is non-degenerate.*
(2) $\mathfrak{s} \perp (\mathfrak{k} + \mathfrak{r})$ *and* $\mathfrak{k} \perp [\mathfrak{r}, \mathfrak{r}]$.
(3) $\mu(\mathfrak{r}) + \mu(\mathfrak{s}) \leq \mu(\mathfrak{g}_{\mathrm{s}})$.

Proof. Note that $\dim \mathfrak{s} \cap \mathfrak{s}^{\perp} \leq \mu(\mathfrak{g}_{\mathrm{s}}) \leq 2$. Since $\langle \cdot, \cdot \rangle_{\mathfrak{s}}$ is invariant, $\mathfrak{s} \cap \mathfrak{s}^{\perp}$ is an ideal in $\mathfrak{s}$. We conclude that $\mathfrak{s} \cap \mathfrak{s}^{\perp} = \mathbf{0}$. This shows (1).

Since $\langle \cdot, \cdot \rangle$ is invariant by $\mathfrak{r}$ and $\mathfrak{s}$, $[\mathfrak{k} \times \mathfrak{s}, \mathfrak{r}] = \mathbf{0}$ implies $\mathfrak{k} \perp [\mathfrak{g}, \mathfrak{g}_{\mathrm{s}}] = \mathfrak{s} + [\mathfrak{r}, \mathfrak{r}]$ and $\mathfrak{s} \perp \mathfrak{r}$. Hence, (2) and (3) hold. □

7. Lie algebras with nil-invariant scalar products of small index

Partially summarizing the results from Proposition 6.25 and Corollary 6.17 we obtain a first structure theorem for metric Lie algebras of relative index $\ell \leq 2$.

Theorem D. *Let* $\mathfrak{g}$ *be a real finite-dimensional Lie algebra with nil-invariant symmetric bilinear form* $\langle \cdot, \cdot \rangle$ *of relative index* $\ell \leq 2$, *and assume that* $\mathfrak{g}^{\perp}$ *does not contain a non-trivial ideal of* $\mathfrak{g}$. *Then:*

(1) *The Levi decomposition* (5.1) *of* $\mathfrak{g}$ *is a direct sum of ideals:* $\mathfrak{g} = \mathfrak{k} \times \mathfrak{s} \times \mathfrak{r}$.
(2) $\mathfrak{g}^{\perp}$ *is contained in* $\mathfrak{k} \times \mathfrak{z}(\mathfrak{r})$ *and* $\mathfrak{g}^{\perp} \cap \mathfrak{r} = \mathbf{0}$.
(3) $\mathfrak{s} \perp (\mathfrak{k} \times \mathfrak{r})$ *and* $\mathfrak{k} \perp [\mathfrak{r}, \mathfrak{r}]$.

We will now study the cases $\ell = 0$, $\ell = 1$ and $\ell = 2$ individually.

7.1. Semidefinite nil-invariant products.

Let $\langle \cdot, \cdot \rangle$ be a nil-invariant symmetric bilinear form on $\mathfrak{g}$.

Proposition 7.1. *If* $\langle \cdot, \cdot \rangle$ *is semidefinite (the case* $\ell = 0$*), then*

(1) $[\mathfrak{g}, \mathfrak{s} + \mathfrak{r}] \subseteq \mathfrak{g}^{\perp}$.

Moreover, if $\mathfrak{g}^{\perp}$ *does not contain any non-trivial ideal of* $\mathfrak{g}$, *then:*

(2) $\mathfrak{g} = \mathfrak{k} \times \mathfrak{r}$ *and* $\mathfrak{r}$ *is abelian.*
(3) *The ideal* $\mathfrak{r}$ *is definite.*

Proof. According to Theorem A, nil-invariance implies that $\mathfrak{g}_{\mathrm{s}}$ acts by skew derivations on $\mathfrak{g}$ and on $\mathfrak{g}/\mathfrak{g}^{\perp}$. By assumption, $\langle \cdot, \cdot \rangle$ induces a definite scalar product on the vector space $\mathfrak{g}/\mathfrak{g}^{\perp}$. Recall that a definite scalar product does not allow nilpotent skew maps. Therefore, $[\mathfrak{s} + \mathfrak{n}, \mathfrak{g}] \subseteq \mathfrak{g}^{\perp}$. Similarly, for $X \in \mathfrak{r}$, $\mathrm{ad}(X)_{\mathrm{n}}(\mathfrak{g}) \subseteq \mathfrak{g}^{\perp}$ and thus also $[\mathfrak{r}, \mathfrak{r}] \subseteq [\mathfrak{r}, \mathfrak{n}] + \mathfrak{g}^{\perp} \subseteq \mathfrak{g}^{\perp}$. Moreover, $[\mathfrak{r}, \mathfrak{k} \times \mathfrak{s}] = [\mathfrak{n}, \mathfrak{k} \times \mathfrak{s}] \subseteq \mathfrak{g}^{\perp}$. This shows (1), while (2) and (3) follow immediately, taking into account Theorem D. □

7.2. **Classification for relative index $\ell \leq 2$.** Now we specialize Theorem D to the two cases $\ell = 1$ and $\ell = 2$ to obtain classifications of the Lie algebras with nil-invariant symmetric bilinear forms in each case.

Theorem E. *Let $\mathfrak{g}$ be a Lie algebra with nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ of relative index $\ell = 1$, and assume that $\mathfrak{g}^\perp$ does not contain a non-trivial ideal of $\mathfrak{g}$. Then one of the following cases occurs:*

(I) $\mathfrak{g} = \mathfrak{k} \times \mathfrak{a}$, *where $\mathfrak{a}$ is abelian and either semidefinite or Lorentzian.*
(II) $\mathfrak{g} = \mathfrak{k} \times \mathfrak{r}$, *where $\mathfrak{r}$ is Lorentzian of oscillator type.*
(III) $\mathfrak{g} = \mathfrak{k} \times \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{a}$, *where $\mathfrak{a}$ is abelian and definite, $\mathfrak{sl}_2(\mathbb{R})$ is Lorentzian and $(\mathfrak{a} \times \mathfrak{k}) \perp \mathfrak{sl}_2(\mathbb{R})$.*

Proof. By Theorem D, $\mathfrak{g}^\perp \cap \mathfrak{r} = \mathbf{0}$. Hence, $\mathfrak{r}$ is a subspace of index $\mu(\mathfrak{r}) \leq 1$.

If $\mathfrak{s} \neq \mathbf{0}$, then by Lemma 6.26, $\mathfrak{s}$ is non-degenerate, so that $\ell(\mathfrak{s}) = 1$, as $\mathfrak{s}$ is of non-compact type. Hence, $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{R})$. Moreover, $\ell(\mathfrak{r}) = 0$, that is, $\mathfrak{r}$ is definite and therefore abelian. The orthogonality is given by Lemma 6.26. This is case (III).

Otherwise, $\mathfrak{s} = \mathbf{0}$. If $\mathfrak{r}$ is semidefinite, then $[\mathfrak{r},\mathfrak{r}] \subseteq \mathfrak{r}^\perp \cap \mathfrak{r}$ by Proposition 7.1. By Lemma 6.26 (2), this implies $[\mathfrak{r},\mathfrak{r}] \subseteq \mathfrak{g}^\perp \cap \mathfrak{r} = \mathbf{0}$. Hence $\mathfrak{r}$ is abelian. This is the first part of case (I). Assume $\mathfrak{r}$ is of Lorentzian type. Then $\mathfrak{r}$ is non-degenerate since $\mu(\mathfrak{r}) \leq 1$. By the classification of invariant Lorentzian scalar products (see remark following Example 3.7), $\mathfrak{r}$ is either abelian or contains a metric oscillator algebra. These are the second part of case (I) or case (II), respectively. □

Theorem F. *Let $\mathfrak{g}$ be a Lie algebra with nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ of relative index $\ell = 2$, and assume that $\mathfrak{g}^\perp$ does not contain a non-trivial ideal of $\mathfrak{g}$. Then one of the following cases occurs:*

(I) $\mathfrak{g} = \mathfrak{r} \times \mathfrak{k}$, *where $\mathfrak{r}$ is one of the following:*
  (a) $\mathfrak{r}$ *is abelian.*
  (b) $\mathfrak{r}$ *is Lorentzian of oscillator type.*
  (c) $\mathfrak{r}$ *is solvable but non-abelian with invariant scalar product of index* 2.
(II) $\mathfrak{g} = \mathfrak{a} \times \mathfrak{k} \times \mathfrak{s}$. *Here, $\mathfrak{a}$ is abelian, $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})$ with a non-degenerate invariant scalar product of index 2. Moreover, $\mathfrak{a}$ is definite and $(\mathfrak{a} \times \mathfrak{k}) \perp \mathfrak{s}$.*
(III) $\mathfrak{g} = \mathfrak{r} \times \mathfrak{k} \times \mathfrak{sl}_2(\mathbb{R})$, *where $\mathfrak{sl}_2(\mathbb{R})$ is Lorentzian, $(\mathfrak{r} \times \mathfrak{k}) \perp \mathfrak{sl}_2(\mathbb{R})$, and $\mathfrak{r}$ is one of the following:*
  (a) $\mathfrak{r}$ *is abelian and either semidefinite or Lorentzian.*
  (b) $\mathfrak{r}$ *is Lorentzian of oscillator type.*

Proof. Write $s = \mu(\mathfrak{s})$ and $r = \mu(\mathfrak{r})$. By Theorem D, $\mathfrak{g}^\perp \cap \mathfrak{g}_{\mathrm{s}} = \mathbf{0}$. By Lemma 6.26, this implies $s + r \leq 2$. Moreover, $\mathfrak{s}$ is non-degenerate and thus has index $s \leq \ell \leq 2$.

First assume $s = 0$, and therefore $\mathfrak{s} = \mathbf{0}$, and $r \leq 2$. For $r = 2$, the following possibilities arise: $\mathfrak{r}$ is non-degenerate with relative index $\ell(\mathfrak{r}) = 2$. This case falls into (I-a) or (I-c). Next, $\mathfrak{r}$ can be degenerate with $\ell(\mathfrak{r}) = 1$, in which case it is either abelian or of oscillator type, the latter yielding part (I-b). In the remaining case $\ell(\mathfrak{r}) = 0$, $\mathfrak{r}$ is semidefinite. As in the proof of Theorem E, this implies that $\mathfrak{r} = \mathfrak{a}$ is abelian. This completes case (I).

Assume $s = 2$ and $r = 0$. Then $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{r}$ is definite and abelian. The orthogonality is Lemma 6.26 (3). This is case (II)

Now assume $s = 1$ and $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{R})$, $r \leq 1$. This yields the two possibilities for $\mathfrak{r}$ in case (III). □

Note that the possible Lie algebras $\mathfrak{r}$ for case (I-c) of Theorem F above are discussed in Section 4.1.

8. Further examples

The examples in this section show that the properties of nil-invariant symmetric bilinear forms with relative index $\ell \leq 2$ given in Theorem D do not hold for higher relative indices. Let $\mathfrak{k}$, $\mathfrak{s}$, $\mathfrak{r}$ and $\mathfrak{g}_{\mathrm{s}}$ be as in the previous sections.

The following standard construction for a Lie algebra with an invariant scalar product (cf. Medina [9]) shows that in general $\mathfrak{g}$ does not have to be a direct product of Lie algebras $\mathfrak{k}$, $\mathfrak{s}$ and $\mathfrak{r}$, and that $\mathfrak{s}$ does not have to be orthogonal to $\mathfrak{r}$.

Example 8.1. Let $\mathfrak{g}$ be a Lie algebra of dimension n, and let ad^* denote the coadjoint representation of $\mathfrak{g}$ on its dual vector space $\mathfrak{g}^*$, and consider the Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*$. The dual pairing defines an invariant scalar product on $\widehat{\mathfrak{g}}$,

$$\langle X_1 + \xi_1, X_2 + \xi_2 \rangle = \xi_1(X_2) + \xi_2(X_1),$$

where $X_i \in \mathfrak{g}$ and $\xi_i \in \mathfrak{g}^*$. The index of $\langle\cdot,\cdot\rangle$ is n. For example, if we choose $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, then the index is 3, and $\mathfrak{k} = \mathbf{0}$, $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{R})$ and $\widehat{\mathfrak{g}}$ has abelian radical $\mathfrak{r} = \mathfrak{sl}_2(\mathbb{R})^* \cong \mathbb{R}^3$. In particular, $\mathfrak{s}$ is not orthogonal to $\mathfrak{r}$ and $[\mathfrak{s}, \mathfrak{r}] \neq \mathbf{0}$.

The next example shows that for relative index $\ell = 3$, the transporter algebra $\mathfrak{l}$ of $\mathfrak{b} = \mathfrak{r}^\perp \cap \mathfrak{r}$ in $\mathfrak{q} = \mathfrak{k}$ (see Section 6.3.1) can be trivial, and as a consequence $\mathfrak{g}^\perp \cap \mathfrak{r}$ is not an ideal in $\mathfrak{g}$. This contrasts the situation for $\ell \leq 2$, compare Corollary 6.15.

Example 8.2. Let $\mathfrak{k} = \mathfrak{so}_3$, and let $\mathfrak{r} = \mathfrak{so}_3 \oplus \mathfrak{so}_3$, considered as a vector space. We write $\mathfrak{so}_3^{\mathrm{L}}$ and $\mathfrak{so}_3^{\mathrm{R}}$ to distinguish the two summands of $\mathfrak{r}$, and for an element $X \in \mathfrak{so}_3$, we write $X^{\mathrm{L}} = (X, 0) \in \mathfrak{so}_3^{\mathrm{L}}$ and $X^{\mathrm{R}} = (0, X) \in \mathfrak{so}_3^{\mathrm{R}}$. Let $\mathfrak{so}_3^{\triangle}$ be the diagonal embedding of $\mathfrak{so}_3$ in $\mathfrak{r}$.

Let $T \in \mathfrak{k}$ act on $X = X_1^{\mathrm{L}} + X_2^{\mathrm{R}} \in \mathfrak{r}$ by

$$\mathrm{ad}(T)X = [T, X_1]^{\mathrm{L}}. \tag{$*$}$$

This makes $\mathfrak{r}$ into a Lie algebra module for $\mathfrak{k}$, and we can thus define a Lie algebra $\mathfrak{g} = \mathfrak{k} \ltimes \mathfrak{r}$ for this action, taking $\mathfrak{r}$ as an abelian subalgebra. Observe also that $\mathfrak{so}_3^{\mathrm{R}}$ is the center of $\mathfrak{g}$.

Let κ denote the Killing form on $\mathfrak{so}_3$. We define a symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ by requiring

$$\langle T, X_1^{\mathrm{L}} + X_2^{\mathrm{R}} \rangle = \kappa(T, X_1) - \kappa(T, X_2), \quad \mathfrak{k} \perp \mathfrak{k}, \quad \mathfrak{r} \perp \mathfrak{r}$$

for all $T \in \mathfrak{k}$, $X_1^{\mathrm{L}} + X_2^{\mathrm{R}} \in \mathfrak{so}_3$. The adjoint operators of elements of $\mathfrak{r}$ are skew-symmetric for $\langle\cdot,\cdot\rangle$. In fact, we have, for all $X, Y \in \mathfrak{r}$, $Z \in \mathfrak{g}$,

$$\langle [X, Y], Z \rangle = 0 = -\langle Y, [X, Z] \rangle$$

and for $T, T' \in \mathfrak{k}$, by $(*)$,

$$\begin{aligned} \langle [T, X], T' \rangle &= \langle [T, X_1^{\mathrm{L}} + X_2^{\mathrm{R}}], T' \rangle = \langle [T, X_1]^{\mathrm{L}}, T' \rangle \\ &= \kappa([T, X_1], T') = -\kappa(T, [T', X_1]) \\ &= -\langle T, [T', X_1]^{\mathrm{L}} \rangle = -\langle T, [T', X] \rangle. \end{aligned}$$

So $\langle\cdot,\cdot\rangle$ is indeed a nil-invariant form on $\mathfrak{g}$, and, since $\mathfrak{r}^\perp = \mathfrak{r}$,

$$\mathfrak{g}^\perp = \mathfrak{k}^\perp \cap \mathfrak{r} = \mathfrak{so}_3^{\triangle} \quad \text{and} \quad \mathfrak{g}_{\mathrm{s}}^\perp \cap \mathfrak{g}_{\mathrm{s}} = \mathfrak{r}^\perp \cap \mathfrak{r} = \mathfrak{r} = \mathfrak{so}_3^{\mathrm{L}} \oplus \mathfrak{so}_3^{\mathrm{R}}.$$

In particular, the index of $\langle\cdot,\cdot\rangle$ is $\mu = 6$ and the relative index is $\ell = 3$. Note that $\langle\cdot,\cdot\rangle$ is not invariant, as $\mathfrak{g}^\perp$ is not an ideal in $\mathfrak{g}$.

Remark. The construction in Example 8.2 works if we replace $\mathfrak{so}_3$ by any other semisimple Lie algebra $\mathfrak{f} = \mathfrak{k}$ of compact type. However, if $\mathfrak{f}$ is not of compact type, then the resulting bilinear form $\langle\cdot,\cdot\rangle$ will not be nil-invariant. Geometrically this means that $\langle\cdot,\cdot\rangle$ cannot come from a pseudo-Riemannian metric on a homogeneous space G/H of finite volume, where G is a Lie group with Lie algebra $\mathfrak{g}$.

Appendix A. Modules with skew pairings

Let $\mathfrak{g}$ be a Lie algebra and let V be a finite-dimensional $\mathfrak{g}$-module. Here we work over a fixed ground field $\Bbbk$ of characteristic 0.

Definition A.1. A bilinear map $\langle\cdot,\cdot\rangle : \mathfrak{g} \times V \to \Bbbk$ such that for all $X, Y \in \mathfrak{g}$, $v \in V$,

$$\langle X, Yv\rangle = -\langle Y, Xv\rangle \tag{A.1}$$

is called a *skew pairing* for V, and V is called a *skew module* for $\mathfrak{g}$.

We make the following elementary observations:

Lemma A.2. *Assume that there exists $X \in \mathfrak{g}$ such that the map $v \mapsto Xv$, $v \in V$, is an invertible linear operator of V. Then every skew pairing for V is zero. More generally, let $X, Y \in \mathfrak{g}$ and $W \subseteq V$ such that $YW \subseteq XV$. Then $Y \perp XW$.*

Proof. Let $w \in W$ and $v \in V$ with $Yw = Xv$. Then

$$\langle Y, Xw\rangle = -\langle X, Yw\rangle = -\langle X, Xv\rangle = 0. \qquad \square$$

Lemma A.3. *If $X \perp V$ then $\mathfrak{g} \perp XV$.*

Proof. Let $Y \in \mathfrak{g}$ and $v \in V$. Then $\langle Y, Xv\rangle = -\langle X, Yv\rangle = 0$. $\square$

A.1. **Skew pairings for $\mathfrak{sl}_2(\Bbbk)$.** The following determines all skew pairings for the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\Bbbk)$.

Proposition A.4. *Let $\langle\cdot,\cdot\rangle : \mathfrak{sl}_2(\Bbbk) \times V \to \Bbbk$ be a skew pairing for the (non-trivial) irreducible module V. If the skew pairing is non-zero, then V is isomorphic to the adjoint representation of $\mathfrak{sl}_2(\Bbbk)$ and $\langle\cdot,\cdot\rangle$ is proportional to the Killing form.*

Proof. We choose a standard basis X, Y, H for $\mathfrak{sl}_2(\Bbbk)$ such that $[X, Y] = H$, $[H, X] = 2X$, $[H, Y] = -2Y$. Let $V = \Bbbk^2$ denote the standard representation. Let e_1, e_2 be a basis such that $Xe_1 = 0$, $Xe_2 = e_1$. Since $He_1 = e_1$ and $He_2 = -e_2$, the operator defined by H is invertible. Hence, it follows that every skew pairing for V is zero by Lemma A.2.

The irreducible modules for $\mathfrak{sl}_2(\Bbbk)$ are precisely the symmetric powers $V_k = \mathrm{S}^k V$, $k \geq 1$. Note that, in V_k, $\operatorname{im} X$ is spanned by the product vectors $e_1^\ell e_2^{k-\ell}$, $k \geq \ell \geq 1$. Similarly, $\operatorname{im} Y$ is spanned by $e_1^{k-\ell} e_2^\ell$, $k \geq \ell \geq 1$.

Consider W, the subspace of $\operatorname{im} Y$ spanned by $e_1^{k-\ell} e_2^\ell$, $\ell \geq 2$. Now $XW \subseteq \operatorname{im} Y$ and from Lemma A.2 we can conclude that $X \perp YW$. Observe that YW is spanned by $e_1^{k-\ell-1} e_2^{\ell+1}$, $\ell \geq 2$, $k \geq \ell + 1$. In particular, $X \perp e_2^k$ if $k \geq 3$. Since also $X \perp \operatorname{im} X$, this shows $X \perp V_k$, $k \geq 3$. By symmetry, we also see that $Y \perp V_k$, $k \geq 3$. Since $\operatorname{im} X$, $\operatorname{im} Y$ together span V_k, we conclude (using Lemma A.3) that $\mathfrak{sl}_2(\Bbbk) \perp V_k$, $k \geq 3$.

Finally, the module V_2 is isomorphic to the adjoint representation. Consider the Killing form $\kappa\colon \mathfrak{sl}_2(\Bbbk) \times \mathfrak{sl}_2(\Bbbk) \to \Bbbk$. Recall that κ is symmetric and skew with respect to the adjoint representation of $\mathfrak{sl}_2(\Bbbk)$ on itself. Therefore, it also defines a skew pairing for V_2. An evident computation using the skew-condition on commutators in $\mathfrak{sl}_2(\Bbbk)$ shows that every skew form $\langle\cdot,\cdot\rangle$ for the adjoint representation is determined by its value $\langle H, H\rangle$. Hence, it must be proportional to the Killing form. □

A.2. **Application to $\mathfrak{so}_3$ over the reals.** Here we consider the simple Lie algebra $\mathfrak{so}_3$ over the real numbers. Since $\mathfrak{so}_3$ has complexification $\mathfrak{sl}_2(\mathbb{C})$, we can apply Proposition A.4 to show:

Proposition A.5. *Let $\langle\cdot,\cdot\rangle : \mathfrak{so}_3 \times V \to \mathbb{R}$ be a skew pairing for the (non-trivial) irreducible module V. If the skew pairing is non-zero, then V is isomorphic to the adjoint representation of $\mathfrak{so}_3$ and $\langle\cdot,\cdot\rangle$ is proportional to the Killing form.*

Proof. Using the isomorphism of $\mathfrak{so}_3$ with the Lie algebra $\mathfrak{su}_2$, we view $\mathfrak{so}_3$ as a subalgebra of $\mathfrak{sl}_2(\mathbb{C})$. We thus see that the irreducible complex representations of $\mathfrak{so}_3$ are precisely the $\mathfrak{su}_2$-modules $\mathrm{S}^k\mathbb{C}^2$.

Now let V be a real module for $\mathfrak{so}_3$, which is irreducible and non-trivial, and assume that $\langle\cdot,\cdot\rangle$ is a non-trivial skew pairing for V. We may extend V to a complex linear skew pairing $\langle\cdot,\cdot\rangle_{\mathbb{C}} : \mathfrak{sl}_2(\mathbb{C}) \times V_{\mathbb{C}} \to \mathbb{C}$, where $V_{\mathbb{C}}$ denotes complexification of the $\mathfrak{su}_2$-module V.

In case $V_{\mathbb{C}}$ is an irreducible module for $\mathfrak{sl}_2(\mathbb{C})$, Proposition A.4 shows that $V_{\mathbb{C}} = V_2$ is the adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$. Hence, V must have been the adjoint representation of $\mathfrak{so}_3$.

Otherwise, if $V_{\mathbb{C}}$ is reducible, V is one of the modules $V_k = \mathrm{S}^{2\ell-1}\mathbb{C}^2$ with scalars restricted to the reals (cf. Bröcker and tom Dieck [4, Proposition 6.6]). It also follows that $V_{\mathbb{C}}$ is isomorphic to a direct sum of $\mathrm{S}^{2\ell-1}\mathbb{C}^2$ with itself. Since we assume that the skew pairing $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ for $V_{\mathbb{C}}$ is non-trivial, Proposition A.4 implies that one of the irreducible summands of $V_{\mathbb{C}}$ is isomorphic to $\mathrm{S}^2\mathbb{C}^2$. This is impossible, since $k = 2\ell - 1$ is odd. □

The Killing form is always a non-degenerate pairing. In the light of the previous two propositions, this give us:

Corollary A.6. *Let $\langle\cdot,\cdot\rangle : \mathfrak{g} \times V \to \Bbbk$ be a skew pairing, where either $\mathfrak{g} = \mathfrak{sl}_2(\Bbbk)$ or $\mathfrak{g} = \mathfrak{so}_3$ and $\Bbbk = \mathbb{R}$. Assume further that $V^{\mathfrak{g}} = \{v \in V \mid \mathfrak{g}v = \mathbf{0}\} = \mathbf{0}$. Define*

$$V^{\perp} = \{X \in \mathfrak{g} \mid \langle X, V\rangle = \mathbf{0}\}\ .$$

Then either $V^{\perp} = \mathbf{0}$ or $V^{\perp} = \mathfrak{g}$.

Proof. The first case occurs precisely if there exists an irreducible summand W of V on which the restricted skew pairing $\mathfrak{g} \times W \to \Bbbk$ induces the Killing form. □

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