

**ISOMETRY GROUPS AND GEODESIC FOLIATIONS OF
LORENTZ MANIFOLDS.
PART II: GEOMETRY OF ANALYTIC LORENTZ
MANIFOLDS WITH LARGE ISOMETRY GROUPS**

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Abstract

This is part II of a series on noncompact isometry groups of Lorentz manifolds. We have introduced in part I, a compactification of these isometry groups, and called “bipolarized” those Lorentz manifolds having a “trivial ” compactification. Here we show a geometric rigidity of non-bipolarized Lorentz manifolds; that is, they are (at least locally) warped products of constant curvature Lorentz manifolds by Riemannian manifolds.

1 Introduction

We continue here our investigation of noncompact isometry groups of compact Lorentz manifolds, started in part I [Z1] which contains dynamical ingredients. Its fundamental tool was the notion of approximate stability. This second part (which is in fact fairly independent of part I) is geometrical, and has the *warped product* construction as a fundamental tool. Recall that this is a construction in the class of pseudo-Riemannian manifolds, defined as follows. Let (L, h) and (N, g) be two pseudo-Riemannian manifolds, and $w : L \rightarrow \mathbf{R}^+$ a (warping) function. The warped product $M = L \times_w N$, is the topological product $L \times N$, endowed with the metric $h \oplus wg$.

The warped product construction is very useful in Riemannian as well as Lorentzian geometry, since it gives sophisticated examples from simple ones. For instance, warped product models are omnipresent in cosmological theories.

Here we are interested in the case where L is Riemannian, N is Lorentzian, and hence $M = L \times_w N$ is Lorentzian. However, from a physical viewpoint, it is more interesting to consider the situation where L is Lorentzian, and N is Riemannian (the warping function is thus a universe expansion function).

There are two key properties of warped products:

1) If $f : N \rightarrow N$ is an isometry, then the trivial extension, $\bar{f} : (x, y) \in L \times N \rightarrow (x, f(y)) \in L \times N$, is an isometry of $L \times_w N$ (see §4.1).

In particular, in the class of Lorentz manifolds with large isometry groups, one can perform warped products by (any) Riemannian manifolds.

In fact, the warped products are reminiscent of semi-direct products in the category of groups, the factor N playing the role of the normal subgroup. One may justify this by the fact that, indeed, $\text{Isom}(N)$ is a normal subgroup of the subgroup of elements of $\text{Isom}(L \times_w N)$, which preserve the topological product $L \times N$ (i.e. the foliations determined by the factors L and N). This suggests to us to call the factor N the *normal* factor of the warped product. (This will be useful for us because we actually need to distinguish between the factors).

2) The second fundamental fact about warped products is that if S is a geodesic submanifold of N , then $L \times S$ is a geodesic submanifold in $L \times_w N$ (see 2.2).

It is very special when a Lorentz manifold (or in general a pseudo-Riemannian manifold, or even just a manifold endowed with a connection) admits many geodesic submanifolds of dimension > 1 (and codimension $\neq 0$). Generically, there is no such submanifold. The degenerate case, when every tangent plane is tangent to a geodesic submanifold, corresponds exactly to Lorentz manifolds of constant curvature (this is also true in the general pseudo-Riemannian case), see §3.

Here, we are especially interested in the case where the factor N has *constant curvature*. So, like N , $L \times_w N$ has many geodesic hypersurfaces.

In this article we investigate the relationships between the following three phenomena: being a warped product, having a large isometry group, and having abundant geodesic hypersurfaces. In particular, we show that in some situations, one of the second two properties may lead to a warped product structure.

1.1 Abundance of geodesic hypersurfaces leads to a warped product structure. Let M be Lorentz manifold. In order to analyze the set of geodesic hypersurfaces in M , we associate to any $x \in M$ a set C_x of tangent directions at x (i.e. 1-dimensional subspaces of $T_x M$) defined as follows. A direction $u \in \mathbf{P}(T_x M)$ belongs to C_x , if it is *isotropic* (this choice is related to our anti-physical preference of signatures of the factors L and N), and the orthogonal u^\perp determines a geodesic hypersurface. Equivalently, there is a *lightlike* geodesic hypersurface H passing through x ,

such that $T_x H = u^\perp$. (Recall here that, if $\langle \cdot, \cdot \rangle$ denotes the Lorentz scalar on $T_x M$, then a vector v is isotropic if $\langle v, v \rangle = 0$, and a hyperplane $E \subset T_x M$ is lightlike if its orthogonal is isotropic, or equivalently, the restriction of $\langle \cdot, \cdot \rangle$ on E is degenerate. A hypersurface is lightlike if its tangent space is everywhere lightlike).

Consider the open set $\mathcal{W}(M)$ of points M having a neighborhood isometric to a warped product with the normal factor being a Lorentz manifold of *constant curvature* and dimension ≥ 3 . That is, $x \in \mathcal{W}(M)$, if and only if there is a neighborhood U of x isometric to a warped product $L \times_w N$, where N is a Lorentz manifold of constant curvature and $\dim N \geq 3$.

Section 3 is devoted to the study of the relationship between the map, $x \rightarrow C_x$, and $\mathcal{W}(M)$. The following statement is a simple corollary of this study, which will be fully proved only in the analytic case, but it needs some results from [Z6] in the smooth case.

Theorem 1.1. *Let M be a Lorentz manifold. Suppose that C_x is infinite for all $x \in M$. Then, $\mathcal{W}(M)$ is dense (and open by definition) in M .*

Observe that this is a local result and that its converse is obviously true. This result admits a kind of generalization to the general pseudo-Riemannian case. Notice that the condition on the existence of geodesic hypersurfaces, cannot be relaxed to an existence condition of geodesic submanifolds of higher codimension. For instance, in the Riemannian case, the symmetric space CP^n admits many geodesic submanifolds of (real) codimension 2, but none of codimension 1 (of course, it is far from being a warped product).

1.2 From the local to the global in the analytic case. In the presence of (real) analyticity, a somewhere local warped product leads to an everywhere local warped product, and then to a global warped product in the universal cover, and finally to full completeness.

Theorem 1.2. *Let M be a compact (real) analytic Lorentz manifold. Suppose that $\mathcal{W}(M)$ is nonempty. Then, the universal cover \tilde{M} is isometric to a warped product of a complete simply connected Lorentz manifold \tilde{N} of constant nonpositive curvature and dimension ≥ 3 , by a complete simply connected Riemannian manifold \tilde{L} . Furthermore, M is (geodesically) complete, and admits another metric for which \tilde{M} is isometric to the direct product $\tilde{L} \times \tilde{N}$.*

The last part of this theorem contains in particular, Carrière's theorem, and its adaptation by B. Klingler, on completeness of compact Lorentz

manifolds of constant curvature [C],[K]. We notice however, that we do not reprove Carrière's theorem here, but rather use it, by observing that its proof may be adapted to the general situation in the theorem above.

1.3 Warped product or local bipolarization, when the isometry group is noncompact. Let M be a compact Lorentz manifold, such that $\text{Isom } M$ is noncompact. From part I, there exists at least one geodesic lightlike codimension 1 foliation of M .

Therefore, for all x in M , $\text{card } C_x \geq 1$. This fact alone may also be proved in a straightforward way, by looking at limits of graphs of the elements of $\text{Isom } M$.

Fuchsian-like behavior of $\text{Isom } M$ was described in part I, having as a consequence a dichotomy (roughly speaking): $\text{card } C_x \leq 2$, or C_x infinite. From the results above, the last situation implies that $\mathcal{W}(M) \neq \emptyset$, and thus M has the nice structure described above, in the analytic case.

Theorem 1.3. *Let (M, g) be a compact (real) analytic Lorentz manifold, such that $\text{Isom } M$ is noncompact. Then, exactly one of the two following possibilities holds:*

- 1) *There exists a new metric g' on M such that*
 - (i) *The universal cover of (M, g') is isometric to a direct product $\tilde{L} \times \tilde{N}$, where \tilde{L} is a complete simply connected Riemannian manifold, and \tilde{N} is a complete simply connected Lorentz manifold of constant nonpositive curvature and dimension ≥ 3 . This decomposition is natural, in particular, $\text{Isom}(\tilde{M}) = \text{Isom}(\tilde{L}) \times \text{Isom}(\tilde{N})$.*
 - (ii) *The universal cover of (M, g) is a warped product $\tilde{L} \times_{\tilde{w}} \tilde{N}$, with a warping function $\tilde{w} : \tilde{L} \rightarrow \mathbf{R}^+$, invariant by the action of the projection of $\pi_1(M)$ on $\text{Isom}(\tilde{L})$.*
 - (iii) *$\text{Isom}(M, g)$ is the subgroup of elements of $\text{Isom}(M, g')$ whose lifts to \tilde{M} preserve \tilde{w} , where $\text{Isom}(\tilde{M})$ acts on \tilde{L} via its projection onto $\text{Isom}(\tilde{L})$.*
- 2) *There are two (not necessarily distinct) codimension 1 lightlike geodesic foliations \mathcal{F}_1 and \mathcal{F}_2 , such that*
 - (i) *Any lightlike geodesic hypersurface in M , is contained in a leaf of \mathcal{F}_1 or \mathcal{F}_2 . In particular, any local isometry of M preserves each of these foliations, or exchanges them.*
 - (ii) *For each i , there is an analytic structure on (the topological manifold) M in respect to which \mathcal{F}_i is an analytic foliation.*

The two cases 1 and 2 in the theorem correspond to the situations

$\mathcal{W}(M) \neq \emptyset$ and $\mathcal{W}(M) = \emptyset$, respectively. The structure of M in the case $\mathcal{W}(M) \neq \emptyset$ follows from Theorem 1.2 (which is proved in §4). The structure of M when $\mathcal{W}(M) = \emptyset$ (and $\text{Isom}(M)$ noncompact) is studied in §5.

Some comments are in order.

Local bipolarization. Recall from part I, that M is called *bipolarized* if its isometry group is noncompact and preserves a pair of lightlike geodesic foliations. The situation 2(ii) in the theorem above, suggests the definition of a local version of this notion (i.e. by means of the pseudogroup of local isometries). We will say that M is *locally bipolarized* if there are two lightlike geodesic foliations \mathcal{F}_1 and \mathcal{F}_2 such that for all $x \in M$, $C_x = \{(T_x\mathcal{F}_1)^\perp, (T_x\mathcal{F}_2)^\perp\}$.

In order to get closer to a classification of compact Lorentz manifolds with noncompact isometry group, the investigation of the geometric and dynamical structure of locally bipolarized manifolds is clearly of interest.

Let us give some examples of locally bipolarized manifolds. Consider $SL(2, \mathbf{R})$ endowed with its Killing form. It has constant negative curvature, and thus, it is by no means bipolarized. Now, endow $SL(2, \mathbf{R})$ with a left invariant Lorentz metric derived from a given Lorentz scalar product $\langle \cdot, \cdot \rangle$ on the Lie algebra $sl(2, \mathbf{R})$. Suppose that $\langle \cdot, \cdot \rangle$ is a kind of “Berger’s metric”, that is, it is given by scaling a hyperbolic element $u \in sl(2, \mathbf{R})$ (i.e. $\exp tu$ is a hyperbolic one parameter group) by a nontrivial factor, and keeping the Killing form on u^\perp . The metric so obtained admits as a local bipolarization the stable and unstable foliations determined by $\exp tu$ (see [GuL] about left invariant metrics on $SL(2, \mathbf{R})$).

The isometry group in the warped product case. Of course, being locally bipolarized is stronger than being bipolarized. For instance, “purely” irrational flat structures (which are of course not locally bipolarized) on the torus, with noncompact isometry group, are bipolarized (see part I, §15). Also, nonhomogeneous 3-anti-de Sitter manifolds are bipolarized, whenever they have a noncompact isometry group (this isometry group is in fact, up to a finite index, isomorphic to \mathbf{R} , and hence it is amenable, which implies from part I that the underlying manifold is bipolarized).

Furthermore, we observed in part I, that if M is not bipolarized, and the factor \tilde{N} in Theorem 1.3 has constant negative curvature (that is, \tilde{N} is an anti-de Sitter space), then, $\dim \tilde{N} = 3$ (i.e. $\tilde{N} = \widetilde{SL(2, \mathbf{R})}$). Equivalently, if $\dim \tilde{N} > 3$, then, M is bipolarized. However, it seems that this situation

never happens. That is, if $\tilde{M} = \tilde{L} \times \tilde{N}$, where \tilde{N} is an anti-de Sitter space of dimension > 3 , then $\text{Isom } M$ is compact (from our definition, we do not call M bipolarized in this case). This was proved in the case that the factor \tilde{L} is trivial, so M is an anti-de Sitter manifold of dimension > 3 [Z2].

Regularity. The lightlike geodesic foliations found in part I are, a priori, only Lipschitz. However, in each of the cases, warped product or locally bipolarized, there are extra reasons leading to higher regularity. Indeed, in the warped product case, we essentially deal with *global* lightlike geodesic foliations of the anti-de Sitter or the Minkowski spaces. They are easy to handle, and can be shown to be analytic.

Here is the idea of the proof of regularity in the locally bipolarized case (which is behind property 2(ii) of Theorem 1.3. Observe that the graph (as a section) of a codimension 1 lightlike geodesic foliation \mathcal{F} on M , is a Lipschitz submanifold $P(\mathcal{F})$, homeomorphic to M , contained in $Gr^0(M)$, the Grassmann bundle of lightlike hyperplanes tangent to M . In fact $P(\mathcal{F})$ is contained in the subset \mathcal{D} , the integrability domain of the tautological plane field (see §3), defined by $\mathcal{D} = \bigcup_{x \in M} C_x^*$, where $C_x^* \subset Gr_x^0(M)$ is the dual of C_x (see above). But, \mathcal{D} is an *analytic set*. So, amusingly, when \mathcal{D} is poor, say it equals $P(\mathcal{F}_1) \cup P(\mathcal{F}_2)$, then we win regularity for \mathcal{F}_1 and \mathcal{F}_2 , because their graphs are open in an analytic set. This implies that \mathcal{F}_1 and \mathcal{F}_2 are “essentially” analytic. But, a priori, \mathcal{D} may have an intrinsic singularity locus, or a vertical locus (where it is regular but tangent to the vertical). However, we guess, none of these singularities may occur in our situation, and the foliations are actually analytic. Anyway, we have the following corollary of Theorem 1.3 (2(ii)).

Theorem 1.4. *Let M be a compact topological manifold which has no codimension 1 analytic foliation, for any analytic structure on M . Then, any analytic Lorentz metric on M has a compact isometry group.*

A classical result of A. Haefliger (see for example [G]) states that compact simply connected manifolds satisfy the condition of the theorem. Therefore, they have compact isometry groups. This gives, another proof of G. D’Ambra’s theorem [D], without using Gromov’s theory of rigid transformation groups.

Recently, T. Barbot [B] has found another example of manifolds satisfying the condition of the theorem above. For instance, a compact manifold with a fundamental group isomorphic to a finite index subgroup of $SL(d, \mathbf{Z})$, with $d \geq 3$, has no analytic foliation. (This is presumably true for all irreducible lattices in semisimple Lie groups of rank ≥ 2). Therefore, such

a manifold has a compact isometry group, when endowed with an analytic Lorentz metric.

A properness theorem. In fact, as was said in part I, one may ask for a more stable compactness of isometry groups. Our method allows us to prove the following result, which we state here without further details.

Theorem 1.5. *Let M be a compact manifold, which is simply connected, or has a fundamental group isomorphic to a finite index subgroup of $SL(d, \mathbf{Z})$, with $d \geq 3$. Denote by $\text{Lor}^{\omega,2}(M)$ (resp. $\text{Diff}^{\omega,2}(M)$) the space of analytic Lorentz metrics on M (resp. analytic diffeomorphisms of M) endowed with the C^2 topology. Then $\text{Diff}^{\omega,2}(M)$ acts properly on $\text{Lor}^{\omega,2}(M)$.*

An application: action of Lie groups. Due to the works [Zi], [Gr], [DGr], [Ko], [ASt1,2], and [Z3,4], many things are now known about isometric actions of connected Lie groups on compact Lorentz manifolds. For example, we know that, if the affine group (of the line) AG acts isometrically on a compact Lorentz manifold M , then, essentially, this action may be extended to $SL(2, \mathbf{R})$ (see [ASt2] or [Z4] for the correct statement of this fact). Let us see how to deduce this fact, in the analytic case, from Theorem 1.3. Of course this would follow by standard algebraic manipulations, if we already knew that the manifold was a warped product, as described in point 1 of Theorem 1.3. So, it suffices to show that a manifold endowed with an AG -action is not bipolarized. For this, let h^t and T^t be two one-parameter groups generating AG such that $h^s T^t h^{-s} = T^{te^s}$, and for a fixed s , denote by ϕ_s the flow $\phi_s^t = \{T^s h^t T^{-s}, t \in \mathbf{R}\}$. Then, for any two different values s_1 and s_2 , the flows ϕ_{s_1} and ϕ_{s_2} have different approximately stable foliations. To see this, let X be the infinitesimal generator of the flow h^t . From the relation $h^s T^t h^{-s} = T^{te^s}$, it follows that the direction of X is the positive Lyapunov space (with exponent 1) for any flow ϕ_s . By volume preservation, there is a nontrivial negative Lyapunov space for ϕ_s , which, since it is isotropic, must have dimension 1, say it is oriented by a vector field Y_s , such that $\langle X, Y_s \rangle = 1$.

The tangent bundles of the approximately stable foliations of the two flows ϕ_{s_1} and ϕ_{s_2} are $Y_{s_1}^\perp$ and $Y_{s_2}^\perp$, respectively, and therefore, they coincide, iff, $Y_{s_1} = Y_{s_2}$. But, by definition, the flows ϕ_{s_1} and ϕ_{s_2} are conjugate by a suitable power $h^{t(s_1, s_2)}$, and in the same way, Y_{s_1} and Y_{s_2} are conjugate, and hence, if they are equal, the flow h^t will preserve the vector field $Y = Y_{s_1} = Y_{s_2}$ (note that because of the equality $\langle X, Y \rangle = 1$, h^t preserves the vector field Y itself and not only its direction). Let us show that this

is impossible. Indeed, the orthogonal of $\mathbf{R}X \oplus \mathbf{R}Y$ is spacelike, which allows one to construct a Riemannian metric on the Lorentz manifold M , preserved by h^t (keep the Lorentz metric on $(\mathbf{R}X \oplus \mathbf{R}Y)^\perp$ and choose X and Y to be orthogonal and having a unit length). Therefore, h^t is precompact, and its closure is a torus \mathbf{T} . The affine group acts by conjugacy, automorphically (and nontrivially) on this torus, which is impossible.

2 Geometry of Warped Products

Here, we will present standard geometric notions related to warped products, and state geometric criteria for the existence of such structures. We will try to avoid the use of local calculus, and instead, we will use synthetic arguments (see [O] for more details).

Umbilic and geodesic submanifolds. Let M be a Lorentz manifold. Let S be a nondegenerate submanifold of M , that is the metric restricted to $T_x S$ is nondegenerate for any $x \in S$. Recall that S is umbilic, if and only if for any $x \in S$, the second fundamental form II_x (which is well defined because of the non-degeneracy hypothesis) has the form $II_x = \langle \cdot, \cdot \rangle n_x$, where n_x is some normal vector to $T_x S$, and where $\langle \cdot, \cdot \rangle$ denotes the Lorentz metric. The geodesic case corresponds to $n_x = 0$, for all $x \in S$.

Let $x \in S$, $u \in T_x S$, and let $\gamma :]-\epsilon, +\epsilon[\rightarrow M$ be the geodesic in M determined by u . For S geodesic, the image of γ is contained in S , for ϵ sufficiently small. This fact is true also when S is umbilic, if in addition u is *isotropic* (this is a remarkable rigidity fact in Lorentz geometry, which has no equivalent statement in Riemannian geometry).

For example, take M to be Minkowski space, i.e. \mathbf{R}^n endowed with a Lorentz form q . The geodesic hypersurfaces are contained in affine hyperplanes. The umbilic hypersurfaces are contained in hyperquadrics $q(x - O) = c$, where $O \in \mathbf{R}^n$ and c is a constant (the proof is formally the same as in the Euclidean case). One can verify that such a hyperquadric is ruled, that is, it contains the isotropic lines which are somewhere tangent to it.

Umbilic and geodesic foliations. A foliation is called geodesic or umbilic, if and only if its leaves are geodesic, or umbilic. The following is a standard fact (see for instance [PR]).

FACT 2.1. *Let \mathcal{F} be a foliation of M such that the orthogonal $T\mathcal{F}^\perp$ is integrable, that is, it determines a foliation \mathcal{F}^\perp , say. Then \mathcal{F} is geodesic*

(resp. umbilic) if and only if the holonomy maps of the foliation \mathcal{F}^\perp , seen as local diffeomorphisms between leaves of \mathcal{F} , preserve the metric (resp. the conformal structure) induced on these leaves (of \mathcal{F}).

Let F and F^\perp be the local leaves through some point for \mathcal{F} and \mathcal{F}^\perp , respectively, leading to a local diffeomorphism of M with $F \times F^\perp$. If \mathcal{F} is geodesic, then the metric has the form $m_{x,y} = h_x \oplus g_{(x,y)}$, where h is a metric on F and for any $x \in F$, $g_{(x,\cdot)}$ is a metric on F^\perp .

If \mathcal{F} is umbilic, then the metric has the form $m_{x,y} = w(x,y)h_x \oplus g_{(x,y)}$, where w is a function on $F \times F^\perp$.

Warped products. As was said in the introduction, a Lorentz manifold M is a warped product of a Lorentz manifold (N, g) by a Riemannian manifold (L, h) , if M is isometric to the product $L \times N$, endowed with a metric of the form $h \oplus wg$, where w is some positive function defined on L . We call N the *normal factor* of the warped product.

The factors L and N define two foliations denoted by \mathcal{L} and \mathcal{N} respectively; \mathcal{N} is called the *normal foliation* of the warped product. From the form of the metric and the above discussion, we infer the following geometric properties: \mathcal{L} is geodesic, and \mathcal{N} is umbilic.

In terms of holonomy pseudogroups, this means that the holonomy of \mathcal{N} (resp. \mathcal{L}) preserves the transverse metric (resp. the transverse conformal structure). In fact, to characterize warped products, we just need that the holonomy maps of \mathcal{L} have constant conformal distortion, that is, they are *homothetic*. In particular the holonomy of \mathcal{L} is projective, i.e. maps geodesics to geodesics. Here is a related stronger property:

FACT 2.2. *A submanifold S of N is geodesic in N , if and only if $L \times S$ is geodesic in $M = L \times N$.*

Proof. It suffices to consider the case $\dim S = 1$.

Suppose that S is geodesic in N . By considering the family of normal geodesic of a hypersurface orthogonal to S , we can locally extend S to a geodesic foliation \mathcal{S} of N , admitting an orthogonal foliation \mathcal{S}^\perp . Consider the foliation \mathcal{F} of M with leaves of the form $L \times \mathcal{S}_y$ where \mathcal{S}_y is a leaf of \mathcal{S} . It has a normal foliation \mathcal{F}^\perp with leaves $\{x\} \times \mathcal{S}_y^\perp$. It then follows that a holonomy map of \mathcal{F}^\perp has the form $\psi : (x, y) \in L \times \mathcal{S}_{y_1} \rightarrow (x, \phi(y)) \in L \times \mathcal{S}_{y_2}$, where ϕ is a holonomy map of \mathcal{S}^\perp , which is isometric by hypothesis. Because the metric on $\{x\} \times N$ is a constant times the metric of N , ϕ sends $\{x\} \times \mathcal{S}_{y_1}$ isometrically onto $\{x\} \times \mathcal{S}_{y_2}$. Therefore ϕ is isometric. Hence from Fact 2.1, \mathcal{F} is geodesic, and in particular $L \times S$ is geodesic in M .

For the converse, that is, $L \times S$ being geodesic implies that S is geodesic, we use the following general fact. Its proof follows from a standard calculation.

Proof.

FACT 2.3. *Let A and B be submanifolds of M , and suppose that B is geodesic. Suppose that B is transverse and orthogonal to A , that is, for all $x \in A \cap B$, $T_x B$ contains $(T_x A)^\perp$. Then $B \cap A$ is geodesic in A .*

Criterion for warped products. Here is the proposition which we will apply to prove the existence of warped products in §3.

PROPOSITION 2.4. *Let $M = L \times N$ be endowed with a metric such that the foliation \mathcal{L} (resp. \mathcal{N}) is geodesic (resp. umbilic).*

Suppose that for all $(x, y) \in L \times N$, there are geodesic hypersurfaces in M , H^1, \dots, H^d , containing (x, y) and such that

- i) H^i is invariant by the foliation \mathcal{L} (i.e. it is a union of leaves of \mathcal{L}), and
- ii) The directions $(T_{(x,y)}H^1)^\perp \cap T_y N, \dots, (T_{(x,y)}H^d)^\perp \cap T_y N$ generate $T_y N$.

Then the leaves $\{x\} \times N$ have constant curvature, and M is a warped product.

Proof. One can write $H^i = L \times S^i$, where S^i is a hypersurface of N . From Fact 2.2, $\{z\} \times S^i$ is a geodesic hypersurface in $\{z\} \times N$, for all $z \in L$.

Thus, $\{z\} \times N$, admits, at each point, geodesic hypersurfaces, whose orthogonal directions generate the tangent space at each point. It will be shown at Proposition 3.2, that this implies that $\{z\} \times N$ has constant curvature. We may assume that the sign of curvature is independent of $z \in L$.

A holonomy map (of \mathcal{L}) taking $\{x\} \times N$ to $\{x'\} \times N$, maps $\{x\} \times S^i$ to $\{x'\} \times S^i$. One may then call it “partially projective”.

The warped product property means that any holonomy map (of \mathcal{L}) $\{x_1\} \times N \rightarrow \{x_2\} \times N$, is homothetic. By hypothesis, \mathcal{N} is umbilic, and hence these holonomy maps are conformal. The question then becomes, is a conformal and “partially projective” map between two Lorentz manifolds, homothetic? In our case, the leaves $\{z\} \times N$ have constant curvature of the same sign, and therefore are homothetic. The proof of the warped product property can then be achieved with help of the following fact. \square

FACT 2.5. *Let N be a Lorentz manifold of constant curvature, with dimension ≥ 3 . Let $\phi : N \rightarrow N$ be a conformal local diffeomorphism,*

which satisfies the following condition. For any $y \in N$, there is S^1, \dots, S^d , geodesic hypersurfaces containing y , such that $(T_x S^1)^\perp, \dots, (T_x S^d)^\perp$ generate $T_x N$, and such that, their images $\phi(S^i)$ are geodesic. Then ϕ is a homothety. (In fact ϕ is an isometry unless M is flat). (Interpretation: a conformal and “partially projective” transformation is homothetic).

Proof. We think that the interpretation of the fact is sufficiently convincing, and so, to avoid complicated notation, we restrict ourselves to the flat case. Also, we will assume that the involved geodesic hypersurfaces S^1, \dots, S^d are lightlike, because, that is what we need for application, in the present paper. Thus M is the Minkowski space $\mathbf{R}^{1,n-1}$. By composing with an isometry, we may suppose that ϕ is a local conformal diffeomorphism fixing 0, and that $D_0\phi$ is a homothety. In particular $D_0\phi$ keeps invariant any tangent line at 0. It then follows that ϕ keeps invariant each isotropic line through 0, since conformal diffeomorphisms preserve isotropic geodesics. Furthermore, ϕ keeps invariant the geodesic hypersurfaces S^1, \dots, S^d (because they are sent by ϕ to geodesic hypersurfaces with the same tangent space). Also, by considering intersection of sub-families of these hypersurfaces, we infer the existence of a basis $\{v_1, \dots, v_n\}$ of $T_0\mathbf{R}^n$, determining lines kept invariant by ϕ . These vectors are spacelike, that is $\langle v_i, v_i \rangle > 0$, since the hypersurfaces S^i are lightlike.

Let P be an affine 2-plane, so $\phi(P)$ is 2-dimensional generalized sphere, i.e. an affine plane, or a quadric defined by means of the Lorentz form (this follows from the fact that ϕ is conformal, as in the Riemannian case).

There are two possibilities for a generalized 2-sphere which is not an affine plane. If it is (somewhere and hence everywhere) timelike (i.e. the induced metric on it is of Lorentz type), then it is ruled, by means of a pair of foliations by isotropic lines. In contrast, if it is spacelike (i.e. the induced metric on it is Riemannian), then it contains no line.

Let P be a spacelike affine plane which contains a line defined by some v_i . Then $\phi(P)$ is a spacelike generalized 2-sphere which contains a line. From the above discussion, $\phi(P)$ must be an affine plane. But, since $D_0\phi$ is a homothety, we have $\phi(P) = P$. Thus all the affine spacelike 2 planes containing a line determined by some v_i are invariant by ϕ . A standard analyticity argument show that this extends to all the affine 2-planes without the spacelike condition. Taking the intersection of these planes, for various v_i , and again by an analyticity argument, we conclude that every line through 0 is invariant by ϕ .

But the same argument works for any point of M (by composing with an

appropriate isometry). This means that ϕ is projective. In particular the restriction of ϕ to an affine plane P through 0 is conformal and projective. This is equivalent to a conformal and projective local transformation of the Euclidean plane \mathbf{R}^2 , if P is spacelike. This is easily seen to be a homothety. Because, there are many such planes, one concludes that ϕ itself is a homothety. \square

3 The Tautological Geodesic Plane Field

An affine connection (e.g. a pseudo-Riemannian structure) on a manifold M , permit to define a tautological geodesic plane fields on the Grassmann bundles of tangent k -planes $Gr_k(M) \rightarrow M$, which generalizes the classical construction of the geodesic flow for $k = 1$. Fix an integer $1 \leq k \leq \dim M$. The connection yields a horizontal bundle H , supplementary to the vertical of the fibration $Gr_k \rightarrow M$. For $p \in (Gr_k)_x$, we identify H_p with $T_x M$. Then $\mathcal{P}_p \subset H_p$ is identified with $p \subset T_x M$. The bundle \mathcal{P} is called a tautological geodesic plane field on $Gr_k(M)$. In general, \mathcal{P} is not integrable. Indeed one may prove (see below), for M pseudo-Riemannian, that if \mathcal{P} is integrable for $k \neq 1$ and $k \neq \dim M$, then M has constant curvature. (It seems that when M is merely affine, then the conclusion is that M is projectively flat).

Observe that (maximal) integral submanifolds of \mathcal{P} project on geodesic submanifolds of dimension k in M . Conversely, if L is a geodesic k -submanifold of M , then the image of the Gauss map $x \rightarrow T_x L$ is an integral submanifold of \mathcal{P} .

We denote by $\exp : TM \rightarrow M$, the exponential map defined on its domain of definition, which is a neighborhood of the zero section. For $p \in Gr_k(M)$, let $\exp p$ denote the image by \exp of the intersection of p with the domain of definition of \exp .

DEFINITION 3.1. *The domain of integrability of \mathcal{P} is the set of $p \in Gr_k(M)$ such that, if $p \subset T_x M$, then a neighborhood of x in $\exp_x p$ is geodesic.*

3.1 The Grassmannian of lightlike hyperplanes of a Lorentz manifold. Let M be a Lorentz manifold and denote by $Gr^0(M) \subset Gr_{n-1}(M)$ the Grassmannian of *lightlike hyperplanes* tangent to M (n is the dimension of M).

The tautological geodesic plane field on $Gr_{n-1}(M)$ is tangent to $Gr^0(M)$. We will denote the tautological plane field restricted to $Gr^0(M)$ by τ , and by \mathcal{D} its *integrability domain*.

Therefore, as in §1.3, the fiber \mathcal{D}_x , is the dual of C_x . Recall that a direction u belongs to C_x , if and only if u^\perp is tangent to a lightlike geodesic hypersurface.

To start with, notice the following rigidity.

PROPOSITION 3.2. *Let M be a Lorentz manifold of dimension ≥ 3 , such that C_x generates T_xM , for any $x \in M$. Then M has constant curvature.*

Proof. The hypothesis means that for any $x \in M$, there are H^1, \dots, H^d , geodesic lightlike hypersurfaces containing x , such that $(T_xH^1)^\perp, \dots, (T_xH^d)^\perp$ generate T_xM .

Fix $x \in M$, denote T_xH^i by B^i , and choose b^i an isotropic vector such that $B^i = (b^i)^\perp$.

For $u \in T_xM$, denote by A_u the curvature operator $A_u : v \in T_xM \rightarrow R(u, v)u \in T_xM$. Then, for $u \in B^i$, A_u preserves B^i (since geodesic submanifolds are “invariant” by the curvature operator). Moreover, $A_u(b^i)$ is collinear to b^i (for $u \in B^i$). Indeed $\langle A_u(b^i), v \rangle = \langle A_u(v), b^i \rangle$. The last quantity equals 0 if $v \in B^i$ since $A_u(v) \in B^i$, and hence $A_u(b^i) \in (B^i)^\perp = \mathbf{R}b^i$.

Choose e_i a unit director vector of $\bigcap_{j \neq i} B^j$, and consider A_{e_i} . Since $e_i \in B^j$, for $j \neq i$, there is $\lambda_{i,j}$, such that $A_{e_i}b^j = \lambda_{i,j}b^j$ (for $i \neq j$).

Since A_{e_i} is symmetric, $\lambda_{i,j}\langle b^j, b^k \rangle = \langle A_{e_i}b^j, b^k \rangle = \langle A_{e_i}b^k, b^j \rangle = \lambda_{i,k}\langle b^j, b^k \rangle$, and we have $\lambda_{i,j} = \lambda_{i,k}$ (for $j \neq k$, $\langle b^j, b^k \rangle \neq 0$, because both b^j and b^k are isotropic). Write $\lambda_i = \lambda_{i,j}$. Thus, the sectional curvature of any nondegenerate plane which contains e_i equals λ_i . From this, we infer that $\lambda_1 = \dots = \lambda_n$ (to see this, consider 2-planes generated by two vectors e_i and e_j). One may use standard algebraic manipulations to show that all the 2-planes in T_xM have the same sectional curvature, and then deduce from Schur’s lemma that M has constant curvature. \square

3.2 Main result

Theorem 3.3. *For $x \in M$, denote by E_x the linear space generated by C_x . Suppose that for an open subset $U \subset M$, we have*

- i) $x \in U \rightarrow E_x$ determines a smooth plane field of dimension ≥ 3 , and
- ii) $\text{card } C_x > \dim E_x$, for x in a dense subset of U .

Then E determines a local warped product structure on U (i.e. E is tangent to the normal foliation of a local warped structure on U), with the leaves of E having constant curvature.

Proof. There are several steps.

STEP 1. E^\perp is integrable and has geodesic leaves. We denote its tangent foliation by \mathcal{L} .

Proof. Let H^1, \dots, H^d , be geodesic lightlike hypersurfaces containing a point $x \in U$, such that $(T_x H^1)^\perp, \dots, (T_x H^d)^\perp$ generate E_x . Denote $L = \bigcap_i H^i$, and let us show that it is a leaf of E^\perp .

Denote by X^i a nonsingular isotropic vector field tangent to H^i (X^i is defined along H^i). Then for $y \in H^i$, $X^i(y) \in C_y$. Hence, if $y \in L = H^1 \cap \dots \cap H^d$, then $X^1(y), \dots, X^d(y) \in C_y$. Thus, by continuity of E , E_y is generated by $X^1(y), \dots, X^d(y)$. Observe now that $T_y L = \bigcap_i T_y H^i = \bigcap_i (X^i(y)^\perp) = (\sum_i \mathbf{R} X^i(y))^\perp = E_y$. Therefore L is a leaf of E^\perp containing x , which (being an intersection of geodesic hypersurfaces) is a geodesic submanifold. \square

STEP 2. Weingarten's endomorphism for plane fields.

The following is a general discussion on the geometry of planes fields, which will be, in particular, applied to E . For a plane field $x \rightarrow G_x \subset T_x M$, such that the metric restricted to G_x is *nondegenerate*, one defines a second fundamental form and Weingarten's endomorphism as follows. For X and Y vector fields tangent to G , and Z a vector field orthogonal to G , $II : G \times G \rightarrow G^\perp$, and $A_Z : G \rightarrow G$, are defined by the equalities: $\langle II(X, Y), Z \rangle = \langle \nabla_X Y, Z \rangle = \langle -A_Z X, Y \rangle$. We have $0 = X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$, and hence $A_Z(X)$ is just the projection of $\nabla_X Z$ on G . It turns out that II and A_Z are tensorial, that is, they depend only on the pointwise values of X, Y and Z . Notice the following property.

FACT 3.4. *A plane field G is integrable if and only if its second fundamental form, or equivalently its Weingarten's endomorphisms, are symmetric.*

In particular, if every Weingarten's endomorphism of G is a homothety (that is, it induces a scalar multiplication on G), then, G is integrable, and has umbilic leaves.

Our plane field E is nondegenerate, since it contains at least two isotropic directions.

STEP 3. End of proof of the theorem, assuming that all Weingarten's endomorphisms are homotheties.

Proof. In this case, from the fact above, E is integrable and has umbilic leaves. Then we use Proposition 2.4, to deduce that E and E^\perp give rise to a warped product (with E and E^\perp corresponding to \mathcal{N} and \mathcal{L} , respectively). Indeed, E is umbilic, E^\perp is geodesic, and furthermore, the condition on the existence of geodesic hypersurfaces saturated by \mathcal{L} , is well satisfied. Indeed,

the hypersurfaces, H^1, \dots, H^d , introduced in the beginning of the proof of the integrability of E^\perp , are saturated by E^\perp . \square

Now follow the steps of the proof that the Weingarten's endomorphisms are actually homothetic.

We will consider eigenspace splittings for E , and then splittings of the factors of the initial splitting, and so on... All these splitting are smooth, if we restrict ourselves to an open dense subset.

Notice that this does not cause any loss of generality. Indeed, if we are able, at the final stage, to prove that the Weingarten's endomorphisms are homothetic, in a dense set, then they will be homothetic everywhere. So, in the sequel, we will always suppose that we are near a generic point.

STEP 4. Let Z be a smooth vector field tangent to $E^\perp (= T\mathcal{L})$. Then, for any $u \in C_x$, $u^\perp \cap E_x$ is invariant by the Weingarten's endomorphism $A_Z(x)$ (or equivalently, u is an eigenvector of the dual Weingarten's endomorphism $A_Z^*(x)$).

Proof. Let $u \in C_x$, and consider H a lightlike geodesic hypersurface such that $T_x H = u^\perp$. Observe that Z is tangent to H (over points of H). This is because E^\perp itself is tangent to H (or equivalently H is saturated by the foliation \mathcal{L}). Since H is geodesic, the covariant derivative $\nabla_X Z(x)$ belongs to $T_x H$, for $X \in T_x H \cap E_x = u^\perp \cap E_x$, and hence, its orthogonal projection $A_Z X(x)$ (onto E_x) belongs to $u^\perp \cap E_x$ (since $E_x^\perp \subset u^\perp$). That is, $A_Z(u^\perp \cap E_x) \subset u^\perp \cap E_x$. \square

STEP 5. The 3-dimensional case.

To start with, let us give the proof when $\dim E = 3$. The cardinality condition in the theorem means that $\text{card } C_x \geq 4$, for any $x \in U$. Hence $A_Z^*(x)$ has at least 4 *isotropic* eigenvectors. Thus, since $\dim E = 3$, the eigenspace decomposition of E_x has at most two factors $E_x = A \oplus B$. If this is nontrivial, then up to a switch of factors, we have $\dim A = 2$ and $\dim B = 1$. Of course our isotropic eigenvectors belong to $A \cup B$. But, a two-dimensional (resp. one-dimensional) subspace of a Lorentz space has at most two (resp. one) isotropic lines (line). This contradiction implies that the decomposition is trivial, that is $A_Z^*(x)$ is a homothety, and thus, also is $A_Z(x)$.

STEP 6. Getting a partial warped product structure.

From Step 4, we infer that all the dual endomorphisms $A_Z^*(x)$, for $Z \in E_x^\perp$ are simultaneously diagonalizable. This determines a splitting $E_x = E_x^1 \oplus \dots \oplus E_x^k$ of common eigenspaces of all the $A_Z^*(x)$. Denote

$C_x^i = C_x \cap E_x^i$, then, we have $C_x = C_x^1 \cup \dots \cup C_x^k$, since the elements of C_x are eigenvectors of $A_Z^*(x)$. Therefore, for some factor, say, E_x^1 , we have $\text{card } C_x^i > \dim E_x^1$. In particular, since E_x^1 contains at least three isotropic lines, we have $\dim E_x^1 \geq 3$.

Observe now that if H is a lightlike geodesic hypersurface containing x , such that $u = T_x H^\perp$ belongs to C_x^i , then for all y near x , $T_y H^\perp$ belongs to C_y^i . Indeed, $T_y H^\perp$ belongs to $C_y = C_x^1 \cup \dots \cup C_x^k$, and by continuity, $T_y H^\perp \in C_x^i$.

This observation allows us to prove the same properties for E^1 , as were already proved for E itself (that is $E^{1\perp}$ is integrable and geodesic, and the dual Weingarten endomorphisms of E^1 admit the elements of C^1 as eigenvectors). Also, by the same argument, the pair $(E^1, (E^1)^\perp)$ would determine a warped product structure, if the Weingarten's endomorphisms of E^1 are homothetic. If not we get in a similar way, a splitting of E^1 . By induction, we arrive at a sub-bundle G of E , with $\dim G \geq 3$, which gives rise to a warped product structure.

We may sum all the intermediate decompositions, and write $E = G \oplus R$, where R is a sub-bundle of E , such that $C_x = (C_x \cap G_x) \cup (C_x \cap R_x)$.

STEP 7. Contradiction

As we said in the beginning of this paper, a (local) isometry, of the of the normal factor, extends to a (local) isometry of the warped product (see §4.1). In our case, a leaf of G has constant curvature. In particular, for any $x \in U$, its local isotropy group contains $O(1, d-1)$, where $d = \dim G$.

The infinitesimal action of $O(1, d-1)$ on $T_x M$ preserves G_x and R_x . In fact, from the exact definition of the extension of the action of $O(1, d-1)$, its action on $T_x M$ is conjugate to that on $\mathbf{R}^{1, d-1} \oplus \mathbf{R}^{n-d}$, where it acts as usual on the first factor, and trivially on the second one. Here G_x corresponds to $\mathbf{R}^{1, d-1}$ and G_x^\perp to \mathbf{R}^{n-d} . Observe that the subspace of fixed vectors of this action is exactly \mathbf{R}^{n-d} .

To R_x corresponds a $O(1, d-1)$ -invariant subspace A intersecting $\mathbf{R}^{1, d-1}$ trivially. This space is not spacelike, since it is generated by isotropic vectors.

To conclude, we just note that this is impossible. Indeed, if $\dim A = 1$, then, $O(1, d-1)$ acts trivially on it, since $O(1, d-1)$ has no nontrivial 1-dimensional representation. If not, A is Lorentzian, and thus A^\perp is spacelike and $O(1, d-1)$ -invariant. But $O(1, d-1)$, as a simple noncompact Lie group, has no nontrivial representation preserving a positive scalar product. Thus $O(1, d-1)$ acts trivially on A^\perp . This contradicts the fact that the space of

fixed elements of the representation is exactly \mathbf{R}^{n-d} . \square

REMARK 3.5. Although it was crucial in our proof (especially at Step 5), the cardinality condition $\text{card } C_x > \dim E_x$ might perhaps be relaxed to the more natural condition $\dim E_x \geq 3$ (of course, by definition we always have $\text{card } C_x \geq \dim E_x$). In fact, this is exactly the content of Proposition 3.2, in the extremal case when $\dim E_x = \dim M$.

3.3 The structure of $\mathcal{W}(M)$. Recall from §1.1 that $\mathcal{W}(M)$ denotes the open set of points of M , having a neighborhood isometric to a warped product of a Lorentz manifold of constant curvature and dimension ≥ 3 , by some Riemannian manifold. Observe that, a priori, a Lorentz manifold may be written as a warped product in many fashions. However, the structure given by the theorem above is unique, because it is associated to the map $x \rightarrow C_x$. It is in a natural sense the maximal warped product structure (among those with a normal factor of constant curvature) on M .

Define $M_{\text{smooth}} = \{x \in M / \text{there is a neighborhood } V \text{ of } x \text{ such that } y \in V \rightarrow E_y \text{ is smooth}\}$, and $M_{\leq k} = \{x \in M / \text{card } C_x \leq k\}$.

Note that, if $x \notin M_{\leq \dim M}$, then $\text{card } C_x > \dim M \geq \dim E_x$, and hence the cardinality condition of Theorem 3.3 is satisfied.

In addition, $M - \text{int}(M_{\leq \dim M})$ contains a dense set of points where $\text{card } C_x > \dim M$ (here int denotes the interior). Therefore, we have the following corollary:

Theorem 3.6. $\text{int}(M_{\text{smooth}} - \text{int}(M_{\leq \dim M})) \subset \mathcal{W}(M)$.

More precisely, $\text{int}(M_{\text{smooth}} - \text{int}(M_{\leq \dim M}))$ has a canonical pair $(\mathcal{N}, \mathcal{L})$ of foliations which determines a local warped product with the leaves of \mathcal{N} having constant curvature. Furthermore, this pair of foliations is invariant under the pseudo-group of local isometries of M .

The analytic case. In this case the integrability domain \mathcal{D} is an analytic set. The smoothness (in fact the analyticity) condition on E , is always satisfied, away from some analytic sets. Indeed, the assignment $x \rightarrow \mathcal{D}_x = C_x^*$, is analytic, in an obvious sense, away from some analytic set. Therefore, we have the following corollary.

COROLLARY 3.7. *Suppose that M is analytic. Then $\mathcal{W}(M) \neq \emptyset$, whenever $\text{int}(M_{\leq \dim M})$ is not dense.*

4 Completeness Properties. Proof of Theorem 1.2

Here, we prove Theorem 1.2, which is essentially that, if M is analytic and $\mathcal{W}(M) \neq \emptyset$, then in fact, $\mathcal{W}(M) = M$.

4.1 Extension of the warped product structure. Here are our two fundamental extension tools.

1) The first one, mentioned in the introduction of this article, is the extension to a warped product of the isometries of its normal factor. Indeed, let $M = L \times_w N$, then, any (local) isometry $f : N \rightarrow N$ induces a (local) isometry $\bar{f} : (x, y) \in L \times N \rightarrow (x, f(y)) \in L \times N$. The fact that \bar{f} is an isometry, follows from the fact that the metric of M has the form $h \oplus wg$, where $w = w(x)$ is a positive function defined (only) on L . Thus \bar{f} preserves and acts isometrically on the leaves of the foliations determined by each of the factors N and L .

By the same rule, Killing fields of N determine Killing fields on M .

2) The second key extension fact is that a Killing field defined on an open subset of a *simply connected analytic* Lorentz manifold, extends (as a Killing field) to the whole manifold ([N] and [Am]).

Now let M be a compact analytic Lorentz manifold such that $\mathcal{W}(M) \neq \emptyset$. Let x_0 be a point of $\mathcal{W}(\tilde{M})$, for which the dimension of the normal factor (with constant curvature) is maximal (among all points of $\mathcal{W}(\tilde{M})$). Denote this dimension by d , and let \tilde{N} be the complete simply connected constant curvature Lorentz manifold of dimension d , and having the same scalar curvature as the leaf of x_0 (in the local warped product).

Denote by \mathcal{G} the Lie algebra of Killing fields of \tilde{N} . From the extension facts recalled above, there is a faithful action of \mathcal{G} on \tilde{M} , that is a monomorphism $X \in \mathcal{G} \rightarrow \bar{X} \in \mathcal{K}$, where \mathcal{K} is the Lie algebra of Killing fields on \tilde{M} . We denote the \mathcal{G} -orbit of a point x by $\mathcal{G}x$.

By definition, near x_0 , the \mathcal{G} -orbits determine a local warped product. The goal is to prove that the \mathcal{G} -orbits determine a local warped product everywhere. That is, firstly, the \mathcal{G} -action gives rise to a regular foliation (i.e. of constant dimension), with leaves locally homothetic to \tilde{N} . Secondly, the orthogonal is integrable, and form together with the \mathcal{G} -foliation a local warped product. The analyticity reduces the proof to the following non-degeneracy fact.

FACT 4.1. *Let U be the set of points of \tilde{M} having a Lorentzian (also called timelike) orbit, i.e. the induced metric on these orbits is of Lorentzian type. Then U is open, and on it, the \mathcal{G} -action determines a local warped product.*

In particular, in order to prove that the \mathcal{G} -action determines everywhere a local warped product, it suffices to prove the equality: $U = \tilde{M}$.

Proof. Observe that $\dim \mathcal{G}x \leq d$, for all $x \in \tilde{M}$, with equality in an open dense set. This follows by analyticity (indeed, if $X_1, \dots, X_{d+1} \in \mathcal{G}$, then $\bar{X}(x) \wedge \dots \wedge \bar{X}_{d+1}(x) = 0$ in an open set).

Recall that, if a pseudo-Riemannian manifold of dimension $\leq d$, has a Killing algebra of the same dimension as that of a manifold of constant curvature and dimension d , then this manifold has dimension d and is of constant curvature, of the same sign. (This is, at least in the Riemannian case, a folk fact, but since it is somewhat difficult to localize in literature, let us give a sketch of proof of it. Recall that all the orthogonal algebras $o(p, q)$, with $p + q = d'$ have the same dimension, which equals in particular the dimension of $o(d')$. Let x be a point of the given pseudo-Riemannian manifold. Its stabilizer algebra can be identified to a subalgebra of some $o(p, q)$, with $p + q \leq d$. But by hypothesis, this stabilizer has a dimension $\geq \dim o(d)$. It then follows that $p + q = d$, and that the stabilizer equals $o(p, q)$ itself. One deduces in particular that the dimension of the manifold equals d . To check that the curvature is constant, one may observe that $O(p, q)$ acts transitively on the space of spacelike 2-planes at x).

This shows in particular that U is open, and that the \mathcal{G} -orbit of any point of U is locally homothetic to \tilde{N} . The orthogonal plane field $x \rightarrow T\mathcal{G}^\perp$ is analytic on U . Consider its second fundamental form, $II : T\mathcal{G}^\perp \times T\mathcal{G}^\perp \rightarrow T\mathcal{G}$. Its vanishing means that the orthogonal is integrable and has geodesic leaves. To check that $II_x = 0$, for $x \in U$, we just use its equivariance under the action of the isotropy algebra $o(1, d-1)$ (in \mathcal{G}) of x . Indeed, $o(1, d-1)$ acts trivially on $T_x\mathcal{G}^\perp$, since it preserves a positive scalar product on it. Therefore, for all $u, v \in T_x\mathcal{G}^\perp$, $II_x(u, v)$ is invariant under the action of $o(1, d-1)$ on $T_x\mathcal{G}$, and hence $II_x = 0$.

Finally, to verify the warped product condition, that is, that any holonomy map of \mathcal{G}^\perp seen as a local diffeomorphisms between two leaves of \mathcal{G} is homothetic, we just observe that this holonomy map commutes with the action of \mathcal{G} on these two leaves. \square

REMARK 4.2. Here (in the fact above) as well as in the next section (§4.1.1) we do not use that \tilde{M} is the universal cover of a compact manifold. We will need this hypothesis at §4.2.

4.1.1 The nonpositively curved case. We have the following stronger result in the nonpositively curved case:

PROPOSITION 4.3. *Let the Lie algebra $\mathcal{G} = \mathcal{G}(\tilde{N})$, where \tilde{N} is a complete simply connected nonpositively curved Lorentz manifold, act isometrically on an analytic Lorentz manifold \tilde{M} (not necessarily the universal cover of a compact manifold), and determine somewhere a local warped product, with a normal factor locally homothetic to \tilde{N} . Then \mathcal{G} determines everywhere a local warped product (\tilde{M} is assumed to be connected).*

Behind the proposition is the existence in the nonpositively curved case, of lightlike Killing fields. They do not exist at all in the positively curved case. The proposition itself is false in this case (see below). Later on (§4.2), we will generalize the proposition to the positively curved case, assuming in addition that \tilde{M} is the universal cover of a compact manifold, and that it is the action related to $\mathcal{W}(M)$.

Lightlike Killing fields. A vector field V on a Lorentz manifold is lightlike (or isotropic) if for all x , $\langle V(x), V(x) \rangle = 0$.

Killing lightlike vector fields have the following remarkable property.

PROPOSITION 4.4 ([BeEM], [ASt1]). *Let V be a nontrivial lightlike Killing field. Then V has no singularity. Furthermore, V has geodesic orbits. (In fact, more generally, V is singularity free, when it is nonspacelike, i.e. $\langle V(x), V(x) \rangle \leq 0$).*

Proof. For the Minkowski space, a Killing vector field vanishing at 0 is a linear Killing vector field. It is tangent to the “pseudo-sphere” $q = \text{constant}$, where q is the Lorentz form. But for a negative constant, this level is spacelike (it is a Riemannian hyperbolic space). Hence, the Killing field is somewhere spacelike, near any neighborhood of 0. The proof in the general case, follows by conjugating by the exponential map \exp_x , where x is assumed to be, by contradiction, a singular point of V .

Let us now show that the orbits of V are geodesic. As a Killing field, V satisfies the following anti-symmetry: $\langle \nabla_V V, U \rangle + \langle \nabla_U V, V \rangle = 0$, for any vector field U . Hence, $\langle \nabla_V V, U \rangle = -(1/2)U \cdot \langle V, V \rangle = 0$, since V is lightlike. Therefore $\nabla_V V = 0$. \square

The following proposition treats lightlike Killing fields on constant curvature Lorentz manifolds. Its proof may be handled by a standard calculation.

PROPOSITION 4.5. *Let $\mathcal{G} = \mathcal{G}(\tilde{N})$ be the Killing algebra of \tilde{N} (as above). Denote by $\mathcal{I} = \mathcal{I}(\tilde{N})$ the subset of \mathcal{G} consisting of lightlike Killing fields. Then,*

- i) If \tilde{N} is the anti-de Sitter space (i.e. it has negative curvature), then \mathcal{I} generates \mathcal{G} as a vector-space.
- ii) If \tilde{N} is the Minkowski space (i.e. it is flat), then the vector space generated by \mathcal{I} equals the radical \mathbf{R}^d of \mathcal{G} . More precisely, $X \in \mathcal{I}$ if and only if X is parallel (i.e. it generates a flow of translations) and is (somewhere) isotropic.
- iii) If \tilde{N} is the de Sitter space, then $\mathcal{I} = \{0\}$.

Beginning. It is crucial to notice that, if $X \in \mathcal{G}$ is lightlike, as a Killing field on \tilde{N} , then the same is true for \bar{X} , as a Killing field on \tilde{M} .

Suppose by contradiction that $U \neq \tilde{M}$ (see Fact 4.1 for notation), and let N_1 be the orbit of a point in the boundary of U . By definition of U , N_1 is lightlike (i.e. the metric on TN_1 is positive nondefinite). If $\dim N_1 \neq 0$, denote by \mathcal{F} its *characteristic* foliation (of dimension 1), i.e. that determined by the direction field $x \in N_1 \rightarrow (T_x N_1)^\perp \cap T_x N$. We denote by Q the (local) quotient space N_1/\mathcal{F} .

FACT 4.6. *If X is a lightlike Killing field, then the restriction of \bar{X} to N_1 is tangent to \mathcal{F} (equivalently, the flow of such a Killing field preserves individually the leaves of \mathcal{F}). Then (from Proposition 4.4), we have $\dim N_1 \geq 1$, and the leaves of \mathcal{F} are lightlike geodesics (in \tilde{M}).*

Proof. A lightlike field X is tangent to \mathcal{F} , because the direction of \mathcal{F} is the unique isotropic direction tangent to N_1 . \square

The anti-de Sitter case. In the case where \tilde{N} is the anti-de Sitter space, \mathcal{I} generates \mathcal{G} , and therefore, \mathcal{G} preserves (individually) the leaves of \mathcal{F} . Hence N_1 has dimension 1 (since N_1 is a \mathcal{G} -orbit). One then verifies that \mathcal{G} cannot act faithfully on such a manifold. One may see this in an easier way in our situation here because N_1 reduces (at least locally) to the orbit of any element $X \in \mathcal{G}$. Thus, it is an isotropic geodesic of \tilde{M} . The action of \mathcal{G} preserves the affine parameter of this geodesic. Therefore, \mathcal{G} would admit an injective homomorphism in the affine group of \mathbf{R} , which is impossible.

The flat case. If N_1 has dimension 1, we get a contradiction as in the anti-de Sitter case. If not, consider the quotient space $Q = N_1/\mathcal{F}$. The \mathcal{G} -action on N_1 , factors through a faithful action of $o(1, d-1)$ ($= \mathcal{G}/\mathbf{R}^d$) on Q . Observe that Q inherits a natural Riemannian metric. Indeed, the Lorentz metric restricted to N_1 is positive degenerate, with kernel $T\mathcal{F}$. But \mathcal{F} is parameterized by any lightlike Killing field $X \in \mathcal{I}$ (this is the meaning of the fact that the flow of \bar{X} preserves individually the leaves of \mathcal{F}). In

particular, the transversal action of the holonomy of \mathcal{F} is equivalent to that of the flow of \bar{X} . Therefore, it preserves the transverse metric, which thus determines a metric on Q . This metric is invariant by the $o(1, d-1)$ -action. As in the proof of Fact 4.1, since $\dim Q \leq d-1$, we have $\dim Q = d-1$, and furthermore, Q has constant curvature. Also, we recognize from the list of Killing algebras of constant curvature manifolds that Q has constant negative curvature, i.e. Q is a hyperbolic space.

It then follows that $\dim N_1 = d$, and in particular that the orbits of \mathcal{G} determine a regular foliation near N_1 . The idea, to find a contradiction, is that, the leaves in U_0 have (intrinsic) constant 0 curvature, but not N_1 , because the quotient space Q is hyperbolic.

To do this in a more rigorous manner, let $X_0 \in \mathcal{I}$ be a lightlike Killing field, and for all $x \in \tilde{M}$, consider Q_x the orbit space of X_0 restricted to $\mathcal{G}x$. In a natural sense, the so obtained quotient Riemannian spaces depend smoothly on $x \in \tilde{M}$. However, for $x \in U_0$, Q_x is Euclidean (because $\mathcal{G}x$ is Minkowskian), but, as stated above for $x \in N_1$, Q_x is hyperbolic. This is a contradiction. Therefore $U = \tilde{M}$.

4.2 The de Sitter case. In this case, $\mathcal{G} = o(1, d)$. Recall the example of the usual action of $o(1, d)$ on the Minkowski space $\mathbf{R}^{1, d}$. This contrasts with the case of nonpositive curvature, because the orbits do not determine a (regular) foliation, since 0 is a fixed point. Observe that there are 3 types of regular orbits: (Lorentzian) de Sitter orbits, (Riemannian) hyperbolic orbits, and the isotropic cone at 0 (without 0), which is lightlike.

In the general case, we have a partition of \tilde{M} into degenerate and nondegenerate orbits. As mentioned in the proof of Fact 4.1, a nondegenerate orbit is a pseudo-Riemannian manifold of dimension $\leq d$, and endowed with a faithful action of $o(1, d)$, and hence, it is (locally and up to a multiple constant) either the (Riemannian) hyperbolic space or the (Lorentz) de Sitter space.

Therefore, we get a partition $D \cup H$ of the set of nondegenerate orbits into de Sitter and hyperbolic orbits, respectively. We write the complementary set as $\tilde{M} - D \cup H = L \cup S$, where S is the subset of singular orbits (i.e. one point orbits), and L is the subset of degenerate but nonsingular orbits.

From Fact 4.1, for all $x \in D$, C_x contains the isotropic cone of $T_x \mathcal{G}x$. We have in fact equality: $C_x = \text{Cone}(T_x \mathcal{G}x)$, by the property of d as a maximal dimension.

Invariance of the \mathcal{G} -foliation under the fundamental group.

Now, our goal is to show that essentially, the \mathcal{G} -foliation passes to M .

Indeed, consider $f \in \pi_1(M)$. Denote by \mathcal{G}' the image by f of the \mathcal{G} -action. It induces an analogous partition $\tilde{M} = D' \cup H' \cup L' \cup S'$. Of course, $C_{fx} = f(C_x)$, and hence $C_y = \text{Cone}(T_y \mathcal{G}'x)$, for $y \in D' = f(D)$. In particular, in the open set $D \cap D'$, the \mathcal{G} and \mathcal{G}' foliations coincide. Therefore, by analyticity, if $D \cap D' \neq \emptyset$, the \mathcal{G} and \mathcal{G}' -foliations coincide everywhere in \tilde{M} . Assume now that $D \cap D'$ is empty and let $x \in D \cap H'$. Consider the stabilizer \mathcal{K} of x for the \mathcal{G}' -action. It is isomorphic to $o(d)$ (the leaf $\mathcal{G}'x$ is isometric to the hyperbolic space \mathbf{H}^d). It preserves C_x and hence $T_x \mathcal{G}x$. Therefore, we get a representation of $o(d)$ in $o(1, d-1)$ (the algebra of orthogonal transformations of the Lorentz space $T_x \mathcal{G}x$). But such a representation must be trivial. Hence \mathcal{K} acts trivially on $T_x \mathcal{G}x$, and thus also on its projection on $T_x \mathcal{G}'x$. This projection is therefore trivial, since the \mathcal{K} -action on $T_x \mathcal{G}'x$ is irreducible. This means that the \mathcal{G} and \mathcal{G}' -orbits of x are orthogonal at x (if $x \in D \cap H'$). This extends by analyticity to all \tilde{M} .

In conclusion, exactly one of two possibilities occurs for the \mathcal{G} and \mathcal{G}' foliations: they coincide everywhere, or they are everywhere orthogonal. But there is a finite number of mutually orthogonal subspaces of dimension d in a tangent space $T_x \tilde{M}$. Therefore, there is a finite index subgroup of $\pi_1(M)$ which preserves the \mathcal{G} -foliation. For the sake of simplicity, we shall suppose that $\pi_1(M)$ itself preserves the foliation.

Structure of orbits. The central flow. A description of orbits, similar to that of the special case of the action of $o(1, d)$ on the Minkowski space $\mathbf{R}^{1,d}$, holds in the general case. Indeed, as was mentioned above, nondegenerate orbits are locally isometric to the de Sitter or to the hyperbolic space of dimension d . It remains to consider the case of lightlike (nontrivial) orbits. Let N_1 be a such orbit. The quotient space Q (of the characteristic foliation of N_1 , see above) is a $o(1, d)$ -homogeneous space of dimension $\leq d-1$. A standard analysis of subalgebras of $o(1, d)$ shows that $d-1$ is the minimal nontrivial dimension of a space on which $o(1, d)$ acts. Furthermore, the minimal dimension is achieved in the case of the usual conformal action on the sphere S^{d-1} .

Therefore, we have either Q is a single point, or Q is (locally) the conformal sphere S^{d-1} . Of course, Q cannot be a single point, since otherwise, $\dim N_1 = 1$, but $o(1, d)$ cannot act nontrivially on \mathbf{R} .

Knowing that Q is the usual conformal sphere, it is not difficult to identify N_1 itself (locally) with the isotropic cone of $\mathbf{R}^{1,d}$, as a $o(1, d)$ -homogeneous *degenerate Riemannian space*.

In the Minkowski space $\mathbf{R}^{1,d}$, the action of $o(1, d)$ on the isotropic cone

commutes with the multiplication flow $(t, x) \rightarrow e^t x$. In a similar way, one constructs a (local) *central flow* $\tilde{\phi}^t$ on D which is tangent the \mathcal{G} -foliation, and commutes with the \mathcal{G} -action. This flow passes to a flow ϕ^t on the projection of D in M .

Structure near the singular set. Let x_0 be a singular point of the \mathcal{G} -action. Then, we get an infinitesimal representation of $o(1, d)$ in $o(1, n-1)$, the orthogonal algebra of $T_{x_0}\tilde{M}$ (n is the dimension of M). In a standard way, one may prove that such a representation is equivalent to the usual inclusion $o(1, d) \subset o(1, n-1)$. In particular, there is an orthogonal decomposition $T_{x_0}M = E \oplus \mathbf{R}^{1,d}$, and the infinitesimal action of $o(1, d)$ is the product of the trivial action on E and the usual action of $o(1, d)$ on $\mathbf{R}^{1,d}$. The exponential map \exp_{x_0} conjugates (locally) the action of $o(1, d)$ on \tilde{M} with its infinitesimal action on $T_{x_0}\tilde{M}$. In particular the set of \mathcal{G} -singular points (near x_0) equals the geodesic spacelike submanifold $F = \exp_{x_0} E$. For $x \in F$, the isotropic cone of $T_x F^\perp$ has two sheets Sh_x^\pm (here we do not mind the possibility of a continuous orientation of these sheets when x runs over the singular set S). The degenerate leaves near x_0 are given by $\exp_x Sh_x^\pm$ for $x \in F$. The (oriented non-parameterized) orbits of $\tilde{\phi}^t$ (near x_0) have the form $\exp_x tu$, $t > 0$, where $u \in Sh_x^\pm$, and $x \in F$. Here, it is essential to observe that F is a repulsor of $\tilde{\phi}^t$. More precisely, for $y \in L$, near x_0 , the orbit $\tilde{\phi}^t(y)$ converges to a point $x \in F$, when $t \rightarrow -\infty$.

From this, one sees that the flow $\tilde{\phi}^t$ can be continuously extended to the \mathcal{G} -singular set S , by letting it act trivially on S . Therefore, the flow $\tilde{\phi}^t$ is now defined on the closed subset $D \cup S$. Its quotient flow ϕ^t is thus defined on a compact space and in particular $\tilde{\phi}^t$ is complete.

Let V be a “conical” neighborhood of S in $S \cup L$, that is, V has the form $V = \cup_{x \in S} V_x$, where V_x is the intersection of the isotropic cone of $T_x S^\perp$, with a ball in $T_x \tilde{S}$ (with respect to any Riemannian metric on \tilde{M}). Suppose that V is π_1 -invariant (choose V to come from a similar neighborhood of the projection of S in M). From the repelling property described above, the complement $L^* = L - V$ is $\tilde{\phi}^t$ -invariant for $t > 0$.

We will get a contradiction by showing that D is (a codimension 1) submanifold of \tilde{M} , and for t big enough, the Jacobian $\text{Jac}(D_x \tilde{\phi}^t)$ is uniformly > 1 , for $x \in D^*$.

Structure of L . From above, the singular set S has codimension at least 3, so $\tilde{M} - S (= D \cup L \cup H)$ is connected. There, the orthogonal plane field of the \mathcal{G} -foliation has constant dimension $n - d$. Therefore, it is analytic, and integrable, and has geodesic leaves (since this is the case

in an open subset of D). We denote by \mathcal{L} its tangent foliation (defined on $\tilde{M} - S$).

For $x \in L$, we have $T_x\mathcal{L} = (T_x\mathcal{G}x)^\perp$. Hence, as the \mathcal{G} -orbit of x , the geodesic leaf \mathcal{L}_x is degenerate. Since it is geodesic, $T_y\mathcal{L}_x$ is degenerate, for all $y \in \mathcal{L}_x$, and thus the \mathcal{G} -orbit of y is degenerate, that is $y \in L$. In conclusion, L is invariant by both the \mathcal{G} and the \mathcal{L} -foliations.

For $x \in L$, the intersection $T_x\mathcal{L} \cap T_x\mathcal{G}x$, is exactly the isotropic direction of $T_x\mathcal{G}x$, which is nothing but the tangent direction of $\tilde{\phi}^t$ at x . In other words, the leaves $\mathcal{G}x$ and \mathcal{L}_x meet along the $\tilde{\phi}^t$ -orbit of x .

From this, we infer that, around any x in \tilde{M} , there is a hypersurface (of \tilde{M}) contained in L . In fact, L which is an analytic set of $\tilde{M} - S$ is a regular hypersurface. To see this, one constructs an analytic distribution Δ , for which L is a union of integral leaves, as follows. Locally, Δ is generated by a family $X_1, \dots, X_d, Y_1, \dots, Y_{n-d}$ of vector fields, where X_1, \dots, X_d (resp. Y_1, \dots, Y_{n-d}) generate the tangent space of the \mathcal{G} -foliation (resp. the tangent space of the foliation \mathcal{L}). From the previous discussion, one sees that L is the union of singular leaves of Δ .

Let g be a π_1 -invariant Riemannian metric on \tilde{M} . Denote by X the vector field generating the flow $\tilde{\phi}$, and for $x \in L$, let $E^u(x)$ (resp. $E^0(x)$) be the orthogonal (with respect to g) of $X(x)$ in $T_x\mathcal{G}x$ (resp. $T_x\mathcal{L}_x$). These two plane fields on L are spacelike (with respect to the Lorentz metric of \tilde{M}). We change the Riemannian metric on L , by equipping E^u and E^0 with the restriction of Lorentz metric, and decreeing that X, E^u and E^0 are orthogonal. With respect to this new Riemannian metric, we have the following relations: $|D\tilde{\phi}^t u| = e^t |u|$ for $u \in E^u$, and $|D\tilde{\phi}^t u| = |u|$ for $u \in E^0$ (and of course $|D\tilde{\phi}^t u| = |u|$ for $u \in \mathbf{R}X$). The first relation comes from the fact that this is the case, in the model space (of $\mathcal{G}x$, which is the isotropic cone of the Minkowski space $\mathbf{R}^{1,d}$, see above).

The second relation follows from the following general property of *lightlike geodesic* submanifolds in Lorentz manifold [Z5]. It is that the (one dimensional) characteristic (isotropic) foliation of such a lightlike submanifold is a transversally Riemannian foliation. A flow parameterizing the characteristic foliation, preserves the degenerate Riemannian metric of the lightlike geodesic submanifold.

Now, the projection of D^* in M is a compact manifold (with boundary), preserved by the (semi-)flow ϕ^t ($t > 0$), and $\text{Jac}(\phi^t) = e^{(d-1)t}$. This is impossible. This means that the lightlike locus L is empty, and therefore, $\tilde{M} = D$, that is all the \mathcal{G} -leaves are of de Sitter type. Thus, by Fact 4.1,

M is everywhere, locally a warped product.

4.3 Completeness.

Global topological product. The foliation \mathcal{G}^\perp is geodesic, in the sense of the Lorentz metric. Therefore, the holonomy of \mathcal{G} preserves the metric on $T\mathcal{G}^\perp$. Thus \mathcal{G} is a transversally Riemannian foliation, since \mathcal{G}^\perp is spacelike.

Transform the Lorentz metric of M into a Riemannian metric, by keeping the same metric on $T\mathcal{G}^\perp$, keeping \mathcal{G} and \mathcal{G}^\perp orthogonal, and choosing any (π_1 -invariant) Riemannian metric on $T\mathcal{G}$.

The foliation \mathcal{G} is still transversally Riemannian in the sense of the new metric, because we have not changed the metric on its orthogonal. It then follows that \mathcal{G}^\perp is geodesic (in the sense of the new Riemannian metric).

Now, we use a result of [BIH] (see also [PR]) which states that if a geodesic foliation \mathcal{L} of a compact Riemannian manifold M admits an orthogonal foliation \mathcal{N} (that is $T\mathcal{L}^\perp$ is integrable), then the pair $(\tilde{\mathcal{L}}, \tilde{\mathcal{N}})$ gives a global topological product in \tilde{M} . More precisely, let $\tilde{\mathcal{L}}_x$ and $\tilde{\mathcal{N}}_x$ be the leaves of a point x of \tilde{M} . Then, the inclusion of $\tilde{\mathcal{L}}_x \cup \tilde{\mathcal{N}}_x$ in \tilde{M} extends to a diffeomorphism $\tilde{\mathcal{L}}_x \times \tilde{\mathcal{N}}_x \rightarrow \tilde{M}$ sending the foliations determined by the factors, to \mathcal{L} and \mathcal{N} , respectively.

A new Lorentz product metric. Let $x_0 \in \tilde{M}$, then \tilde{M} is homeomorphic to $\tilde{\mathcal{L}}_{x_0} \times \tilde{\mathcal{N}}_{x_0}$, endowed with a warped metric $h \oplus wg$, where h (resp. g) is the Riemannian (resp. Lorentzian) metric on $\tilde{\mathcal{L}}_{x_0}$ (resp. $\tilde{\mathcal{N}}_{x_0}$), and $w : \tilde{\mathcal{L}}_{x_0} \rightarrow \mathbf{R}$ is a warping function.

Our aim here is to show that the product metric $h \oplus g$ is $\pi_1(M)$ -invariant (and hence descends to a locally product metric on M). This is equivalent to the invariance of the warping function w by the π_1 -action on $\tilde{\mathcal{L}}_{x_0}$.

If $\tilde{\mathcal{N}}_{x_0}$ is not flat, w is π_1 -invariant, since $w(x) = \kappa(x)^{-2}$, where $\kappa(x)$ is the sectional curvature of $\tilde{\mathcal{N}}_x$.

Now, we give briefly an idea of how to prove the invariance of w in the flat case. Let us start by considering the case $\dim \tilde{\mathcal{L}}_{x_0} = 1$. In M , we have a codimension one transversally Riemannian foliation \mathcal{N} . Let ϕ^t be the flow generated by a unit vector field orthogonal to \mathcal{N} . Then, ϕ^t preserves \mathcal{N} (this is exactly the meaning of \mathcal{N} being transversally Riemannian). In fact, because of the local warped product structure, ϕ^t sends leaves homothetically to leaves. More precisely, let $\tilde{\phi}^t$ be the lift of ϕ^t to \tilde{M} . Then, for $x, y \in \tilde{\mathcal{L}}_{x_0}$, we have $\text{Jac}(\tilde{\phi}_x^t) = \text{Jac}(\tilde{\phi}_x^t | \tilde{\mathcal{N}}) = (w(y)/w(x))^{n-1/2}$, where t is such that $\tilde{\phi}^t(x) = y$ ($n = \dim M$).

There are two possibilities. The first is that all the leaves of \mathcal{N} are

dense. In this case, $\text{Jac}(\phi^t) = \text{constant}$, and hence equals 1, and thus, w is constant. The second case is that all the leaves of \mathcal{N} are closed. We then have: for $x, y \in \tilde{\mathcal{L}}_{x_0}$, $\text{Vol}\mathcal{N}_{\pi(x)}/\text{Vol}\mathcal{N}_{\pi(y)} = (w(x)/w(y))^{n-1/2}$, where $\pi : \tilde{M} \rightarrow M$ is the projection. Therefore, w is π_1 -invariant.

The same proof works in the higher dimensional case if we suppose that there is a parallelism, i.e. a frame of vector fields X^1, \dots, X^k , tangent to \mathcal{L} , preserving the foliation \mathcal{N} , and also preserving the volume along \mathcal{L} . But, such a parallelism is exactly what Molino's theory on Riemannian foliations [Mo] yields, up to passing to another foliation naturally associated to \mathcal{N} .

Completeness along the constant curvature factor. Recall that compact Lorentz manifolds of constant curvature, are (geodesically) complete. This result was proved by Y. Carrière [C], in the flat case, and then the proof was adapted to the general case by B. Klingler [K]. Our observation here is that the Carrière's proof may be easily updated to handle the case of compact Lorentz manifolds whose universal cover is a global (direct) product of a Lorentz manifold of constant curvature by a Riemannian manifold. The point is to develop, \tilde{M} which is the product $\tilde{\mathcal{L}}_{x_0} \times \tilde{\mathcal{N}}_{x_0}$, into the product $\overline{M} = \tilde{\mathcal{L}}_{x_0} \times \tilde{N}$ where \tilde{N} is the simply connected Lorentz manifold with the same curvature as $\tilde{\mathcal{N}}_{x_0}$. Then, instead of triangles, as used in the Carrière's proof, we use subsets of \overline{M} of the form $B \times \Delta$, where B is a ball of $\tilde{\mathcal{L}}_{x_0}$ and Δ is a triangle in \tilde{N} . This leads to the conclusion that \tilde{M} is isometric to \overline{M} .

Completeness. We infer from [RoS], that M , endowed with the old warped Lorentz metric, is complete. Indeed the warping function w is bounded, since it is π_1 -invariant.

End of the proof of Theorem 1.2. It remains to show that the factor \tilde{N} has nonpositive (constant) curvature. This fact was observed in part I, §15, as being a slight generalization of the Calabi-Markus phenomena.

5 Analytic Bi-polarized Manifolds. Proof of the Second Half of Theorem 1.3

The possibilities (1) and (2) of Theorem 1.3, correspond to the cases $\mathcal{W}(M) \neq \emptyset$, and $\mathcal{W}(M) = \emptyset$, respectively. The structure of M when $\mathcal{W}(M) \neq \emptyset$ follows from Theorem 1.2. The goal of the present section is to study the other situation. So, let M be a compact Lorentz manifold, with $\text{Isom } M$

noncompact, and $\mathcal{W}(M) = \emptyset$. We will show that, in the analytic case, M satisfies the description stated in point (2) of Theorem 1.3. As was said in the introduction, the noncompactness of $\text{Isom } M$ implies that $\text{card } C_x \geq 1$ for all $x \in M$ (this follows from part I, or a direct proof).

Bi-polarization. In what follows, except for Proposition 5.1, M will be supposed to be analytic. In fact, in the proof of this proposition itself, we will use the fact that M_{smooth} is dense in M . This was observed to be true in the analytical case (see 3.7), but its proof in the smooth case is harder ([Z6]). Also, from the latter reference, we infer that one may slightly change the definition of M_{smooth} so that it remains open and dense, and not only the map $x \rightarrow E_x$ (the space generated by C_x) is smooth, but the map $x \rightarrow C_x$ itself is semi-continuous in an obvious manner. Again, in the analytic case, this fact follows from that the integrability domain is an analytic set.

PROPOSITION 5.1. *Let M be a compact Lorentz manifold, with $\text{Isom } M$ noncompact, and $\mathcal{W}(M) = \emptyset$. Then, $1 \leq \text{card } C_x \leq 2$, for all $x \in M_{smooth}$.*

In particular M is bipolarized, that is, $\text{Isom } M$ preserves two (perhaps identical) lightlike geodesic foliations \mathcal{F}_1 and \mathcal{F}_2 (see [Z1, §11] for details).

In fact, in M_{smooth} , we have $C_x = \{T_x\mathcal{F}_1^\perp, T_x\mathcal{F}_2^\perp\}$.

Proof. From Theorem 3.6, we infer, since $\mathcal{W}(M) = \emptyset$, that $\text{int}(M_{\leq \dim M})$ is dense in M_{smooth} . Consider $A = \cup\{C_x/x \in \text{int}(M_{\leq \dim M})\}$. This is an $\text{Isom } M$ -invariant subset of the projective isotropic tangent bundle \mathbf{PT}^0M , with finite fibers A_x (over M). Furthermore, it is measurable, since $x \rightarrow C_x$ is semi-continuous [Z6]. From the barycenter construction (Theorem 2.6 of part I), $\text{Isom } M$ would be compact, if we didn't have $\text{card } A_x \leq 2$ almost everywhere. This implies that almost everywhere, $\dim E_x \leq 2$. This extends to all M_{smooth} , since the map $x \rightarrow E_x$ is smooth, and hence $\text{card } C_x \leq 2$, for all $x \in M_{smooth}$.

In particular, the fibers of the limit set Λ of $\text{Isom } M$ in \mathbf{PT}^0M also satisfy: $\text{card } \Lambda_x \leq 2$. Therefore $\text{Isom } M$ is elementary, or equivalently, M is bipolarized (see part I, §11).

Finally, let us check the equality $C_x = \{T_x\mathcal{F}_1^\perp, T_x\mathcal{F}_2^\perp\}$, along M_{smooth} . We have seen that, $\dim E_x \leq 2$, for $x \in M_{smooth}$. Therefore, if x belongs to the transversality set $T = \{x \in M \mid T_x\mathcal{F}_1^\perp \neq T_x\mathcal{F}_2^\perp\}$, then, $\text{card}\{T_x\mathcal{F}_1^\perp, T_x\mathcal{F}_2^\perp\} = 2$, and hence $C_x = \{T_x\mathcal{F}_1^\perp, T_x\mathcal{F}_2^\perp\}$. This extends by continuity to the closure of T in M_{smooth} .

In the coincidence set $C = M - T$, we have $\text{card}\{T_x\mathcal{F}_1^\perp, T_x\mathcal{F}_2^\perp\} = 1$. Suppose that in some (open) component U of $\text{int}(M_{smooth} \cap C)$, we have

card $C_x = 2$. Then, we get a continuous $\text{Isom } M$ -invariant isotropic direction field X along U , such that $C_x = \{T_x\mathcal{F}_1^\perp = T_x\mathcal{F}_2^\perp, X(x)\}$. This contradicts the “source-sink” dynamical behavior of the derivative action of $\text{Isom } M$ on \mathbf{PT}^0M . Indeed, on U , the dynamics has only one “pole” $T\mathcal{F}_1^\perp|U = T\mathcal{F}_2^\perp|U$. Thus, for an $\text{Isom } M$ -recurrent point x (which exists since U is open), we must have $X(x) = T_x\mathcal{F}_1^\perp = T_x\mathcal{F}_2^\perp$, which contradicts the definition of X . This finishes the proof of the proposition. \square

Local bipolarization in the analytic case. The foliations \mathcal{F}_1 and \mathcal{F}_2 determine two sections of the Grassmann bundle $Gr^0(M) \rightarrow M$. Their images $P(\mathcal{F}_1)$ and $P(\mathcal{F}_2)$ are (topological) submanifolds of $Gr^0(M)$, homeomorphic to M . Let \mathcal{D}^0 denote $P(\mathcal{F}_1) \cup P(\mathcal{F}_2)$.

Consider \mathcal{D} , the integrability domain of τ on $Gr^0(M)$. Obviously, $\mathcal{D}^0 \subset \mathcal{D}$.

FACT 5.2. *Keep the hypotheses of the previous proposition, and assume that M is analytic. Let $p \in \mathcal{D} - \mathcal{D}^0$, and $\Sigma \subset Gr^0(M)$, a small transversal to the tautological plane field τ , containing p . Then p is isolated in $\Sigma \cap \mathcal{D}$.*

Proof. Suppose the contrary. Then, by analyticity, $\Sigma \cap \mathcal{D}$ contains a curve, which is in fact contained in $\mathcal{D} - \mathcal{D}^0$ if Σ is small enough (since $\mathcal{D} - \mathcal{D}^0$ is open in \mathcal{D}). In other words, we obtain a one parameter family of lightlike geodesic hypersurfaces, which are not leaves of \mathcal{F}_1 or \mathcal{F}_2 . This one parameter family fills, at least, some open set U , say. This contradicts the validity of the equality $C_x = \{T_x\mathcal{F}_1^\perp, T_x\mathcal{F}_2^\perp\}$, in the open dense set M_{smooth} . \square

COROLLARY 5.3. *We have: $\mathcal{D} = \mathcal{D}^0$ (equivalently, $C_x = \{T_x\mathcal{F}_1^\perp, T_x\mathcal{F}_2^\perp\}$, for all $x \in M$).*

Proof. We deduce from the fact above that $\mathcal{D} - \mathcal{D}^0$ is closed in \mathcal{D} . Indeed, let $p \in \mathcal{D}^0$, and Σ a transversal as above. Then, from the previous fact, $\Sigma \cap (\mathcal{D} - \mathcal{D}^0)$ is open and discrete in the analytic set $\Sigma \cap \mathcal{D}$. This latter set has finitely many connected components. Therefore, $\Sigma \cap (\mathcal{D} - \mathcal{D}^0)$ must be finite. In particular p cannot be an accumulation point of $\mathcal{D} - \mathcal{D}^0$, so $\mathcal{D} - \mathcal{D}^0$ is closed.

It then follows that $\mathcal{D} - \mathcal{D}^0$ consists of a finite union of closed leaves of τ . By projecting in M , we get closed lightlike geodesic hypersurfaces S^1, \dots, S^k . Up to a subgroup of finite index, we may suppose that $\text{Isom } M$ preserves each hypersurface S^i . Observe that S^i is nowhere tangent to \mathcal{F}_1 (or \mathcal{F}_2) since otherwise, it would be a leaf of this foliation.

As in the proof of the proposition above, for $f \in \text{Isom}(M)$, the action of Df on the projective isotropic cone $\mathbf{PT}^0M|S^i$, along S^i , preserves $(TS^i)^\perp$,

which contradicts the fact that it is a source-sink dynamics determined by the attractor-repulsor pair $((T\mathcal{F}_1)^\perp|S^i, (T\mathcal{F}_2)^\perp|S^i)$. \square

Regularity. The analytic set \mathcal{D} equals $P(\mathcal{F}_1) \cup P(\mathcal{F}_2)$, a union of two topological manifolds. As above, let Σ be a transversal to τ , at a point $p \in \mathcal{D}$, and consider the 1-dimensional (local) analytic set $A = \mathcal{D} \cap \Sigma$. Topologically, A is a union of one or two topological curves, depending on the projection of p in M belongs to the transversality set T , or to the coincidence set C respectively (recall that C is the closed subset of M where \mathcal{F}_1 and \mathcal{F}_2 are tangent).

From the structure theory of 1-dimensional (real) analytic sets ([Ch]), A is a union finitely many branches, i.e. images of analytic curves. Here, an analytic curve means an analytic injective map from an interval $] - \epsilon, +\epsilon[$ to $Gr^0(M)$, and sending 0 to p . In our situation, there are two analytic curves c_1 and c_2 , such that $A = \text{Image}(c_1)$ if p projects on T , and $A = \text{Image}(c_1) \cup \text{Image}(c_2)$ if p projects on C .

One starts (if necessary) by permuting branches, or equivalently changing the foliations \mathcal{F}_1 and \mathcal{F}_2 , so that, above the coincidence set C , we have: $\Sigma \cap P(\mathcal{F}_1) = \text{Image}(c_1)$ and $\Sigma \cap P(\mathcal{F}_2) = \text{Image}(c_2)$. One sees that this manipulation gives rise to new foliations (always denoted \mathcal{F}_1 and \mathcal{F}_2) which coincide with the former ones in each side of the coincidence set C . Observe furthermore that the new decomposition $\mathcal{D} = P(\mathcal{F}_1) \cup P(\mathcal{F}_2)$ is canonical, since the decomposition $A = \text{Image}(c_1) \cup \text{Image}(c_2)$ is canonical (although there are no canonical parameterizations c_1 and c_2).

Now, let us consider one of the foliations, say \mathcal{F}_1 , and perform some change of the induced analytic structure of $P(\mathcal{F}_1)$, so that it becomes an analytic manifold. Consider as above a transversal set $A = \Sigma \cap P(\mathcal{F}_1) = \text{Image}(c_1)$. Then, p is a regular point of $P(\mathcal{F}_1)$, if and only if p is a regular point of A , if and only if we may choose c_1 having 0 (the pre-image of p) as an immersion point.

If p is a singular point, we choose a less singular curve c_1 , and define an (abstract) analytic structure on A such that c_1 is a parameterization (i.e. c_1^{-1} is a chart). One then changes the analytic structure of $P(\mathcal{F}_1)$ near p accordingly. More precisely, in $P(\mathcal{F}_1)$, p admits a neighborhood which is the image of a foliated homeomorphic analytic map $\phi : (t, u) \in I \times \mathbf{R}^{n-1} \rightarrow (c_1(t), f(u)) \in P(\mathcal{F}_1)$, where f is an analytic diffeomorphism. Then we decree that ϕ^{-1} is a chart for the new analytic structure. One then can verify that this is a well defined analytic structure on $P(\mathcal{F}_1)$, for which \mathcal{F}_1 (or equivalently the plane field τ) is analytic. Endow M with the pull-back

of the so defined structure on $P(\mathcal{F}_1)$ by the section map $x \rightarrow T_x\mathcal{F}_1$. Then \mathcal{F}_1 is analytic with respect to this structure. This finishes the proof of the second half of Theorem 1.3.

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