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ABSTRACT

Kundt spacetimes are of great importance to general relativity. We show that a Kundt spacetime is a Lorentz manifold with a non-singular isotropic geodesic vector field having its orthogonal distribution integrable and determining a totally geodesic foliation. We give the local structure of Kundt spacetimes and some properties of left invariant Kundt structures on Lie groups. Finally, we classify all left invariant Kundt structures on three-dimensional simply connected unimodular Lie groups.

I. INTRODUCTION

Kundt spacetimes are of great importance to general relativity, as well as alternative gravity theories. To begin with, let us say that Kundt spacetimes constitute a natural generalization of pp-wave spacetimes. Roughly speaking, a Kundt spacetime is defined by the fact that it supports a vector field; all of its scalar invariants vanish, without being Killing (see, for instance, Ref. 1, Chapter 6). One of our motivations here is to provide a coordinate-free treatment of Kundt spacetimes, which is hard to find in the general relativity literature. More precisely, let us define a Kundt spacetime as a Lorentz manifold (M, g) having the following property (see, for instance, Refs. 2, 4, 5, and 12):

(K1) There exists a non-singular vector field V on M such that

$$g(V,V) = g(\nabla_V V, \nabla_V V) = \mathrm{tr}(A^V) = g(B^V, B^V) = g(C^V, C^V) = 0, \tag{1}$$

where ∇ is the Levi-Cività connection, $A^V : TM \to TM$ denotes the (1,1) tensor field given by $A^V(X) = \nabla_X V$, B is its symmetric part, and C is its skew-symmetric part, and if $F : TM \to TM$ is a (1,1)-tensor field, then $g(F,F) = \mathrm{tr}(F \circ F^*)$, where F^* is its adjoint with respect to g.

It turns out (see Proposition 2.1) that this property is equivalent to the following:

(K2) There exist a non-singular vector field V on M and a differential one-form α such that

$$g(V,V) = 0, \quad \nabla_X V = \alpha(X)V, \quad \text{and} \quad \nabla_V V = 0 \tag{2}$$

for any vector field X orthogonal to V.

A fundamental observation for us was (see Proposition 2.1 and more details in Secs. II, III, and IV) that Kundt properties (K1) or (K2) imply the following property (see also Ref. 3 for a somewhat similar approach):

(LK) There exists on M a codimension one totally geodesic foliation, which is degenerate with respect to g. More precisely, there exists a codimension one foliation $\mathcal{F}$ such that each leaf L of $\mathcal{F}$ is a lightlike (totally) geodesic hypersurface, that is, $T^{\perp}L \subset TL$ and any geodesic $\gamma : [a, b] \to M$, $a < 0 < b$, somewhere tangent to L, is locally contained in L: if $\gamma'(0) \in T_{\gamma(0)}L$, then there exists $\epsilon > 0$ such that $\gamma([-\epsilon, +\epsilon]) \subset L$.

Actually, until assuming the direction field $T^{\perp}\mathcal{F}$ orientable (which is always possible up to passing to a double covering), the property (LK) is equivalent to the following:

(LKbis) there exist a non-singular vector field V on M and a differential one-form α such that

$$g(V, V) = 0, \nabla_X V = \alpha(X)V \tag{3}$$

for any vector field X orthogonal to V.

We will refer to a Lorentz manifold satisfying (LK) as a *local Kundt spacetime*. Indeed, (LK) implies (K1) locally near any point in M. In other words, a spacetime admitting a codimension one lightlike geodesic foliation is locally Kundt.

Another major motivation to study Kundt Lorentz manifolds lies in their relation to CSI-spaces; those having all of their scalar curvature invariants are constant. The simplest one is the scalar curvature Scal_g, but one can also consider eigenvalues of the Ricci operator Ric_g or the g-norm of the Riemann tensor Rm_g. All those are scalar curvature invariants of order 1. Higher order ones are obtained by considering covariant derivatives of Rm_g. Hence, CSI means, in particular, that all these quantities are constant functions on M.

Locally homogeneous spaces are CSI, and the existence of CSI spaces that are not locally homogeneous spaces is a non-Riemannian phenomenon, which makes a one major difference between the positive and non-definite cases in pseudo-Riemannian structures. A typical example is given by the (conformally flat) plane wave metric $g = dx^2 + dy^2 - 2dvdu - 2f(u)(x^2 + y^2)du^2$, which is VSI (i.e., has vanishing scalar invariants) for any f, but locally homogeneous for only few f's. This example is Kundt; in fact, $V = \frac{\partial}{\partial v}$ is a parallel vector field.

It is believed that a CSI Lorentz space, if it is not locally homogeneous, must be of Kundt type. This conjecture has been proved in some cases, e.g., in lower dimensions 3 and 4; see, for instance, Refs. 6–8. In another direction, there is a notion of $\mathcal{I}$-degenerate metrics, meaning that they have non-trivial (i.e., non-isometric) deformations keeping all the scalar curvature invariant functions the same (not-depending on the deformation parameter, but maybe depending on the point of M); those are believed to be Kundt too.[9,10]

Not all locally homogeneous spacetimes are Kundt; neither all Kundt spacetimes are locally homogeneous, but it is worthwhile to consider locally homogeneous Kundt spaces as a special class of both the homogeneous and Kundt categories. Our project is to study Kundt structures on three-dimensional Lie groups G endowed with a left invariant Lorentzian metric g. It is natural, in this special homogeneous framework, to introduce a stronger Kundt property as follows. We call (G, g) a *Kundt Lie group* (respectively, *locally Kundt Lie group*) if it admits a non-singular left invariant vector field V satisfying (K2) [respectively, (LKbis)]. In other words, we assume here compatibility between the (K2) or (LKbis) property and the algebraic structure of G.

A. Results

One of our principal results, Theorem 4.1, states, essentially, that a three-dimensional Lorentz group, which is Kundt as a spacetime, is, in fact, a Kundt group. We also classify, up to isometric isomorphism, all unimodular three-dimensional Kundt groups.

This paper is organized as follows. In Sec. II, we provide a synthetic (coordinate-free) account on Kundt spacetimes emphasizing on their relationship with lightlike geodesic foliations. We introduce Kundt groups and general facts about them in Sec. III. The Proof of Theorem 4.1 and further results are given in Sec. IV. Section V contains the classification up to automorphism of Kundt Lorentz groups.

II. KUNDT SPACETIMES AND GEODESIC FOLIATIONS

Recall from the Introduction that a Kundt spacetime is a Lorentz manifold satisfying the property (K1). Let us prove that this property is equivalent to (K2) and implies (LK). Moreover, (LK) implies (K1) locally near any point of the Lorentz manifold.

Proposition 2.1. Let (M, g) be a Lorentz manifold. Consider the following assertions:

(i) *(M, g) is a Kundt spacetime.*
(ii) *There exist on M an isotropic non-singular vector field V and a differential one-form α such that, for any $X \in \Gamma(V^{\perp})$,*

$$\nabla_X V = \alpha(X)V \quad \text{and} \quad \nabla_V V = 0.$$

(iii) *There exists on M a totally geodesic codimension one foliation, which is degenerate with respect to g. This means that there exists a vector sub-bundle $F \subset TM$ of rank $(\dim M) - 1$, where the restriction of g to F is degenerate, and for any $X, Y \in \Gamma(F)$, $\nabla_X Y \in \Gamma(F)$, where ∇ is the Levi-Cività connection of g.*

Then, (i) and (ii) are equivalent and both imply (iii). Moreover, (iii) implies that (ii) holds in a neighborhood of any point in M.

Proof. $(i) \Rightarrow (ii)$. Assume that (M, g) is a Kundt spacetime. This means that there exists a non-singular vector field V satisfying (1). Fix a point $p \in M$, and denote by A the endomorphism given by $Au = \nabla_u V$ for any $u \in T_pM$. Choose an isotropic vector $U \in T_pM$ such that $g(U, V) = 1$ and an orthonormal basis $(e_1, \ldots, e_{n-2})$ of $\{U, V_p\}^\perp$. Note that

$$g(A(V), A(V)) = g(V_p, V_p) = g(A(V), V) = 0.$$

Thus, the vector subspace span$\{A(V), V\}$ is totally isotropic; hence, its dimension equals 1, which means $A(V) = \alpha_0 V$ for some $\alpha_0 \in \mathbb{R}$. On the other hand, since $g(V, V) = 0$, for any $u \in T_pM$, $g(A(u), V) = g(\nabla_u V, V) = 0$; hence, $A(T_pM) \subset V^\perp$. This implies that for any $u \in T_pM$, $g(A(u), A(u)) \geq 0$ and $g(A(u), A(u)) = 0$ if and only if $A(u) = \alpha(u)V$. Now, from (1),

$$0 = \mathrm{tr}(A^* A) = 2g(A(V), AU) + \sum_{i=1}^{n-2} g(A(e_i), A(e_i)) = \sum_{i=1}^{n-2} g(A(e_i), A(e_i)).$$

Moreover, we have

$$0 = \mathrm{tr}(A) = g(AV, U) + g(AU, V) + \sum_{i=1}^{n-2} g(Ae_i, e_i) = \alpha_0.$$

Hence, $\nabla_V V = 0$. This completes the proof of $(i) \Longrightarrow (ii)$.

Let us prove $(ii) \Longrightarrow (i)$. Fix a point p, and consider A and $(U, V, e_2, \ldots, e_{n-2})$ as defined above. We have $A(T_pM) \subset \mathbb{R}V$. Hence, $V^\perp \subset \ker A^*$. With this fact in mind, we get

$$\mathrm{tr}(A) = g(AU, V) + g(AV, U) + \sum_{i=2}^{n} g(Ae_i, e_i) = 0,$$

$$\mathrm{tr}(BB^*) = 2g(BU, BV) + \sum_{i=2}^{n} g(Be_i, Be_i) = 0,$$

$$\mathrm{tr}(CC^*) = 2g(CU, CV) + \sum_{i=2}^{n} g(Ce_i, Ce_i) = 0.$$

This completes the proof of $(i) \Longrightarrow (ii)$.

Let us prove now that $(ii) \Longrightarrow (iii)$. Let $F = V^\perp$. We have for any $X, Y \in \Gamma(F)$,

$$g(\nabla_X Y, V) = -g(Y, \nabla_X V) = -\alpha(X) g(Y, V) = 0.$$

Hence, $\nabla_X Y \in \Gamma(F)$. This shows that F is integrable and defines a degenerate codimension one totally geodesic foliation.

Now, we prove that if (iii) holds, then for any $p \in M$, there exists a vector field V near p satisfying (1). Hence, suppose that there exists an integrable degenerate codimension one sub-bundle $F \subset TM$, which defines a totally geodesic foliation. Fix a point $p \in M$, and choose a non-singular vector field $V \in \Gamma(F^\perp) \subset \Gamma(F)$ near p. It is obvious that V is isotropic. Moreover, since $\nabla_V V \in \Gamma(F)$ and $g(\nabla_V V, V) = 0$, then $\nabla_V V = \alpha V$. As above, denote by A the endomorphism A_p^V and choose a basis $(u, V_p, e_1, \ldots, e_{n-2})$. We have, obviously, that $A(T_pM) \subset F_p$. Moreover, since F is totally geodesic for any $X, Y \in \Gamma(F)$,

$$0 = g(\nabla_X Y, V) = -g(Y, \nabla_X V),$$

which implies that $A(F_p) \subset \mathbb{R}V$. So far, we have shown that locally near p for any $X \in \Gamma(F)$,

$$AV = \alpha_0 V, \qquad A(X) = \alpha(X) V.$$

To finish the proof, we look for a vector field $V' = e^f V$, where f is a function such that V' satisfies (ii). This is equivalent to $V(f) = -\alpha_0$, and such a function exists locally. □

A. Kundt coordinates

Another way to compare the Kundt property with the existence of a codimension one lightlike geodesic foliation is given by the following fact, which asserts the existence of adapted local coordinates associated with lightlike geodesic foliations, where the metric has a special form.[12] The same adapted coordinates are known to characterize Kundt spacetimes.

Proposition 2.2. Let (M,g) be a Lorentz manifold satisfying (iii) of Proposition 2.1. Then, near any point in M, there exists a local coordinates system $(v,u,x=(x^2,\ldots,x^n))$ where the metric has the following form:

$$g = 2dudv + H(v,u,x)du^2 + \sum_{i=2}^{n} W_i(v,u,x)dudx^i + \sum_{i,j} h_{ij}(u,x)dx^i dx^j.$$

Remarks 2.1.

- *Observe that the functions h_{ij} do not depend on v.*
- *The foliation in Proposition 2.1 corresponds to the (local) u-levels.*
- *One can also show the converse that a foliation admitting an adapted chart where the metric has such a form is lightlike geodesic.*

Proof. Suppose that there exists a vector sub-bundle F of TM of rank $(\dim M)-1$ such that the restriction of the metric to F is degenerate and $\Gamma(F)$ is stable by the Levi-Cività product.

Fix a point $p\in M$, and let Σ be a local hypersurface containing p and transversal to $F^\perp$. Then, $F_\Sigma = F\cap T\Sigma$ determines a foliation on Σ. Hence, there exists a coordinates system $(x^2,\ldots,x^n,u)$ on Σ such that the leaves of F_Σ are the u-levels. There is a section $T:\Sigma\longrightarrow (F^\perp)_{|\Sigma}$ such that $g(T,\frac{\partial}{\partial u})=2$. Choose an injective immersion $\phi:\mathbb{R}^{n-1}\longrightarrow M$ such that $\phi(\mathbb{R}^{n-1})=\Sigma$. Then, there exists $\epsilon>0$ such that the map $\Phi:\mathbb{R}^{n-1}\times(-\epsilon,\epsilon)\longrightarrow M$ given by $\Phi(t,s)=\exp_{\phi(t)}(sT)$ is a diffeomorphism into its image. Denote by V the image by Φ of the vector field $\frac{\partial}{\partial s}$. Since $F^\perp$ is totally geodesic, then V is tangent to $F^\perp$ and, hence, satisfies $g(V,V)=0$. By construction, we have $\nabla_V V=0$, and according to the Proof of Proposition 2.1 for any $X\in\Gamma(F)$, $\nabla_X V=\alpha(X)V$.

On the other hand, the vector fields $\frac{\partial}{\partial u},\frac{\partial}{\partial x^2},\ldots,\frac{\partial}{\partial x^n}$ on Σ define a family of vector fields on $\mathbb{R}^{n-1}$ and, hence, define a family of vector fields on $\mathbb{R}^{n-1}\times(-\epsilon,\epsilon)$, which commute with $\frac{\partial}{\partial s}$. Let $U,X_2,\ldots,X_n$ be their images by Φ. We deduce that $V,U,X_2,\ldots,X_n$ are commuting and give rise to a local coordinate system $(v,u,x^2,\ldots,x^n)$ on M such that

$$V=\frac{\partial}{\partial v},\qquad U=\frac{\partial}{\partial u},\quad \text{and}\quad X_i=\frac{\partial}{\partial x_i},\quad i=2,\ldots,n.$$

Observe now that, for any vector field Z commuting with V, the scalar product $g(V,Z)$ is constant along the V-trajectories. Indeed, since $g(V,V)=0$ and $[Z,V]=0$, we get

$$V.g(V,Z)=g(\nabla_V V,Z)+g(V,\nabla_V Z)=g(V,\nabla_Z V)=0.$$

We deduce that for any $i=2,\ldots,n$, $g(V,X_i)$ and $g(U,V)$ are constant along the trajectories of V, and since they are constant along Σ, we get that $g(U,V)=2$ and $g(V,X_i)=0$. Moreover, we have

$$V.g(X_i,X_j)=g(\nabla_V X_i,X_j)+g(X_i,\nabla_V X_j)=g(\nabla_{X_i}V,X_j)+g(X_i,\nabla_{X_j}V)=\alpha(X_i)g(V,X_j)+\alpha(X_j)g(X_i,V)=0.$$

This completes the proof. □

Remark 1. For a general codimension one lightlike foliation, not necessarily geodesic, we have similar adapted coordinates, but with the functions h_{ij} also depending on v.

III. KUNDT GROUPS

A *Lorentz Lie group* is a Lie group G endowed with a left invariant Lorentzian metric g. Denote by $\mathfrak{g}$ the Lie algebra of G and $\langle\ ,\ \rangle = g(e)$. We call $(\mathfrak{g},\langle\ ,\ \rangle)$ a Lorentz Lie algebra. The Levi-Cività product is the product $\bullet$ on $\mathfrak{g}$ given by

$$2\langle u\bullet v,w\rangle=\langle[u,v],w\rangle+\langle[w,u],v\rangle+\langle[w,v],u\rangle,\quad u,v,w\in\mathfrak{g}. \tag{4}$$

A *Kundt Lie group* (respectively, *locally Kundt Lie group*) is a Lorentz Lie group (G,g) having an isotropic left invariant vector field satisfying (2) [respectively, (3)].

Proposition 3.1. Let (G,g) be a connected Lorentz Lie group. Then, the following are equivalent:

(i) *(G,g) is a locally Kundt Lie group.*
(ii) *There exists a codimension one subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, which is degenerate and stable by the Levi-Cività product.*

Moreover, if (G,g) is a locally Kundt Lie group, then it is a Kundt Lie group if and only if, for any generator e of $\mathfrak{h}^\perp$, $e\bullet e=0$.

Proof. Let us prove that (i) implies (ii). (G, g) is a locally Kundt Lie group if and only if there exists a left invariant vector field V satisfying (3). The codimension one subspace $\mathfrak{h} = V(e)^{\perp}$ is degenerate, and for any $u, v \in \mathfrak{h}$, denote by u^l and v^l the corresponding left invariant vector fields. Then,

$$g(\nabla_{u^l} v^l, V) = u^l.g(v^l, V) - g(v^l, \nabla_{u^l} V) = -\alpha(u^l) g(v^l, V) = 0.$$

Hence, $\nabla_{u^l} v^l(e) = u \bullet v \in \mathfrak{h}$. This means that $\mathfrak{h}$ is stable by the Levi-Cività product, which completes the proof of $(i) \Rightarrow (ii)$.

Let us show now that $(ii) \Rightarrow (i)$. Suppose that there exists $\mathfrak{h}$, a codimension one degenerate subalgebra of $\mathfrak{g}$, which is stable by the Levi-Cività product, and consider v as a generator of $\mathfrak{h}^{\perp}$. Denote by V the left invariant vector field associated with v. Then, according to the Proof of Proposition 2.1, we have, for any $x \in \mathfrak{h}$,

$$\nabla_{x^l} V = \alpha(x^l) V \quad \text{and} \quad \nabla_V V = \alpha_0 V,$$

where α_0 is a constant. The last assertion is obvious. □

Definition 3.1. Let $\mathfrak{g}$ be a Lie algebra. A Kundt pair on $\mathfrak{g}$ is a pair $(\langle\ ,\ \rangle, \mathfrak{h})$, where $\langle\ ,\ \rangle$ is a Lorentzian product on $\mathfrak{g}$ and $\mathfrak{h}$ is a $\langle\ ,\ \rangle$-degenerate codimension one subalgebra stable by the Levi-Cività product $\bullet$ given by (4), and for any $e \in \mathfrak{h}^{\perp}$, $e \bullet e = 0$.

There is a large class of Lie groups, which cannot carry a locally Kundt group structure.

Lemma 3.1. Let $\mathfrak{g}$ be a semi-simple compact Lie algebra. Then, $\mathfrak{g}$ cannot have a codimension one subalgebra.

Proof. Suppose that $\mathfrak{g}$ has a codimension one Lie subalgebra $\mathfrak{h}$. Since $\mathfrak{g}$ is compact, it carries a bi-invariant scalar product $\langle\ ,\ \rangle$, i.e., ad_x is skew-symmetric for any $x \in \mathfrak{g}$. For any $x \in \mathfrak{h}$, ad_x leaves $\mathfrak{h}$ invariant, and since it is skew-symmetric, it leaves $\mathfrak{h}^{\perp} = \mathbb{R}e$ invariant, so $[x, e] = 0$ and e is central, which contradicts the fact that $\mathfrak{g}$ is semi-simple. □

Corollary 3.1. Let G be a compact semi-simple Lie group. Then, G cannot carry any structure of the locally Kundt Lie group.

The oscillator group is named so by Streater in Ref. 13 as a four-dimensional connected, simply connected Lie group, whose Lie algebra (known as the oscillator algebra) coincides with the one generated by the differential operators, acting on functions of one variable, associated with the harmonic oscillator problem. The oscillator group has been generalized to any even dimension $2n \geq 4$, and several aspects of its geometry have been intensively studied, both in differential geometry and in mathematical physics (see Refs. 14–19). Oscillator Lie groups have a natural left invariant Kundt structure.

Example 1. For $n \in \mathbb{N}^*$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ with $0 < \lambda_1 \leq \cdots \leq \lambda_n$, the λ-oscillator group, denoted by G_λ, is the Lie group with the underlying manifold $\mathbb{R}^{2n+2} = \mathbb{R} \times \mathbb{R} \times \mathbb{C}^n$ and product

$$(t, s, z).(t', s', z') = \left(t + t', s + s' + \frac{1}{2} \sum_{j=1}^{n} Im[\bar{z}_j \exp(it\lambda_j) z_j'], \ldots, z_j + \exp(it\lambda_j) z_j', \ldots \right).$$

Its Lie algebra $\mathfrak{g}_\lambda$ is $\mathbb{R} \times \mathbb{R} \times \mathbb{C}^n$ with its canonical basis $\mathbb{B} = \{e_{-1}, e_0, e_j, \check{e}_j\}_{j=1,\ldots,n}$ such that

$$e_{-1} = (1, 0, 0), \qquad e_0 = (0, 1, 0), \qquad e_j = (0, 0, (0, \ldots, 1, \ldots, 0)), \quad \text{and} \quad \check{e}_j = (0, 0, (0, \ldots, \imath, \ldots, 0)),$$

and the Lie brackets are given by

$$[e_{-1}, e_i] = \lambda_i \check{e}_i, \qquad [e_{-1}, \check{e}_i] = -\lambda_i e_i, \qquad [e_i, \check{e}_i] = e_0 \tag{5}$$

for $i = 1, \ldots, n$. The unspecified products are either given by antisymmetry or zero. For $x \in \mathfrak{g}_\lambda$, let

$$x = x_{-1} e_{-1} + x_0 e_0 + \sum_{i=1}^{n} (x_i e_i + \check{x}_i \check{e}_i).$$

The non-degenerate symmetric bilinear form

$$\boldsymbol{k}_\lambda(x, x) := 2x_{-1} x_0 + \sum_{j=1}^{n} \frac{1}{\lambda_j} (x_j^2 + \check{x}_j^2) \tag{6}$$

satisfies

$$\boldsymbol{k}_\lambda([x, y], z) + \boldsymbol{k}_\lambda(y, [x, z]) = 0 \quad \text{for any} \quad x, y, z \in \mathfrak{g}_\lambda$$

and, hence, defines a Lorentzian bi-invariant metric g on G_λ. The Levi-Cività connection of g is given by $\nabla_X Y = \frac{1}{2}[X, Y]$ for any left invariant vector fields X, Y. Hence, the left invariant vector field associated with e_0 is parallel and defines a Kundt group structure on G_λ.

IV. KUNDT SPACETIMES VS KUNDT GROUPS

In dimension 3, we have the following result.

Theorem 4.1. *Let G be a Lie group of dimension 3 endowed with a left invariant Lorentz metric g. Assume that there exists a degenerate totally geodesic hypersurface Σ in (G, g) (not necessarily complete).*

(i) *Then, either (G, g) has a constant sectional curvature or (G, g) is a locally Kundt Lie group.*
(ii) *Furthermore, if the isotropy group of 1 in the isometry group $\mathsf{Iso}(G, g)$ is non-compact, then such a hypersurface Σ exists.*

To prove this theorem, we need the following lemma, which also appears in Refs. 20–22.

Lemma 4.1. Let (G, g) be a Lorentz Lie group of dimension 3. If there are three non-tangent (i.e., having different tangent planes at 1) lightlike geodesic hypersurfaces through 1, then (G, g) has a constant curvature.

Proof. Assume that there exist three different lightlike geodesic hypersurfaces Σ_1, Σ_2, and Σ_3 through $1 \in G$. Let T_1, T_2, T_3 be their tangent spaces and V_1, V_2, V_3 be non-vanishing vectors in their orthogonals $T_1^\perp, T_2^\perp$, and $T_3^\perp$.

Since Σ_i's are geodesic, each T_i is invariant under the Riemann curvature: if $u, v, w \in T_i$, then $R(u, v)w \in T_i$.

Let e_3 be a unit (spacelike) vector generating $T_1 \cap T_2$, and consider $A_{e_3} : V \to R(e_3, V)e_3$. Then, $\langle A_{e_3}(V_1), e_3\rangle = \langle R(e_3, V_1)e_3, e_3\rangle = 0$, that is, $A_{e_3}(V_1)$ is orthogonal to e_3. However, it is also orthogonal to V_1 by curvature-invariance of $V_1^\perp$. Therefore, $A_{e_3}(V_1)$ is collinear to V_1, say, $A_{e_3}(V_1) = \lambda_{12}V_1$. Similarly, $A_{e_3}(V_2) = \lambda_{21}V_2$.

Consider $\langle R(e_3, V_1)e_3, V_2\rangle$, which equals $\langle R(e_3, V_2)e_3, V_1\rangle$. It also equals $\lambda_{12}\langle V_1, V_2\rangle = \lambda_{21}\langle V_1, V_2\rangle$. Since V_1 and V_2 are both null, $\langle V_1, V_2\rangle \neq 0$, then $\lambda_{12} = \lambda_{21}$. In conclusion, A_{e_3} is a homothety on $(e_3)^\perp$, of ratio, say, λ ($= \lambda_{12} = \lambda_{21}$).

Since $A_{e_3}(e_3) = 0$, we conclude that $R(e_3, W)e_3 = \lambda(W\langle e_3, e_3\rangle - \langle W, e_3\rangle e_3)$ for any W.

In a similar way, one defines e_2 and e_1 unit vectors generating $T_1 \cap T_3$ and $T_2 \cap T_3$, respectively, and deduces a similar formula for $R(e_i, W)e_i$ with the same λ.

In fact, up to orientation, e_i are uniquely defined. More precisely, in a Lorentz linear three-space $(E, \langle\, ,\, \rangle)$, there is, up to isometry, a unique system of three null directions. Indeed, with the notation above, we can choose generators V_1, V_2, and V_3 of these directions such that $\langle V_i, V_j\rangle = 1$ for any $i \neq j$, so the matrix coefficients $\langle V_i, V_j\rangle_{ij}$ are fully given. This implies that any two such systems are related by an isometry.

Now, the system $\{e_1, e_2, e_3\}$ is given (up to orientation) by $\{V_1, V_2, V_3\}$, so all scalar products $\langle e_i, e_j\rangle$ are given, i.e., computable by means of $\langle, \rangle$.

It follows that $R(e_i, e_j)e_i$ are given for any i, j.

Consider now $X = R(e_i, e_j)e_k$ with $k \neq i, j$ and $i \neq j$. Then, $\langle X, e_k\rangle = 0$ and $\langle X, e_i\rangle = \langle R(e_i, e_j)e_i, e_k\rangle$ and, hence, computable and similarly for $\langle X, e_j\rangle$, and therefore, X is computable.

From all this, it follows that all the curvatures $R(e_i, e_j)e_k$ are computable, exactly as in the case of a space of constant curvature λ. □

Proof of Theorem 4.1

Proof.

(i) It follows from the lemma that if (G, g) does not have constant curvature, then through any point pass exactly one or two (germs of) lightlike geodesic hypersurfaces.

For the sake of clarity, let us consider first the case where there exists exactly one germ of such hypersurfaces. More precisely, there exists a geodesic lightlike hypersurface Σ containing 1. Uniqueness means that for any S, a geodesic lightlike hypersurface, with $1 \in S$, then $\Sigma \cap S$ is a neighborhood of 1 in both Σ and S.

For any $x \in G$, the translated hypersurface $x\Sigma$ is the unique geodesic lightlike hypersurface passing through x.

Let us see that the tangent space of Σ is left invariant: if $x \in \Sigma$, then $T_x\Sigma = xT_1\Sigma$ (the last notation means the left translation by x of $T_1\Sigma$). Indeed, both Σ and $x\Sigma$ are geodesic lightlike hypersurfaces containing x; hence, they coincide near x and, thus, have the same tangent space: $T_x\Sigma = T_x(x\Sigma) = xT_1\Sigma$.

This left invariance of $T\Sigma$ means that Σ is a "local subgroup." Its "maximal extension" will be a subgroup, which is a geodesic lightlike hypersurface. To be more formal, one defines a plane field E on G, with $E(x)$ being the tangent space of the unique geodesic lightlike hypersurface through x. Therefore, $E(x) = T_x(x\Sigma)$, which implies that E is left invariant. Uniqueness implies that E is integrable: Σ is a local leaf of E through 1. The global leaf of E is a subgroup.

Let us now consider the case where we have two geodesic lightlike hypersurfaces Σ^1 and Σ^2 through 1, which do not coincide near 1. This is equivalent to $E(1) = T_1\Sigma^1 \neq T_1\Sigma^2 = F(1)$.

Let $x \in \Sigma^1$. Then, through x, we have three geodesic lightlike hypersurfaces: Σ^1, $x\Sigma^1$, and $x\Sigma^2$. Therefore, two among them coincide locally. Let us assume that x is close to 1, and deduce that Σ^1 and $x\Sigma^2$ cannot coincide near x. For this, it is enough to show they have different tangent spaces $A = T_x\Sigma^1 \neq T_x(x\Sigma^2) = B$. If they were equal, they would have the same translation to 1, $x^{-1}A = x^{-1}B$. On the one hand, $x^{-1}B = T_x\Sigma^2 = F(1)$, and on the other hand, $x^{-1}A = x^{-1}T_x\Sigma$ is close to $E(1) = T_1\Sigma^1$ by continuity of the tangent space of Σ^1 and the fact that x is close to 1. Since $E(1)$ and $F(1)$ are transversal, the same is true for $x^{-1}A$ and $x^{-1}B$ for x sufficiently close to 1, and in particular, they are not equal; hence, $A \neq B$. From all of this, we infer that Σ^1 and $x\Sigma^1$ coincide near x.

As in the case of a unique geodesic lightlike hypersurface, one deduces that Σ^1 is a "local group." More precisely, one defines two left invariant plane fields E and F, extending $E(1)$ and $F(1)$, respectively. Our previous argument implies that Σ^1 is a local leaf at 1 of E, and so E is integrable, and the same applies for F. The E and F leaves of 1 are, therefore, two subgroups, which are geodesic lightlike hypersurfaces.

(ii) Now, assume, I, that the isotropy group of 1 in the isometry group $\mathrm{Iso}(G, g)$ is non-compact.

Let $f_n \in I$ be a diverging sequence (it has non-convergent sub-sequence), and consider their graphs $F_n = Gr(f_n) \subset G \times G$. Endow $G \times G$ with the metric $g \oplus (-g)$. Then, F_n's are isotropic and totally geodesic. Then, we will consider a limit L of a subsequence of F_n. To give a formal meaning of this, consider a small convex neighborhood C of $(1, 1)$ in $G \times G$. This means any two points of C can be joined by a unique geodesic segment contained in C. Consider $F_n \cap C$, and note F_n^0, the connected component of $(1, 1)$ in $F_n \cap C$. Now, one can give sense to convergence of F_n, exactly as in the situation of affine subspaces in an affine flat space. More precisely, F_n^0 converge to L if the tangent spaces $T_{(1,1)}F_n^0$ converge to $T_{(1,1)}L$.

Such a limit L is a geodesic isotropic submanifold in $G \times G$ of dimension equal to $\dim G$, but it is no longer a graph of some map $f : G \to G$ since otherwise f_n will converge to f. Thus, L intersects non-trivially the vertical $\{1\} \times G$ and, hence, projects onto a degenerate geodesic submanifold Σ in $G \times \{1\}$. The intersection $L \cap (\{1\} \times G)$ has dimension one since it is isotropic; therefore, Σ is a hypersurface. □

A. Comments on the constant curvature case

It is natural to ask what happens if (G, g) has a constant sectional curvature, say, c? Let us give here some examples and hints; details will appear elsewhere. The Proof of Theorem 4.1 does not apply since there are infinitely many "germs" of lightlike geodesic hypersurfaces through 1: any lightlike hyperplane in T_1G is tangent to a lightlike geodesic hypersurface exactly as in the universal Lorentz space $\tilde{M}(c)$ of constant curvature c.

- It turns out that if the isotropy group I is non-compact and has dimension 1 or 2, then (G, g) is a Kundt group. Indeed, the Proof of Theorem 4.1 can be adapted well to this situation where $\dim I = 1$ or 2. Consider for this the derivative action of I on $\mathfrak{g} = T_1G$. It preserves exactly one or two hyperplanes, which are, in fact, lightlike. Indeed, let L be a closed connected non-compact subgroup of $\mathsf{O}(1, 2)$. If $\dim L = 1$, then it is either a hyperbolic one parameter group; in this case, it preserves exactly two lightlike hyperplanes or it is a unipotent one parameter group, and in this case, it preserves exactly one lightlike hyperplane. In case $\dim L = 2$, it is conjugate to the triangular subgroup of $\mathsf{SL}(2, \mathbb{R})$ and preserves exactly one lightlike hyperplane. All these claims can be confirmed by a direct check-in. In summary, if $\dim I = 1, 2$, the argument in the Proof of Theorem 4.1 can be adapted and yields a left invariant plane field E and, thus, a lightlike geodesic subgroup H.
  Let us give the following example with $c = 0$. Consider on $\mathbb{R}^3$ the Lorentz metric $dx^2 + dydz$. The plane $E = \{z = 0\}$ is lightlike. Its linear stabilizer is a subgroup S of dimension 2 in $\mathsf{SO}(1, 2)$, and hence, its stabilizer in the full Poincaré group $\mathsf{SO}(1, 2) \ltimes \mathbb{R}^3$ is $S \ltimes E$. It contains, in particular, the three-dimensional (non-unimodular) group G of elements $(t, a, b) \in \mathbb{R} \times E$ acting by $(x, y, z) \to (x + a, e^t y + b, e^{-t} z)$. This action is free and transitive on the upper half space $\{z > 0\}$, and hence, G inherits a left invariant (non-complete) flat metric. It is Kundt since its (Abelian) subgroup E has lightlike geodesic orbits. Observe here that the full isometry group of this left invariant metric is $S \ltimes E$. In particular, the isotropy group has dimension one.
- Assume now that the isotropy group has dimension 3. Thus, $\dim Iso(G, g) = 6$, and (G, g) is locally isometric to the universal space $\tilde{M}(c)$ of constant curvature c. However, a subgroup of dimension 3 in $\mathsf{O}(1, 2)$ contains at least its identity component, and also, a subgroup of dimension 6 in $\mathsf{Iso}(\tilde{M}(c))$ contains at least its identity component $\mathsf{Iso}^0(\tilde{M}(c))$. Therefore, as a homogeneous space, (G, g) is globally isometric to $\tilde{M}(c) = \mathsf{Iso}^0(\tilde{M}(c))/\mathsf{O}^0(1, 2)$. In other words, G acts transitively and freely on $\tilde{M}(c)$ [or equivalently, (G, g) has constant curvature c and is complete].
  Let us give the example of the Euclidean group Euc_2. Its universal cover $\widetilde{\mathsf{Euc}_2}$ acts simply transitively isometrically on $(\mathbb{R}^3, dx^2 + dy^2 - dz^2)$ by $((x, y), z) \to ((R_t(x, y) + (a, b)), z + t)$, where R_t is the rotation of angle t and $(t, a, b) \in \widetilde{\mathsf{Euc}_2}$. Its unique two-dimensional subgroup is $\mathbb{R}^2$, which acts by translations $(x, y, z) \to (x + a, y + b, z)$. It has spacelike geodesic orbits. Therefore, $\widetilde{\mathsf{Euc}_2}$ is a flat complete Lorentz group that is not a Kundt group.
  We believe this is the unique complete Lorentz group of constant curvature, which is not a Kundt group.
- Finally, there are examples of flat groups (G, g) with isotropy group I of dimension 0, which are not Kundt groups. To see an example, consider as above the metric $dx^2 + dydz$. Let $S \subset \mathsf{SO}(1, 2)$ be the stabilizer of the isotropic direction $\mathbb{R}\frac{\partial}{\partial y}$ and T be the subgroup of translations in this direction. Take $G = S \ltimes T$. It has an open orbit on which it acts freely, which allows one to endow it with a flat (non-complete) metric. One can show that it is not a Kundt group.

V. CLASSIFICATION OF THREE-DIMENSIONAL UNIMODULAR SIMPLY CONNECTED KUNDT LIE GROUPS

In this section, we give a complete classification of Kundt Lie group structures on three-dimensional unimodular Lie groups. According to Proposition 3.1, the classification of Kundt Lie group structures on a simply connected Lie group G is equivalent to the classification of

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Kundt pairs $(\mathfrak{h}, \langle\ ,\ \rangle)$ on its Lie algebra $\mathfrak{g}$. Two Kundt pairs $(\mathfrak{h}_1, \langle\ ,\ \rangle_1)$ and $(\mathfrak{h}_2, \langle\ ,\ \rangle_2)$ are called equivalent if there exists an automorphism of Lie algebra $\phi : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$ and $\phi^*(\langle,\rangle_2) = \langle,\rangle_1$.

In dimension 3, we have the following useful characterization of Kundt pairs.

Proposition 5.1. Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a Lorentzian Lie algebra, and let $\mathfrak{h}$ be a codimension one subalgebra. Then, we have the following:

(i) *If $\mathfrak{h}$ is Abelian, then $(\mathfrak{h}, \langle\ ,\ \rangle)$ is a Kundt pair if and only if $\mathfrak{h}$ is degenerate, and if e is a generator of $\mathfrak{h}^\perp$, then $\mathbf{ad}_e(\mathfrak{g}) \subset \mathfrak{h}$.*

(ii) *If $\mathfrak{h}$ is non-Abelian, then $(\mathfrak{h}, \langle\ ,\ \rangle)$ is a Kundt pair if and only if $\mathfrak{h}^\perp = [\mathfrak{h}, \mathfrak{h}]$, and if e is a generator of $\mathfrak{h}^\perp$, then $\mathbf{ad}_e(\mathfrak{g}) \subset \mathfrak{h}$.*

Proof. Let e be a generator of $\mathfrak{h}^\perp$. Then, $(\mathfrak{h}, \langle\ ,\ \rangle)$ is a Kundt pair if and only if $e \in \mathfrak{h}^\perp$, $\langle e, e\rangle = 0$, and $e \bullet e = 0$, and for any $u, v \in \mathfrak{h}$,

$$0 = 2\langle u \bullet v, e\rangle = \langle [e, u], v\rangle + \langle [e, v], u\rangle. \tag{7}$$

Note first that $e \bullet e = 0$ and $\langle e, e\rangle = 0$ if and only if, for any $u \in \mathfrak{g}$,

$$0 = \langle e \bullet e, x\rangle = \langle [u, e], e\rangle,$$

which is equivalent to $\mathbf{ad}_e(\mathfrak{g}) \subset \mathfrak{h}$.

If $\mathfrak{h}$ is Abelian, then (7) holds trivially.

Suppose now that $\dim \mathfrak{g} = 3$ and $\mathfrak{h}$ is not Abelian. Then, there exists a basis (u, v) of $\mathfrak{h}$ such that $\langle u, v\rangle = 0$ and $[u, v] = u$. Put $e = au + bv$. Then, from the relation above, we get

$$0 = -b\langle u, v\rangle + a\langle u, u\rangle = a\langle u, u\rangle \quad \text{and} \quad 0 = \langle [e, u], u\rangle = b\langle u, u\rangle.$$

This implies that $\langle u, u\rangle = 0$, and hence, $\mathfrak{h}^\perp = [\mathfrak{h}, \mathfrak{h}]$. The converse is obviously true. □

There are five simply connected three-dimensional unimodular non-Abelian Lie groups:

1. The nilpotent Lie group Nil known as Heisenberg group whose Lie algebra will be denoted by $\mathfrak{n}$. We have

$$\mathrm{Nil} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\} \quad \text{and} \quad \mathfrak{n} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{n}$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$, where

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the non-vanishing Lie bracket is $[X_1, X_2] = X_3$.

2. $\mathrm{SU}(2) = \left\{ \begin{pmatrix} a + bi & -c + di \\ c + di & a - bi \end{pmatrix}, a^2 + b^2 + c^2 + d^2 = 1 \right\}$ and $\mathfrak{su}(2) = \left\{ \begin{pmatrix} iz & y + ix \\ -y + xi & -zi \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$. The Lie algebra $\mathfrak{su}(2)$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$,

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad X_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

where the non-vanishing Lie brackets are

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad \text{and} \quad [X_3, X_1] = X_2.$$

3. The universal covering group $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ of $\mathrm{SL}(2, \mathbb{R})$ whose Lie algebra is $\mathrm{sl}(2, \mathbb{R})$. The Lie algebra $\mathrm{sl}(2, \mathbb{R})$ has a basis $\mathbb{B}_0 = (e, f, h)$, where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where the non-vanishing Lie brackets are

$$[e, f] = h, \qquad [h, e] = 2e, \quad \text{and} \quad [h, f] = -2f.$$

4. The solvable Lie group $\mathrm{Sol} = \left\{ \begin{pmatrix} e^x & 0 & y \\ 0 & e^{-x} & z \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$ whose Lie algebra is $\mathfrak{sol} = \left\{ \begin{pmatrix} x & 0 & y \\ 0 & -x & z \\ 0 & 0 & 0 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$. The Lie algebra $\mathfrak{sol}$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$, where

$$X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and}$$

where the non-vanishing Lie brackets are

$$[X_1, X_2] = X_2 \quad \text{and} \quad [X_1, X_3] = -X_3.$$

5. The universal covering group $\widetilde{\mathrm{E}_0}(2)$ of the Lie group

$$\mathrm{E}_0(2) = \left\{ \begin{pmatrix} \cos(\theta) & \sin(\theta) & x \\ -\sin(\theta) & \cos(\theta) & y \\ 0 & 0 & 1 \end{pmatrix}, \theta, x, y \in \mathbb{R} \right\}.$$

Its Lie algebra is

$$\mathrm{e}_0(2) = \left\{ \begin{pmatrix} 0 & \theta & x \\ -\theta & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, \theta, y, z \in \mathbb{R} \right\}.$$

The Lie algebra $\mathrm{e}_0(2)$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$, where

$$X_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

where the non-vanishing Lie brackets are

$$[X_1, X_2] = X_3 \quad \text{and} \quad [X_1, X_3] = -X_2.$$

Let us find the two-dimensional subalgebras of the three-dimensional unimodular Lie algebras.

Proposition 5.2.

1. *Let $\mathfrak{h}$ be a two-dimensional subalgebra of $\mathfrak{n}$. Then, $\mathfrak{h} = \mathrm{span}\{X_3, aX_1 + bX_2\}$, $(a, b) \neq (0, 0)$.*
2. *$\mathfrak{su}(2)$ has no subalgebra of dimension 2.*
3. *Let $\mathfrak{h}$ be a two-dimensional subalgebra of $\mathfrak{sol}$. Then, either $\mathfrak{h} = \mathrm{span}\{X_2, X_3\}$, $\mathfrak{h} = \mathrm{span}\{X_2, X_1 + aX_3\}$, or $\mathfrak{h} = \mathrm{span}\{X_3, X_1 + aX_2\}$ $(a \in \mathbb{R})$.*
4. *Let $\mathfrak{h}$ be a two-dimensional subalgebra of $\mathrm{e}_0(2)$. Then, $\mathfrak{h} = \mathrm{span}\{X_2, X_3\}$.*
5. *Let $\mathfrak{h}$ be a two-dimensional subalgebra of $\mathrm{sl}(2, \mathbb{R})$. Then, there exists an automorphism of $\mathrm{sl}(2, \mathbb{R})$, which sends $\mathfrak{h}$ to $\mathrm{span}\{h, e\}$.*

Proof.

1. A two-dimensional subalgebra $\mathfrak{h}$ of $\mathfrak{n}$ must be Abelian and contains the center. Hence, $\mathfrak{h} = \mathrm{span}\{X_3, aX_1 + bX_2\}$ and $(a, b) \neq (0, 0)$.
2. It is a consequence of Lemma 3.1.
3. Denote $\mathfrak{h}_0 = \mathrm{span}\{X_2, X_3\}$. If $\mathfrak{h}$ is Abelian and $\mathfrak{h} \neq \mathfrak{h}_0$, then $\mathfrak{h} = \mathrm{span}\{X_1 + U, V\}$, where $U, V \in \mathfrak{h}_0$, and hence, $[X_1, V] = 0$, which is impossible. Hence, if $\mathfrak{h}$ is Abelian, then $\mathfrak{h} = \mathfrak{h}_0$.

   If $\mathfrak{h}$ is not Abelian, then $\mathfrak{h} \neq \mathfrak{h}_0$. Thus, $\mathfrak{h} = \mathrm{span}\{X_1 + U, V\}$, where $U, V = aX_2 + bX_3 \in \mathfrak{h}_0$ and $[\mathfrak{h}, \mathfrak{h}] = \mathbb{R}V$. Now,

$$[X_1 + U, V] = aX_2 - bX_3.$$

   Hence, the vectors $aX_2 + bX_3, aX_2 - bX_3$ must be linearly dependent; hence, $ab = 0$, which completes the proof.
4. We can use the same argument as above and get $a^2 + b^2 = 0$.

5. Note first that $[e,f]=h,[h,e]=2e,[h,f]=-2f$. Let $\mathfrak{h}$ be a two-dimensional subalgebra of $\mathrm{sl}(2,\mathbb{R})$. Then, there exists a basis (u,v) of $\mathfrak{h}$ such that $[u,v]=2v$. The endomorphism ad_u is skew-symmetric with respect to the Killing form; hence, $\mathrm{tr}(\mathrm{ad}_u)=0$. It has 2 and 0 as eigenvalues, so the third eigenvalue is -2. Hence, there exists $w\in\mathrm{sl}(2,\mathbb{R})$ such that $[u,w]=-2w$. Now,
$$[u,[v,w]]=2[v,w]-2[v,w]=0.$$
Hence, $[v,w]=\alpha u$. By replacing w by $\frac{1}{\alpha}w$, we get that the automorphism ϕ, which sends (u,v,w) to (h,e,f) sends $\mathfrak{h}$ to $\mathrm{span}\{h,e\}$, which completes the proof. □

Theorem 5.1. *Let $(\mathfrak{h},\langle\ ,\ \rangle)$ be a Kundt pair of $\mathfrak{n}$. Then, $(\mathfrak{h},\langle,\rangle)$ is equivalent to $(\mathfrak{h}_0,\langle,\rangle_0)$, where*

1. $\langle,\rangle_0=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&\mu\end{bmatrix},\mu>0$, *and* $\mathfrak{h}_0=\mathrm{span}\{X_1\pm X_2,X_3\}$;
2. $\langle,\rangle_0=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$ *and* $\mathfrak{h}_0=\mathrm{span}\{X_1,X_3\}$.

Proof. Let $(\mathfrak{h},\langle\ ,\ \rangle)$ be a Kundt pair on $\mathfrak{n}$. According to Ref. 23, Theorem 3.1, there exists an automorphism ϕ of $\mathfrak{n}$ such that the matrix of $(\phi^{-1})^*(\langle\ ,\ \rangle)$ in the basis (X_1,X_2,X_3) has one of the following forms:
$$\mathfrak{n}_1=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&\mu\end{bmatrix},\quad \mathfrak{n}_2=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-\mu\end{bmatrix},\quad\text{or}\quad \mathfrak{n}_3=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\quad \mu>0.$$
By virtue of Proposition 5.2, $\phi(\mathfrak{h})=\mathfrak{h}_0=\mathrm{span}\{X_3,aX_1+bX_2\}$ with $(a,b)\neq(0,0)$. According to Proposition 5.1, $(\phi(\mathfrak{h}),(\phi^{-1})^*(\langle\ ,\ \rangle))$ is a Kundt pair if and only if $\mathfrak{h}_0$ is degenerate and $\mathbf{ad}_e(\mathfrak{n})\subset\mathfrak{h}_0$, where e is a generator of $\mathfrak{h}_0^\perp$. Since $[\mathfrak{n},\mathfrak{n}]\subset\mathfrak{h}_0$, then the last condition holds.

Now, $\mathfrak{h}_0$ cannot be $\mathfrak{n}_2$-degenerate, and it is $\mathfrak{n}_3$-degenerate if and only if $b=0$. Finally, $\mathfrak{h}_0$ is $\mathfrak{n}_1$-degenerate if and only if $a^2-b^2=0$, which completes the proof. □

Theorem 5.2. *Let $(\mathfrak{h},\langle\ ,\ \rangle)$ be a Kundt pair of $\mathfrak{sol}$. Then, $(\mathfrak{h},\langle,\rangle)$ is equivalent to $(\mathfrak{h}_0,\langle\ ,\ \rangle_0)$, where either*

1. $\langle\ ,\ \rangle_0=\begin{pmatrix}\lambda&0&0\\0&0&-1\\0&-1&0\end{pmatrix},\lambda>0$, *and* $\mathfrak{h}_0=\mathrm{span}\{X_2,X_1\}$ *or* $\mathfrak{h}_0=\mathrm{span}\{X_3,X_1\}$;
2. $\langle\ ,\ \rangle_0=\begin{pmatrix}\lambda^2&0&0\\0&\lambda&1\\0&1&0\end{pmatrix}$ *and* $\mathfrak{h}_0=\mathrm{span}\{X_3,X_1\}$;
3. $\langle\ ,\ \rangle_0=\begin{pmatrix}0&0&-\frac{2}{b}\\0&1&1\\-\frac{2}{b}&1&1\end{pmatrix},\quad b>0$, *and* $\mathfrak{h}_0=\mathrm{span}\{X_2,X_3\}$;
4. $\langle,\rangle_0=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$ *and* $\mathfrak{h}_0=\mathrm{span}\{X_2,X_3\}$.

Proof. Let $(\mathfrak{h},\langle\ ,\ \rangle)$ be a Kundt pair on $\mathfrak{sol}$. Then, according to Ref. 23, Theorem 3.4, there exists an automorphism ϕ of $\mathfrak{sol}$ such that the matrix of $(\phi^{-1})^*(\langle\ ,\ \rangle)$ in the basis (X_1,X_2,X_3) has one of the following forms:
$$\begin{cases}\mathrm{sol}_1=\begin{pmatrix}\frac{4}{u^2-v^2}&0&0\\0&1&\frac{u}{v}\\0&\frac{u}{v}&1\end{pmatrix},v>0,u<v,\quad \mathrm{sol}_2=\begin{pmatrix}\frac{4}{v^2-u^2}&0&0\\0&\frac{u}{v}&-1\\0&-1&\frac{u}{v}\end{pmatrix},v>0,u<v,\quad \mathrm{sol}_3=\begin{pmatrix}\frac{1}{u+v}&0&0\\0&-\frac{v}{u}&1\\0&1&1\end{pmatrix},\ u>0,v>0,\\ \mathrm{sol}_4=\begin{pmatrix}\frac{1}{u}&0&0\\0&-1&0\\0&0&1\end{pmatrix},\ u>0,\quad \mathrm{sol}_5=\begin{pmatrix}0&0&-\frac{2}{b}\\0&1&1\\-\frac{2}{b}&1&1\end{pmatrix},\quad b>0,\quad \mathrm{sol}_6=\begin{pmatrix}\lambda^2&0&0\\0&\lambda&1\\0&1&0\end{pmatrix},\quad \lambda\neq0,\quad \mathrm{sol}_7=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}.\end{cases}$$

By virtue of Proposition 5.2, $\phi(\mathfrak{h}) = \mathfrak{h}_0$, where either $\mathfrak{h}_0 = \mathrm{span}\{X_2, X_3\}$, $\mathfrak{h}_0 = \mathrm{span}\{X_2, X_1 + aX_3\}$, or $\mathfrak{h}_0 = \mathrm{span}\{X_3, X_1 + aX_2\}$ $(a \in \mathbb{R})$.

If $\mathfrak{h}_0 = \mathrm{span}\{X_2, X_3\}$, then it is Abelian, and it is sol_5-degenerate and sol_7-degenerate. For sol_5, $\mathfrak{h}_0^\perp = \mathbb{R}(X_2 - X_3)$, and we have $\mathbf{ad}_{X_2 - X_3}(\mathfrak{sol}) \subset \mathfrak{h}_0$. We have the same situation for sol_7. Thus, $(\mathfrak{h}_0, \mathrm{sol}_5)$ and $(\mathfrak{h}_0, \mathrm{sol}_7)$ are Kundt pairs.

If $\mathfrak{h}_0 = \mathrm{span}\{X_2, X_1 + aX_3\}$, then $[\mathfrak{h}_0, \mathfrak{h}_0] = \mathbb{R}X_2$. We have obviously $\mathbf{ad}_{X_2}(\mathfrak{sol}) \subset \mathfrak{h}_0$, and according to Proposition 5.1, $(\mathfrak{h}_0, \langle\ ,\ \rangle)$ is a Kundt pair if and only if $\langle X_2, X_2\rangle = \langle X_2, X_1 + aX_3\rangle = 0$. This is possible if and only if $\langle , \rangle = \mathrm{sol}_2$ with $u = 0$ and $a = 0$.

If $\mathfrak{h}_0 = \mathrm{span}\{X_3, X_1 + aX_2\}$, then $[\mathfrak{h}_0, \mathfrak{h}_0] = \mathbb{R}X_3$. We have obviously $\mathbf{ad}_{X_3}(\mathfrak{sol}) \subset \mathfrak{h}_0$, and according to Proposition 5.1, $(\mathfrak{h}_0, \langle\ ,\ \rangle)$ is a Kundt pair if and only if $\langle X_3, X_3\rangle = \langle X_3, X_1 + aX_2\rangle = 0$. This is possible if and only if $\langle , \rangle = \mathrm{sol}_2$, $u = 0$, and $a = 0$ or $\langle , \rangle = \mathrm{sol}_6$ and $a = 0$. □

Theorem 5.3. *Let $(\mathfrak{h}, \langle\ ,\ \rangle)$ be a Kundt structure on $e_0(2)$. Then, $(\mathfrak{h}, \langle\ ,\ \rangle)$ is equivalent to $(\mathfrak{h}_0, \langle\ ,\ \rangle_0)$, where* $\langle\ ,\ \rangle_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix}$, $\mu > 0$, *and* $\mathfrak{h}_0 = \mathrm{span}\{X_2, X_3\}$.

Proof. Let $(\mathfrak{h}, \langle\ ,\ \rangle)$ be a Kundt pair on $e_0(2)$. Then, according to Ref. 23, Theorem 3.5, there exists an automorphism ϕ of $e_0(2)$ such that the matrix of $(\phi^{-1})^*(\langle\ ,\ \rangle)$ in the basis (X_1, X_2, X_3) has one of the following forms:

$$\langle\ ,\ \rangle_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & u & 0 \\ 0 & 0 & v \end{pmatrix}, \ u > 0, v > 0, \qquad \langle\ ,\ \rangle_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & u & 0 \\ 0 & 0 & v \end{pmatrix}, \ u > 0, v > 0, \qquad \langle\ ,\ \rangle_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & u \end{pmatrix}, u > 0.$$

By virtue of Proposition 5.2, $\phi(\mathfrak{h}) = \mathfrak{h}_0 = \mathrm{span}\{X_2, X_3\}$. According to Proposition 5.1, $(\phi(\mathfrak{h}), (\phi^{-1})^*(\langle\ ,\ \rangle))$ is a Kundt pair if and only if $\mathfrak{h}_0$ is degenerate and $\mathbf{ad}_e(\mathfrak{n}) \subset \mathfrak{h}_0$, where e is a generator of $\mathfrak{h}_0^\perp$. Now, $\mathfrak{h}_0$ cannot be $\langle , \rangle_1$-degenerate neither $\langle , \rangle_2$-degenerate. Finally, $\mathfrak{h}_0$ is $\langle , \rangle_3$-degenerate, $\mathfrak{h}_0^\perp = \mathbb{R}X_2$, and $\mathbf{ad}_{X_2}(e_0(2)) \subset \mathfrak{h}_0$. □

Theorem 5.4. *Let $(\mathfrak{h}, \langle\ ,\ \rangle)$ be a Kundt pair on $\mathrm{sl}(2, \mathbb{R})$. Then, there exists an automorphism ϕ of $\mathrm{sl}(2, \mathbb{R})$ such that $\phi(\mathfrak{h}) = \mathrm{span}\{e, h\}$ and the matrix of $\phi^*(\langle , \rangle)$ in the basis (e, f, h) has one of the following forms:*

$$\begin{bmatrix} 0 & 4\alpha & 0 \\ 4\alpha & 0 & 0 \\ 0 & 0 & 8\beta \end{bmatrix}, \begin{bmatrix} 0 & 4\alpha & 0 \\ 4\alpha & 1 & 0 \\ 0 & 0 & 8\beta \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 & 4\alpha & 0 \\ 4\alpha & 0 & 2\sqrt{2} \\ 0 & 2\sqrt{2} & 8\alpha \end{bmatrix}, \ \beta > 0, \ \alpha \in \mathbb{R}^*.$$

Proof. Let $(\mathfrak{h}, \langle\ ,\ \rangle)$ be a Kundt pair on $\mathrm{sl}(2, \mathbb{R})$. According to Proposition 5.2, we can suppose that $\mathfrak{h} = \mathrm{span}\{e, h\}$. A direct computation using the software Maple shows that the automorphisms of $\mathrm{sl}(2, \mathbb{R})$ leaving $\mathfrak{h}$ invariant are of the form

$$T = \begin{bmatrix} a & -ab^2 & -2ab \\ 0 & a^{-1} & 0 \\ 0 & b & 1 \end{bmatrix}, \ a, b \in \mathbb{R}.$$

Denote by B the Killing form of $\mathrm{sl}(2, \mathbb{R})$. Its matrix in the basis (e, f, h) is given by

$$M = \begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

We consider the isomorphism A, symmetric with respect to B, and it is given by $B(Au, v) = \langle u, v\rangle$ for any $u, v \in \mathrm{sl}(2, \mathbb{R})$. According to Proposition 5.1, the pair $(\mathfrak{h}, \langle\ ,\ \rangle)$ is Kundt if and only if $\langle e, e\rangle = \langle e, h\rangle = 0$ and $\mathbf{ad}_e(\mathrm{sl}(2, \mathbb{R})) \subset \mathfrak{h}$. This is equivalent to Ae, which is a generator of the orthogonal of $\mathfrak{h}$ with respect to B, which is equivalent to the existence of $\alpha \neq 0$ such that $Ae = \alpha e$. The normal form of isomorphisms, which are symmetric with respect to a Lorentzian scalar product, is known. We give here the normal form of those having an isotropic eigenvector. According to Ref. 24, there exists a basis $\mathbb{B} = (f_1, f_2, f_3)$ of $\mathrm{sl}(2, \mathbb{R})$ such that

(i) $Mat(A,\mathbb{B}) = \begin{bmatrix} \beta & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$ and $Mat(B,\mathbb{B}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$;

(ii) $Mat(A,\mathbb{B}) = \begin{bmatrix} \beta & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{bmatrix}$ and $Mat(B,\mathbb{B}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$;

(iii) $Mat(A,\mathbb{B}) = \begin{bmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{bmatrix}$ and $Mat(B,\mathbb{B}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Let P be the passage matrix from $\mathbb{B}_0 = (e,f,h)$ to $\mathbb{B}$.

- The case (i). The relation $A(e) = \alpha e$ implies that we can choose $f_2 = e$, and the relation $P^t MP = Mat(B,\mathbb{B})$ gives that $P = \begin{bmatrix} -\sqrt{2} & 1 & -1 \\ 0 & 0 & 1/4 \\ 1/4\sqrt{2} & 0 & 1/2 \end{bmatrix}$. The matrix of $\langle , \rangle$ in the basis B_0 is given by $Mat(A,\mathbb{B}_0)^t M$ and $Mat(A,\mathbb{B}^0) = PMat(A,\mathbb{B})P^{-1}$. Hence, we get

$$Mat(\langle\ ,\ \rangle,\mathbb{B}_0) = \begin{bmatrix} 0 & 4\alpha & 0 \\ 4\alpha & 32\beta - 32\alpha & -16\beta + 16\alpha \\ 0 & -16\beta + 16\alpha & 8\beta \end{bmatrix}.$$

  Now, the automorphism

$$T_1 = \begin{bmatrix} 1 & -4 & -4 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

  satisfies

$$T_1^t Mat(\langle\ ,\ \rangle,\mathbb{B}_0)T_1 = \begin{bmatrix} 0 & 4\alpha & 0 \\ 4\alpha & 0 & 0 \\ 0 & 0 & 8\beta \end{bmatrix}.$$

- The case (ii). We have

$$P = \begin{bmatrix} -\sqrt{2} & 1 & -1 \\ 0 & 0 & 1/4 \\ 1/4\sqrt{2} & 0 & 1/2 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 4 & -1 & -4 \\ 0 & 1/4 & 0 \\ 0 & 1/2 & 1 \end{bmatrix},$$

$$T_2^t Mat(\langle\ ,\ \rangle,\mathbb{B}_0)T_2 = \begin{bmatrix} 0 & 4\alpha & 0 \\ 4\alpha & 1 & 0 \\ 0 & 0 & 8\beta \end{bmatrix}.$$

- The case (iii). We have

$$P = \begin{bmatrix} 1 & -\sqrt{2} & -1 \\ 0 & 0 & 1/4 \\ 0 & 1/4\sqrt{2} & 1/2 \end{bmatrix} \quad \text{and} \quad T_3 = \begin{bmatrix} 4 & -1 & -4 \\ 0 & 1/4 & 0 \\ 0 & 1/2 & 1 \end{bmatrix},$$

$$T_3^t Mat(\langle\ ,\ \rangle,\mathbb{B}_0)T_3 = \begin{bmatrix} 0 & 4\alpha & 0 \\ 4\alpha & 0 & 2\sqrt{2} \\ 0 & 2\sqrt{2} & 8\alpha \end{bmatrix}.$$

□

A. Kundt vs locally Kundt Lie groups

In fact, it turns out from the previous proofs that we have shown that any three-dimensional unimodular locally Kundt Lie group is, in fact, a Kundt Lie group. This result is not true, in general, as the following example shows.

Example 2. Consider $\mathbb{R}^4$ endowed with the Lie algebra structure where the only non-vanishing Lie bracket is given by $[e_1, e_2] = e_2$ and the Lorentzian scalar product is given by

$$\langle \, , \, \rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Lie subalgebra $\mathfrak{h} = \mathrm{span}\{e_1, e_3, e_4\}$ is Abelian and satisfies $\mathfrak{h}^\perp = \mathbb{R}e_1$ and, hence, according to Proposition 3.1, defines a local Kundt Lie group structure on the corresponding simply connected Lie group. However, $\mathbf{ad}_{e_1}(\mathbb{R}^4) \not\subset \mathfrak{h}$, and hence, according to Proposition 5.1, this structure is not global.

We think, however, it is worthwhile to investigate the natural question: Is a locally Kundt Lie group, without being a Kundt group, still a (globally) Kundt spacetime?

VI. CONCLUSIONS

The starting point of this article is the reformulation of the notion of a Kundt spacetime in a more geometric and synthetic language. This gives, in fact, a local version of the Kundt propriety and leads naturally to ask in which circumstances the local property implies the global one.

However, our most important contribution is to introduce this notion of a Kundt Lie group to mean a Lie group with a left invariant Kundt structure, that is, not only the Lorentz metric is left invariant but also the codimension one geodesic foliation involved in the definition of Kundt spacetimes is left invariant.

We investigated Kundt groups in the three-dimensional case by giving an explicit classification.

More importantly, we proved that a Lorentz group (i.e., a Lie group with a left invariant Lorentz metric) is, in fact, a Kundt group, unless, it is very special, that is, more precisely, it has constant sectional curvature, or, equivalently, it acts isometrically with some open orbit on Minkowski, de Sitter, or anti de Sitter spacetimes of dimension 3.

This rigidity fact seems to admit a higher-dimensional version that is worthwhile to investigate, but classification seems more complicated since the (practical Bianchi) classification of Lie algebras is available only in dimension 3.

The four-dimensional case is particularly interesting for physical applications. It was very recently considered in Ref. 25 where the authors studied spacetimes admitting Killing spinors with a non-zero real constant. They call them supersymmetric Kundt configurations since they are automatically Kundt and are supersymmetric configurations of supergravity.

Kundt groups and supersymmetric Kundt spacetimes (and also plane waves, pp-waves, Brinkmann spacetimes, etc.) are manageable subclasses of Kundt spacetimes, defined by compatibility with an extra geometric structure. Their study is a natural step toward the understanding of the general Kundt case, which is of great importance in general relativity.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Mohamed Boucetta: Writing – original draft (equal). **Aissa Meliani**: Writing – original draft (equal). **Abdelghani Zeghib**: Writing – original draft (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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