Geometriae Dedicata **107:** 57–83, 2004.

Lipschitz Regularity in Some Geometric Problems

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(Received: 23 August 2002; accepted in final form: 7 April 2003)

Abstract. The Lipschitz regularity is perhaps the most natural, and surely the most geometrical among all the types of regularities. For example, the Lipschitz character of an ordinary differential equation (vector field) is the natural classical sufficient condition for the (unique) integrability of this equation. The goal here is to show that, in some sense, the Lipschitz regularity is also necessary, if one assumes (geometric) individual conditions on the trajectories. In other words, we show that tangential rigidity leads to a transversal regularity.

Mathematics Subject Classifications (2000). 34A26, 53A20, 57R30.

Key words. Lipschitz, line, parabola, causal, codimension one foliation.

1. Introduction

Recall that a mapping f between two metric spaces (X, d_X) and (Y, d_Y) is Lipschitz if there exists a constant c such that $d_Y(f(x), f(y)) \leqslant c d_X(x, y)$, for any $x, y \in X$. It is called locally Lipschitz, if every point of X admits a neighborhood where the restriction of f is Lipschitz. Being defined in this broad context, this gives Lipschitz regularity an advantage in flexibility over the other classical C^k-differentiability. On the other hand, Lipschtiz regularity is omnipresent; this is seen, for instance, in piecewise linear objects in nature. We are going here to bring out and analyze, in an elementary way, a Lipschitz regularity phenomenon whose starting point is the following:

STARTING FACT 1.1 (Lipschitz regularity of planar partition by line segments). *Let U be an open subset of the Euclidean plane $\mathbb{R}^2$. Let $\mathcal{F}$ be a family of* disjoint (*straight*) *line segments which are* complete *in U i.e. any element of $\mathcal{F}$ equals the open straight line segment joining two points of the boundary of U.* (*Equivalently, an element of $\mathcal{F}$ is the connected component of an intersection $l \cap U$, where l is a line*).

Then, $\mathcal{F}$ is locally Lipschitz, that is, the map $u: S \to S^1$, defined in the support $S \subset U$ of $\mathcal{F}$, which associates to a line segment its unit tangent vector, is locally Lipschitz. Furthermore, the Lipschitz constant (*which is a function on U*) *depends only on U* (*not on $\mathcal{F}$*).

Proof. We will focus on the case where the family $\mathcal{F}$ consists of exactly two line segments. One might say, there is nothing to prove here: obviously, any partition with two elements, which are closed submanifolds of any manifold, is locally

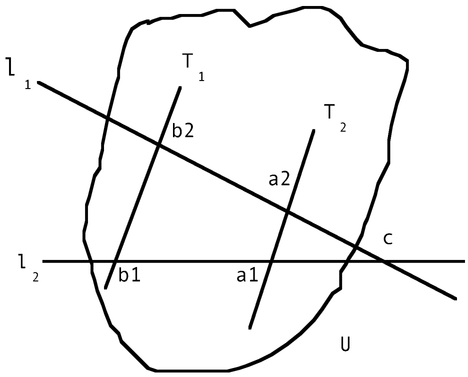

Figure 1.

Lipschitz. The point here is to show this is uniform, that is the involved (local) Lipschitz constants do not depend on the partition, but, as stated above, only on the geometry of U. Then, the proof in the two elements case yields a one for partitions with three elements, and so and so, in fact, for partitions with any cardinality. Therefore, we will restrict ourselves to the case of two elements.

Because the problem is local, we will assume that U is convex. Our family consists of two line segments $l_1 \cap U$ and $l_2 \cap U$, where l_1 and l_2 are lines in the Euclidean plane $\mathbb{R}^2$, which by hypothesis intersect outside U. In order to evaluate the Lipschitz distortion of the family, one just evaluate the same for the holonomy mappings obtained from any smooth curves T and T' transversal to the family. They are mappings defined on sets of two points by: $T \cap l_i \to T' \cap l_i$. In fact, there is no loss to generality in assuming that T and T' are parallel line segments. In notation, let T_1 and T_2 be two parallel (compact) line segments contained in U, such that T_1 (resp. T_2) cuts l_1 and l_2 on a_1 and a_2 (resp. b_1 and b_2) respectively.

Let c be the intersection point of l_1 and l_2 (the following discussion is trivial if $l_1 \cap l_2 = \emptyset$). The triangles ca_1a_2 and cb_1b_2 are similar and hence

$$\frac{d(b_1, b_2)}{d(a_1, a_2)} = \frac{d(c, b_1)}{d(c, a_1)}$$

(Thales Theorem). Therefore, we have the following estimate:

$$\begin{aligned}\frac{d(b_1, b_2)}{d(a_1, a_2)} = \frac{d(c, b_1)}{d(c, a_1)} &\leqslant \frac{d(c, a_1) + d(a_1, b_1)}{d(c, a_1)} \leqslant 1 + \frac{d(a_1, b_1)}{d(c, a_1)} \\ &\leqslant 1 + \frac{d_{\mathrm{Haus}}(T_1, T_2)}{d(l_1 \cap l_2, T_1)} \leqslant 1 + \frac{d_{\mathrm{Haus}}(T_1, T_2)}{d(\partial U, T_1)}\end{aligned}$$

In this last line, d denotes the distance between subsets (the inf of distances of their points), and d_{Haus} denotes the Hausdorff distance between subsets.

This shows that $\mathcal{F}$ is uniformly locally Lipschitz when composed with two elements, and hence for any $\mathcal{F}$ as explained above. □

The Fact applies in particular to partitions of U by closed (in U) line segments, i.e. when $\mathcal{F}$ has a total support. It says that such a partition is a Lipschitz foliation. Observe that we don't assume a priori that a foliation is continuous.

In the opposite case where the support of $\mathcal{F}$ is a proper subset S of U, the Fact tells that $\mathcal{F}$ extends to a Lipschitz lamination on the closure of S.

In this text, we will generally focus on foliations instead of partitions with a proper support.

A foliation $\mathcal{F}$ of U may be parameterized by a vector field X (our interests here are local in nature, and so we don't mind on global nonorientability problems). For us here, the foliation $\mathcal{F}$ is *Lipschitz*, if one can choose such a X to be Lipschitz (this is equivalent to the fact that the unit tangent vector field is Lipschitz). The flow of such a vector field is Lipschitz, i.e. a one parameter family of (local) Lipschitz homeomorphisms. In fact, more generally, any holonomy map, partially defined between two transversals τ and τ', is Lipschitz. The transversals are required to be C^1 submanifolds (or merely Lipschitz submanifolds, suitably defined). In general, having a Lipschitz holonomy is weaker than being oriented by a Lipschitz vector field, but the two notions are equivalent in the case of the foliations considered here.

Let's mention one corollary of Fact 1.1 which shows how non-Lipschitz family of lines are 'confluent'.

COROLLARY 1.2. *Let $\mathcal{D}$ be the space of lines of $\mathbb{R}^2$, and $\mathcal{C} \subset \mathcal{D}$ a topological curve. Suppose that $\mathcal{C}$ is purely unrectifiable, that is no sub-curve of it is rectifiable (i.e. has finite length). Then $\mathcal{C}$ foliates nowhere in the plane. More exactly, let $\mathcal{C}'$ be the corresponding set in the unit tangent bundle $T^1\mathbb{R}^2$, then, the projection $\mathcal{C}' \to \mathbb{R}^2$ is injective on no open set of $\mathcal{C}'$.*

1.1. A GENERALIZATION TO 'NON-CONNECTED LEAVES'

Consider a partition as in the starting fact above. In general, the (complete) lines determined by the line segments of the partition to intersect, outside of U. It may happen that these extended lines determine an analogous partition in some other open subset U'. In some sense, by following the extended lines, the partition in U regenerates somewhere else in the plane. Our proof above applies and yields that the holonomy maps from U to U' are locally bi-Lipschitz.

PROPOSITION 1.3. *Let U be an open subset of the Euclidean plane, and $\mathcal{F}$ a partition of a subset N of U, with classes of the form $l \cap U$, where l is a line (do not take the connected component of $l \cap U$).*

Then, the 'global holonomy' of $\mathcal{F}$ is locally bi-Lipschitz. More exactly, let τ_1 and τ_2 be two 1-dimensional C^1-submanifolds transverse to $\mathcal{F}$, the partially defined holonomy map $x \in \tau_1 \to f(x) = \mathcal{F}_x \cap \tau_2 \in \tau_2$ is locally bi-Lipschitz. (τ_1 and τ_2 are not necessarily in the same connected component of U).

1.1.1. *Topological Caustic = Differentiable Caustic*

Observe the following amazing fact (see also Section 9.2).

COROLLARY 1.4. *Let τ be a one-dimensional C^1-submanifold of the plane. Consider the normal exponential map*

$$\pi\colon (x, t) \in \tau \times \mathbb{R} \to x + tN(x) \in \mathbb{R}^2$$

where N is a unit normal field on τ. If for an open subset $V \subset \tau \times \mathbb{R}$, $\pi_{|V}$ is injective, then it is a locally bi-Lipschitz homeomorphism onto its image.

If τ is C^2 (in which case π is C^1), a point (x, t) is regular, i.e. it admits a neighborhood diffeomorphically mapped to its image, iff, (x, t) is topologically regular, that is, it admits a neighborhood homeomorphically mapped to its image.

Proof. The proof follows essentially from Proposition 1.3. In the C^2 case, apply the fact that if a C^1 homeomorphism is bi-Lipschitz, then it is a diffeomorphism, that is, its derivative is nonsingular (since otherwise, the inverse cannot be Lipschitz). □

Remark 1.5. Recall that the (differential or analytic) caustic of τ is the set of singular values of π. This is well defined in the case π is C^1, that is, when τ is C^2. In the case when τ is C^1, one may define a topological caustic as the set of topologically singular values of π (which simply means that π fails to be injective at these values). The Corollary states the equality between topological and analytic caustics.

It follows that if τ is everywhere non $C^{1+\mathrm{Lip}}$ (i.e. N is nowhere Lipschitz) then τ is contained in its caustic!

Remark 1.6. The property of the exponential map stated in the Corollary is reminiscent of properties of holomorphic functions on the complex field. There is in fact a well developed theory of singularities of 'optical Lagrangian submanifolds', which confirms analogies but also differences with the holomorphic situation (see for instance [1]).

The corollary above seems to be generalizable to other situations (see Section 9.2).

1.2. SHARPNESS

The following shows that the Lipschitz condition in the previous facts is optimal.

FACT 1.7. *Let τ be a compact one-dimensional C^1-submanifold in the plane, and $\vec{n}\colon \tau \to S^1$ a unit transversal vector field along τ ($\vec{n}(x)$ is transverse to $T_x\tau$ for any $x \in \tau$). For a positive real T, consider,*

$$D_T(x) = \{x + t\vec{n}(x)/|t| \leqslant T\}$$

In order that there exists $T > 0$ such that the family $\{D_T(x), x \in \tau\}$ foliates, i.e. $x \neq x' \Longrightarrow D_T(x) \cap D_T(x') = \emptyset$, it is necessary and sufficient that $\vec{n}$ is Lipschitz.

The necessity part is essentially the content of the Starting Fact. The proof of sufficiency is similar.

1.3. INTERPOLATION OF HOMEOMORPHISMS

Let $\mathcal{B}$ denotes the closed strip in the plane delimited by the lines (parallel to the y-axis) $\{0\} \times \mathbb{R}$ and $\{1\} \times \mathbb{R}$. Any increasing homeomorphism $f: \mathbb{R} \to \mathbb{R}$ determines a foliation $\mathcal{I}_f$ by line segments on $\mathcal{B}$. Leaves are obtained by interpolating pairs $(0, x)$ and $(1, f(x))$ (by line segments). The increasing condition on f is equivalent to the non-intersection in $\mathcal{B}$ of these line segments. So, we have a foliation for which f is a holonomy map.

This construction seems to contradict our previous statements because an increasing homeomorphism f is not necessarily Lipschitz! The point is that, our results apply to foliations of open sets, that is, the unit tangent vector field of $\mathcal{I}_f$ is locally Lipschitz on the interior of $\mathcal{B}$, but fails to be locally Lipschitz on the whole of $\mathcal{B}$.

FACT 1.8 (Characterization of locally bi-Lipschitz homeomorphisms). *$\mathcal{I}_f$ extends to a foliation by line segments to a neighborhood of $\mathcal{B}$ iff f is locally bi-Lipschitz.*

Remark 1.9. Denote by $\partial^- \mathcal{B}$ (resp. $\partial^+ \mathcal{B}$) the component $\{0\} \times \mathbb{R}$ (resp. $\{1\} \times \mathbb{R}$) of the boundary of $\mathcal{B}$. Then, $\mathcal{I}_f$ can be extended to an open set containing $\partial^- \mathcal{B}$ (resp. $\partial^+ \mathcal{B}$), iff, f (resp. f^{-1}) is locally Lipschitz. A (local) Lipschitz defect (that is a kind of function measuring the modulus of Lipschitz continuity) of f can be expressed as a rate of confluence of elements of $\mathcal{I}_f$ near $\partial^- \mathcal{B}$ (outside of $\mathcal{B}$).

We also have the following more general fact where f is not assumed to be increasing, or even homeomorphic. Here, the disjointness condition concerns only a neighborhood of the source.

FACT 1.10. *Let τ_1 and τ_2 be two disjoint one-dimensional C^1-submanifolds in the plane, and $f: \tau_1 \to \tau_2$ a map (not necessarily homeomorphic). Consider the family of lines $\{x, f(x)\}$ (this denotes the line joining x and $f(x)$), for $x \in \tau_1$. There exists a neighborhood U of τ_1 on which the family of subsets $\{x, f(x)\} \cap U, x \in \tau_1$, are disjoint, iff, f is locally Lipschitz.*

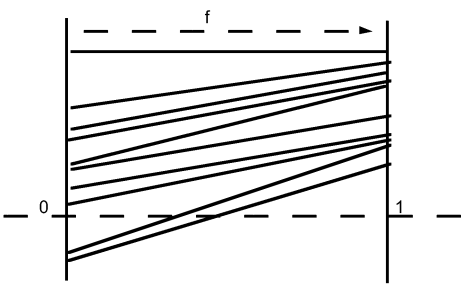

Figure 2. The foliation $\mathcal{I}_f$ on $\mathcal{B}$.

1.3.1. *Realization of a Homeomorphism as a Holonomy of a Foliation by Rigid Curves.*

It is very suggestive to ask whether the facts above extend to foliations by 'rigid' curves.

In particular, one may ask if, in the construction of $\mathcal{I}_f$, changing straight line segments by graphs of polynomials (each graph interpolates two points $(0, x)$ and $(1, f(x))$), one may get a foliation defined in a neighborhood of $\mathcal{B}$ (for f non-Lipschitz)?

One may try the following naive construction. Consider the polynomial of degree 3 (Lagrange-)interpolating $(-1, x)$; $(0, x)$; $(1, f(x))$ and $(2, f(x))$. One expects that the family of cubics obtained for $x \in \mathbb{R}$ gives rise to a foliation of the strip delimited by $\{-1\} \times \mathbb{R}$ and $\{2\} \times \mathbb{R}$ having obviously f as a holonomy map.

However, the non-intersection of cubics (over the interval $[-1, 2]$) is not automatic. The Lagrange interpolation is not monotonic! In fact, it seems that none of the interpolation methods can be (monotonic). So, we formulate the question of realization of f as follows:

QUESTION 1.11. What is the required regularity for a homeomorphism of $\mathbb{R}$ to be a holonomy map of a foliation of an open set of the plane by graphs of polynomials with bounded degree (or more weakly by graphs of polynomials with unbounded degree, or even by real analytic leaves)?

1.4. CODIMENSION ONE FOLIATIONS

All the statements here generalize to codimension 1 foliations. For example, Fact 1.1 is valid for foliations of open sets of the Euclidean space $\mathbb{R}^n$ by pieces of hyperplanes.

1.5. CONTENT OF THE ARTICLE

The objective of this article is to try to understand Fact 1.1 and to see to what extent it generalizes to codimension 1 foliations by 'rigid' leaves. In particular to answer Question 1.11 (Theorem 5.1).

Our result is that there is a mild regularity for all codimension 1 foliations by rigid leaves (Corollary 4.2), that we will call *graph-Lipschitz* regularity (Section 4.1.1).

Our approach to the graph Lipschitz regularity is based on the following construction. Locally, the leaves of a codimension 1 foliation may be seen as graphs of functions on an open set U of a Euclidean space. The foliation (locally) corresponds to a curve in the functional space $C(U, \mathbb{R})$. The nonintersection of leaves implies that this curve is *causal*, i.e. it is directed by the cone of *positive* functions on U. The graph-Lipschitz regularity follows from this description, in the case of leafwise rigid foliations, that is essentially when this causal curve lies in a finite-dimensional space.

The article also contains comments on related topics, in particular some special generalizations of Lipschitz regularity to the higher codimension case.

2. Counter-Examples in Higher Codimension

The automatic Lipschitz regularity is a codimension 1 phenomenon. For example, in $\mathbb{R}^3$ there is place for (global) nonmeasurable partition by line segments. Indeed, endow $\mathbb{R}^3$ with coordinates (x, y, z) and consider the unit vector field

$$X(x, y, z) = (\cos\theta(z), \sin\theta(z), 0),$$

where $\theta\colon \mathbb{R} \to \mathbb{R}$ is any function.

For z fixed, X determines a (global) foliation of $\mathbb{R}^2 \times \{z\}$ by parallel lines with slope $\theta(z)$. Therefore, X determines a partition of $\mathbb{R}^3$ by lines, which has the same regularity as θ. In particular, X may be non-Lipschitz, even nonmeasurable, etc.

Observe that when X is measurable, its flow

$$\phi^t\colon (x, y, z) \to (x, y, z) + tX(x, y, z)$$

preserves the volume. In the case where θ is regular, say C^1, this follows from the unipotent form of $D\phi^t$. For a nonregular θ one uses approximation by C^1 ones.

In particular, such an X is a special solution of the Euler equations of hydrodynamics, which are:

$$\partial X/\partial t = (-X.\nabla)X - \nabla p,$$
$$\operatorname{div} X = 0.$$

Here, $X = X(t, x, y, z)$ is a nonautonomous vector field, p is called the pressure, $(-X.\nabla)X = DX.X$ is the (geodesic) curvature of the trajectories (of particles).

Our example above is stationary ($\partial X/\partial t = 0$) and special in the sense that $\nabla p = 0$, i.e. the pressure is constant.

Although, our example may be nonregular, it has a simple geometry. One dares ask if this is a general fact (or may be not):

QUESTION 2.1. Find all the (global) foliations (with a fixed regularity, for example C^ω) of $\mathbb{R}^3$ by lines. For example, assuming further that they are volume preserving (and hence special solutions of Euler equations).

3. Another Approach. Lipschitz Curves and Directing Cones

Here we start by considering another (qualitative) approach to the Lipschitz regularity. To simplify notations, we restrict ourselves to foliations of the unit (open) disc

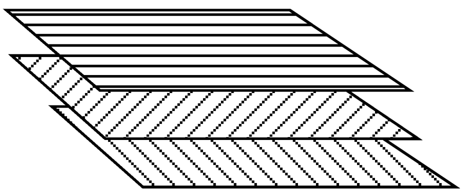

Figure 3.

of the plane. We identify a foliation by line segments, to its quotient space, viewed as a subset, in fact a curve, in the space of line segments of the disc.

The nonintersection condition translates to that the quotient space is 'totally ordered', or equivalently that it is a 'causal' curve in the sense of the canonical conformal Lorentz structure on the space of line segments of the disc. Here follow the details.

3.0.1. *The Space of Line Segments of the Disc*

When a disc is seen (as a convex subset) in the (real) projective plane, there is a polar correspondence between the space of line segments of this disc and the interior of its complementary set, which is homeomorphic to a Möbius strip.

Here, to avoid global topological complications, we will describe a partial correspondence between the space of line segments of the usual disc $\mathbf{H}^2$ (i.e. the unit disc in the Euclidean plane) and its exterior $\mathbf{dS}^2$ (see below for justification of the notation).

To a line segment $s \subset \mathbf{H}^2$, associate $p(s) \in \mathbf{dS}^2$, the intersection point of the two lines (in $\mathbf{dS}^2$) which are tangent to the unit circle (the boundary of the disc) at the endpoints of the given line segment s. This is not defined for segments passing through the origin 0 (that is why one needs to consider the projective plane).

The inverse map will be denoted $p \in \mathbf{dS}^2 \to s(p) \subset \mathbf{H}^2$.

3.0.2. *Cones*

The notation $\mathbf{H}^2$ refers to the hyperbolic structure of the disc, which is a Riemannian metric compatible with its projective structure. The notation $\mathbf{dS}$ reads de Sitter and refers to a canonical Lorentz metric on $\mathbf{dS}^2$. The associated Lorentz conformal structure, i.e. its isotropic cone field, is easy to describe.

In dimension 2, a Lorentz conformal structure consists in giving two transverse direction fields. For $p \in \mathbf{dS}^2$, let L_p^1 and L_p^2 be the line segments emanating from p and tangent to the boundary $\partial\mathbf{H}^2$. One may choose L_p^1 and L_p^2 to be continuous on p, by imposing that their tangent (half)-directions (at p) l_p^1 and l_p^2 determine the canonical orientation of the Euclidean plane.

The isotropic cone at p is the union of the direction of l_p^1 and l_p^2.

The positive infinitesimal causality cone at p is: $C_p^+ = \mathbb{R}^+ l_p^1 + \mathbb{R}^+ l_p^2$. In other words, it is the union of all the half directions determining (half) line segments which meet $\partial\mathbf{H}^2$.

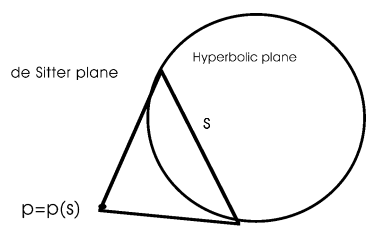

Figure 4.

The geometric causality cone $\mathcal{C}_p^+$ is the part of the plane delimited by L_p^1, L_p^2 and $\partial\mathbf{H}^2$, i.e. the union of half segments joining p to points of $\partial\mathbf{H}^2$.

One defines analogously the negative cones C_p^- and $\mathcal{C}_p^-$, and nonoriented cones $C_p = C_p^+ \cup C_p^-$ and $\mathcal{C}_p = \mathcal{C}_p^+ \cup \mathcal{C}_p^-$

3.0.3. *Orders*

In $\mathbf{dS}^2$, we have a partial order $p \leqslant q \Longleftrightarrow \mathcal{C}_p^+ \subset \mathcal{C}_q^+ \Longleftrightarrow p \in \mathcal{C}_q^+ \Longleftrightarrow \mathcal{C}_q^- \subset \mathcal{C}_p^- \Longleftrightarrow q \in \mathcal{C}_p^-$.

Observe that the set of points comparable with p, i.e. those points q satisfying one of the relations $p \leqslant q$, or $q \leqslant p$, is $\mathcal{C}_p$.

To interpret this for line segments of $\mathbf{H}^2$, let us restrict our considerations to the set $\mathcal{S}$ of line segments contained in the (open) lower half disc (delimited by the x-axis).

There is an order $\leqslant$ on $\mathcal{S}$, $s \leqslant s' \Longleftrightarrow s'$ is above s, i.e. s' separates s from the upper half disc.

The subset of $\mathbf{dS}^2$ corresponding to $\mathcal{S}$ is the lower half strip $\mathcal{P}$ delimited by the lines tangent to $\partial\mathbf{H}^2$ and parallel to the y-axis.

For $p, q \in \mathcal{P}$ (equivalently $s(p)$, $s(q) \in \mathcal{S}$), we have $p \leqslant q \Longleftrightarrow s(p) \leqslant s(q)$.

3.0.4 *Foliations*

Observe that for *nearby* segments $s, s' \in \mathcal{S}$, if $s \cap s' = \emptyset$, then, s and s' are comparable. Equivalently:

FACT 3.1. *A point $p \in \mathcal{P}$ admits a neighborhood V_p such that, if $p' \in V_p$, and the corresponding segments $s(p)$ and $s(p')$ are disjoint, then: $p' \in \mathcal{C}_p$ (equivalently $p \in \mathcal{C}_{p'}$).*

Consider now a foliation $\mathcal{F}$ of the lower half disc by segments, and let τ be a transversal. This defines a map $Q: x \in \tau \to p(\mathcal{F}_x) \in \mathcal{P}$. Its image is the quotient space of $\mathcal{F}$. Identify τ with $\mathbb{R}$ via $I: \mathbb{R} \to \tau$.

The curve $c = Q \circ I: \mathbb{R} \to \mathbf{dS}^2$ is causal, in the sense that, $\forall t, t'$, $c(t') \in \mathcal{C}_{c(t)}$. In fact, up to a switch of orientation (of $\mathbb{R}$) we may assume that c is order preserving: $t' \geqslant t \Longrightarrow c(t') \geqslant c(t)$, that is, $t' \geqslant t \Longrightarrow c(t') \in \mathcal{C}_{c(t)}^+$

3.0.5. *Causal Versus Lipschitz*

We are now in position to prove a variant of Fact 1.1 by showing that the curve c is ‘geometrically Lipschitz’, that is it can be parametrized to become Lipschitz.

This fact is well known in Lorentz geometry (and other fields). Its proof is very natural, being causal is essentially the geometric counterpart of being geometrically Lipschitz. We summarize the situation in the following statement.

FACT 3.2. *Let $\mathcal{P}'$ be the sub-strip in $\mathcal{P}$ of points with y-coordinates $\leqslant -2$ (that is, $\mathcal{P}' = \{(x, y) \in \mathbb{R}^2, |x| \leqslant 1, y \leqslant -2\}$).*

Let $p = (x_p, y_p)$, $q = (x_q, y_q) \in \mathcal{P}'$ two comparable points, say, $p \leqslant q$. Then:

$$d(y_p, y_q) \leqslant d(p, q) \leqslant \frac{1}{\cos(\pi/4)} d(y_p, y_q)$$

Here d denotes the Euclidean distance.

In particular, if A is a totally ordered subset in $\mathcal{P}'$, then the (Euclidean) projection pr: $A \to y$-axis, is injective, and bi-Lipschitz, with a bi-Lipschitz constant $\leqslant \frac{1}{\cos(\pi/4)}$.

In particular, a causal curve lying in $\mathcal{P}'$ can be re-parameterized, by projecting it onto the y-axis (which is also a causal curve), to be (uniformly) bi-Lipschitz.

Any causal curve in $\mathbf{dS}^2$ (not necessarily in $\mathcal{P}'$) can be parametrized by projecting onto a causal line segment, to become locally bi-Lipschitz.

Proof. The proof of the first point follows by observing that the maximal angle between the line joining p and q and the y-axis, is $\pi/4$. □

3.1. CURVES DIRECTED BY CONE FIELDS. LINEAR VERSION

Here, we generalize the previous fact to cone structures not necessarily derived from conformal Lorentz structures. However, to simplify, we will only consider a linear situation.

3.1.1. *Data*

Let E be a finite-dimensional vector space and C a cone (usually convex but *nonclosed*), such that the closure of its convex envelope $[\overline{C}]$ is a convex *proper* cone, i.e. it doesn't contain a half vector space: if $u \in [\overline{C}]$ and $-u \in [\overline{C}]$, then $u = 0$.

This determines an order on $E, p \geqslant q \Longleftrightarrow p \in q + C \Longleftrightarrow p - q \in C$.

A (vector) line D is called causal, if $D = \mathbf{R}u_0$, where $u_0 \in C$.

Let F be a support hyperplane for $[\overline{C}]$ such that $F \cap [\overline{C}] = \{0\}$, then, we have a decomposition $E = D \oplus F$, and a projection *pr*: $E \to F$.

DEFINITION 3.3. We say that a totally ordered subset B is causally *parameterized* with respect to a decomposition $E = D \oplus F$, if A is the graph $\{(s, d(s)), s \in A\}$, of a map $d: A \to F$, where A is a subset of D.

We say that B is causally parameterized, if this happens for some decomposition.

LEMMA 3.4. *Given a decomposition $E = D \oplus F$, then, any totally ordered subset B can be causally parameterized with respect to it, in a uniform Lipschitz manner. More exactly, the projection pr: $B \to D$ is injective and bi-Lipschitz onto its image. The bi-Lipschitz constant depends only on C, D and F.*

In particular, identifying D with $\mathbb{R}$, any totally ordered subset B, is the image of an order preserving Lipschitz map $c: A \subset \mathbb{R} \to E$.

Proof. The proof is the same as that of the fact above. We assume that E is endowed with a Euclidean metric for which D and F are orthogonal.

The bi-Lipschitz constant is $1/\cos\alpha$, where α is the maximal angle formed by D and half lines contained in C. The angle α is $< \pi/2$ by the hypothesis that the closed convex envelope of C is a proper convex cone. □

3.1.2. *Causal Curves*

We say that a map $c: I \subset \mathbb{R} \to E$, where I is an interval, is a *causal curve* if it is continuous injective and its image is totally ordered (or equivalently, c is continuous and order preserving). In a standard way, one proves.

FACT 3.5. *A totally ordered set corresponds to a causal curve, iff, its projection on D is connected* (*an interval*).

In particular, a causal curve is a (*uniform*) *Lipschitz graph over D.*

Conversely, an absolutely continuous curve, that is $c(s) = \int_0^s c'(\xi)\,\mathrm{d}\xi$, *and* $c' \in L^1_{\mathrm{Loc}}(\mathbb{R}, E)$, *is a causal curve, iff, almost everywhere,* $c'(s) \in C$.

Remark 3.6. We are not stating here that a continuous order-preserving map $\mathbb{R} \to E$ is Lipschitz, but rather than that it can be (causally) reparameterized (over some decomposition) to become Lipschitz (canonically and uniformly). In fact, one naturally meets causal but not causally parameterized curves. The notion of causal curves is geometric and not parametric.

3.1.3. *Infinite Dimensional Case*

All the previous discussion generalizes to the case where E is an infinite-dimensional Banach space, if the closed envelope $[\overline{C}]$ has a *compact section,* i.e. its intersection with the unit sphere is compact (of course, always assuming that it is proper) (actually, the point is to suppose there exist $\epsilon > 0$, such that if $v = v_F + v_D \in C$, then $\|v_D\| \geqslant \epsilon \|v\|$).

4. Analytic Formulation. Codimension One Foliations by Graphs

Here, we show that a foliation structure is essentially equivalent to giving a causal curve in a suitable functional space. This allows us to derive Lipschitz properties for such a foliation, when its associated functional space is finite-dimensional, and conversely, to construct examples of such foliations.

4.0.4. *Order on* $C(M, \mathbb{R})$

Let M be a topological space. Consider $C(M, \mathbb{R})$ its space of continuous numerical functions. At this stage, we don't mind on the choice of a topology on it. This functional space has a canonical cone $C(M, \mathbb{R}^\bullet_+)$, that of (strictly) positive functions. This is a convex cone, and its closure for any reasonable topology is $C(M, \mathbb{R}_+)$, the cone of non-negative functions. It is proper.

The vector space D of constant functions is a *canonical causal* line.

A support hyperplane for $\overline{C}$ will be a supplementary F of D, such that no $f \in F$ has a definite sign (i.e. f is positive or negative). There is no canonical choice of such a support hyperplane.

4.1. FROM PARTITIONS TO CAUSAL CURVES

PROPOSITION 4.1. *Let M be a connected metric space, and $\mathcal{F}$ a partition of a subset N of $M \times \mathbb{R}$ such that every class (we will also call it leaf) is the graph of an element of $C(M, \mathbb{R})$. Let $B(\mathcal{F})$ be the set of involved functions, i.e. the functions whose graphs are the leaves of $\mathcal{F}$.*

Suppose that the vector subspace E of $C(M, \mathbb{R})$ generated by $B(\mathcal{F})$, together with the constant functions, is finite-dimensional. Also, suppose that the elements of $B(\mathcal{F})$ (and hence E) are locally Lipschitz (as functions on M).

Then, $B(\mathcal{F})$ is totally ordered and therefore, can be (casually re-) parameterized (over the constants) by a Lipschitz map: $c: A \to B(\mathcal{F}) \subset E$, where A is a subset of $\mathbb{R}$.

The mapping $\Phi_c: (x, s) \in M \times A \to (x, c(s)x) \in N$ is a locally Lipschitz (global) parameterization of $\mathcal{F}$, i.e. it sends bijectively the partition $M \times \{\cdot\}$ (on $M \times A$) to $\mathcal{F}$.

$B(\mathcal{F})$ corresponds to a causal curve, iff, N (the support of $\mathcal{F}$) intersects the fibers $\{x\} \times \mathbb{R}, x \in M$, in connected sets. In this case, if furthermore M is a manifold, then, Φ_c is a homeomorphism onto its image.

Proof. The connectedness of M is exactly required to ensure that if two functions u_1 and u_2, have nonintersecting graphs, then, one of the possibilities $u_1 < u_2$, or $u_2 < u_1$ holds. Therefore $B(\mathcal{F})$ is totally ordered. Lemma 3.4 gives a Lipschitz parametrization of $B(\mathcal{F})$ as claimed.

The true meaning of c being Lipschitz is that, for any basis $f_0, \dots, f_k$ of E, we have $c(s) = \Sigma a_i f_i$, with $a_i: A \to \mathbb{R}$ Lipschitz. It then follows, that, if the f_i are locally Lipschitz, then $\Phi_c: (x, s) \to (x, \Sigma a_i(s) f_i(x))$ is locally Lipschitz.

It is clear that Φ_c is injective and that its image is the support of $\mathcal{F}$.

For any fixed x, $s \in A \to c(s)x$ is an injective continuous mapping from A onto the fiber $N \cap (\{x\} \times \mathbb{R})$. If this fiber is connected, then A is a countable disjoint union of intervals. One then shows that A is actually an interval, if all the fibers are connected.

If A is an interval, and M is a manifold, then $M \times A$ is also a manifold, which ensures that Φ_c is a homeomorphism onto its image. □

4.1.1. *Graph-Lipschitz Regularity of Partitions by Rigid Graphs*

See the Remark below for a justification of the word 'graph-Lipschitz'.

COROLLARY 4.2. *Let M be a manifold. A partition of $M \times \mathbb{R}$ by graphs of elements of $\mathrm{Lip}_{\mathrm{Loc}}(M, \mathbb{R})$, all belonging to the same finite-dimensional space E, can be parametrized by a locally Lipschitz homeomorphism.*

Let's specialize to the simplest situation where M is an interval of $\mathbb{R}$, and $M \times \mathbb{R}$ is foliated by C^1-graphs.

FACT 4.3 (A variant of Fact 1.1 for leafwise rigid planar vector fields). *Let M be an interval of $\mathbb{R}$, $V(x, y) = (1, Y(x, y))$ a vector field on $M \times \mathbb{R}$ for which there exist $f_0, \ldots, f_k \in C^1(M, \mathbb{R})$, such that, any trajectory of V is the graph of a linear combination of the f_i.*

Then, there is a homeomorphism $\Phi: M \times \mathbb{R} \to M \times \mathbb{R}$ sending the horizontal $M \times \{\cdot\}$ to the foliation determined by V, and

(i) *Φ is locally Lipschitz, and*
(ii) *$V \circ \Phi$ is locally Lipschitz*

Proof. Take $\Phi = \Phi_c$, where c is as above. Write $c(s) = \Sigma a_i f_i$, with $a_i: \mathbb{R} \to \mathbb{R}$ Lipschitz. Then, $\Phi(x, s) = (x, \Sigma a_i(s) f_i(x))$ and $V \circ \Phi = \partial\phi/\partial x = \left(1, \Sigma a_i \partial f/\partial x\right)$ which is therefore locally Lipschitz. □

Remark 4.4 [Graph-Lipschitz regularity]. One may refer to the property $V \circ \Phi$ locally Lipschitz, as that V is graph-Lipschitz. By this we mean that, when V is seen as a section of the tangent bundle, its image (or graph) is a 'Lipschitz submanifold' (see Section 9.1.1 for various definitions of each objects). We think, the notion of graph regularity, e.g. graph-C^1, graph-$C^\omega \ldots$, is natural, and may be useful. As example, in the one-dimensional case, a mapping f: $[a, b] \to \mathbb{R}$ is graph-Lipschitz, iff, it has a bounded variation. Indeed, the curve Graph(f) can be parameterized in a Lipschitz way, iff, it is rectifiable, i.e. it has a finite length. Writing this length as a limit of sums over subdivisions of $[a, b]$, shows that it is finite, iff, the analogous sums corresponding to the total variation of f are finite.

4.2. FROM CAUSAL CURVES TO FOLIATIONS

So far, we have proved Lipschitz regularities by showing that a partition structure leads to causality, and then to Lipschitz, properties. The goal now is to construct foliations by means of causal curves. To begin with, observe that, if $c: \mathbb{R} \to C(M, \mathbb{R})$ is a causal curve, then the map $\Phi = \Phi_c: (x, s) \to (x, c(s)x)$ is injective. Therefore, Φ_c would parameterize a foliation, whenever it is surjective and bi-continous.

More exactly, we have the following Proposition and Fact, whose proofs follow from a standard manipulation as in Section 3.1 and Proposition 4.1.

PROPOSITION 4.5. *Let M be an open subset of a Euclidean space and $c: s \in \mathbb{R} \to (s, d(s)) \in D \oplus F = C(M, \mathbb{R})$ a continuous curve, causally parameterized over the constants, and with a finite-dimensional range E. Analytically, suppose that:*

- *There exist $f_1, \ldots, f_k \in Lip_{\mathrm{Loc}}(M, \mathbb{R})$, generating a support hyperplane F for the positive cone of E, that is, no element of F has a definite sign.*

- *$c(s) = s + a_1(s)f_1 + \cdots + a_k(s)f_k$, with a_i $\mathbb{R} \to \mathbb{R}$ continuous.*
- *c is causal, i.e. $s' > s \Longrightarrow \forall x \in M, c(s')x - c(s)x > 0$.*

Then, $\Phi_c: (x, s) \to (x, c(s)x) = (x, s + a_1(s)f_1(x) + \cdots + a_k(s)f_k(x))$ is a homeomorphism form $M \times \mathbb{R}$ to an open subset of it, which is locally Lipschitz. In particular on the image of Φ_c, there is a foliation $\mathcal{F}$ (with leaves the graphs of the $c(s)$) parameterized by Φ_c.

FACT 4.6. *A curve c as in the previous proposition is obtained by causally reparameterizing over the constants, any causal curve c_0. The infinitesimal ingredients for such a curve (i.e. c_0) are:*

- *$f_0, \ldots, f_k$ a basis of E (the finite-dimensional space generated by the $c(s)$, or equivalently the $c_0(s)$).*
- *$a_0, \ldots, a_k \in L^1_{\mathrm{Loc}}(\mathbb{R}, \mathbb{R})$, such that, for almost all s, $\Sigma a_i(s)f_i$ is a (strictly) positive function on M, i.e. $\Sigma a_i(s)f_i(x) > 0, \forall x \in M$.*
- *$c_0(s) = \Sigma\left(\int_0^s a_i(\xi)\mathrm{d}\xi\right)f_i$.*

4.3. EXAMPLE. CONE STRUCTURES ON POLYNOMIAL SPACES

Let M be an open subset of $\mathbb{R}^n$ and denote by $P_k(M)$ the space of polynomial functions, with degree $\leqslant k$ on $\mathbb{R}^n$ restricted to M. The positive polynomials give a cone structure on $P_k(M)$. Let us describe some examples.

- $M =]-\alpha_1 + \alpha[$, and $k = 1$. A polynomial function of degree 1 has a representation $f: x \to ax + b$. Such a function has a constant sign on M, iff, $f(-\alpha)$ and $f(+\alpha)$ have the same sign, that is, $(-a\alpha + b)(a\alpha + b) = -\alpha^2 a^2 + b^2 > 0$. Therefore, the cone structure is equivalent to the conformal structure defined by the Lorentz metric $-\alpha^2 a^2 + b^2$ on the (a, b)–plane.
- $M =]-\infty, +\infty[$, and $k = 1$. When, $\alpha \to \infty$, the Lorentz structures degenerate to a direction field, that of constant functions.
- $M = \mathbb{R}$, and $k = 2$. A polynomial of degree 2 is written $ax^2 + bx + c$. It is positive on $\mathbb{R} \Longleftrightarrow a > 0$ and $b^2 - 4ac < 0$. Therefore, we obtain the conformal structure of the time oriented Minkowski space $\{(a, b, c)\}$ endowed with the Lorentz form $b^2 - 4ac$.

5. Anti-Lipschitz Facts

Despite all the already proved Lipschitz properties, we have the following answer to Question 1.11.

THEOREM 5.1. *Any increasing homeomorphism of $\mathbb{R}$ can be realized as a holonomy map of a foliation of $\mathbb{R}^2$ by parabolas.*

More precisely, let f be a increasing homeomorphism of $\mathbb{R}$. *There are functions* $a(s)$ *and* $b(s)$ *such that*

(i) *the family of graphs of the parabolas* $\{a(s)x^2 + b(s)x + s\}_{s\in\mathbb{R}}$ *foliates* $\mathbb{R}^2$, *and*
(ii) $f(s) = s + a(s) + b(s)$, *i.e.* $(1, f(s)) = H_1(0, s)$ *where* H_1 *is the holonomy map* $\{0\} \times \mathbb{R} \to \{1\} \times \mathbb{R}$ (*Alternatively,* H_1 *is the time* 1 *of the flow of the vector field* V *such that* $V \circ \Phi: (x, y) \to (1, 2a(y)x + b(y))$, *where* $\Phi(x, y) = (x, a(y)x^2 + b(y)x + y)$.)

Proof. For the reader convenience, we will give here a proof essentially independent of the previous developments.

We firstly treat the case where f is absolutely continuous, that is, f has a locally integrable derivative $f' \in L^1_{\mathrm{Loc}}(\mathbb{R})$, and $f(s) = \int_0^s f'(\xi)\, \mathrm{d}\xi$.

In this case the family $\{a(s)x^2 + b(s)x + s\}_{s\in\mathbb{R}}$ of parabolas, will satisfy that a and b are absolutely continuous. In fact, we will define their derivative a' and b', which we verify they are in $L^1_{\mathrm{Loc}}(\mathbb{R})$, and define a and b to be the integrals $\int a'$ and $\int b'$, respectively.

Take,

$$a'(s) = \begin{cases} f'(s) & \text{if } f'(s) > 1, \\ f'(s) + 1 & \text{if } f'(s) \leqslant 1 \end{cases}$$

and

$$b'(s) = \begin{cases} -1 & \text{if } f'(s) > 1, \\ -2 & \text{if } f'(s) \leqslant 1. \end{cases}$$

We easily verify that $a', b' \in L^1_{\mathrm{Loc}}(\mathbb{R})$, and $(b')^2 - 4a' < 0$, that is, for almost all s, $a'(s)x^2 + b'(s)x + 1$ is a positive polynomial. This means that the so- obtained curve

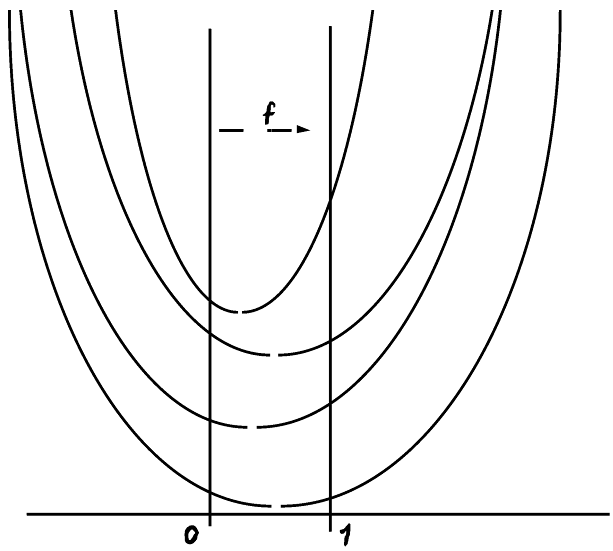

Figure 5. A foliation by parabolas interpolating f.

in the (Minkowki) space of parabolas (4.3) is causal, or in other words, if Par(s) is the graph of $a(s)x^2 + bx + s$, then Par(s) is strictly above $Par(s')$ whenever $s > s'$.

For any fixed x, consider the mapping

$$g^x: s \to a(s)x^2 + b(s)x + s$$

If $g^x(s) = y$, then $(x, y) \in Par(s)$. Therefore, in order to prove that we get a foliation of the whole plane, it suffices to show that g^x is surjective for all x.

We have:

$$\frac{\partial g^x(s)}{\partial s} = \begin{cases} f'(s)x^2 - x + 1 & \text{if } f'(s) > 1, \\ (f'(s)+1)x^2 - 2x + 1 = f'(s)x^2 + (x-1)^2 & \text{if } f'(s) \leqslant 1 \end{cases}$$

and, hence,

$$\frac{\partial g^x(s)}{\partial s} > \min(x^2 - x + 1, (x-1)^2)$$

This shows that for any $x \neq 1, g^x$ is uniformly expanding, that is its derivative is uniformly bounded from below by a positive constant, and therefore it is surjective (since it is continuous).

Now, for $x = 1, g^x$ is f itself, which is surjective by hypothesis.

Finally, by construction, $f' = a' + b' + 1$, and thus, $f(s) = a(s) + b(s) + s$ (by absolute continuity), that is the foliation by parabolas interpolates f as desired.

This finishes the proof in the absolutely continuous case.

If f is not absolutely continuous, approximate it by a sequence (f_n) of absolutely continuous (or even if we want Lipschitz) homeomorphisms, in the sense of uniform convergence on compact sets.

For each n, consider functions a_n and b_n, foliations $\{\mathrm{Par}_n(s), s \in \mathbb{R}\}$, and mappings g_n^x.

We have the following estimates: $f_n(s) \leqslant a_n(s) \leqslant f_n(s) + |s|$, and $|b_n(s)| \leqslant 2|s|$.

These lead to estimates of g_n^x by means of f_n. The same is true for $h_n^x = (g_n^x)^{-1}$.

In particular for any fixed x and any $s_0 = 0$, $h_n^x(0)$ is bounded. Therefore, because of the uniform expanding estimate for g_n^x, the sequence h_n^x satisfies Ascoli Theorem, and hence has a convergent subsequence, say h_n^x itself converges to some mappings h^x.

Let us show that h^x is injective. If not, there would exist $a > b$, such that $h^x(a) = h^x(b) = s$.

It then follows that there exist a decreasing sequence s_n, an increasing sequence s'_n such that $s = \lim s_n = \lim s'_n$, and $a = \lim g_n^x(s_n) > b = \lim g_n(s'_n)$.

In words, let $P = \lim Par_n(s_n)$, and $P' = \lim \mathrm{Par}_n(s'_n)$, then P is strictly above P' (limits exist after passing to subsequences). But these parabolas have at least two common points $(0, s)$ and $(1, f(s))$. This is a contradiction: the difference of the defining function of P and P' is positive, and has two different zeros.

Therefore, h^x is injective, and by the above estimates relating h_n^x and f_n, h^x is also surjective.

This implies that g_n^x converges to $(h^x)^{-1}$. One easily sees how this determines a foliation by parabolas as desired. □

Remark 5.2. Although, it seems somehow canonical, the construction above is by no means unique. There is a large space of foliations answering the problem. It is amazing to see how these foliations degenerate.

Remark 5.3. As we discussed in the introduction, the construction of Theorem 5.1 may be interpreted as an interpolation of a homeomorphism by parabolas. One may generalize this to an interpolation of finite systems $H_1, H_2, \ldots, H_n$ of increasing homeomorphisms of $\mathbb{R}$. The statement is that there is a foliation of $\mathbb{R}^2$ by graphs of polynomials of degree $\leqslant n+2$, such that, for any $s \in \mathbb{R}$, the $n+1$ points $(0, s), (1, H_1(s)), (2, H_2H_1(s)), \ldots, (n_1H_n, \ldots, H_1(s))$ are in the same leaf.

6. Explanation

So far, we have obtained three seemingly conflicting results: a (plain) Lipschitz regularity for foliations by line segments (Fact 1.1), a graph-Lipschitz regularity (i.e. a Lipschitz parameterization) for foliations by rigid curves (Corollary 4.2), and finally, a construction of non-Lipschitz foliations by parabolad (Theorem 5.1).

The reason of the non-Lipschitz regularity is simply that the Lipschitz parameterizations are not necessarily *bi-Lipschitz*.

We are interested here in understanding the difference between the cases of line segments and the other rigid curves (say, parabolas). In particular, this will give a new approach to Fact 1.1, by showing that the Lipschitz parameterizations are actually bi-Lipschitz in this case.

To simplify notation, we restrict ourselves to the case where M is a bounded open interval of $\mathbb{R}$. Let c be a causally parameterized curve with a finite-dimensional linear range $E \subset C(M, \mathbb{R})$. Again to simplify notation, we suppose that E is generated by $f_0, \ldots, f_k$, C^1 functions defined on the whole of $\mathbb{R}$.

The causally parameterized curve c is bi-Lipschitz, say it satisfies an inequality $0 < a < \|c'(s)\| < b$. It is convenient here to take the sup norm on E, $\|u\| = \sup_{x \in M} |u(x)|$. (This is well defined since the elements of E are defined on $\mathbb{R}$ and M is bounded.)

To begin with, let us suppose that c is C^1, that is essentially, c is everywhere differentiable. The general case, will be treated (at the end) by approximation.

Consider $\Phi_c\colon (x, s) \to (x, c(s)x)$, the parameterization of the foliation determined by c. We have:

$$D\Phi_c = \begin{pmatrix} 1 & 0 \\ * & c'(s)x \end{pmatrix}.$$

If c has the form $c(s)x = \Sigma a_i(s) f_i(x)$, then, the $*$ term is $\Sigma a_i(s) f_i'(x)$.

It then follows in particular that det $D\Phi_c = c'(s)x$. Therefore, Φ_c is singular at (x, s), iff, $c'(s)x = 0$.

6.0.1. *Maximum Principle*

Recall that, for almost all s, $c'(s)$ is a positive function on M, but it may happen for some s that c' is (only) nonnegative, that is, $c'(s)x$ vanishes somewhere in M.

We will say that E satisfies the *maximum* principle (more exactly here a minimum principle) if for $u \in E - \{0\}$, u nonnegative $\Longleftrightarrow u$ is positive. In other words, the positive cone on E has 0 alone as an accumulation point.

In the situation above, $c'(s) \neq 0$ (since $a < \|c'(s)\|$) and, hence,

FACT 6.1. *If E satisfies the maximum principle, then Φ_c is nonsingular.*

The following obvious statement expresses one difference between line segments and parabolas:

FACT 6.2. *For $E = P_k(M)$ the space of polynomial functions on M of degree $\leqslant k$, the maximum principle is satisfied, iff, $k \leqslant 1$, that is when E consists of affine functions.*

6.0.2. *Harnack Principle*

Observe that $(\|D\Phi_c(x,s)\|)^{-1}$ would be estimated from above, iff, one can do so far the term $c'(s)x$. One needs to estimate this last term a qualitative maximum principle. We will say that E satisfies a *Harnack* principle, if for any compact $K \subset M$, there exists a constant δ_K such that, for any $u \in E$ positive, $\|u\|_K = \sup_{y\in K} |u(y)| \leqslant \delta_K |u(x)|$, for any $x \in K$.

As above, we have:

PROPOSITION 6.3. *If E satisfies a Harnack principle, then Φ_c is locally bi-Lipschitz.*

Proof. The proof is already given in the case where c is C^1. But the estimates for $(\|D\Phi_c\|)^{-1}$ are uniform on c, and locally uniform on (x,s), and therefore, these estimates extend to the general Lipschitz case, almost everywhere (on x,s)). It is classical that almost everywhere boundness of derivatives implies Lipschitz. Therefore Φ_c is bi-Lipschitz. □

Remark 6.4. Since E is finite-dimensional, there is an equivalence between the maximum and the Harnack principle.

6.0.3. *A New Proof of Fact* 1.1

This reduces to the claim that $P_1(M)$ satisfies a Harnack principle.

Of course, none of the $P_k(M), k > 1$ satisfies a Harnack principle, one may however show that the foliations constructed in the proof of Theorem 5.1 are 'almost everywhere' Lipschitz. More exactly:

THEOREM 6.5. *The foliations by parabolas constructed in the proof of Theorem* 5.1 *are Lipschitz* (*i.e. their tangent direction is Lipschitz*) *away from* $\{0\} \times \mathbb{R}$ *and* $\{1\} \times \mathbb{R}$.

In other words there are foliations of the plane by parabolas, which are Lipschitz away from two parallel lines, and the holonomy between these lines may be any given homeomorphism.

7. Nonlinear and Infinite-Dimensional Situations

In the previous sections, we have considered foliations on $M \times \mathbb{R}$ by graphs of functions, which generate a linear subspace of finite dimension in $C(M, \mathbb{R})$. We will in the present section relax both the conditions 'linear' and 'finite'.

7.0.4. *Final Notion of Leafwise Rigid Foliations*

Recall the notation of Proposition 4.1. We have a partition $\mathcal{F}$ of a subset $N \subset M \times \mathbb{R}$, by the graphs of a set $B(\mathcal{F})$ of functions on M.

Previously, we assumed that the linear subspace of $C(M, \mathbb{R})$ generated by $B(\mathcal{F})$ has finite dimension. Now, we will say that $\mathcal{F}$ is a partition by *rigid leaves* (or a leafwise rigid partition) if $B(\mathcal{F})$ is contained in a finite dimensional C^1 manifold E.

The theory works in the same way as in the linear case. On E, there is a positive cone field. It is true that this cone field is not necessarily continuous but rather semi-continuous, but this doesn't matter.

One may extend this notion to general foliations, whose leaves are not globally graphs. (One can also generalize the notion of leafwise rigidity to higher codimension foliations, but only the codimension one case interests us here).

As in the linear case, one may define a maximum principle, which will be equivalent to a Harnack principle, due to the finiteness of the dimension.

THEOREM 7.1. *A codimension one leafwise rigid foliation can be locally parameterized by Lipschitz homeomorphisms. The foliation is Lipschitz if its leaves satisfy a maximum principle.*

COROLLARY 7.2 [14]. *A codimension one geodesic foliation on a manifold endowed with a C^1 connection, is locally Lipschitz.*

Remark 7.3. From this, one deduces that the foliations by parabolas constructed in Theorem 5.1 cannot become geodesic for any C^2 metric on the plane. Nevertheless, there exist C^0 metrics for which such a foliation is geodesic. This is the case of any foliation by graphs. Take any metric for which the projection on the x-axis is a Riemannian submersion with the tangent space of the foliation as a horizontal space.

Remark 7.4. Another case where rigidity and maximum principle are fulfilled, is that of codimension one umbilical foliations on Riemannian manifolds.

7.0.5. *Infinite-Dimensional Linear Case*

As was said in Remark 3.6, all the theory extends to the infinite dimensional case, with the condition that the nonnegative cone on E is 'projectively' compact, that is its intersection with the unit sphere is compact.

This applies to the situation of foliations on $M \times \mathbb{R}$ by graphs of harmonic functions, with respect to a Riemannian metric on M. It is classic that the restriction to a compact subset of M, of positive harmonic functions on M, is a cone with a compact section. Also, maximum principle and Harnack inequalities are valid, they actually originated in this situation.

COROLLARY 7.5. *A foliation on $M \times \mathbb{R}$ by graphs of harmonic functions, is locally Lipschitz.*

7.0.6. *Minimal Foliations*

Minimal hypersurfaces in a Riemannian manifold can be written locally as graphs of functions satisfying a 'special' second-order elliptic semi-linear equation. There is a well developed theory of such equations, leading to that their solutions enjoy most of the properties of the linear case (regularity, compactness of positive cones, maximum principle . . .)[9]. Here, one may mix the nonlinear and the infinite-dimensional discussions above to show:

THEOREM 7.6 (B. Solomon [12]). *A codimension* 1 *foliation on a Riemannian manifold by minimal leaves, is locally Lipschitz.*

7.0.7. *Other Situations*

J. Moser [9] and V. Bangert (see, for instance, [2]) studied foliations (more exactly laminations) by graphs of functions, which are solutions of a variational problem. They assumed that the associated Euler equation is elliptic of second order, and satisfies some estimates. Solomon's Lipschitz regularity result is generalized to this situation in [9].

Surely, there are some technical difficulties in adapting our approach to this nonlinear infinite-dimensional situation, but we think that it yields at least an explanation of this Lipschitz regularity phenomenon.

8. The Complex Case

8.0.8. *Geodesic Foliations*

By multiplying the examples of Section 2 by $\mathbb{R}$, one gets (global) codimension 2 foliations of $\mathbb{R}^4$ by 2-planes, which are not Lipschitz (even nonmeasurable). However, if $\mathbb{R}^4$ is identified with $\mathbb{C}^2$, then any foliation of an open set, with leaves contained in complex lines, is Lipschitz. More exactly, Fact 1.1 extends straightforwardly to the complex case. Indeed, its proof is essentially algebraic. It can be rewritten to cover the case of all plane geometries (over general fields), endowed with 'compatible' metrics. In particular, Fact 1.1 extends also to the p-adic case.

The extension to the complex codimension 1 case is also straightforward.

8.0.9. *Holomorphic Motions*

Let M be a complex manifold with a base point m_0. A holomorphic motion of $X \subset \mathbb{C}$ with parameter space M, is a partition of a subset $N \subset M \times \mathbb{C}$, by graphs of holomorphic functions on M, such that the fiber $N \cap (\{m_0\} \times \mathbb{C})$ equals X.

Any $x \in X$ determines a motion $\phi^x: M \to \mathbb{C}$, $\phi^x(m) = u_x(m)$, where u_x is the function whose graph is the leaf of (m_0, x). The nonintersection condition of graphs means that motions of different points of X don't meet, at any (time) $m \in M$.

Our investigation in this article may be seen as a study of (real) rigid motions. Let us however say that our original motivation was not the study of real versions of holomorphic motions, but rather, to understand Fact 1.1 and related results.

There is no (natural) order on $\mathbb{C}$, and thus we can't speak easily about causal structures. For example, the nonintersection of graphs induces of the space of complex polynomials a 'weakly complex causal cone' structure. This structure is somewhat complicated, and we don't see how to exploit it, for example to find a graph-Lipschitz regularity for foliations by graphs of such polynomials. We think however, it is worth investigating this structure, at least to give a new approach to the following quasi-conformal regularity of holomorphic motions which, in fact, does not require rigidity!

FACT 8.1 (see, for instance, [13] and [4]). *A holomorphic motion is transversally quasi-conformal, that is, for any $m \in M$, the map $x \in X \to \phi^x(m) \in \mathbb{C}$ is quasi-conformal.*

Conversely, any (orientation preserving) quasi-conformal map of $\mathbb{C}$ is a holonomy of a holomorphic motion.

Remark 8.2. It then follows that, for some real homeomorphisms f, the foliation by parabolas constructed in Theorem 5.1, cannot be extended by complexification, to any neighborhood of $\mathbb{R}^2$ in $\mathbb{C}^2$, as a holomorphic motion, i.e. for any neighborhood of $\mathbb{R}^2$, the complexified parabolas must intersect. Indeed, if this extension were possible, then the holonomy would be quasiconformal, and therefore, its restriction to $\mathbb{R}^2$ would be quasi-symmetric. It suffices to take f nonquasi-symmetric.

9. Comments on the Higher Codimension Case

So far, all the obtained regularity results concern codimension 1 situations, which was necessary as shown in Section 2. Here, we discuss some special higher codimension cases. We will also ask some questions concerning Lipschitz regularity.

9.1. LIPSCHITZ FOLIATIONS. FROBENIUS THEOREM.

When working on [16], I met the following problem. We have on a Lorentz manifold $(M, \langle,\rangle)$, a Lipschitz plane field E, which is Lorentzian (that is, the metric on it is of Lorentzian type, i.e. with signature $- + \cdots +$) for which we know that there exists a measurable locally bounded section n of $E^{\perp}$ (the orthogonal of E), such that, for any Lipschitz vector fields X, Y tangent to $E, (\nabla_X Y)^{E^{\perp}} = \langle X, Y\rangle n$, almost everywhere, where $()E^{\perp}$ means orthogonal projection on $E^{\perp}$. We hope to deduce from this that E is integrable, with umbilical leaves.

Observe that since, $[X, Y] = \nabla_X Y - \nabla_Y X$, the bracket of any two Lipschitz vector fields tangent to E, is tangent to E (almost everywhere). The integrability question we are asking is thus a Lipschitz version of Frobenius Theorem. We then found that this was recently proved by S. Simic.

THEOREM 9.1 ([11]). *Assume that a Lipschitz k-plane field E is involutive, in the sense that for all Lipschitz vector fields X, Y tangent to E, the bracket* $[X, Y]$ *is almost everywhere tangent to E. Then E is integrable, in the sense that through every point passes a leaf which is a k-submanifold of class* C^{1+Lip}*, tangent to E.*

9.1.1. *An Invariance Problem*

A leaf F of our E has a second fundamental form of the type $II(X, Y) = \langle X, Y\rangle n$, where n is a measurable normal vector field. This is the classical definition of umbilical submanifolds, but usually in a sufficiently smooth context, say C^2. Here, since F is of Lorentz type, we have the following extra-regularity:

FACT 9.2. *Suppose that the Lorentz metric on M is* C^{∞}*, and let F be a* C^2 *umbilical Lorentzian submanifold of dimension* $\geqslant 3$*. Then F is* C^{∞}*.*

Proof. Observe that this is standard if F is geodesic. The general umbilical case follows from the fact that F is isotropically geodesic, that is an isotropic geodesic tangent to F is locally contained in it. The proof of this goes as follows. Consider a solution of the equation $\nabla^F_{\gamma'(t)}\gamma'(t) = 0$, where ∇^F is the connection on F, and $\gamma(t)$ is a curve contained in F, with an isotropic initial data. Solutions exist since F is C^2, and hence, the coefficients of the equation are C^0. The curve $\gamma(t)$ will be everywhere isotropic, and thus $II(\gamma'(t), \gamma'(t)) = \langle\gamma'(t), \gamma'(t)\rangle n = 0$, that is $\gamma(t)$ is an isotropic geodesic of M. Now, to see that F is C^{∞}, observe that it contains many C^{∞} codimension 1 submanifolds inside it. Indeed, for any x, let $\exp_x$ be the exponential map at x, and $T^0_x F$ the isotropic cone at x. The previous discussion says that $\exp_x(T^0_x F)$ is locally contained in F. Observe now that $\exp_x(T^0_x F) - \{x\}$ is a C^{∞} hypersurface contained in F. □

This kind of regularity seems quite surprizing, since the involved PDE problem is of hyperbolic nature (because of the Lorentz condition). Classically, only elliptic equations have regularizing effects, but there is here an extra rigidity phenomenon.

The goal now is to try to generalize the fact to the case where F is C^{1+Lip}.

Let ϕ^t be the geodesic flow on TM, and G be its infinitesimal generator vector field. Denote by T^0M (resp. T^0F) the space of isotropic vectors tangent to M (resp. F). The geodesic flow preserves T^0M. The umbilicity equation $II = \langle, \rangle n$, which gives $II(u, u) = 0$ for $u \in T^0F$, implies that G is tangent (almost everywhere) to T^0F. If F is C^2, then T^0F is C^1, and therefore the tangency integrates to a local invariance: T^0F is locally invariant by ϕ^t, which exactly means that F is isotropically geodesic. This gives another proof of the previous fact. If F is merely C^{1+Lip}, then F is a 'Lipschitz submanifold', and we are naturally led to ask.

QUESTION 9.3. Let G be a C^∞ nonsingular vector field, with a flow ϕ^t on a manifold V and W a Lipschitz submanifold tangent to G. Is it true that W is locally invariant by ϕ^t, that is, for any $x \in W$, there exist U a neighborhood of x in W, and $\epsilon > 0$, such that $\phi^t(U) \subset W$, for $|t| < \epsilon$?

There is some subtlety in defining Lipschitz submanifolds. The strongest notion is that, locally, in some bi-Lipschitz chart, W corresponds to a Euclidean subspace. The weakest notion is that, locally, W is the image of an injective Lipschitz map defined on an open subset of a Euclidean space, with maximal rank almost everywhere.

The uniqueness of solutions of smooth vector fields yields an affirmative answer to the question in the case $\dim W = 1$, even with a weakest possible definition, that is, W is the image of a Lipschitz (not necessarily injective) curve (not necessarily injective, but not reduced to a point, in order to have $\dim W = 1$).

Remark 9.4. An analogous easier question may be posed for geodesic submanifolds of Riemannian manifolds, that is, let F be a C^{1+Lip} submanifold in a C^∞ Riemannian manifold M, such that almost everywhere, the second fundamental form vanishes, then, F is geodesic, that is, F contains, locally, the geodesics which are somewhere tangent to it, and therefore, F is C^∞. In the case, where M is the three-dimensional Euclidean space and F is a surface, one translates the infinitesimal condition to that the Gauss map $F \to S^2$ has rank 0, and is therefore constant, that is, F is geodesic.

9.2. NORMALS OF SUBMANIFOLDS

The Lipschitz regularity extends to vector fields (more exactly direction fields) with line segment orbits, on open sets of Euclidean spaces of dimension > 2, if they satisfy additional differential relations, which are automatic in dimension 2. One such a condition is that the orthogonal of the direction field is integrable. This is essentially equivalent to saying that the direction field is a gradient. More precisely, I learned from A. Fathi, when writing a first version of the present paper, the following statement (which is now published in [5]). Let H be a C^1 hypersurface in the Euclidean space $\mathbb{R}^n$, then, the family of its normals foliates a neighborhood of it, iff, H is C^{1+Lip}, i.e. a unit normal vector field along H, is Lipschitz. Another equivalent statement is that, if the distance function $x \to d(x, H)$ is C^1 near H, then it is in fact C^{1+Lip}. With this formulation, Fathi's result generalizes to a wide class of solutions of the Hamilton–Jacobi equation (the distance functions are a special case of them). Also, the result on normals of hypersurfaces generalizes to normals of submanifolds of any codimension, and in any (smooth) Riemannian manifold. As in Section 1.1.1, from this one deduces the equality of analytic and topological caustics of C^2 submanifolds in Riemannian manifolds.

9.3. 'ANOSOV' FOLIATIONS

The following Lipschitz regularity was crucial in [15] to prove nonexistence of 1-dimensional continuous geodesic foliations on compact hyperbolic 3-manifolds.

THEOREM 9.5. *Let C and C' be two continuous curves in S^2, the unit sphere of the Euclidean space $\mathbb{R}^3$. Suppose that for any $x, x' \in C, y, y' \in C'$ with $(x, x') \neq (y, y')$, we have $]x, y[\cap]x', y'[= \emptyset$. (In other words, the join of C and C' foliates a subset of the open unit ball).*

Then, C and C' are (more exactly, can be parameterized as) Lipschitz curves.

9.3.1. *Almost example of geodesic foliations on hyperbolic manifolds*

A candidate for a one-dimensional geodesic foliation on a hyperbolic manifold was discussed in [15]. The general construction goes as follows. Let $\phi: N \to S^1$ be a closed manifold fibering over the circle, with fiber type S and monodromy $\sigma: S \to S$, that is $N = S \times [0, 1]/(x, 1) \sim (\sigma(x), 0)$.

We have a suspension flow ϕ^t, that is that generated by the vector field $\partial/\partial t$ on [0, 1], and a dual 1-form ω, which vanishes on the factor S, and such that $\omega(\partial/\partial t) = 1$.

Let $\bar{N}$ denote the cyclic covering corresponding to ω. It is nothing but $S \times \mathbb{R}$.

Let's call a Riemannian metric on N basic, if $\partial/\partial t$ has a constant length and is orthogonal to the factor S.

One can prove that the orbits of ϕ^t are geodesic for any basic metric g_0. Furthermore, when lifted to $\bar{N}$, they are *globally minimizing*.

Suppose now that N is endowed with another metric g which has *negative* curvature.

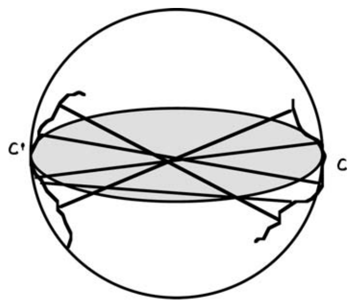

Figure 6. Nonintersection of line segments $\Longrightarrow C$ and C' are Lipschitz.

The orbits of ϕ^t lifted to the universal covering $\bar{N}$ are quasi-geodesic (in the sense of g). Indeed, along such an orbit, the intrinsic and extrinsic g-distances are comparable with the same constants comparing the Riemannian metrics g and g_0.

There is a continuous straightness homotopy $f: N \to N$, whose lift to $\bar{N}$ associates to a $\tilde{\phi}^t$-orbit its asymptotic g-geodesic.

Suppose now that σ is a pseudo-Anosov diffeomorphism on a surface S of genus $\geqslant 2$. In this case, by a theorem of Thurston [10], N has a *hyperbolic* (i.e. with constant curvature -1) metric.

It was proved in [15] that, for such a hyperbolic metric g, the image of f can not generate a foliation by g-geodesics, that is, essentially, $f: N \to N$ cannot be injective.

On the other hand, it was proved that the boundary map which associates to a $\tilde{\phi}^t$-orbit its endpoints at infinity, is injective. This implies in particular that the set of corresponding g-geodesics determines a topological 3-manifold $\hat{N}$ in the unit tangent bundle T^1N, which is invariant under the geodesic flow ($\hat{N}$ is homeomorphic to N and projects surjectively on it). However, $\hat{N}$ is not Lipschitz. In fact, one can prove (for instance by improving technics of [15]), that $\hat{N}$ has a Hausdorff dimension >3.

QUESTION 9.6. Calculate the Hausdorff dimension of $\hat{N}$. In particular, is it possible to calculate it by means of the (pseudo-Anosov) monodromy σ?

Remark 9.7. The geodesic flow restricted to $\hat{N}$ is pseudo-Anosov. The Hausdorff dimension of $\hat{N}$ equals 1+ the sum of Hausdorff dimensions of the stable and unstable leaves. This last sum equals the sum of Hausdorff dimensions of the stable and unstable quotient spaces of the stable and unstable foliations. These quotient spaces are realized as one-dimensional topological subsets of the sphere at infinity $\partial_\infty \mathbb{H}^3$. They are images of the stable and unstable ending maps $p^s, p^u: \mathbb{H}^3 \to \partial_\infty \mathbb{H}^3$, which associate to a point of $\mathbb{H}^3$, the positive and negative ends of its ϕ^t-orbit.

9.4. MATHER LAMINATIONS

So far, we have been concerned with various Lipschitz regularities which are local in nature. Here we will mention results on Lipschitz regularity due to global reasons. (We won't discuss the proofs here, but the phenomenon is in reality based on a local 'crossing Lemma' [8]). There are many contributors to this theory, we quote for our purpose here [2] and [8]. The most synthetic approach seems to be that of Mañé [6]. I learned a lot from D. Massart's thesis [7].

Here follows a brief review of the theory. We are given a compact manifold N endowed with a Riemannian metric g and a closed 1-form ω. They induce functions on the tangent bundle TN, by the rules $v \to g(v, v)$, and $v \to \omega(v)$.

To a compactly supported probability measure μ on TN, associate its action $A(\mu) = \int (g - \omega)\, d\mu$.

Consider the critical value $c(g, \omega) = \inf_\mu A(\mu)$. Measures which minimize A, i.e. $A(\mu) = c(g, \omega)$ are called minimizing (for (g, ω)) and are automatically invariant under the geodesic flow of g.

The Mather set $\mathcal{M}_{(g,\omega)}$ is the closure of the union of the supports of minimizing measures. It is a compact subset of TM invariant under the geodesic flow of g. It depends only on the cohomology class of ω (for fixed g). In particular, if ω is a coboundary, then $\mathcal{M}_{(g,\omega)}$ is the 0-section. Otherwise, $\mathcal{M}_{(g,\omega)}$ does not meet the 0-section, a vector u in $\mathcal{M}_{(g,\omega)}$ has no nontrivial multiple λu which belongs to $\mathcal{M}_{(g,\omega)}$. Therefore $\mathcal{M}_{(g,\omega)}$ can be identified with a closed subset $\mathcal{PM}_{(g\omega)}$ of the unit tangent bundle T^1N, invariant under the geodesic flow.

GRAPH THEOREM 9.8 (Mather, [8]). *The projection* $\pi: TN \to N$ *maps* injectively $\mathcal{PM}_{(g,\omega)}$ *in* N*, and has a Lipschitz inverse. In particular,* $\pi(\mathcal{PM}_{(g,\omega)})$ *is the support of a transversally measured one-dimensional Lipschitz geodesic lamination on* N*, we denote it by* $\mathcal{L}_{(g,\omega)}$ *and call it the Mather lamination on* N *associated to* (g, ω).

9.4.1. *Geometric-dynamical Description of Mather Geodesic Laminations*

The lamination $\mathcal{L}_{(g,\omega)}$ has the following geometric characterization (see [3] and [8]). Let $\bar{N}$ be the cyclic covering associated to ω. Consider $M_{(g,\omega)} \subset T^1N$, the set of vectors which determine a geodesic whose lift to $\bar{N}$ is minimizing (i.e. it minimizes the distance in $\bar{N}$ between any two points lying on it). Then, $\mathcal{PM}_{(g,\omega)}$ is contained in $M_{(g,\omega)}$.

$\mathcal{PM}_{(g,\omega)}$ is obtained from $M_{(g,\omega)}$ by purifying it from 'dissipative' geodesics. More exactly, $\mathcal{PM}_{(g,\omega)}$ is what is sometimes called the Poincaré recurrence set of the geodesic flow on $M_{(g,\omega)}$, that is the union of supports of invariant probability measures on $M_{(g,\omega)}$ (this union is closed since there is a single measure whose support equals the union). Mané introduced weaker geometric purifications which give rise to larger Lipschitz geodesic laminations [6].

9.4.2. *Lipschitz Geodesic Laminations on Hyperbolic* 3*-manifolds*

Keep the notations of Section 9.3.1: N, ϕ^t and $\sigma: S \to S$ a pseudo-Anosov diffeomorphism of a surface of genus $\geqslant 2$.

If g is a basic metric on N, then $\mathcal{L}_{(g,\omega)}$ is nothing but the orbit foliation of the suspension flow ϕ^t.

In contrast, if g is a hyperbolic metric on N, then the support of $\mathcal{L}_{(g,\omega)}$ is a proper subset of N, i.e. $\mathcal{L}_{(g,\omega)}$ is not a foliation [15]. In fact, one may prove that the support of $\mathcal{L}_{(g,\omega)}$ is not a rectifiable set, unless it is a finite union of closed geodesics.

From the previous Section, the description of $\mathcal{L}_{(g,\omega)}$ reduces to understand the 'nondissipative' minimizing geodesics in $\bar{N} = S \times \mathbb{R}$. One may for instance wonder if $\mathcal{L}_{(g,\omega)}$ is a 'fragment' of the suspension flow, when transformed by the straightness homotopy (Section 9.3.1), that is, is $\mathcal{L}_{(g,\omega)}$ the image by the straightness homotopy of a sublamination of the suspension flow?

There are only a finite number of isotopy classes of hyperbolic metrics on N. Therefore, we get a canonical finite collection of critical values $c(g, \omega)$, and a finite collection of isotopy classes of laminations $\mathcal{L}_{(g,\omega)}$, where g is a hyperbolic metric.

It fact, it seems that $\mathcal{L}_{(g,\omega)}$ is not a foliation of N, when g is merely a metric of negative curvature. This indicates how a basic metric is far from being of negative curvature. The fibration picture is well known to be destroyed with respect to hyperbolic metrics (in the 3-hyperbolic space, the fibres look like to the sheets of a cauliflower). It is very suggestive to describe the lamination $\mathcal{L}_{(g,\omega)}$ as a measurement of this distortion (for g hyperbolic). In particular, how are these objects related to the dynamics of the pseudo-Anosov σ, and can they be described by means of σ only?

The philosophy behind these questions is to see whether some characteristics of the hyperbolic 3-manifold are accessible by means of the pseudo-Anosov monodromy, and thus what kind of invariants of pseudo-Anosov diffeomorphims are so obtained? This possibility would give further evidences to the hyperbolicity Theorem. I shared these thoughts with J. P. Otal, some years ago (before he wrote his full proof of the hyperbolicity Theorem for fibering 3-manifolds [10]).

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