Actions of discrete groups on stationary Lorentz manifolds

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(joint work with Paolo Piccione)
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Linear Dynamics
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Introduction
Global invariants of Lorentz metrics

$M$ a differentiable manifold (today everywhere compact)

$\text{Diff}^k(M)$ acts on

$\text{Rie}^{k-1}(M)$ (resp. $\text{Lor}^{k-1}(M)$) = space of $C^{k-1}$ Riemannian (resp. Lorentz) metrics on $M$.

Endow them with the Banach topology (or Frechet for $k = \infty$)

It is known that $\text{Diff}(M)$ acts properly on $\text{Rie}(M)$

i.e. The quotient $X = \text{Riem}(M)/\text{Diff}(M)$ is Hausdorff = modular space of $M$.

• A function on $F: g \in X \rightarrow F(g) \in \mathbb{R}$ is a Riemannian invariant: diameter, volume, integral curvature, injectivity radius...
Global invariants of Lorentz metrics

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(SUPER-) QUESTION: **When is the** $\text{Diff}(M)$-action on $\text{Lor}(M)$ **proper?**

Recall $G$ acts properly on $X$ if: $\forall K \subset X$ compact, the set (of return times)

$$G_K = \{g \in G, gK \cap K \neq \emptyset\}$$

is compact

– Gromov: the difficulty in the global studying of Lorentz manifolds lies in the fact that $\text{Lor}(M)/\text{Diff}(M)$ does not exist (as a Hausdorff space).
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The Question

The $\text{Diff}(M)$-action on $\text{Lor}(M)$ proper $\implies \forall \ g \in \text{Lor}(M)$, Stabilizer($g$) is compact,
But Stabilizer($g$) $=$ Isom($g$)

Question

*When is the isometry group of a compact Lorentz manifold non-compact?*

In the non-compact case:

Question: Classify Lorentz manifolds $(M, g)$ for which $G = \text{Isom}(M, g)$ acts non-properly
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\[ G = \text{Isom}(M, g) \]

\( G^0 \) its identity component (i.e the connected component of 1)

Cases:

- \( G^0 \) non-compact (strongest hypothesis)
- \( G^0 \) compact and non-trivial
- \( G^0 \) trivial

\[ \Gamma = G/G^0 \] the “discrete part of \( G \)”

\( \Gamma \) acts by conjugacy: \( \Gamma \to \text{Aut}(G^0) \to \text{Out}(G^0) \)

(Conditions on this action)
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Paradigmatic example: Flat Lorentz tori

$q$ a Lorentz form on $\mathbb{R}^n$
$\rightarrow$ a Lorentz flat torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$.

The linear isometry group of $(\mathbb{T}^n, q)$:

$$O(q, \mathbb{Z}) = GL(n, \mathbb{Z}) \cap O(q)$$

Full isometry group: the semi-direct product: $O(q, \mathbb{Z}) \ltimes \mathbb{T}^n$

For generic $q$, $O(q, \mathbb{Z})$ is trivial.
$q$ rational $\iff q(x) = \alpha(\sum a_{ij}x_ix_j)$, and $a_{ij}$ are rational numbers,

Harich-Chandra-Borel theorem ($n \geq 3$)

$O(q, \mathbb{Z})$ is big in $O(q)$;

It is a lattice in $O(q)$.

$O(q, \mathbb{Z})$ is a "standard" arithmetic (real) hyperbolic group

For $q_0 = -x_1^2 + x_2^2 + \ldots + x_n^2$: $O(q, \mathbb{Z})$ has finite covolume

$O(q, \mathbb{Z})$ may be co-compact for other $q$, say in dimension $n = 3$

When $q$ is not rational, many intermediate situations are possible.
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When $q$ is not rational, many intermediate situations are possible.
Recall: \( \text{PSL}(2, \mathbb{R}) \rightarrow SO(1, 2) \)  
(Acton of \( \text{SL}(2, \mathbb{R}) \) on polynomials of degree 2)  
\( \text{SL}(2, \mathbb{Z}) \rightarrow O_{\mathbb{Z}}(1, 2) \)

Some elements of \( O_{\mathbb{Z}}(1, 2) \)  
Hyperbolic:

\[
\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \ldots
\]

Parabolic (unipotent):

\[
\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \rightarrow \ldots
\]
Dimension 2

\[ q_0 = x^2 - y^2 \]

\[ \text{SO}(1, 1) = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \right\} \]

\[ \text{SO}_\mathbb{Z}(1, 1) = \{1\} ? \]
(Avez: observed that Anosov diffeomorphisms on the 2-torus preserve Lorentz metrics ?)

\[ A \in SL(2, \mathbb{Z}) \] hyperbolic, e.g.

\[
\begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix}
\]

\( x^u \) and \( x^s \) coordinates along eigen-directions

\( q = x^u x^s \).

\( A \) preserves \( q \)

\( \text{Isom}(\mathbb{T}^2, q) = (\text{essentially}) \mathbb{Z} \ltimes \mathbb{T}^2, \mathbb{Z} \) generated by \( A \).

\( A \) preserves some rational \( q = ax^2 + cxy + by^2 \) (with all coefficients \( \neq 0 \))
An arithmetico-dynamical Remark

\( A \in O(q, \mathbb{Z}) \)

A hyperbolic means it has an eigenvalue of norm \( \neq 1 \)

Thus Spectrum \((A) = \{\lambda, \lambda^{-1}, \sigma_1, \ldots \sigma_k\}\), algebraic integers, \(\lambda\) real and \(> 1\)

\(\sigma_i \in S^1\)

The corresponding diffeomorphism on \(\mathbb{T}^n\) is partially hyperbolic with one dimensionnal stable and unstable foliation.

These foliations may be minimal (all leaves dense)

In this case, \(\lambda\) is a Salem number,

Conversely, any Salem number occur as a leading eignevalue for some \(A \in O(q, \mathbb{Z})\) for some \(q\).
Connected examples

Suspension $\mathbb{T}^3_A$

The suspension of $A$ gives a flat Lorentz manifold endowed with an isometric flow which is Anosov (chaotic)

$\mathbb{T}^3_A = SO\ell/\Gamma$,

$SO\ell$: the 3-dimensional unimodular solvable non-nilpotent group.

(Compare with Bianchi)
Non-suspension examples

Instead of \textit{SOL}

take $G = SL(2, \mathbb{R})$,

$M = SL(2, \mathbb{R})/\Gamma$, $\Gamma$ a co-compact lattice

The $G$ action on $G/\Gamma$ preserves a Lorentz metric,

This metric has constant negative curvature (locally AdS)

Explanation: at the origin $1 \in G/\Gamma$, take the Killing form

$\kappa : G \times G \to \mathbb{R}$ (it has a Lorentz signature).
Another examples: Oscillator (or Warped Heisenberg) groups

There is a (family of) groups $G$, solvable but looking like $SL(2, \mathbb{R})$:

- they are solvable, so their Killing form is degenerate
- they have a bi-invariant Lorentz form on their Lie algebra
- they have lattices
Results
Hypotheses

\( M \) compact Lorentz
\( G \) acts isometrically
\( G^0 \) compact
\( G \) non-compact
\( \Gamma = G/G^0 \) acts on Aut(\( G^0 \)).

**A geometric hypothesis:** The \( G^0 \) action is not everywhere non-timelike: there is \( x_0 \) such that \( G^0 x_0 \) is timelike (the induced metric is Lorentz).

Example, strong situation: \( M \) is [stationary]: there is an everywhere timelike Killing field.
Essentially: the conjugacy action of $\Gamma$ on $G^0$ is not equicontinuous.

**Fact** (non-trivial): The algebraic and geometric hypotheses are equivalent.
First formulation of results, corollaries

Up to finite cover for $M$ and finite index subgroup for $G$ (everywhere),

$G^0$ has a toral $\Gamma$-invariant factor $T$ (of some dimension $d$)

- The action of $\Gamma$ on $T$ preserves some Lorentz form $q$ and $\Gamma = O(q, \mathbb{Z})$.
- The action of $T$ on $M$ is everywhere free
- The orbits are all timelike: the identification of any orbit $T \times \mathbb{T}$ with $T$ gives a $\Gamma$-invariant Lorentz form $q_x$ (on $T$)
Corollary

*If a Lorentz manifold has a non compact isometry group and a somewhere timelike Killing field, then $M$ is stationary.*
Corollary

A compact simply connected STATIONARY Lorentz manifold has compact isometry group.

(this will become from the next precise theorem)

D’Ambra Theorem: A compact simply connected ANALYTIC Lorentz manifold has compact isometry group.

Here the metric is $C^2$
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D’Ambra Theorem: A compact simply connected ANALYTIC Lorentz manifold has compact isometry group.

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Challenge: Generalize D’Ambra Theorem to the smooth case

- Why it is important to deal with the non-analytic case?
- reminiscent to the case of codimension 1 foliations: they may exist on the smooth case but not the analytic one (Heafliger).

Why simply connected manifolds?
- Because it is generally thought that dynamics, at least in a rigid geometric background, is encoded in the fundamental group.
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Precise statements

**Theorem**

\( \text{Iso}_0(M, g) \) contains a torus \( \mathbb{T} = \mathbb{T}^d \), endowed with a Lorentz form \( q \), such that \( \Gamma \) is a subgroup of \( \text{O}(q, \mathbb{Z}) \).

There is a new Lorentz metric \( g_{\text{new}} \) on \( M \) having a larger isometry group than the original \( g \), such that \( \Gamma = \text{O}(q, \mathbb{Z}) \).

Geometrically:
- \( M \) is metric direct product \( \mathbb{T} \times N \), where \( N \) is a compact Riemannian manifold,
- or \( M \) is an amalgamated metric product \( \mathbb{T} \times S^1 L \), where \( L \) is a lightlike manifold with an isometric \( S^1 \)-action.

The last possibility holds when \( \Gamma \) is a parabolic subgroup of \( \text{O}(q) \).
Remarks

- Having this description of $g^{new}$, one can understand $g$: the metric on the $\mathbb{T}$ orbits varies in the modular space of $\Gamma$-invariant Lorentz metrics on $\mathbb{T}$.

- The difference between the direct product and amalgamated case lies in the fact that the orthogonal distribution of $\mathbb{T}$ is integrable and has closed leaves.

- The statement is optimal: giving data: $\Gamma, N..., one constructs $M$.

- Consideration of finite covers is necessary...
Amalgamated products
The connected case

Theorem

(Zimmer, Gromov, Adams-Stuck, Zeghib) Let $G$ be a connected non-compact Lie group acting isometrically on a compact Lorentz manifold.

Then the Lie algebra $\mathcal{G}$ is isomorphic to a direct sum

$$\mathcal{K} + \mathbb{R}^k + S,$$

where $\mathcal{K}$ is the Lie algebra of a compact semi-simple Lie group, $k \geq 0$ is an integer and $S$ is trivial or isomorphic to:

- a Heisenberg algebra (of some dimension),
- a warped Heisenberg algebra, or
- $sl(2, \mathbb{R})$.

Conversely, any such algebra is isomorphic to the Lie algebra of the isometry group of some compact Lorentz manifold.
In particular if the $G$-orbits are somewhere timelike, the the factor $S$ is non-trivial, and we have a (local) warped product...
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Lorentz Dynamics
Recurrence vs homogeneity: A Gauß map

$G$ acts on $(M, g)$, $g$ a pseudo-Riemannian metric

Each orbit $G.x$ is a $G$-homogeneous pseudo-Riemannian space: $G/H$.
- So the metric is left invariant by $G$
- and also right invariant by $H$.

In particular if $G/H$ is compact, the metric is bi-invariant by a big subgroup, essentially bi-invariant,

Zimmer-Gromov ... Philosophy: Since $M$ is compact, $G.x$ looks like a compact space:... the metric is essentially bi-invariant
A Gauß map $Ga : M \rightarrow \text{Sym}(\mathcal{G})$,

$Ga(x)$ is the quadratic form on $\mathcal{G}$ obtained via $\mathcal{G} \rightarrow T_x(Gx)$, the derivative at 1 of the map $G \rightarrow Gx$

$Ga(U, V) = g_x(\bar{U}(x), \bar{V}(x))$

$= g_x(\frac{\partial}{\partial t} (\exp tU)(x), \frac{\partial}{\partial t} (\exp tV)(x))$

$\bar{U}$ the vector field on $M$ associated to $U$

$(\bar{U}(x) = \frac{\partial}{\partial t} (\exp tU)(x))$
Equivariance

\[Ga(g.x) = g.Ga(x)\]

The system \(X = Ga(M) \subset Sym(G)\) is a factor of \(M\).

Opposition:

\((G, M)\) a conservative (general) \(G\)-dynamical system

\((G, X)\) a dissipative (linear) dynamical system

Goal: The action on \(X\) is trivial!

Interpretation: the metric on orbits is bi-invariant.
What is special for linear systems

“Furstenberg lemma”, Illustration (in a radically simple situation)
Let
\[ H = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \]
act on \( \mathbb{R}^2 \).
Let \( z = (x, y) \).
If \( z \) is \( H \)-recurrent, then \( z = 0 \)
If \( z \) is non-escaping, then \( x = 0 \), or \( y = 0 \).
Recall:
- \( z \) recurrent, if there is \( n_i \to \infty \), and \( H^{n_i}z \to Z \)
- \( z \) is non-escaping if there is \( K \) a compact set and \( H^{n_i}z \in K \)
\[ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \]

Any \( U \)-recurrent point is fixed.

\[ \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \]

All points are recurrent...
Furstenberg: A **recurrent** linear dynamical system is made up to elliptic elements only.

- If $X \subset \mathbb{R}^N$ admits a finite $G$-invariant measure, then $G$ acts on its support via a homomorphism in a compact group in $GL(N)$.

Warning: One also needs linear actions on projective spaces, and “meromorphic” Gauß maps...
Case of semi-simple groups

$G$ a simple Lie group,
$X \subset \text{Sym}(G)$ a compact $G$-invariant subset $\implies G$ acts trivially on $X$.

(Typical case: $SL(2, \mathbb{R})$)

Embedding theorems (Zimmer...): If $G$ acts on $M$ preserving a pseudo-Riemannian metric of type $(p, q)$, then $G$ embeds in $O(p, q)$.
In fact, the embedding is made via the adjoint representation $Ad : G \to GL(G)$.

The standard homogeneous example is $G/\Gamma$.
(The general case is a “non-commutative” $G/\Gamma$)
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Generalities on Toral actions

Notations: $\Gamma = G/G^0$ acts by automorphism on $G^0$

The action is non-equicontinuous $\implies G^0$ is not semi-simple (since for a compact semisimple $G^0$, $Aut(G^0) \cong G$ is compact)

$\mathbb{T}_1$ the toral factor

$\mathbb{T} = \mathbb{T}^k \subset \mathbb{T}_1$ a minimal $\Gamma$-invariant sub-torus

$\rho : \Gamma \to Aut(\mathbb{T}^k) = GL(k, \mathbb{Z})$

$\Gamma$ acts on $Sym(\mathbb{R}^k)$
Almost Lorentz implies Lorentz

**Lemma**

(Case $\Gamma = \{A^n, n \in \mathbb{Z}\}$)

Let $F = \text{Sym}(\mathcal{E})$, $(\mathcal{E} = \mathbb{R}^k)$

and assume $A = EHU$ non-elliptic (i.e., either $H$ or $U$ is non-trivial).

Suppose there is a Lorentz form $q_0$ which is $A$-recurrent, and let $K \subset \text{GL}(\mathcal{E})$ be the torus generated by the powers of $E$.

Then, $\int_K B^F(q_0) \, d\mu(B)$ is an $A$-invariant Lorentz form, where $\mu$ is the Haar measure on $K$.

Remarks:

- This fact is trivial in the case of Euclidean (positive) forms...
Proposition

Let \( \rho : \Gamma \to GL(\mathcal{E}) \) be such that \( \rho(a) \) is non-elliptic for any \( a \in \Gamma \).

Let \( F = \text{Sym}(\mathcal{E}) \), and assume that the associated action \( \rho^F \) preserves a compact set of \( F \) contained in the (open) subset of Lorentz forms, and that \( \rho^F \) leaves invariant a finite measure on such compact set. Then, \( \rho(\Gamma) \) preserves some Lorentz form.
Corollary

Let $\Gamma$ be a subgroup of $GL(k,\mathbb{Z})$ which acts on $\text{Sym}(\mathbb{R}^k)$ by preserving a finite measure supported in the open set of Lorentz forms. Then, up to a finite index, $\Gamma$ preserves a Lorentz form.
Goal: uniformity and no-singularity
Prototypes of Lorentz isometries: hyperbolic and parabolic
which qualitative properties unify them?

Let $\phi$ be a diffeomorphism of a compact manifold $M$.

**Definition**

A vector $v \in T_x M$ is called approximately stable if there is a sequence $v_n \in T_x M$ such that:

- $v_n \to v$
- $D_x \phi^n v_n$ is bounded in $TM$.

The set of approximately stable vectors in $T_x M$ is denoted $\text{AS}(x, \phi)$

Their union over $M$ is denoted $\text{AS}(\phi)$,

The vector $v$ is called **strongly approximately stable** if $D_x \phi^n v_n \to 0$.

Similar notations: $\text{SAS}(x, \phi)$ and $\text{SAS}(\phi)$
Theorem (Zeghib)

Let $\phi$ be an isometry of a compact Lorentz manifold $(M, g)$ such that the powers $\{\phi^n\}_{n \in \mathbb{N}}$ of $\phi$ form an unbounded set (i.e., non precompact in $\text{Iso}(M, g)$). Then:

- $\text{AS}(\phi)$ is a Lipschitz condimension 1 vector subbundle of $TM$ which is tangent to a condimension 1 foliation of $M$ by geodesic lightlike hypersurfaces;

- $\text{SAS}(\phi)$ is a Lipschitz 1-dimensional subbundle of $TM$ contained in $\text{AS}(\phi)$ and everywhere lightlike.