Lorentz Geometries in dimension 3

Abdelghani Zeghib

joint work with Sorin Dumitrescu

La quatrième géométrie

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Henri Poincaré

Meaning

1. Euclidean space
2. Hyperbolic space
3. Elliptic space (sphere)
4. Minkowski space :

$$\mathbb{R}^{1+n} : q = -x_0^2 + x_1^2 + \dots x_n^2$$

(Later, the mathematical framework of the special Relativity, a Poincaré-Einstein invention ?)

En guise de préambule

- La Géométrie Lorentzienne est Belle

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- Re-Belle !

- Essayons de l'apprivoiser !

Principal Result

Theorem

(Complete classification, in particular Uniformization)

Let M be : a compact **locally homogeneous** *Lorentz* <u>*3-manifold*</u>
(and of non-Riemannian type...).

Then, M admits a Lorentz metric of **constant** *sectional non-positive curvature (say 0 or -1).*

The talk

- Explanations of involved notions,

- Why this is not obvious ?

- Give more comments around fundamental concepts of geometry

- Ask various questions,....

Contents

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Isometry

(Most explanations in the simpler Riemannian case)

• Metric spaces : (M, d_M), (N, d_N) two metric spaces :
$f : N \to N$ isometry, if f is bijective and
$d_N(f(x), f(y)) = d_M(x, y), \forall\, x, y \in M$

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$f : (M, g) \to (N, h) \iff f_* g = h$

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- The group of isometries is a Lie group...

Homogeneous spaces

(M, g) **homogeneous** $\iff \forall x, y \in M, \exists\, f$ isometry of M, and $f(x) = y$
$Isom(M, g)$ the group of isometries of M,
M homogeneous $\iff Isom(M)$ acts transitiveley on M.

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(M, g) **locally homogeneous :** $\forall x, y \in M$, there is f a local isometry of M, such that $f(x) = y$:

$$f : U_x \to V_y, f(x) = y$$

isometry, where U_x and V_y neighborhoods of x and y

Examples

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If $U \subset M$ an open set, then the vector field generating the G-action (on M) are defined (by restriction) on U, but are not necessarily complete ! (Exercise : in which case U is homogeneous ?)

• Subtile example : Quotients : A quotient (by isometries) of a homogeneous space is locally homogeneous (but not necessarily homogeneous) : example : if the universal cover $(\tilde{M}, \tilde{g})$ is homogeneous, then (M, g) is locally homogeneous. ((Reason : small open subsets of M are identified to small open subsets of $\tilde{M}$)).

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3. M is abtained as : $M = \Gamma \setminus X$
   - (this is an arithmetico-dynamical problem...)

Some recalls on Curvature, constant curvature

• Curvatures can be defined for pseudo-Riemannian metrics

• Hierarchy :
- sectional curvature,
- Ricci curvature,
- scalar curvature (and others...)

• Constant sectional curvature, up to constant : 0, $+1$ or -1
- Flat (parabolic)
- Elliptic (spherical)
- Hyperbolic

(classical terminology, Euclidean, Spherical (or Riemman ?), Non-Eucildean or Labatchosky)

- Terminology in the Lorentz case :

- Minkowski,
- de Sitter,
- Anti de Sitter

Non-positive curvature = Minkowski and Anti de Sitter

Counter-example to the principal theorem in the Riemannian case

$M = \mathbb{S}^2 \times \mathbb{S}^1$ is homogeneous under the action of $SO(3) \times SO(2)$ (not only locally !)

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Proof. For Topological reasons. The universal cover of M is $\mathbb{S}^2 \times \mathbb{R}$,
- A flat manifold is covered by $\mathbb{R}^3$
- A hyperbolic manifold is covered by $\mathbb{H}^3$, homeomorphic to $\mathbb{R}^3$
- An elliptic manifold is covered by $\mathbb{S}^3$

Contents

Thom's catastrophe Theory viewpoint : Self-centerdness of homogeneous (non-random) objects

• Even, if a real function is defined as a general mapping..., one deals ONLY with few classes of them, e.g. polynomials... (say those with a geometric flavor)

• Similarly are (locally) homogeneous spaces within Riemannian spaces....

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• Similarly are (locally) homogeneous spaces within Riemannian spaces....

• Approximation (à la Taylor) of Riemannian metrics by (locally) homogeneous ones :
- At order 1 : Euclidean (tangent) spaces
- <u>Higher order : ? ? ?</u>

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21th Century

- .... ?

In the vein of F. Klein (Erlangen Program)

(following Klein) a **Geometry**
(($\bullet$ (G, X)-structure in modern terminology))
consists in :
Giving (G, X), where :
G a Lie group acting transitively continuously on X,
so $X = G/I$: a homogeneous space

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$\bullet$ A Riemannian geometry (or a geometry of Riemannian type) :
If the G-action on X preserves some Riemannian metric.

In other words, a Riemannian geometry $\Longleftrightarrow$ A homogeneous Riemannian manifold.

$\bullet$ A Lorentz geometry :.....

First examples

EUCLIDEAN PLANAR GEOMETRY

$X = \mathbb{R}^2$

$G =$ The group of planar Euclidean displacements = rotation-translation = the semi-direct product :

$$O(2) \ltimes \mathbb{R}^2$$

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Scholar geometry :

- Figure = subset of X
- Equality : Equivalence relation on the space of figures : $F_1 \cong F_2 \iff \exists A \in G, A(F_1) = F_2$ ....
- Goal : Numerical invariants of $\cong$ on some subspaces of figures (e.g. space of triangles...) (Invariant Theory for kids)

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TRANSLATION GEOMETRY (the most rigid)
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PROJECTIVE GEOMETRY
$X = \mathbb{R}P^n$
$G = GL(n, \mathbb{R})$

Contents

Notion of **geometric structure** (definition by means of examples) :

- Riemannian metric
- pseudo-Riemannian metric,
- Connection,
- Conformal pseudo-Riemannian structure,
- projective structure,
- Symplectic form,
- Contact structure,
- CR structure
- Cartan connection...

A (modelization) Problem

• Find a definition "geometric structure" unifying all the examples (tensors, tensors up to scalar function, kind of Christoffel symbols, ...),
– and sub-definitions making hierarchy of them...

• There is a notion of rigidity of a geometric structure, e.g. a Riemannian metric is rigid
but not is a symplectic structure.

Preserved Geometric structure associated to a geometry

Geometry $=$ (G, X)-structure $\neq$ Geometric structure

(Klein $\neq$ Riemann)

However, there is a “functor”

(G, X), a geometry $\rightarrow$ g, a geometric structure =
the best (the most natural, the most rigid) geometric structure
on X which is G-invariant,

- A drawback : this is "multi-valued " !

One general problem on geometries

Problem : Find all geometric structures on X which are preserved by the G-action and specify a canonical one ?

Fact : *Any geometry has an associated* **rigid** *geometric structure.*

• Case of the previous examples :
- Euclidean planar geometry → Riemannian (flat) metric,
- Translation geometry → Parrallelism,
- Affine planar → (flat) Connection,
- Conformal planar geometry → Conformal Riemannian structure,
- Projective geometry → Cartan (projective) connection...

More examples, exercises

- (G, G)-structure $\rightarrow$ parallelism
- $(Sym(n), \mathbb{R}^n) \rightarrow$ symplectic structure $+$ connection...

The dynamically trivial Riemannian case

Topological criterion : $(G, X = G/I)$ has an associated Riemannian structure $\iff$ the isotropy group is compact.

- No criterion for other geometric structures...

Contents

Problem

The Lie group G is given,
Describe all the geometries (G, X),
and their associated geometric structures,
i.e. all G-homogeneous spaces $X = G/I$...

Example : Geometries associated to $SL(2, \mathbb{R})$ (How Lorentz geometry appears naturally)

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• $\dim X = 1$, $X = S^1$, a projective structure
I is the affine group :

$$\begin{pmatrix} \exp s & t \\ 0 & \exp -s \end{pmatrix}$$

$\dim X = 2$, Case 1

$X = \mathbb{H}^2$, hyperbolic plane (Riemannian structure),
I an elliptic one parameter group

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$\dim X = 2$, Case 2

$X = dS_2$, de Sitter plane (Lorentz structure),
$X =$ the space of geodesics of $\mathbb{H}^2$, I a one parameter hyperbolic (semi-simple) group,

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

$\dim X = 2$, Case 3

$X = \mathbb{R}^2 - \{0\}$ (unimodular) affine (punctured) plane (endowed with the linear action of $SL(2, \mathbb{R})$),
I a one parameter parabolic (unipotent) group : $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

$\dim X = 3$

$X = G/I$, $I = \Gamma$ discrete,
X is a G-homogeneous space

Here : *The (unique !) preserved geometric structure is a Lorentz metric (which turns out to be of constant negative curvature)*

General construction : left invariant metrics on Lie groups (a Huge class of homogeneous pseudo-Riemannian spaces)

G a Lie group,

• G acts on itself by the left $(g, x) \in G \times G \to g.x = gx$ (there is another different action on the right) ?

• $T_1 G = \mathcal{G}$ the Lie algebra,

•• There is a bijection :

$$\{ \text{ scalar (pseudo-)Euclidean products on } \mathcal{G} \text{ } \}$$
$$\leftrightarrow$$
$$\{ \text{ pseudo-Riemannian metrics on } G\text{, invariant under the left } G\text{-action } \}$$

Explanation

$g \in G$
$L_g : x \in G \mapsto gx \in G$
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• $x \to m_x$ is a pseudo-Riemannian metric $\bar{m}$ on G

By construction : $\bar{m}$ is left-invariant : $(L_g)_*(\bar{m}) = \bar{m}$

Bi-invariance

In general the right action is not isometric : $(R_g)_*(\bar{m}) \neq \bar{m}$

Precisely : $\bar{m}$ **is bi-invariant** $\iff$ $m = \langle , \rangle$ **is** $Ad(G)$**-invariant.**
(reminiscent of unimodularity)

Examples :
• $G = SO(3)$ (essentially $\mathbb{S}^2$) :
bi-invariant metric $\iff$ Killing form on the Lie algebra $\iff$ constant sectional curvature

- $G = SL(2, \mathbb{R})$

$\kappa =$ the Killing form on

$sl(2, \mathbb{R}) = \{A, tr(A) = 0,\ A\ \ 2 \times 2\text{matrix}\}$

$\kappa(A) = -\det A = a^2 + bc$

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It is bi-invariant : $\kappa(gAg^{-1}) = \kappa(A)$, with signature $-++$ (maybe $+--$?)

Metrics on the quotients

• $\Gamma \subset G$ a discrete group
- A left invariant metric on G descends to a well defined metric on $X = \Gamma \setminus G$

• If $\bar{m}$ is bi-invariant, then the right G-action on $X = \Gamma \setminus G$ is isometric.

$$\text{An } Ad(G)\text{-invariant scalar pseudo-product on } \mathcal{G}$$
$$\leftrightarrow$$
$$(G, \Gamma \setminus G) \text{ a geometry of pseudo-Riemannian type}$$

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Question (exercise) : are the above all the $SL(2,\mathbb{R})$-homogeneous spaces ?

A first list of examples of Lorentz geometries in dimension 3

(Recall the goal is to get the complete list...)

• Minkowski : $X = \mathbb{R}^{1,2} : -t^2 + x^2 + y^2$
$G = SO(1,2) \ltimes \mathbb{R}^3$ Poincaré-Lorentz group (in dimension 3)

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$X = \widetilde{SL(2,\mathbb{R})}$, with the Killing form.
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• de Sitter
$X = SO(1,3)/SO(1,2)$, the space of geodesic hyperplanes of $\mathbb{H}^3$
$G = SO(1,3) \cong PSL(2,\mathbb{C})$

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(G, X)-structures on manifolds

Spaces locally modeled on (G, X) :

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• If M is simply connected, a (G, X)-structure on M :
- a developing map (a “global-local” chart :
$d : M \to X$, i.e. a local diffeomorphism (allowing one to pull-back the geometry of X).

- In general (M not simply connected) :

$$d : \tilde{M} \to X$$

such that the action of $\pi_1 M$ transforms to an action of G, i.e. There is a holonomy representation :

$$\rho : \pi_1(M) \to G, \text{ and } d \circ \gamma = \rho(\gamma) \circ d$$

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• Equivalently : an atlas of charts with target open subsets of X, and as transition, restrictions of transformations $\in G$.

Example : Pairs of pants can not be sewn up Euclidean flat patches, but hyperbolic ones...

Hyperbolic rules allow one to create very complicated objects...

Example and nuisance

Double quotient : $X = G/I$, Γ discrete subgroup of G
$M = \Gamma \setminus X = \Gamma \setminus G/I$ has a (G, X)-structure
(since Γ is discrete, M is locally $\cong G/I$)

However : **Problematic of quotients of locally homogenous non-Riemannian spaces**

- Find condition so that $\Gamma \setminus X$ exists, i.e. it is a standard manifold... $\iff$ the Γ-action is proper, so Γ is **thin** !

- Get $M = \Gamma \setminus X$, so Γ is **thick** !

- Equilibrium in the game G vs I vs Γ

Challenges, the “simplest case”, Markus and Auslander conjectures

$(G, X) = (SL(n, \mathbb{R}) \ltimes \mathbb{R}^n, \mathbb{R}^n)$ = affine flat (unimodular) geometry,
Markus conjecture : any compact manifold with a (G, X)-structure is a quotient $\mathbb{R}^n/\Gamma$ (where Γ is a discrete group of unimodular affine transformations acting properly uniformly on $\mathbb{R}^n$)
Auslander conjecture : if Γ exists, then it is solvable (variant of Biebarbach theorem on crystallographic groups)

• Some names : Margulis, Goldman, Hirsch, Fried, Abels, Soifer, Benoist, Dumitrescu, Klingler, Schlenker, Barbot, Labourie,...

The Lorentz flat case

Instead of $SL(3,\mathbb{R}) \ltimes \mathbb{R}^3$, restrict to $G =$ Poincaré group in dimension $3 = O(1,2) \ltimes \mathbb{R}^3 \longrightarrow$ Minkowski geometry

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Minkowski geometry
Examples of compact manifolds supporting such a geometry :
Nilmanifolds and Solmanifolds,
They are "flat",
They cannot be Euclidean flat (since not covered by a torus),
but are Lorentzian flat

<u>Lorentz geometry is not only beautiful, but also useful !</u>

Nil-manifolds

•• The Heisenberg group :
- $q = xy + t^2$ on $\mathbb{R}^3$
- Three one-parameter groups of $O(q)$:

1. Translation along x and y
   $(x, y, z) \to (x + a, y + b, z)$, $a \in \mathbb{R}$
2. Transvection-Translation one parameter group :

$$(x, y, z) \to \begin{pmatrix} 1 & t & -t^2/2 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + (0, 0, t)$$

• This gives a representation $Heis \to O(q) \ltimes \mathbb{R}^3$
– $Heis$ acts freely transitively on $\mathbb{R}^3$.

Sol-manifolds

Add a Boost-Translation (instead of the transvection-translation)
$(t, a, b) \in SOL$

$$(x, y, z) \to \begin{pmatrix} \exp t & 0 & 0 \\ 0 & \exp -t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + (a, b, t)$$

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– Γ a lattice : $t \in \mathbb{Z}$, $x, y \in A$, $A \subset \mathbb{R}^2$ a lattice, preserved by the digonal action...

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– Γ a lattice : $t \in \mathbb{Z}$, $x, y \in A$, $A \subset \mathbb{R}^2$ a lattice, preserved by the digonal action...

– Metric distorsion : the t-translation flow is an Anosov dynamical system on $\Gamma \setminus SOL$!

Contents

Position of the problem

Classify (i.e. find G and I), such that : - $X = G/I$, $\dim X = 3$,
– G acts by preserving a Lorentz metric on X. And :

1) The geometry is **essential**, i.e. **non-Riemannian** $\iff$
the action of G is not proper $\iff$
the action does not preserve an auxiliary Riemannian metric
$\iff$
The istropy group I is not compact $\iff$
the adjoint action of I on $\mathcal{G}/\mathcal{I}$ is not equicontinous.

2) **Maximality** : An order on Lorentz geometries : (G, X) is bigger than (= most beautiful) (G', X) if $G' \subset G$.
Maximal : there is no other Lorentz metric with more symmetries...(in particular G is the full group of isometries).

3) **Existence of a compact model** : $\exists$ M compact supporting a (G, X)-structure.

Riemannian case : the 8 geometries of Thurston

Riemannian geometries in dimension 3 : **maximal and having a compact model.**

1. $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$
2. $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$

   (The most symmetric) Left invariant metrics of 3-Lie groups :
3. $X = SL(\tilde{2}, \mathbb{R})$, $G = SL(\tilde{2}, \mathbb{R}) \times S^1$
4. $X = Heis$, $G = Heis \times S^1$
5. $X = SOL$, $G = SOL$

A COMPACT MANIFOLD CAN NOT HAVE MORE THAN ONE GEOMETRY (contrast with the Lorentz case) : **WHY** (Exercise) ?

Uniformization

Theorem

There are 4 $(= \frac{8}{2})$ Lorentz geometries in dimension 3 (maximal, non-Riemannian and having a compact model).
If a compact M possesses an anti de Sitter geometry, then, it has no other one.
In all the other cases, M has a Minkowski geometry (and sometimes one other geometry).

Remarks, On maximality, example of the Solid

X = the configuration space of a solid with a fixed rotating point

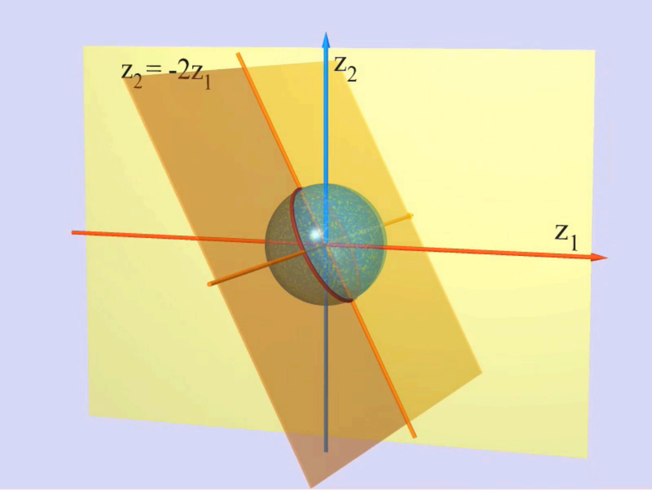
• (by definition of a solid), $G = SO(3)$ acts simply transitively on X (preserving the physical structure)

• The (symmetry group) $G = SO(3)$
$\cong X$ (the geometric substratum)

• The Physics of the solid is encoded in a G-invariant Riemannian metric on $X \iff$ left invariant metric on $G = SO(3)$

Topology
- $X \cong SO(3)$ ($= \mathbb{S}^3$ up to $\mathbb{Z}/2\mathbb{Z}$)

Left invariant metrics $\iff$ scalar products on the Lie algebra



$z_2 = -2z_1$
z_2
z_1

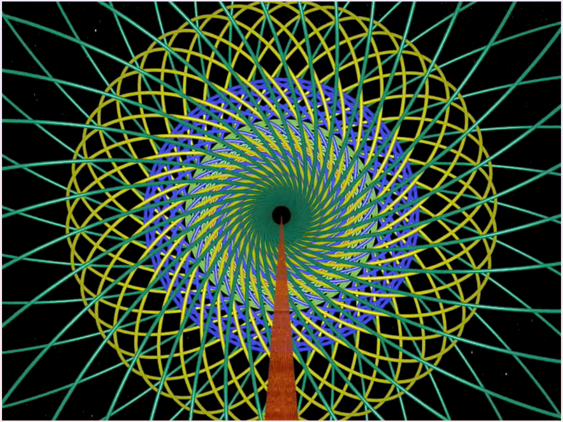
Diagonalization $\rightarrow$ 3 parameters= inertia moments,

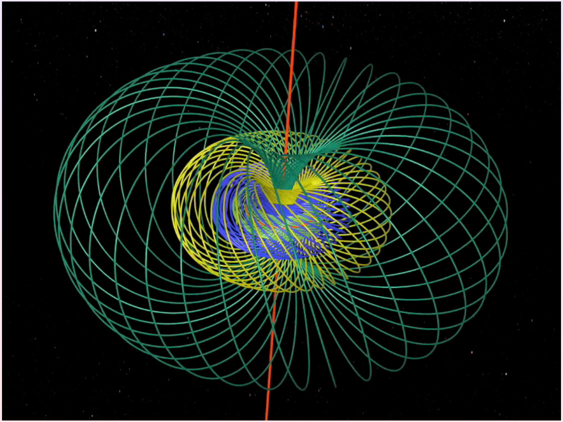
The full group $H = Isom(X)$ contains $SO(3)$
– Case of 3 different inertia moments : $H = SO(3)$
– Equality of two moments : $H = SO(3) \times S^1$

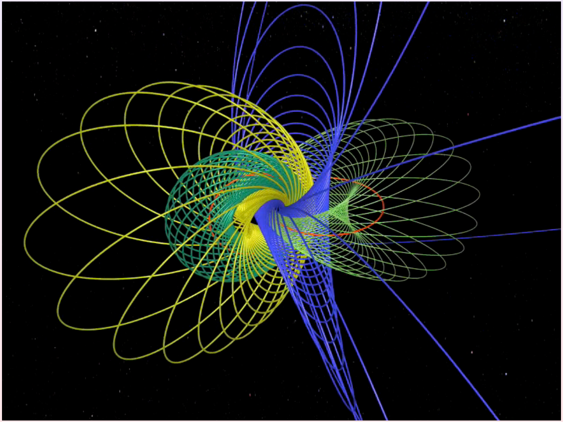
– Maximal case $H = SO(4)$ (up to finite objects), X has constant (positive) curvature, i.e. its universal cover is the round sphere.

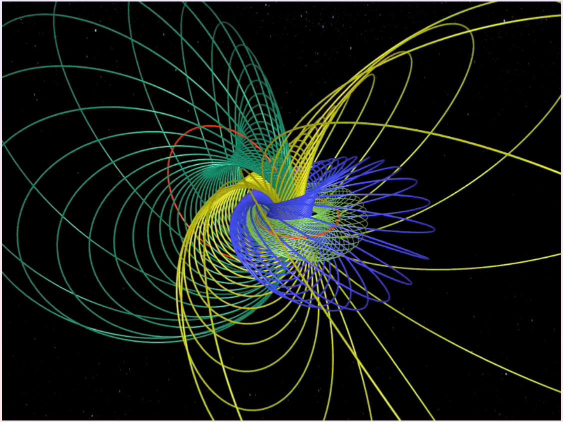
Choosing the maximal case $\iff$ among all geometries on the sphere, we choose the most symmetric one.

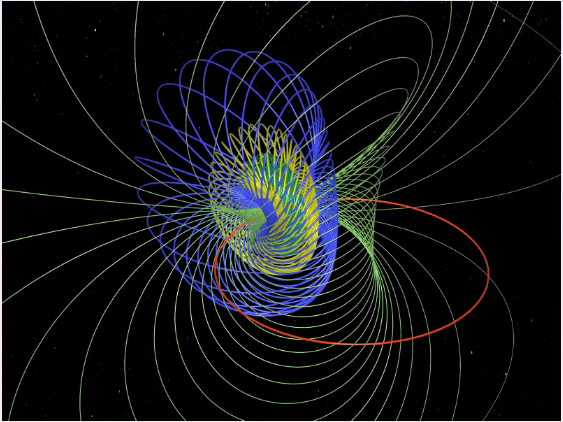
Remark, the intermediate case, 2 equal eigenvalues $\iff$ Berger spheres.











Uniformization, Perelman

In very few words :

Ricci flow in a locally homogeneous framework

Suggestion : a Ricci flow technique (which converges in the Lorentz but not the Riemannian case).

Contents

Lorentz-Heisenberg Geometry (Quantification ?)

heis $= \{X, Y, Z\}$, $[X, Y] = Z$
Lorentz-Heisenberg : any Lorentz metric : $\langle Z, Z\rangle > 0$ (say $= 1$)

Fact

1. *Up to an automophism (of Heis) and a multiplicative constant, there is a unique such a metric.*
2. *Its isometry group is $\mathbb{R} \ltimes$ Heis. This geometry is maximal, non-Riemannian.*
3. *The isotropy $I \cong \mathbb{R}$. If $\{X, Y\}$ is orthogonal to Z, and X, Y are isotopic, then I acts by semi-simple automorphism : $X \to e^t X$, $Y \to e^{-t} Y$.*

Theorem

(Unique rigidity) The holonomy group of a compact Lorentz-Heisenberg manifold is a lattice $\Gamma \subset Heis$.

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Assuming completeness of the (G, X)-structure, the Theorem reduces to :
If $\Gamma \subset \mathbb{R} \ltimes$ *Heis* acts properly co-compactly on *Heis*, then $\Gamma \subset$ *Heis*

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$\Gamma \subset$ *Heis*
(Left as an exercise)

Lorentz-*SOL* Geometry, ...

$$sol = \{T, Y, Z\},$$
$$[T, Y] = Y,\ [T, Z] = -Z$$
$$[sol, sol] = \{Y, Z\} \cong \mathbb{R}^2$$

Lorentz-*SOL* Geometry, ...

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Metric :
- Z is isotropic
- $\{Y, Z\}$ degenerate ($\Longrightarrow \langle Y, Z \rangle = 0$, $\langle Y, Y \rangle > 0$)

Say :
$0 = \langle T, T \rangle = \langle T, Y \rangle$, and $\langle Z, T \rangle = 1$

Fact

- Such a metric is unique up to ...

- Unique rigidity...

- The isotropy is unipotent, but not automorphic !

- The Killing algebra $\mathcal{G} = \{X, sol\} = \{X;\ Y, Z, T\}$,

$$ad_X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

in the basis $\{Z, Y, T\}$

- In fact $G = \mathbb{R} \ltimes$ *Heis :* $\mathbb{R} \to T$ *and Heis* $\to \{X, Y, Z\}$ *(the direct computation of G is not abvious !)*

•• $Z \to a^2 Z$, $Y \to aY$, $T \to T$ *is a homothetic automorphism with dilation a.*

- However, X *is not conformally flat !*

- X is a plane-wave spacetime...

Contents

- Origin of pathologies : (metric) distortion of the model ?

- Origin of pathologies : (metric) distortion of the model ?

– A typical lack : Compactness $\nrightarrow$ completeness !

Contents

The Lorentz case

• Goedesic completeness of compact Lorentz manifolds ?
- No :
– Bohl Torus :
$(\mathbb{R}^2 - \{0\}, \frac{dxdy}{x^2+y^2})$. Homotheties act isometrically,
$f : (x, y) \to 2(x, y)$ generates an isometry group $\cong \mathbb{Z}$
- $T^2 = \mathbb{R}^2 - \{0\}/\mathbb{Z}$

Explanation : Why and where T^2 is incomplete ?

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Explanation : Why and where T^2 is incomplete ?
- It turns out **the hole is at ∞ :** the x-axis is light geodesic in $\mathbb{R}^2 - \{0\}$ which goes to infinity in a finite time !
- (Its projection in T^2 is a closed (periodic) geometric curve !)

The geodesic flow of the Lorentz surface $(\mathbb{R}^2 - \{0\}, \frac{dxdy}{x^2+y^2})$ is **completely integrable**, since it admits 1 symmetry : homotheties give a Killing field...

(Left) Invariant metrics on Lie groups

• • Worse, a **generic** left invariant pseudo-Rienmannian (but not Riemannian) metric is **not** geodesically complete !

• Remark : left Riemannian metrics are complete (since homogeneous).
However, their geodesic flow are generically a **chaotic** dynamical system.

Geodesics of let invariant metrics

From the dictionary $\{$Lie algebras $\} \leftrightarrow \{$Lie groups $\}$
and

$$\{\text{ scalar (pseudo-)products on } \mathcal{G} \text{ }\}$$
$$\leftrightarrow$$
$$\{\text{ left invariant metrics on } G \text{ }\}$$

the geodesics of G are determined by solutions in $\mathcal{G}$ of the differential equation :

$$\dot{x} = ad_x^* x$$

$ad_x u = [x, u]$,
ad_x^* its adjoint (by means of the given scalar product)

General form :

$$\dot{z}^i = \Sigma \Gamma^i_{jk} z^j z^k$$

- A quadratic differential equation (with constant coefficients)

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((Recall the general equation of geodesics : $\ddot{x}^i = \Sigma\Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k$

In our case : Γ^i_{jk} does not depend on $x \rightarrow z^i = \dot{x}^i$))

Explanation, curves in $\mathcal{G}$, curves in G

Dictionary “development” (and its reciprocal)

$$\text{“Curves in } G \leftrightarrow \text{Curves in } \mathcal{G}$$

To $t \to C(t) \to G$ associate $t \to D(t) = C(t)^{-1} \frac{\partial C}{\partial t} \in \mathcal{G}$

• Reciprocal, case of $GL(n)$:
Given $D(t)$ a curve in $\mathcal{G}l(n) = M_{n\times n}$
$C(t)$ is solution of the differential equation

$$C'(t) = C(t)D(t), \quad C(0) = 1$$

Contents

• Hopf manifolds ($\subset$ affine manifolds,)

- Hopf torus : $S = \mathbb{R}^2 - \{0\}/(x \sim \gamma x)$
- $\gamma(x) = 2x$,
- $GL(2, \mathbb{R})$ acts affinely on S.

- The action of the radial ("contracting") flow : $x \to e^t x$, has the all its orbits periodic, it can be made isometric !
- Remark : the $SL(2, \mathbb{R})$-action cannot preserve a volume.

- Holonomy : $\rho_0 : \mathbb{Z}^2 \to GL(2, \mathbb{R})$, the image of ρ_0 is generated by γ
– There are very complicated other holonomies (for other affine structures)

• Translation surfaces
$(\mathbb{R}^2, \mathbb{R}^2)$
S a Riemann surface, ω a (closed) holomorphic 1-form.
locally, $d : U \subset S \to d(z) = \int_{z_0}^{z} \omega \in \mathbb{C}$
It is defined up to a constant,
$d : \tilde{S} \to \mathbb{C}$
$d(\gamma(z)) = d(z) + C(\gamma)$
- “Singular” translation structure
- Holonomy group $\Gamma =$ group of periods
-Not discrete and acts "ergodically” on $\mathbb{C}$ despite d is a local diffeomorphism (outside the singularities)...

Contents

Contents

The maximal dimension, 6

This corresponds to the constant curvature case :
The isotropy group equals $O(1,2) \Longrightarrow$ all the 2-planes have the same curvature.

The minimal dimension 3, Bianchi classification

– Here the isotropy is trivial $\rightarrow$ the Lorentz geometry is inessential.

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Recall

• Bianchi Work : classification (+...) of Lie algebras of dimension 3 :

- The are two simple algebras : $sl(2,\mathbb{R})$ and $so(3)$
- If not $\mathcal{G}$ is solvable
- It contains and ideal of dimension 1 or 2 $\rightarrow \exists$ one $\mathcal{B}$ of dimension 2.
- G is a semi-direct product of $\mathbb{R} \ltimes_{A^t} B$

– Classification, up to conjugacy of one parameter groups of $GL(2,\mathbb{R})$....

Dimension 5 is not possible !

- $L = O(1,2)$ is represented in the space of jets (at any fixed order) of the metric at x_0.

-The isotropy I is exactly the stabilizer of this jet.

– The unique subgroup of L of dimension 2, is the affine group, it is co-compact.

- If a stabilizer of u in L is co-compact, then it equals L,
Proof : the orbit of u is compact, but any semi-simple one parameter subgroup has no non-trivial bounded orbits...

The hard dimension : 4

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The linear (rather quadratic) problem :

Algebraic Classification Problem

Classify G, solvable of dimension 4, I a closed non-compact subgroup of dimension one, such that $Ad(I)$ preserves a Lorentz scalar product on $\mathcal{G}/\mathcal{I}$

$\Longleftrightarrow$ The algebraic classification of homogeneous Lorentz 3-spaces ?

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$\Longleftrightarrow$ The algebraic classification of homogeneous Lorentz 3-spaces ?
— Not available ! ?

Contents

A well defined vector-field V on X

$$\begin{pmatrix} 1 & t & -t^2/2 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \exp t & 0 & 0 \\ 0 & \exp -t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- V corresponds to the 1-eigenvalue

$\rightarrow$ a vector-field on X which commutes with G (but not necessarily a Killing field !)
$\rightarrow$ a vector field on M

- Unipotent isotropy $\rightarrow$ V is isotropic
- Semi-simple isotropy : V is spacelike

A lightlike geodesic hypersurface F

- Unipotent case : $V^{\perp}$: a field of degenerate (lightlike) hyperplane field
- Semi-simple case : Two hyperplane-fields : $V \oplus V^{-}$ and $V \oplus V^{+}$

Fact

These hyperplane-fields are integrable. They define a G-invariant codimension one lightlike geodesic foliation.

Proof : Get the leaf F through the base point x_0 :
$s \in I$,
$E_s = Graph(s) = \{(x, s(x)), x \in X\} \subset X \times X$:
a geodesic isotropic 3-submanifold in $(X \times X, g \oplus (-g))$.

E a limit of E_s, $s \to \infty$
$F =$ the projection of E

(Affine and lightlike) Geometry of F, a codimension 1 subgroup H

G preserves the foliation.
$H \subset G$ the stabilizer of F,
F is H- homogeneous,
H has codimension 1 in G.

F has an induced connection,
F has an induced degenerate (positive) metric :
• A 1-dimensionnal transversally Riemannian foliation (the Kernel of the metric).

Classification in dimension 2 : the connection is symmetric $\rightarrow$...
$H = \mathit{Heis}$, or
$H = \mathbb{R} \times AG$, $AG =$ the affine group

The new enriched algebraic problem

- G is solvable of dimension 4,
- $H = Heis$ or $= \mathbb{R} \times AG$ has codimension 1 in G
- $I \subset H$, $\dim I = 1$
- A normal form for the $Ad(I)$-action on $\mathcal{G}/\mathcal{I}$
- V a G-invariant vector field on X...

Contents

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Contents

One step : partial completeness

- In case V is isotropic $\rightarrow$ V is the characteristic (Kernel) field of F
- V is complete on M (by compactness)
- $\tilde{F} = \mathbb{R}^2$ endowed with a (auxiliary) complete Riemannian metric, and a transversally Riemannian 1-foliation
– The F-leaves on M have a complete $(H, H/I)$-structure $\rightarrow$ The F-leaves in M are geodesically complete...