The Minkowski problem in the Minkowski space

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Introduction: Minkowski

Hermann Minkowski (1864 – 1909)

Minkowski

Minkowski space Mink_n:
$\mathbb{R}^n$ with the quadratic form $q(t,x) = -t^2 + x_1^2 + \dots x_{n-1}^2$

Spheres: $S(r) = \{(t,x), q(t,x) = r^2\}$

e.g. the hyperbolic space $\mathbb{H}^{n-1}$ = sphere of radius $\sqrt{-1}$

- Mink_4 is the spacetime of special Relativity

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Isometry groups

Poincaré group $Poi_n = Isom(\mathrm{Mink}_n)$:

It contains linear isometries: the Lorentz group

$Lor_n = O(1, n-1)$

and

Translations: $\mathbb{R}^n$

Poi_n is a semi-direct product $Lor_n \ltimes \mathbb{R}^n$

Remark, in Convex and Finsler geometry

In convex geometry...

Finsler metric on a manifold M: a norm on each tangent space $T_x M$
$\mathbb{R}^n$ endowed with a constant norm (i.e a Finsler metric invariant by translation): a Minkowski space!

Classical Minkowski problem

$\Sigma \subset \mathbb{R}^3$ a compact convex surface (topological sphere)
$G : \Sigma \to \mathbb{S}^2$ Gauß map,
$K^\Sigma : \Sigma \to \mathbb{R}^+$

$f : K^\Sigma \circ (G^\Sigma)^{-1}$ is a function on $\mathbb{S}^2$

Question: which functions on $\mathbb{S}^2$ have this form?

Necessary condition $\int_{\mathbb{S}^2} \frac{x}{f(x)} dx = 0$

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Its solution

Minkowski, Lewy, Alexandrov, Pogorelov, Nirenberg, Gluck, Yau, Cheng...:

Σ exists for any f on $\mathbb{S}^2$ satisfying the necessary condition.
It is unique up to translation.

Steps:
- Polyhedral case – analytic case – generalized solution – regularity....

– Rigidity...

Polyhedral case

Σ polyhedra
$G : \Sigma \to \mathbb{S}^2$ multivalued Gauß map
μ the (Hausdorff) volume measure on Σ
$\nu = G^*\mu$ its image: a measure on $\mathbb{S}^2$

• Which measure on the sphere has the form $\nu = G^*\mu$ for some Σ?

Necessary condition $\int_{\mathbb{S}^2} x d\nu = 0$

In dimension 2

$u_1, \ldots, u_k$ unit vectors
$l_1, \ldots, l_k$ lengths

Construct a polygon P with edges $e_1, \ldots, e_k$ (non-ordered) parallel to the directions of $u_1, \ldots, u_k$ and having lengths $l_1, \ldots, l_k$

This consists in choosing the right order?

Chasles relation $\Sigma l_i u_i = 0$ (non-ordered)

Equivalent formulation with v_i normal to u_i

In higher dimension: $e_i \rightarrow$ facets of dimension $n-1$
$l_i \rightarrow$ volume of e_i...

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Because the great expansion of the mathematical literature makes it so hard to follow the developments, an author who treats well known problems has the duty to acquaint himself with the literature, refer the reader to the best sources, and state clearly in which respect his contribution transcends the existing results. **The present paper is quite irresponsible in all these respects.**

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Variants

• Minkowski problem in Higher dimension:
The Gauß-Kronecker-Lipschitz-Killing curvature = product of eigenvalues of the second fundamental form = Jacobian of the Gauß-map

•**Weyl problem**: Which metric g of positive curvature on $\mathbb{S}^2$ admits an isometric immersion in $\mathbb{R}^3$?

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•**Weyl problem**: Which metric g of positive curvature on $\mathbb{S}^2$ admits an isometric immersion in $\mathbb{R}^3$?

- **Nirenberg Problem** Which function f on $\mathbb{S}^2$ has the form $f = Scal^{\Sigma} \circ \Phi$, where $\Phi : \mathbb{S}^2 \to \Sigma$ is a conformal diffeomorphism?
—— Higher dimensional case?

- Other curvatures

- Intrinsic variants

Hyperbolic case

Hyperbolic surfaces?

– Hilbert, Effimov...: $\mathbb{R}^3$ contains no complete surface with negative curvature bounded away from 0.

• $\mathbb{R}^3 \to \mathrm{Mink}_3$

Let $\Sigma \subset \mathrm{Mink}_3$ be **spacelike**
i.e. the induced metric (from Mink_3) is Riemannian

— Examples: $\mathrm{Mink}_3 : q = -t^2 + x^2 + y^2$
$\mathbb{R}^2 = \{t = 0\}, \quad \mathbb{H}^2 = \{q = -1\},$
— Counter-examples, timelike surfaces $\mathrm{Mink}_2 = \{y = 0\}$,
de Sitter $dS_2 = \{q = +1\}$

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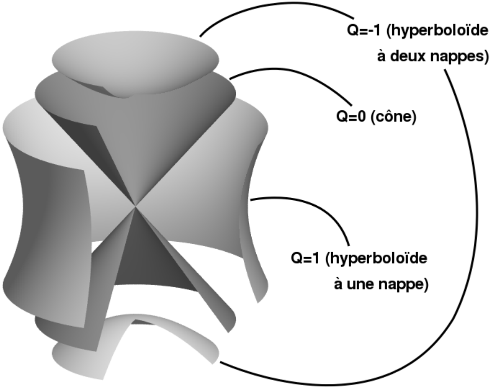

Q=-1 (hyperboloïde
à deux nappes)
Q=0 (cône)
Q=1 (hyperboloïde
à une nappe)

Remark: a spacelike surface in Mink_3 can not be closed (compact without boundary)

There is a Gauß map: $G(= G^{\Sigma}) : \Sigma \to \mathbb{H}^2$
Gaussian curvature is defined similarly: $K^{\Sigma} : \Sigma \to \mathbb{R}$

$K^{\Sigma}(x) = \det(D_x G^{\Sigma})$

If $K^{\Sigma} < 0$, and some "properness condition", G is a global diffeomorphism,

• **(naive) Minkowski problem:** Given $f : \mathbb{H}^2 \to \mathbb{R}$ negative, find Σ such that $K^{\Sigma} \circ (G^{\Sigma})^{-1} = f$

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Non-rigidity of $\mathbb{H}^2$

Hano-Nomizu:

There is (exactly) 1 one-parameter family of revolution surfaces (around the x-axis)

- which contains the hyperbolic space $\mathbb{H}^2$,
- all of them have constant curvature -1 but are not congruent to $\mathbb{H}^2$ (up to $\mathrm{Iso}(\mathrm{Mink}_3)$)

Remark

$\mathbb{H}^n$ *is rigid in* Mink_{n+1} *for* $n \geq 3$

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Equivariant immersions

Giving a surface $(S, g) \iff$ giving $(\tilde{S}, \tilde{g})$ equipped with the isometric $\pi_1(S)$-action

Equivariant isometric immersion of S: (f, ρ) with
$f : \tilde{S} \to \mathrm{Mink}_3$ isometric immersion
$\rho : \pi_1(S) \to \mathrm{Iso}(\mathrm{Mink}_3)$
$f \circ \gamma = \rho(\gamma) \circ f$, for any $\gamma \in \pi_1(S)$

Example: any metric of curvature -1 has a canonical equivariant isometric immersion with image $\mathbb{H}^2$

References: Gromov, Labourie, Schlenker, Fillastre, ...

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The images $\Sigma = f(\tilde{S})$; $\Gamma = \rho(\pi_1)$

Setting:

The interesting case is when Γ acts properly on Σ, i.e. Σ/Γ is a Hausdorff space

Better: $f : \tilde{S} \to \Sigma$ diffeomorphism, that induces a diffeomorphism
$S = \tilde{S}/\pi_1 \to \Sigma/\Gamma$

In particular, as an abstract group $\Gamma \cong \pi_1$

Example: a hyperbolic structure determines an isometry $\tilde{S}/\pi_1 \to \mathbb{H}^2/\Gamma$

$\{$ Hyperbolic structures $\} \cong$
$\{$ Fuschian representations of π_1 in $O(1,2)\}$

• Generalization: Here we deal with representations $\pi_1 \to Poi_3 = O(1,2) \ltimes \mathbb{R}^3$, with image Γ acting properly on some Σ...

Results

$G : \Sigma \to \mathbb{H}^2$ the Gauß map

Γ acts on Σ and its **linear part** Γ^L acts on $\mathbb{H}^2$

The "Linear part" projection: affine $\to$ linear,
$lin : Affin(\mathbb{R}^3) \to GL(\mathbb{R}^3)$
$(lin : Poi_3 \to Lor_3)$
$\Gamma^L = lin(\Gamma)$

G is *lin*-equivariant: $G \circ \gamma = lin(\gamma) \circ G$

Direct problem:
$(\Sigma, \Gamma, K^\Sigma) - -- \to (\mathbb{H}^2, \Gamma^L, f = K^\Sigma \circ (G^\Sigma)^{-1})$

Inverse problem

• Data:
Γ^L subgroup of $O(1,2)$
$f : \mathbb{H}^2 \to \mathbb{R}$ negative and Γ^L-invariant

• Hypotheses: Γ^L fuschian co-compact (i.e. Γ^L discrete and $\mathbb{H}^2/\Gamma^L$ compact)

•• Problem: find all the pairs (Σ, Γ) such that:
– Γ has a linear part projection Γ^L
– Σ is a spacelike Γ-invariant surface
and such that:
$K^\Sigma \circ (G^\Sigma)^{-1} = f$

Theorem

Let Γ^L be a co-compact fuschian group in $O(1,2)$, and $f : \mathbb{H}^2 \to \mathbb{R}$ a negative Γ^L-invariant function.
For any subgroup Γ in the Poincaré group Poi_3, with linear part Γ^L, there is exactly one Γ-invariant solution of the Minkowski problem.

Introduction 2: Foliations and time functions

Γ vs Γ^L

$\Gamma^L \subset Lor_3$ given, what are the $\Gamma \subset Poi_3$ having Γ^L as a linear projection

Γ is an affine deformation of Γ^L

General setting: $\Gamma^L \subset GL(\mathbb{R}^3)$ given, consider its affine representations
$\rho : \gamma \in \Gamma^L \rightarrow (\gamma, t(\gamma)) \in Aff(\mathbb{R}^3)$

$t(\gamma)$ translational part of $\rho(\gamma)$

• ρ is a homomorphism $\iff$ $t : \Gamma^L \to \mathbb{R}^3$ is a cocycle:
$t(\gamma_1\gamma_2) = \gamma_1(t(\gamma_2)) + t(\gamma_2)$

$\rho \sim \rho' \iff$ they are conjugate via a translation

The quotient space: $H^1(\Gamma^L)$ (or $H^1(\Gamma^L, \mathbb{R}^3)$)

Identification of the cohomology

$\mathbb{R}^3$ is identified to the Lie algebra $o(1,2) \cong sl_2(\mathbb{R})$
The representation of $\Gamma^L \subset O(1,2) \cong PSL_2(\mathbb{R})$ is identified to its adjoint representation

H^1 is the tangent space to the space of representation of Γ^L in $O(1,2)$ up to conjugacy

Equivalently,

if $\Gamma^L \cong \pi_1(S)$, then,

– $\Gamma^L \in Teic(S)$

– and $H^1 = T_{\Gamma^L} Teic(S)$

$\dim H^1 = 6g - 6, \; g = \text{genus } (S)$

Geometric counterpart

• The geometric counterpart of Γ^L Fuschian is a hyperbolic structure on S

• Now the **geometric counterpart** of Γ a subgroup of Poi_3 acting properly co-compactly on some Σ is a **Lorentz 3-manifold** M_Γ such that:

– M_Γ is flat, i.e. locally isometric to Mink_3

– M_Γ is diffeomorphic to $\mathbb{R} \times S$

– M_Γ contains "any" Σ/Γ as previously...

– M_Γ is maximal with respect to these properties

The conformally static case

In the case $\Gamma = \Gamma^L \iff \Gamma$ contained in $O(1,2)$ up to a conjugacy, $\iff \Gamma$ has a global fixed point,
then $M_\Gamma = Co^3/\Gamma$
Co^3 the 3-dimensional (solid) light-cone $=$ $\{x, y, t)/x^2 + y^2 - t^2 < 0\}$

$S = \mathbb{H}^2/\Gamma$
$M_\Gamma = \mathbb{R}^+ \times S$ with the warped product metric $-dr^2 + r^2 ds^2$
where ds^2 is the hyperbolic metric on S

Conformally static: homotheties act conformally

REM: Big-bang models: warped products $-dr^2 + w(r)ds^2$ where ds^2 is a metric of constant sectional curvature on a 3-manifold.

Compare with hyperbolic ends:

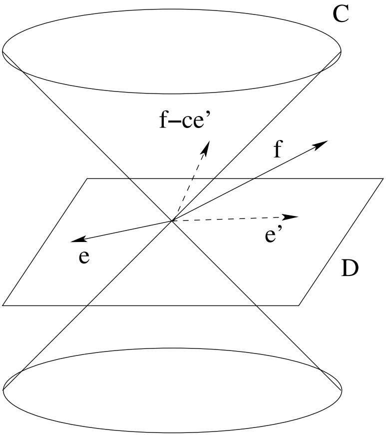
Fuschian case $\cong$ conformally static

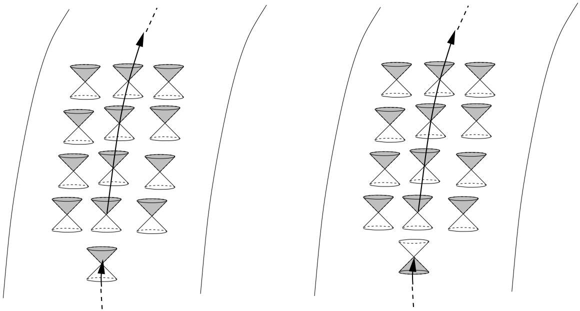
Domains of dependence

If Γ is not linear, then the light-cone is replaced by $\mathcal{D}$ the **domain of dependence** of Σ

A little bit Causality theory:

Timelike curves (in Minkowski), figure





Global hyperbolicity

M is **globally hyperbolic** if it contains a **Cauchy hypersurface** Σ:

– Σ spacelike

– A timelike curve meet Σ at most on 1 point

- Any timelike curve can be extended to meet Σ

Domains of dependence

$\Sigma \subset \mathrm{Mink}_3$ (or any M)
$\mathcal{D} = \mathcal{D}(\Sigma)$= domain of dependence of Σ = the maximal open set in which Σ is a Cauchy surface

$x \in \mathcal{D}^+ =$ any futur oriented timelike curve from x meets Σ
$x \in \mathcal{D}^-$ ...

Examples:
$\mathbb{H}^2 - - - - - \rightarrow$ the light-cone Co^3
The spacelike $\mathbb{R}^2 - - - \rightarrow$ the full Mink_3

Flat MGHC

Abstract approach: Witten question, initialized by Mess: Classify MGHC flat spacetimes M of dimension 3:

F: M is a (locally) flat Lorentz 3-manifold (i.e. locally isometric to Mink_3)

GH: M is globally hyperbolic

C: M is spatially compact, i.e. it has a compact Cauchy surface, say homeomorphic to a surface S of genus ≥ 2 (so M is homeomorphic to $\mathbb{R} \times S$)

M: M is maximal with respect to these properties (i.e. if M isometrically embeds in a similar M', then $M \cong M'$)

The theory

Mess,... Bennedeti, Guadagnini, Bonsante....
$M = \mathcal{D}/\Gamma$ as previously

$\mathcal{D}$ is the domain of dependence of some Σ spacelike in Mink_3,
– but not necessarily with negative curvature (smooth and convex)
– Σ is not "privileged"

Canonical times

Time function $T : M \to \mathbb{R}$: the levels of T are Cauchy surfaces

There is a canonical time: the **cosmological time** $T^C : \mathcal{D} \to \mathbb{R}^+$ (or $M \to \mathbb{R}^+$)

$T^C(x) =$ sup of lengths of timelike curves having x as a terminal extremity

Example: $\mathbb{H}^2$: $T^C(x) = \sqrt{-q(x)}$ q is the Lorentz form (T^C is intrinsic, so the quadratic form q can be recovered from the solid light-cone, without reference to the ambient Minkowski)

Remarks:

— For Mink_3, $T^C = \infty$

— By definition, $T^C < \infty$ for big-bang models

— Relativity and abolition of time?

Geometrically, $T^C : \mathcal{D} \to \mathbb{R}^+$ is the time distance to $\partial\mathcal{D}$
$T^C(x) = d(x, \partial\mathcal{D})$

As in the Euclidean case, the gradient ∇T is Lipschitz and has straight lines trajectories

The levels of T are equidistant, and are $C^{1,1}$-submanifolds

Fact (smooth rigidity)

T^C is C^2 (and hence C^∞) $\iff$ $\mathcal{D}$ is the light-cone Co^3

Question: existence of geometric smooth times?

In the light-cone case, and more generally warped products $-dt^2 + w(t)ds^2$, the time $T(t,x) = t$ has "**rigid**" geometrical levels:
They are **umbilical**,

Question:Does M have a geometrical time, i.e. with levels satisfying some extrinsic condition?

This is motivated by the fact that M_Γ is a deformation of $M_{\Gamma^L} = Co^3/\Gamma^L$
– what remains from the warped product structure after deformation?

Hope: existence of times with levels satisfying one PDE (there are many in the umbilical case)

Principal theorem leading to solution of Minkowski problem

Theorem (Barbot-Béguin-Zeghib, Existence of K-time)

M_Γ admits a unique time function $T^K : M \to]-\infty, 0[$, such that the level ${T^K}^{-1}(c)$ has constant Gaussian-curvature c. Furthermore any compact spacelike surface with constant Gaussian curvature in M_Γ coincides with some level of T^K

The CMC-case

This is done, for any dimension

Andersson-Barbot-Béguin-Zeghib

de Sitter and anti de Sitter MGHC

K-times and CMC-times exist in for MGHC spacetimes of constant curvature, i.e. locally isometric to the de Sitter of the anti de Sitter spaces.

Rem: more authors for the structure of MGHC spacetimes locally modelled of de Sitter or anti de Sitter: Scannel...

Use in the solution of Minkowski problem

The leaves of the K-time are used as barriers....

Convex Geometry of domains of dependence, Regular light domains

(as a complement)

Let Hyp be the set of affine hyperplanes in Mink_3

$Hyp^+ \subset Hyp$ spacelike hyperplanes

$Hyp^0 \subset Hyp$ lightlike

Fact

$\mathcal{D}$ is Hyp^0- convex: any $y \in \partial\mathcal{D}$ has support hyperplane $\in Hyp^0$

– In particular, if x is regular, then the tangent plane $T_x\mathcal{D} \in Hyp^0$

REM: Define $\mathcal{L}$-convex sets, fro $\mathcal{L}$ a (nice) subset of Hyp.

For $P \in Hyp^0$ let $I^+(P)$ its future $= \cup\{I^+(x), x \in P\}$

$\mathcal{D} = \cap_{\mathcal{F}} I^+(P)$
$\mathcal{F} \subset Hyp^0$

Example: Misner strip: $I^+(P_1) \cap I^+(P_2)$
$Co^3 = \cap\{I^+(P)$ such that $0 \in P\}$

Any $\mathcal{D}$ is a "fractured" cone...

Eikonal equation

Fact

$\partial\mathcal{D}$ is the graph (contained in $\mathrm{Mink}_3 = \mathbb{R}^2 \times \mathbb{R}$) of a global **continuous** *solution of the Hamilton-Jacobi equation* $\| d_x f \| = 1$, $f : \mathbb{R}^2 \to \mathbb{R}$

Compactness theorems

Plan of the proof: "method of continuity"

(of existence of K-time)

– Assume existence of time function (or foliation) on an open set

– Study what happens at the boundary: show it can be extended excluding degeneracy, There are three compactness facts

1. Compactness of spacelike surfaces?
2. Uniform convexity
3. No loos of smoothness.

Spacelike accessibility set

M Lorentz, say time-oriented
$J^+(x)$ = causal future of x,
$J^-(x)$ = past of x
$J^+(x) \cup J^-(x)$ is the set of points accessible from x by timelike curves

– Spacelike-variant?

Fact: If $\dim M > 2$: any two points can be joined by a spacelike curve

Example: Mink_3

$c : s \rightarrow (x(s), y(s), t(s))$

c spacelike $\iff t'^2(s) \leq x'^2 + y'^2$,

Example:

$s \in \mathbb{R}$, $x(s) = r\cos(s/r)$, $y(s) = r\sin(s/r)$,

and $t'(s) < 1$, arbitrary

One can join any (x, y, t_1) to any (x, y, t_2)

Alternative notion:
K compact in M

$Sp(K) = \cup\{S,\ S$ Cauchy surface with $S \cap K \neq \emptyset\}$

In particular $Sp(x)$ is the union of Cauchy surfaces containing x,

Remark: One considers here also rough topological Cauchy surfaces: topological sub-manifolds meeting exactly once any non-extensible timelike curve

Conformally static case

Theorem

Let $M = Co^3/\Gamma^L$ be a conformally static MGHCF.
Let $K \subset M$ compact.
Then $Sp(K)$ (The set of Cauchy surfaces meeting K) is compact (when endowed with the Hausdorff topology)

Question: for which spaces this compactness property holds?
- Yes for higher dimensional conformally static MGHCF
- Maybe, never in the non-conformally static case?

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Proof

$x \in M$
$NC(x) = M - (I^+(x) \cup I^-(x))$: set of non-comparable points with x (in the sense of the causality order)

Fact: $NC(x)$ is compact for any x

Because M is conformally static, this does not depend on the height of x

Consider x high enough.

Γ^L has a compact fundamental domain in $\mathbb{H}^n$
– Homethetic images of this domain are fundamental for the action on the homethetics of $\mathbb{H}^n$

Thus, there is a closed cone C strictly contained in Co^3, a covering domain for Γ^L: iterates of $\mathbf{C}$ cover Co^3

Now: $\mathbf{C} - (I^+(x) \cup I^-(x))$ is compact for any $x \in \mathbf{C}$

In the general case

Theorem

Let $M = \mathcal{D}/\Gamma$ be a MGHCF.
Let $K \subset M$ compact, then the diameter of the elements of $Sp(K)$ is uniformly bounded.

Let $\epsilon > 0$, then, the set of elements of $Sp(K)$ having a systole $\geq \epsilon$ is compact.
In other words, if $S_n \in Sp(K)$ leave any compact subset of M, then their systole $\rightarrow 0$.

Uniform convexity

Theorem

Along a compact $K \subset M$, all the convex Cauchy surfaces are uniformly spacelike: there exists ϵ such that $d(T_x S, C_x) \geq \epsilon$, for any $x \in K$ and S a Convex Cauchy surface (where d is any auxiliary metric).
In particular, the volume of S is (locally) uniformly bounded (from below).

Proof

Otherwise, we get a convex Cauchy surface containing a complete lightlike (isotropic) half-line.

Smooth regularity of limits

Theorem

Let M be a MGHCF with Cauchy surface homeomorphic to (the topological surface) S endowed with the C^∞-topology. and $Met^\infty_{Curv\leq 0}(S)$ those of non-positive scalar curvature.

— **Let $Emb(S, M)$ be the set of metrics g having an isometric embedding in M.**

• *Then $Emb(S, M)$ is closed in $Met^\infty(S)$*

Furthermore, let $K \subset M$ compact, and $C \subset C^\infty(S)$ compact, and consider $Emb(S, M; K, C)$ the space of $g \in Emb(S, M)$ whose curvature belongs to C and image of their embedding meets K.

• *Then, $Emb(M, S; K, C)$ is compact in $Met^\infty(S)/Diff^\infty(S)$.*

• *This applies in particular if C consists of constant functions in a bounded interval.*

Proof

(S_n, g_n)
$f_n : S \to M$ isometric immersions,

If $g_n \to g$, we can see the f_n as isometric immersions of the same (S, g)

Question: What happens for the limit of $f_n : (S, g) \to M$, knowing that the images $f_n(S)$ converge geometrically to a surface S_∞

– What happens if f_n does not converge in the C^∞-topology,
– Equivalently, if S_∞ is not a smooth surface?

Answer: if S_∞ is not smooth, then it is a "pleated" surface, More precisely, S_∞ contains a **complete ambiant geodesic**, i.e. a straight line.

This is impossible in our case

References: ... Labourie, Schlenker

One synthetic approach: this isometric immersion problem can be formulated as a pseudo-holomorphic curve problem for a suitable almost complex "symplectic" structure, Degeneration of pseudo-holomorphic curves: Gromov's compactness theorem

Example: in $\mathbb{C}P^1 \times \mathbb{C}P^1$, let S_n be the graph of $h_n \in SL_2(\mathbb{C})$...

Last part of the theorem

In the case of constant curvature: boundness of curvature and diameter implies compactness

Quick use

Steps to get a K-foliation (i.e. by constant Gaussian curvature surfaces)

1. A constant Gaussian curvature surface generates a K-foliation in its neighborhood (the Gauß flow creates barriers, and the maximum principle puts all K-surfaces in order, and hence foliate)

2. By the compactness and regularity theorems, the foliation extends to the boundary...

A preliminary step

Find one Σ of constant Gaussian curvature?

High levels of the cosmological time have almost 0 curvature

The same is true for CMC-levels

By Treibergs: the CMC-leaves are convex

Push by the Gauß flow (i.e. normals) to create barriers

Standard Facts: Geometric times on globally hyperbolic spacetimes

Causality Theory

Causality

- at the infinitesimal level, causal characters:

(E, q) a Lorentz vector space: q has type $- + \ldots +$

$\mathbb{R}^{n+1}$: $q = -x_0^2 + x_1^2 + \ldots x_n^2$

$u \in E$

spacelike: $q(u) > 0$

timelike $q(u) < 0$

lightlike (isotropic, null): $q(u) = 0$

$F \subset E$ subspace
- spacelike: $q|F$ Euclidean scalar product
- timelike $q|F$ Lorentz product
- lighluike $q|F$ degenerate (and thus positive with Kernel of dimension 1)

(M, g) Lorentz manifold
$x \to C_x$ isotropic cone at x: a field of cones

Two Lorentz metrics are conformal iff they have the same cone field.

Temporal orientation: a continuous choose of one component, say C_x^+

Figure

• convex cone-fields

- positive time cone: $x \to T_x^+$
- its closure is $\bar{T}_x^+ = T_x^+ \cup C_x^+$

• temporal curve: an integral curve of the cone-field T^+:
$c : I \subset \mathbb{R} \to M$, Lipschitz, and almost everywhere
$c'(t) \in T^+_{c(t)}$

- Causal curve: $\bar{T}$ instead of T

• (chronological) Future $I^+(x) = \{y \in M$ such that there exists $c : [0, 1]$ temporal, $c(0) = x$, $c(1) = y$ $\}$
Similarly: $J^+(x)$ causal future:

Past: $I^-(x)$, $J^-(x)$

Figure

A (topological) function $t : M \to \mathbb{R}$ is a time (function) if t is (strictly) increasing along any (positive) timelike curve.

- A smooth time function is a submersion along any smooth timelike curve $\iff$ The gradient of t is in the negative time cone.

A hypersurface $\Sigma \subset M$ is a Cauchy hypersurface if:
- any time curve cut it at most once
- any time curve can be extended (as a time curve) in order to cut it

Remark: with a dynamical systems language, Σ is a cross section of the cone field.

(M, g) is globally hyperbolic (GH) if it admits a Cauchy hypersurface.

This implies M is diffeomorphic to a product $N \times \mathbb{R}$, such that any leaf $\{.\} \times N$ is a Cauchy hyeprsurface.

Some extrinsic Lorentz geometry

Second fundamental form

(M, g) a time-oriented Lorentz manifold
$S \subset M$ spacelike,
$x \to \nu(x)$ unit timelike positive normal
$\langle \nu, \nu \rangle = -1$

$\nabla_X Y = II_x(X, Y)\nu \iff II_x(X, Y) = -\langle \nabla_X \nu, Y \rangle$
$x \to A_x \in End(T_x S)$,
$A_x(X) = -\nabla_X \nu$ Weingarten map

$\lambda_1(x), \ldots, \lambda_{n-1}$ eigenvalues of A_x

H_k symmetric function of degree k on the λ_i.

- Mean curvature $H_1(x) = H^S(x) = \lambda_1 + \dots \lambda_{n-1}$

Recall the Gauß equation for the sectional curvature
$\langle R^S(X,Y)X,Y\rangle =$
$\langle II(X,X)\nu, II(Y,Y)\nu\rangle - \langle II(X,Y), II(X,Y)\rangle$
There is a sign $-$, since $\langle \nu,\nu\rangle = -1$

- Scalar curvature $Scal^S = -(1/2)H_2$

- Gaussian (or Lipschitz-Killing...) curvature:
$K^S = -(\lambda_1 \cdot \ldots \cdot \lambda_{n-1}) = -\det(A_x)$

- $\mathbb{H}^2$: $A_x = -Id_{T_x\mathbb{H}^2}$

Maximum principle

- F-curvature: any function of the λ_i

Fact

Let $x \in S \cap S'$, and S' in the future of S, say
$(S, x) \leq (S', x)$,
Then, $II_x^{S'} \leq II_x^S$

Corollary

$$(S, x) \leq (S', x) \Longrightarrow H_1^{S'}(x) \leq H_1^S(x)$$

Corollary

By definition, S is convex if $II_x \leq 0$ (iff $\lambda_i \leq 0$)

Assume S and S' **convex**, *then*

$(S,x) \leq (S',x) \Longrightarrow K^{S'}(x) \leq K^{S}(x)$ *and* $Scal^{S'}(x) \leq Scal^{S}(x)$

Gauß flow

Restrict to $M = \mathrm{Mink}_n$,

S convex,

$x \in S \to \phi^t(x) = x + t\nu(x) \in M; \quad \phi^t(S) = S_t$

$t \to S_t$ is an increasing family: $t \leq s \Longrightarrow S_t \leq S_s$

The second fundamental form of S^t increases with t:
$A^{S_t}_{\phi^t(x)} = A_x(1 + tA_x)^{-1}$
(the tangent spaces are identified by parallel translation)

Monotony

Fact

Assume:

– $S \leq S'$,

– There is a minimal t such that $S' \leq S_t$, and the contact of S' and S^t is realized at $y = \phi^t(x)$.

Then, $II_x^S \leq II_y^{S'}$

In particular, if S and S' have constant F-curvature, then $F^S \leq F^{S'}$

Compact Cauchy surfaces

Assume

– M is locally as $Mink_n$ (in order to use Gauß flow)

– M is globally hyperbolic

– In addition, Cauchy surfaces of M are compact (in order that t and x in the previous fact exist)

Fact

In this case, if S and S' have constant F-curvature, and (say) $F^S \leq F^{S'}$, then $S \leq S'$.

F-foliations

F-times