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Pseudo-Conformal actions of the Möbius group

M. Belraouti [a,*], M. Deffaf [a], Y. Raffed [a], A. Zeghib [b]

[a] Faculté de Mathématiques, USTHB, BP 32, El-Alia, 16111 Bab-Ezzouar, Algeria
[b] Abdelghani Zeghib, UMPA, ENS de Lyon, France

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We study compact connected pseudo-Riemannian manifolds (M, g) on which the conformal group $\mathrm{Conf}(M, g)$ acts essentially and transitively. We prove, in particular, that if the non-compact semi-simple part of $\mathrm{Conf}(M, g)$ is the Möbius group, then (M, g) is conformally flat.

Contents

* Corresponding author.
E-mail addresses: mbelraouti@usthb.dz (M. Belraouti), mdeffaf@usthb.dz (M. Deffaf), yazidsaid.raffed@usthb.edu.dz (Y. Raffed), abdelghani.zeghib@ens-lyon.fr (A. Zeghib).

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1. Introduction

A pseudo-Riemannian manifold is a differentiable manifold M endowed with a pseudo-Riemannian metric g of signature (p, q). Two metrics g_1 and g_2 on M are said to be conformally equivalent if and only if $g_1 = \exp(f)g_2$ where f is C^∞ function. A conformal structure is then an equivalence class $[g]$ of a pseudo-Riemannian metric g and a conformal manifold is a manifold endowed with a pseudo-Riemannian conformal structure. A remarkable family of conformal manifolds is given by the conformally flat ones. These are pseudo-Riemannian conformal manifolds that are locally conformally diffeomorphic (i.e. preserving the conformal structures) to the Minkowski space $\mathbb{R}^{p,q}$ i.e. the vector space $\mathbb{R}^{p+q}$ endowed with the pseudo-Riemannian metric $-dx_0^2 - ... - dx_{p-1}^2 + dy_0^2 + ... + dy_{q-1}^2$.

The conformal group $\mathrm{Conf}(M, g)$ is the group of transformations that preserve the conformal structure $[g]$. It is said to be essential if there is no metric in the conformal class of g for which it acts isometrically. In the Riemannian case, the sphere $\mathbb{S}^n$ is a compact conformally flat manifold with an essential conformal group.

The Einstein universe $\mathrm{Ein}^{p,q}$ is the equivalent model of the standard conformal sphere in the pseudo-Riemannian setting. It admits a two-fold covering conformally equivalent to the product $\mathbb{S}^p \times \mathbb{S}^q$ endowed with the conformal class of $-g_{\mathbb{S}^p} \oplus g_{\mathbb{S}^q}$. It is conformally flat and its conformal group, which is in fact the pseudo-Riemannian Möbius group $\mathrm{O}(p+1, q+1)$, is essential. Actually the Einstein universe is the flat model of conformal pseudo-Riemannian geometry. This is essentially due to the fact that the Minkowski space embeds conformally as a dense open subset of the Einstein universe $\mathrm{Ein}^{p,q}$ and in addition to the Liouville theorem asserting that conformal local diffeomorphisms on $\mathrm{Ein}^{p,q}$ are unique restrictions of elements of $\mathrm{O}(p + 1, q + 1)$. Hence a manifold is conformally flat if and only if it admits a $(\mathrm{O}(p + 1, q + 1), \mathrm{Ein}^{p,q})$-structure.

In the sixties A. Lichnérowicz conjectured that among compact Riemannian manifolds, the sphere is the only essential conformal structure. This was generalised and proved independently by Obata and Ferrand (see [19], [16]). In the pseudo-Riemannian case, a similar question, called the pseudo-Riemannian Lichnérowicz conjecture, was raised by D'Ambra and Gromov [1]. Namely, if a compact pseudo-Riemannian conformal manifold is essential then it is conformally flat. This was disproved by Frances see [7], [9].

1.1. Examples

The following examples will be, as predicted by our results, conformally flat. In the classical Riemannian case, $\mathrm{Ein}^{0,n}$ is the usual sphere $\mathbb{S}^n$ endowed with the conformal action of the Möbius group $\mathrm{O}(1, n)$.

For $p, q > 1$, the double covering $\mathbb{S}^p \times \mathbb{S}^q \to \mathrm{Ein}^{p,q}$ is non-trivial and yields another conformal essential action of $\mathrm{O}(p + 1, q + 1)$.

One can also ask which subgroups $G \subset \mathrm{O}(p + 1, q + 1)$ act transitively and essentially on $\mathrm{Ein}^{p,q}$. Let us mention here the natural example of orthogonal groups (preserving the quadratic form $\mathbb{R}^{p+1,q+1}$) together with a complex, quaternionic or an octonian structure. More precisely, we have $\mathrm{U}(m, n)$ (resp. $\mathrm{Sp}(m, n)$) act transitively on $\mathrm{Ein}^{2m-1,2n-1}$ (resp. on $\mathrm{Ein}^{4m-1,4n-1}$). Also, the exceptional group F_4^{-20} acts naturally on $\mathrm{Ein}^{7,15}$. All these actions are transitive and essential.

There is also a non-obvious action of $\mathrm{O}(1, 4)$ on $\mathrm{Ein}^{3,3}$. It can be topologically described as follows. The Möbius group $\mathsf{O}(1, 4)$ acts on the sphere $\mathbb{S}^3$. It also acts on the space of orthonormal systems of vectors tangent to $\mathbb{S}^3$ (since the action is conformal). It also acts on the projectivization of this space, that is the space of tangent orthogonal bases up to a constant. This is topologically $\mathrm{Ein}^{3,3}$, up to a cover, and the $\mathrm{O}(1,4)$ action is transitive. Now, the point is that this preserves a conformal structure of type $(3,3)$. This can be showed, but not trivially, by describing the isotropy action. The safest way is to make use of the classical fact that $\mathsf{O}(1,4)$ is up to a cover isomorphic to $\mathrm{Sp}(1,1)$. The latter acts naturally on $\mathrm{Ein}^{3,3}$.

Let us finally observe that $\mathrm{Sp}(1)$ also acts on $\mathrm{Ein}^{3,3}$, and this commutes with $\mathrm{Sp}(1,1)$, and hence $\mathrm{Sp}(1,1)\times \mathrm{Sp}(1)$ acts on $\mathrm{Ein}^{3,3}$.

1.1.1. The 2-dimensional case

Conformal pseudo-Riemannian structures in dimension 2 are not rigid, in particular the group of local conformal transformations has infinite dimension. Conformal actions on compact surfaces of Lie groups are however relevant even in this dimension. In the Riemannian case, the conformal group of a compact surface is in fact a Lie group, e.g. $\mathrm{Conf}(\mathbb{S}^2) = \mathrm{SO}(1,3) = \mathrm{PSL}(2,\mathbb{C})$. This is no longer the case of $\mathrm{Ein}^{1,1}$. This latter surface is in fact the torus $\mathbb{S}^1 \times \mathbb{S}^1$, with the conformal structure given by the fact that the two factors are isotropic. So, $\mathrm{Conf}(\mathrm{Ein}^{1,1})$ coincides with diffeomorphisms preserving the product structure. This is therefore $\mathsf{Diff}(\mathbb{S}^1) \times \mathsf{Diff}(\mathbb{S}^1)$ augmented with the involution $(x,y) \to (y,x)$. On the other hand, from the quadric model of $\mathrm{Ein}^{1,1}$, one gets the restricted conformal group $\mathsf{PSO}(2,2)$ which is identified to $L = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. Therefore L as well as $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{SO}(2)$ are two examples of Lie groups acting transitively and essentially on $\mathrm{Ein}^{1,1}$.

It turns out that L is a maximal Lie group in $\mathrm{Conf}(\mathrm{Ein}^{1,1})$, that is there is no bigger Lie group $L' \subset \mathrm{Conf}(\mathrm{Ein}^{1,1})$ which contains L.

Consider two finite covers of degrees m and n: $\mathbb{S}^1 \to \mathbb{S}^1$, and let $\mathrm{PSL}^k(2,\mathbb{R})$ denote the k-cover of $\mathrm{PSL}(2,\mathbb{R})$. Then $L^{m,n} = \mathrm{PSL}^m(2,\mathbb{R}) \times \mathrm{PSL}^n(2,\mathbb{R})$ acts conformally on $\mathbb{S}^1 \times \mathbb{S}^1$ endowed with the pull back of the conformal structure of $\mathrm{Ein}^{1,1}$ via the covering of degree mn: $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathrm{Ein}^{1,1}$. Observe however that this conformal structure is also just given by the product structure, and so it is isomorphic to $\mathrm{Ein}^{1,1}$. From all this, we infer that the Lie groups $L^{m,n}$ act faithfully transitively and essentially on $\mathrm{Ein}^{1,1}$. In sum, on the same $\mathrm{Ein}^{1,1}$, we have different actions of these Lie groups $L^{m,n}$, but they all finitely cover the $L^{1,1}$-action (so we can say they are all the same up a finite covers).

1.2. Results

The present article is the first of a series on the pseudo-Riemannian Lichnérowicz conjecture in a homogeneous setting [5,4]. The general non homogeneous case, but with signature restrictions, was amply studied by Zimmer, Bader, Nevo, Frances, Zeghib, Melnick and Pecastaing (see [24], [2], [10], [21], [22], [20], [17]). Let us also quote [15] as a recent work in the Lorentz case. More exactly, we prove the following classification result.

Theorem 1.1. *Let $(M,[g])$ be a conformal connected compact pseudo-Riemannian manifold. We suppose that there exists G a Lie subgroup of the conformal group $\mathrm{Conf}(M,g)$ acting essentially and transitively on $(M,[g])$. We suppose moreover that the non-compact semi-simple part of G is locally isomorphic to the Möbius group $\mathrm{SO}(1,n+1)$. Then $(M,[g])$ is conformally flat. More precisely $(M,[g])$ is conformally equivalent to*

- *The conformal Riemannian n-sphere or;*
- *Up to a finite cover, the Einstein universe $\mathrm{Ein}^{1,1}$ or;*
- *Up to a double cover, the Einstein universe $\mathrm{Ein}^{3,3}$.*

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2. Preliminaries

2.1. Notations

Throughout this paper (M, g) will be a compact connected pseudo-Riemannian manifold of dimension n endowed with a transitive and essential action of a Lie subgroup G of the conformal group $\mathrm{Conf}(M, g)$.

Lemma 2.1. *We can assume that G is connected.*

Proof. The connected component G^0 acts transitively. Let us show that it acts essentially. If not, it will preserve a metric g_0 in the conformal class. If $f \in G$, then f^*g_0 is another G^0-invariant metric since G^0 is normal in G. But any other G^0-invariant metric in the conformal class has the form αg_0, with α a G^0-invariant function and hence constant. It follows that $f^*g_0 = \alpha(f)g_0$, where $\alpha : G \to \mathbb{R}^*$ is a homomorphism. In other words G act by g_0-homotheties. Since M is compact, this implies $\alpha = \pm 1$, and hence G acts non-essentially. □

Fix a point x in M and denote by $H = \mathrm{Stab}(x)$ its stabilizer in G. Denote respectively by $\mathfrak{g}$, $\mathfrak{h}$ the Lie algebras of G and H. Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ be a Levi decomposition of $\mathfrak{g}$, where $\mathfrak{s}$ is semi-simple and $\mathfrak{r}$ is the solvable radical of $\mathfrak{g}$. Denote by $\mathfrak{s}_{nc}$ the non-compact semi-simple factor of $\mathfrak{s}$, by $\mathfrak{s}_c$ the compact one and let $\mathfrak{n}$ be the nilpotent radical of $\mathfrak{g}$. Note that $\mathfrak{n}$ is an ideal of $\mathfrak{g}$. Let us denote respectively by S, S_{nc}, S_c, R and N the connected Lie sub-groups of G associated to $\mathfrak{s}$, $\mathfrak{s}_{nc}$, $\mathfrak{s}_c$, $\mathfrak{r}$ and $\mathfrak{n}$.

Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{s}$ associated with a Cartan involution Θ. Consider $\mathfrak{s} = \mathfrak{s}_0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{s}_\alpha = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{s}_\alpha$ the root space decomposition of $\mathfrak{s}$, where Δ is the set of roots of $(\mathfrak{s}, \mathfrak{a})$. Denote respectively by Δ^+, Δ^- the set of positive and negative roots of $\mathfrak{s}$ for some chosen notion of positivity on $\mathfrak{a}^*$. Then $\mathfrak{s} = \mathfrak{s}_- \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_+$, where $\mathfrak{s}_+ = \bigoplus_{\alpha\in\Delta^+} \mathfrak{s}_\alpha$ and $\mathfrak{s}_- = \bigoplus_{\alpha\in\Delta^-} \mathfrak{s}_\alpha$.

For every $\alpha \in \mathfrak{a}^*$, consider

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}, \forall H \in \mathfrak{a} : ad_H(X) = \alpha(H)X\}.$$

We say that α is a weight if $\mathfrak{g}_\alpha \neq 0$. In this case $\mathfrak{g}_\alpha$ is its associated weight space. As $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n}$ (see [13, Theorem 13]) then, for every $\alpha \neq 0$, $\mathfrak{g}_\alpha = \mathfrak{s}_\alpha \oplus \mathfrak{n}_\alpha$, where

$$\mathfrak{n}_\alpha = \{X \in \mathfrak{n}, \forall H \in \mathfrak{a} : ad_H(X) = \alpha(H)X\}.$$

Moreover, the commutativity of $\mathfrak{a}$ together with the fact that finite dimensional representations of a semi-simple Lie algebra preserve the Jordan decomposition implies that elements of $\mathfrak{a}$ are simultaneously diagonalisable in some basis of $\mathfrak{g}$. Thus $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\neq 0} \mathfrak{g}_\alpha$.

Finally we will denote respectively by A, S_+ the connected Lie subgroups of G corresponding to $\mathfrak{a}$ and $\mathfrak{s}_+$.

2.2. General facts

We will prove some general results about the conformal group G. We start with the following general fact:

Proposition 2.2. *We have that* $[\mathfrak{s}, \mathfrak{n}] = [\mathfrak{s}, \mathfrak{r}]$. *In particular the sub-algebra* $\mathfrak{s} \ltimes \mathfrak{n}$ *is an ideal of* $\mathfrak{g}$.

Proof. For this, let us consider the semi-simple S-representation in $GL(\mathfrak{r})$. It preserves $\mathfrak{n}$ and thus has a supplementary invariant subspace. But $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n}$ so automorphisms of $\mathfrak{r}$ act trivially on $\mathfrak{r}/\mathfrak{n}$ and hence $[\mathfrak{s}, \mathfrak{g}] \subset \mathfrak{s} \oplus [\mathfrak{s}, \mathfrak{n}] \subset \mathfrak{s} \ltimes \mathfrak{n}$. We deduce that $\mathfrak{s} \ltimes \mathfrak{n}$ is an ideal of $\mathfrak{g}$. □

Next we will prove:

Proposition 2.3. *The non-compact semi-simple factor* S_{nc} *of* S *is non trivial.*

Let us first start with the following simple observation:

Proposition 2.4. *If a conformal diffeomorphism* f *of* (M, g) *preserves a volume form* ω *on* M, *then it preserves a metric in the conformal class of* g.

Proof. Let f be a diffeomorphism preserving the conformal class $[g]$ and a volume form ω on M. Denote by ω_g the volume form defined on M by the metric g. On the one hand, there exists a C^∞ real function ϕ such that $\omega = e^{\phi}\omega_g$. Hence ω is the volume form defined by the metric $e^{\frac{2\phi}{n}} g$. On the other hand, we have $f^* e^{\frac{2\phi}{n}} g = e^{\psi} e^{\frac{2\phi}{n}} g$, for some C^∞ function ψ. Thus $f^*\omega = e^{\frac{n}{2}\psi}\omega$. But, f preserves the volume form ω, so $\psi = 0$ which means that f preserves the metric $e^{\frac{2\phi}{n}} g$. □

As a consequence we get:

Corollary 2.5. *The conformal group* G *preserves no volume form on* M.

Assume that the non-compact semi-simple factor S_{nc} is trivial. Then by [23, Corollary 4.1.7] the group G is amenable. So it preserves a regular Borel measure μ on the compact manifold M. It is in particular a quasi-invariant measure with associated rho-function $\rho_1 = 1$ (in the sense of [3]). Let now ω_g be the volume form corresponding to the metric g. As the group G acts conformally and the action is C^∞, the measure ω_g is also quasi-invariant with C^∞ rho-function ρ_2 (see [3, Theorem B.1.4]). Again by [3, Theorem B.1.4], the measures μ and ω_g are equivalent and $\frac{d\mu}{d\omega_g} = \frac{1}{\rho_2}$. This shows that μ is a volume form. Then one uses Corollary 2.5 to get the Proposition 2.3.

Alternative proof. The previous proof says that, in general, for a homogeneous space G/H, if the G-action preserves a Radon measure μ, then this measure is smooth. An alternative (self-contained) proof consists in lifting this measure to a left invariant measure of G, and use uniqueness (up to a constant) of the Haar measure. Let ν_H be the Haar measure of H. If $A \subset G$ is a Borel subset (say with a compact closure), $x \in G$ let $F_A(x)$ be the H-measure of $A \cap xH$, that is $\nu_H(x^{-1}A \cap H)$. Observe that $F_A(xh) = \nu_H((h^{-1}x^{-1})A \cap H) = \nu_(x^{-1}A \cap H) = F(x)$, and thus F is well defined on G/H. Define now $\nu(A) = \int F d\mu$. For $g \in G$, an explicit computation of F_{gA} shows that $\nu(gA) = \nu(A)$, that is ν is a left G-invariant measure on G, and so it a Haar measure, in particular it is defined by a volume form.

In the general case the essentiality of the action ensures the non discreteness of the stabilizer H.

Proposition 2.6. *The stabilizer* H *is not discrete.*

Proof. If it was not the case then H would be a uniform lattice in G. But as the action is essential, there is an element $h \in H$ that does not preserve the metric on $\mathfrak{g}/\mathfrak{h}$. So $|\det(\mathrm{Ad}_h)| \neq 1$ contradicting the unimodilarity of G. □

To finish this part let us prove the two following important Lemmas that will be used later in the paper:

Lemma 2.7. *Let $\pi : S_{nc} \longrightarrow \mathrm{GL}(V)$ be a linear representation of S_{nc} into a linear space V. Then, the compact orbits of S_{nc} are trivial.*

Proof. We can assume without loss of generality that the linear representation π is irreducible. Assume that S_{nc} has a compact orbit $\mathcal{C} \subset V$. Then the convex envelope $\mathrm{Conv}(\mathcal{C} \cup -\mathcal{C})$ is an S_{nc}-invariant compact convex symmetric set with non empty interior. Thus the action of S_{nc} preserves the Minkowski gauge $\|.\|$ (which is in fact a norm) of $\mathrm{Conv}(\mathcal{C} \cup -\mathcal{C})$. But $\mathrm{Isom}\,(\mathrm{Conv}(\mathcal{C} \cup -\mathcal{C}), \|.\|)$ is compact. So the restriction of the representation π to $\mathrm{Conv}(\mathcal{C} \cup -\mathcal{C})$ gives rise to an homomorphism from a semi-simple group with no compact factor to a compact group and hence is trivial. □

Lemma 2.8. *A linear representation $\pi : \mathfrak{s}_{nc} \longrightarrow \mathrm{gl}(V)$ of $\mathfrak{s}_{nc}$ into a linear space V is completely determined by its restriction to $\mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_+$. More precisely, $\pi_{\mathfrak{s}_{nc}}(V) = \mathrm{Vect}\left(\pi_{\mathfrak{a}\oplus\mathfrak{m}\oplus\mathfrak{s}_+}(V)\right)$.*

Proof. It is in fact sufficient to show that $\pi_{\mathfrak{s}_-}(V) \subset \mathrm{Vect}\left(\pi_{\mathfrak{a}\oplus\mathfrak{s}_+}(V)\right)$. For that, fix $x \in \mathfrak{s}_{-\alpha} \subset \mathfrak{s}_-$ and let $a \in \mathfrak{a}$ such that $\mathbb{R}x \oplus \mathbb{R}a \oplus \mathbb{R}\Theta(x) \cong \mathfrak{sl}(2, \mathbb{R})$ (see for example [14, Proposition 6.52]). Thus the restriction of π to $\mathbb{R}x \oplus \mathbb{R}a \oplus \mathbb{R}\Theta(x)$ is isomorphic to a linear representation of $\mathfrak{sl}(2, \mathbb{R})$ into V. Using Weyl Theorem we can assume without loss of generality that this last is irreducible. But irreducible linear representations of $\mathfrak{sl}(2, \mathbb{R})$ into V are unique up to isomorphism (see for instance [12, Theorem 4.32]). It is then easy to check that they verify $\pi(x)(V) \subset \mathrm{Vect}\left(\pi_{\mathbb{R}a\oplus\mathbb{R}\Theta(x)}(V)\right)$ (see [12, Examples 4.2]). This finishes the proof. □

3. Lie algebra formulation

3.1. Enlargement of the isotropy group

As the manifold G/H is compact, the isotropy subgroup H is a uniform subgroup of G. If H was discrete then it is a uniform lattice and in this case G would be unimodular. In the non discrete case, this imposes strong restrictions on the group H. When H and G are both complex algebraic it is equivalent to being parabolic i.e. contains maximal solvable connected subgroup of H. In the real case, Borel and Tits [6] proved that an algebraic group H of a real linear algebraic group G is uniform if it contains a maximal connected triangular subgroup of G. Recall that a subgroup of G (respectively a sub-algebra of $\mathfrak{g}$) is said to be triangular if, in some real basis of $\mathfrak{g}$, its image under the adjoint representation is triangular.

Let $H^* = \mathrm{Ad}^{-1}\left(\overline{\mathrm{Ad}(H)}^{Zariski}\right)$ be the smallest algebraic Lie subgroup of G containing H. By [11, Corollary 5.1.1], the Lie algebra $\mathfrak{h}^*$ of H^* contains a maximal triangular sub-algebra of $\mathfrak{g}$. The sub-algebra $(\mathfrak{a} \oplus \mathfrak{s}_+) \ltimes \mathfrak{n}$ being triangular, we get the following fact:

Fact 3.1. *Up to conjugacy, the sub-algebra h^* contains $(\mathfrak{a} \oplus \mathfrak{s}_+) \ltimes \mathfrak{n}$.*

Consider the vector space $\mathrm{Sym}(\mathfrak{g})$ of bilinear symmetric forms on $\mathfrak{g}$. The group G acts naturally on $\mathrm{Sym}(\mathfrak{g})$ by $g.\Phi(X, Y) = \Phi(Ad_{g^{-1}}X, Ad_{g^{-1}}Y)$. Let $\langle ., . \rangle$ be the bilinear symmetric form on $\mathfrak{g}$ defined by

$$\langle X, Y \rangle = g\left(X^*(x), Y^*(x)\right),$$

where g is the pseudo-Riemannian metric, X^*, Y^* are the fundamental vector fields associated to X and Y and x is the point fixed previously. It is a degenerate symmetric form with kernel equal to $\mathfrak{h}$.

Let P be the subgroup of G preserving the conformal class of $\langle ., . \rangle$. It is an algebraic group containing H and normalizing the sub-algebra $\mathfrak{h}$. In particular, it contains H^*: the smallest algebraic group containing H. Using Fact 3.1 we get that up to conjugacy, the Lie algebra $\mathfrak{p}$ of P contains $(\mathfrak{a} \oplus \mathfrak{s}_+) \ltimes \mathfrak{n}$.

Proposition 3.2. *The Cartan sub-group A does not preserve the metric $\langle ., . \rangle$.*

Proof. First as $\mathfrak{h}$ is an ideal of $\mathfrak{p}$ then by taking quotient of both P and H by H, we can suppose that H is a uniform lattice of P and in particular that P is unimodular.

Assume that A preserves the metric $\langle .,. \rangle$. On the one hand, the groups S_+ and N preserve the conformal class of $\langle .,. \rangle$. On the other hand, they act on $\mathrm{Sym}(\mathfrak{g})$ by unipotent elements. So the groups A, S_+, and N preserve the metric $\langle .,. \rangle$. But by Iwasawa decomposition $(A \ltimes S_+)$ is co-compact in S. Thus the S_{nc}-orbit of $\langle .,. \rangle$ is compact in $\mathrm{Sym}(\mathfrak{g})$ and hence trivial by Lemma 2.7. Therefore S_{nc} and N are subgroups of P. This implies that for any $p \in P$, $\left|\det (\mathrm{Ad}_p)_{|\mathfrak{g}/\mathfrak{p}}\right| = 1$. Indeed, the action of G on $(\mathfrak{s}_c + \mathfrak{r})/\mathfrak{n}$ factors through the product of the action of S_c on $\mathfrak{s}_c$ by the trivial action on $\mathfrak{r}/\mathfrak{n}$. As P contains S_{nc} and N, its action on $\mathfrak{g}/\mathfrak{p}$ is a quotient of the action of S_c on $\mathfrak{s}_c$. But S_c is compact, thus it preserves some positive definite scalar product and hence the determinant $\left|\det (\mathrm{Ad}_p)_{|\mathfrak{g}/\mathfrak{p}}\right| = 1$.

Now let $h \in H$ such that Ad_h does not preserve $\langle .,. \rangle$. We have that

$$1 \neq \left|\det (\mathrm{Ad}_h)_{|\mathfrak{g}/\mathfrak{h}}\right| = \left|\det (\mathrm{Ad}_h)_{|\mathfrak{g}/\mathfrak{p}}\right| \left|\det (\mathrm{Ad}_h)_{|\mathfrak{p}/\mathfrak{h}}\right|$$

Finally we get $\left|\det (\mathrm{Ad}_h)_{|\mathfrak{p}/\mathfrak{h}}\right| \neq 1$ which contradicts the unimodularity of P. □

3.2. Distortion

The group P preserves the conformal class of $\langle .,. \rangle$. There exists thus an homomorphism $\delta : P \to \mathbb{R}$ such that: for every $p \in P$ and every $u, v \in \mathfrak{g}$,

$$\langle \mathrm{Ad}_p(u), \mathrm{Ad}_p(v) \rangle = e^{\delta(p)} \langle u, v \rangle = \left|\det (\mathrm{Ad}_p)_{|\mathfrak{g}/\mathfrak{h}}\right|^{\frac{2}{n}} \langle u, v \rangle \tag{1}$$

In particular if $p \in P$ preserves the metric then $\delta(p) = 0$ and

$$\langle \mathrm{Ad}_p(u), \mathrm{Ad}_p(v) \rangle = \langle u, v \rangle \tag{2}$$

Or equivalently

$$\langle \mathrm{ad}_p(u), v \rangle + \langle u, \mathrm{ad}_p(v) \rangle = 0 \tag{3}$$

It follows that if the action of $p \in P$ on $\mathfrak{g}$ is unipotent then $\delta(p) = 0$. Therefore, the homomorphism δ is trivial on S_- and N but not on A by Proposition 3.2. We continue to denote by δ the restriction of δ to A. We can see it alternatively as a linear form $\delta : \mathfrak{a} \to \mathbb{R}$, called **distortion**, verifying: for every $a \in \mathfrak{a}$ and every $u, v \in \mathfrak{g}$,

$$\langle \mathrm{ad}_a(u), v \rangle + \langle u, \mathrm{ad}_a(v) \rangle = \delta(a) \langle u, v \rangle \tag{4}$$

Definition 3.1. Two weights spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are said to be paired if they are not $\langle .,. \rangle$-orthogonal.

Definition 3.2. A weight α is a non-degenerate weight if $\mathfrak{g}_\alpha$ is not contained in $\mathfrak{h}$.

It turns out that in all our forthcoming proofs we can forget our initial group data and instead use (only) the Lie algebra data above, that is:

- There is a weight decomposition as above,
- There is a distortion $\delta : \mathfrak{a} \longrightarrow \mathbb{R}$,
- The pairing condition of two weight spaces implies their sum is δ,

- $\mathfrak{h}$ is normalized by $(\mathfrak{a} \oplus \mathfrak{s}_+) \ltimes \mathfrak{n}$.
- The essentiality is translated into the fact that $\delta \neq 0$, and the compactness of G/H is replaced by that $(\mathfrak{a} \oplus \mathfrak{s}_+) \ltimes \mathfrak{n}$ normalizes $\mathfrak{h}$.

Definition 3.3. We say that a subalgebra $\mathfrak{g}'$ is a modification of $\mathfrak{g}$ if $\mathfrak{g}'$ projects surjectively on $\mathfrak{g}/\mathfrak{h}$ and in addition $\mathfrak{g}'$ contains the non-compact semi-simple factor $\mathfrak{s}_{nc}$ of $\mathfrak{g}$.

In this case $\mathfrak{g}'/\mathfrak{h}' = \mathfrak{g}/\mathfrak{h}$, where $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h}$. Since $\mathfrak{g}'$ contains $\mathfrak{s}_{nc}$, the pair (g', h') satisfies all the previous Lie algebraic requirements. (This however doesn't ensure (a priori) that $G'/H' = G/H$, and thus compact).

Proposition 3.3. *If the weight space* $\mathfrak{g}_0$ *is degenerate then up to modification,* $\mathfrak{g}$ *is semi-simple and* $M = G/H$ *is conformally flat.*

Proof. On the one hand, $\mathfrak{g}_0 \subset \mathfrak{h}$ implies that $\mathfrak{a} \subset \mathfrak{h}$. As $\mathfrak{h}$ is an ideal of $\mathfrak{p}$, we get that $\mathfrak{s}_+ = [\mathfrak{s}_+, \mathfrak{a}] \subset \mathfrak{h}$. On the other hand, $\mathfrak{r} \subset \mathfrak{g}_0 + \mathfrak{n} \subset \mathfrak{g}_0 + [\mathfrak{n}, \mathfrak{a}] \subset \mathfrak{h}$. Thus, up to modification, we can assume that $\mathfrak{g}$ is semi-simple and that $\mathfrak{h}$ contains $\mathfrak{a} + \mathfrak{m} + \mathfrak{s}_+$.

Now let α_{max} be the highest positive root and let $X \in \mathfrak{g}_{\alpha_{max}}$. Then $d_1 e^X : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ is trivial. Yet e^X is not trivial. We conclude using [8, Theorem 1.4]. □

A direct consequence of Equation (4), is that if $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are paired then $\alpha + \beta = \delta$. This shows that if α is a non-degenerate weight then $\delta - \alpha$ is also a non-degenerate weight. In particular if 0 is a non-degenerate weight, then $\mathfrak{g}_0$ and $\mathfrak{g}_\delta$ are paired and hence δ is a non-degenerate weight. In fact:

Proposition 3.4. *If* 0 *is a non-degenerate weight then* δ *is a root. Moreover* $\mathfrak{s}_\delta \not\subset \mathfrak{h}$.

Proof. First we will prove that the subalgebras $\mathfrak{a}$ and $\mathfrak{n}_\delta$ are $\langle .,. \rangle$-orthogonal. Let $a \in \mathfrak{a}$ such that $\delta(a) \neq 0$. Using Equation (4) for a, $u = a$ and $v \in \mathfrak{n}_\delta$, we get, $\langle a, \mathrm{ad}_a(v) \rangle = \delta(a) \langle a, v \rangle$. But v preserves $\langle .,. \rangle$, thus by Equation (3), $\delta(a) \langle a, v \rangle = 0$. Hence $\langle a, v \rangle = 0$, for every $v \in \mathfrak{n}_\delta$. We conclude by continuity.

Now if δ was not a root then $\mathfrak{s}_\delta = 0$ and $\mathfrak{g}_\delta = \mathfrak{n}_\delta$. Thus $\mathfrak{a}$ and $\mathfrak{g}_\delta$ are orthogonal. Which implies that $\mathfrak{a} \subset \mathfrak{h}$. But $\mathfrak{h}$ is an ideal of $\mathfrak{p}$, so $\mathfrak{g}_\delta = [\mathfrak{g}_\delta, \mathfrak{a}] \subset \mathfrak{h}$. This contradicts the fact that $\mathfrak{g}_\delta$ is paired with $\mathfrak{g}_0$.

To finish we need to prove that $\mathfrak{s}_\delta \not\subset \mathfrak{h}$. If this was not the case then $\mathfrak{a}$ would be orthogonal to $\mathfrak{g}_\delta$. Hence $\mathfrak{g}_\delta \subset \mathfrak{h}$ which contradicts again the fact that $\mathfrak{g}_\delta$ is paired with $\mathfrak{g}_0$. □

3.3. The isotropy group is big

From now and until the end we will suppose that the non-compact semi-simple part S_{nc} of G is locally isomorphic to the Möbius group $\mathrm{SO}(1, n+1)$. In this case the Cartan Lie algebra $\mathfrak{a}$ is one dimensional and we have $\mathfrak{s}_{nc} = \mathfrak{s}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_\alpha$, where α is a positive root, $\mathfrak{a} \cong \mathbb{R}$, $\mathfrak{m} \cong \mathfrak{so}(n)$, and $\mathfrak{s}_{-\alpha} \cong \mathfrak{s}_\alpha \cong \mathbb{R}^n$. Moreover, $\mathfrak{g}_{\pm\alpha} = \mathfrak{s}_{\pm\alpha} \oplus \mathfrak{n}_{\pm\alpha}$, $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_c \oplus \mathfrak{r}_0$, $\mathfrak{g}_\beta = \mathfrak{n}_\beta$ for every $\beta \neq 0, \pm\alpha$ and $\mathfrak{r} = \mathfrak{r}_0 \oplus \bigoplus_{\beta \neq 0} \mathfrak{n}_\beta$.

In section 3.1 we saw that the isotropy group H is contained in the algebraic group P which turn out to be big i.e. to contain the connected Lie groups A, S_α and N. Our next result shows that the group H itself is big:

Proposition 3.5. *The Lie algebra* $\mathfrak{h}$ *contains* $\mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \bigoplus_{\beta \neq 0} \mathfrak{n}_\beta$.

Proof. We have that $\mathfrak{a} \subset \mathfrak{h}$. Indeed, if 0 is a degenerate weight then we are done. If not, then δ is a root and $\mathfrak{a} \subset \mathfrak{g}_0$ is orthogonal to every $\mathfrak{g}_\beta$ with $\beta \neq \delta$. From the proof of Proposition 3.4 we know that $\mathfrak{a}$ and $\mathfrak{n}_\delta$ are orthogonal. Thus it remains to show that $\mathfrak{a}$ and $\mathfrak{s}_\delta$ are orthogonal. For that, let $x \in \mathfrak{s}_\delta$ then $\Theta(x) \in \mathfrak{s}_{-\delta}$

and $[x, \Theta(x)] \neq 0$ in $\mathfrak{a}$. Now using Equation (3) and the fact that one of x or $\Theta(x)$ preserve $\langle ., . \rangle$, we get $\langle \mathrm{ad}_x(\Theta(x)), x \rangle = 0$. But $\mathfrak{a}$ is one dimensional so it is orthogonal to $\mathfrak{s}_\delta$.

To end this proof, we have that $\mathfrak{h}$ is an ideal of $\mathfrak{p}$ and so

$$\mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \bigoplus_{\beta \neq 0} \mathfrak{n}_\beta = \mathfrak{a} \oplus \left[\mathfrak{a}, \mathfrak{s}_\alpha \oplus \bigoplus_{\beta \neq 0} \mathfrak{n}_\beta \right] \subset \mathfrak{h} \oplus [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{h}. \quad \square$$

As a consequence we get:

Corollary 3.6. *If 0 is a non-degenerate weight, then $\delta = -\alpha$.*

3.4. A suitable modification of $\mathfrak{g}$

We will show that $\mathfrak{g}$ admits a suitable modification $\mathfrak{g}'$. This allows us to considerably simplify the proofs in the next section. More precisely, we have:

Proposition 3.7. *The solvable radical decomposes as a direct sum $\mathfrak{r} = \mathfrak{r}_1 \oplus \mathfrak{r}_2$, where $\mathfrak{r}_1$ is a subalgebra commuting with the semi-simple factor $\mathfrak{s}$ and $\mathfrak{r}_2$ is an $\mathfrak{s}$-invariant linear subspace contained in $\mathfrak{h}$. In particular $\mathfrak{g}' = \mathfrak{s} \oplus \mathfrak{r}_1$ is a modification of $\mathfrak{g}$.*

To prove Proposition 3.7, we need the following lemma:

Lemma 3.8. *We have $[\mathfrak{s}, \mathfrak{n}] = [\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{h}$.*

Proof of Lemma 3.8. First we prove that $[\mathfrak{n}, \mathfrak{g}_0] \subset \mathfrak{h}$. For this, note that by the Jacobi identity and the fact that $\mathfrak{n}$ is an ideal of $\mathfrak{g}$, we have $\left[\bigoplus_{\beta \neq 0} \mathfrak{n}_\beta, \mathfrak{g}_0\right] = \bigoplus_{\beta \neq 0} \mathfrak{n}_\beta$ which in turn is a subset of $\mathfrak{h}$ by Proposition 3.5. Thus one need to prove that $[\mathfrak{n}_0, \mathfrak{g}_0] \subset \mathfrak{h}$. We know that $\mathfrak{n}$ preserve the metric $\langle ., . \rangle$. So using Equation (3) for $p \in \mathfrak{n}_0$, $u \in \mathfrak{g}_0$ and $v \in \mathfrak{g}_\delta$ gives us: $\langle \mathrm{ad}_p(u), v \rangle + \langle u, \mathrm{ad}_p(v) \rangle = 0$. But once again by Jacobi identity, the fact that $\mathfrak{n}$ is an ideal of $\mathfrak{g}$ and Proposition 3.5 we have $\mathrm{ad}_p(v) \in \mathfrak{g}_\delta \cap \mathfrak{n} = \mathfrak{n}_\delta \subset \mathfrak{h}$. So $\langle \mathrm{ad}_p(u), v \rangle = 0$, which means that $[\mathfrak{n}_0, \mathfrak{g}_0]$ is orthogonal to $\mathfrak{g}_\delta$. Using the fact that $[\mathfrak{n}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0$ and that $\mathfrak{g}_0$ is orthogonal to every $\mathfrak{g}_\beta$ for $\beta \neq \delta$ we get that $[\mathfrak{n}_0, \mathfrak{g}_0] \subset \mathfrak{h}$.

Next we have that $\mathfrak{s}_c \subset \mathfrak{g}_0$ thus $[\mathfrak{s}_c, \mathfrak{n}] \subset [\mathfrak{g}_0, \mathfrak{n}] \subset \mathfrak{h}$.

Finally we finish by proving that $[\mathfrak{s}_{nc}, \mathfrak{n}] \subset \mathfrak{h}$. On the one hand we have,

$$[\mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \mathfrak{m}, \mathfrak{n}] \subset [\mathfrak{a} \oplus \mathfrak{s}_\alpha, \mathfrak{n}] + [\mathfrak{g}_0, \mathfrak{n}] \subset \mathfrak{h} + \mathfrak{h} \subset \mathfrak{h}.$$

On the other hand, as $\mathfrak{s}_{nc}$ is semi-simple we have by Lemma 2.8 that $[\mathfrak{s}_{nc}, \mathfrak{n}] \subset \mathrm{Vect}\left([\mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_\alpha, \mathfrak{n}]\right) \subset \mathfrak{h}$.

Proof of Proposition 3.7. The subalgebra $[\mathfrak{s}, \mathfrak{n}] = [\mathfrak{s}, \mathfrak{r}]$ is $\mathfrak{s}$-invariant, so it admits an $\mathfrak{s}$-invariant supplementary subspace $\mathfrak{r}_1'$ in $\mathfrak{r}$. But $\mathfrak{s}$ acts trivially on $\mathfrak{r}/[\mathfrak{s}, \mathfrak{n}]$ and thus it acts trivially on $\mathfrak{r}_1'$. We take $\mathfrak{r}_1$ to be the $\mathfrak{s}$-invariant subalgebra generated by $\mathfrak{r}_1'$ (in fact the action of $\mathfrak{s}$ on $\mathfrak{r}_1$ is trivial).

It is clear that $\mathfrak{r}_1$ is a direct sum of $\mathfrak{r}_1'$ and $\mathfrak{r}_1''$: an $\mathfrak{s}$-invariant subspace of $[\mathfrak{s}, \mathfrak{n}]$. Consider $\mathfrak{r}_2$ to be the supplementary of $\mathfrak{r}_1''$ in $[\mathfrak{s}, \mathfrak{n}] = [\mathfrak{s}, \mathfrak{r}]$. It is $\mathfrak{s}$-invariant and by Lemma 3.8 we have $\mathfrak{r}_2 \subset \mathfrak{h}$.

4. The Möbius conformal group: a classification theorem

This section is devoted to prove Theorem 1.1. We distinguish two situations: when $\mathfrak{m}$ is contained in $\mathfrak{h}$ and when it is not. In this last one, we first consider the case where only the non-compact semi-simple part

S_{nc} is non trivial. Then deduce from it the general case. From now and until the end we will assume, up to modification, that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}_1$.

4.1. The Frances-Melnick case

We suppose that the sub-algebra $\mathfrak{m}$ is contained in $\mathfrak{h}$. Then we have the following proposition:

Proposition 4.1. *M is conformally equivalent to the standard sphere $\mathbb{S}^n$ or the Einstein universe* $\mathrm{Ein}^{1,1}$.

Proof. Assume first that $\mathfrak{g}_0$ is contained in $\mathfrak{h}$. Then by Proposition 3.3, M is conformally flat and after modification, $\mathfrak{r} = 0$. Moreover, $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{s}_{-\alpha}$. This is because $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_c$ and $[\mathfrak{m}, X] = \mathfrak{s}_{-\alpha}$ for every $X \neq 0$ in $\mathfrak{s}_{-\alpha}$. Thus M is conformally equivalent to $\mathrm{SO}(1, n+1)/\mathrm{CO}(n) \ltimes \mathbb{R}^n \cong \mathbb{S}^n$.

Now suppose that $\mathfrak{g}_0$ is not in $\mathfrak{h}$. In this case $\mathfrak{g}_{-\alpha} = \mathfrak{g}_\delta$ is paired with $\mathfrak{g}_0$. But $\mathfrak{a}$, $\mathfrak{m}$ and $\mathfrak{n}_\delta$ are contained in $\mathfrak{h}$ so $\mathfrak{s}_{-\alpha}$ is paired with $\mathfrak{s}_c \oplus (\mathfrak{r}_0 \cap \mathfrak{r}_1)$. Note that $\mathfrak{m}$ acts on $\mathfrak{s}_{-\alpha} \oplus (\mathfrak{s}_c \oplus (\mathfrak{r}_0 \cap \mathfrak{r}_1))$ by preserving the pairing (in fact the action of $\mathfrak{m}$ preserves the metric $\langle ., . \rangle$). On the contrary for $n \geq 2$, $\mathfrak{m} \cong \mathfrak{so}(n)$ acts trivially on $\mathfrak{r}_0 \cap \mathfrak{r}_1$ and transitively on $\mathfrak{s}_{-\alpha} - \{0\}$, so $n = 1$. As the metric is of type (p, q), we conclude that the projection of $\mathfrak{s}_c \oplus (\mathfrak{r}_0 \cap \mathfrak{r}_1)$ on $\mathfrak{g}/\mathfrak{h}$ is $\cong \mathbb{R}$. Thus, after modification $\mathfrak{g} = \mathfrak{so}(1, 2) \oplus \mathbb{R} = \mathfrak{u}(1, 1)$, $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{s}_\alpha = \mathbb{R} \oplus \mathbb{R}$. In this case, $\mathfrak{g}/\mathfrak{h}$ is the direct sum of two isotropic 1-dimensional subspaces. This implies in particular that the metric on $\mathfrak{g}/\mathfrak{h}$ is unique up to constant. One then recognizes the usual action of $\mathrm{U}(1, 1)$ on $\mathrm{Ein}^{1,1}$. Hence M is, up to finite cover, conformally equivalent to $\mathrm{Ein}^{1,1}$. □

4.2. The non-compact semi-simple case

Here we suppose that $\mathfrak{m}$ is not contained in $\mathfrak{h}$, the compact semi-simple part $\mathfrak{s}_c$ and the radical solvable part $\mathfrak{r}_1$ are both trivial. We will show:

Proposition 4.2. *The pseudo-Riemannian manifold M is conformally equivalent to* $\mathrm{Ein}^{3,3}$

By Corollary 3.6, δ is a negative root. In particular $\delta = -\alpha$ and $\mathfrak{g}_{-\alpha}$ is paired with $\mathfrak{g}_0$. In addition $\mathfrak{g} = \mathfrak{s}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_\alpha$ and $\mathfrak{a} \oplus \mathfrak{s}_\alpha \subset \mathfrak{h}$. We have:

Proposition 4.3. *The root space $\mathfrak{s}_\delta$ does not intersect $\mathfrak{h}$. In particular the metric is of type (n, n).*

Proof. If it was the case then let $0 \neq X \in \mathfrak{s}_\delta \cap \mathfrak{h}$. We have $[[X, \mathfrak{s}_{-\delta}], X] = \mathfrak{s}_\delta$ so $\mathfrak{s}_\delta \subset \mathfrak{h}$. This contradicts the fact that $\mathfrak{g}_\delta$ is paired with $\mathfrak{g}_0$. □

Consider the bracket $[., .] : \mathfrak{s}_\alpha \times \mathfrak{s}_{-\alpha} \longrightarrow \mathfrak{a} \oplus \mathfrak{m}$. Denote by $. \wedge . : \mathfrak{s}_\alpha \times \mathfrak{s}_{-\alpha} \longrightarrow \mathfrak{m}$ and $. \vee . : \mathfrak{s}_\alpha \times \mathfrak{s}_{-\alpha} \longrightarrow \mathfrak{a}$ its projections on $\mathfrak{m}$ and $\mathfrak{a}$ respectively. Direct computations give us:

Lemma 4.4.

(1) $\forall X \in \mathfrak{s}_\alpha$, $\forall x \in \mathfrak{s}_{-\alpha}$: $X \vee x = \Theta(x) \vee \Theta(X)$.
(2) $\forall X \in \mathfrak{s}_\alpha$, $\forall x \in \mathfrak{s}_{-\alpha}$, $\forall y \in \mathfrak{s}_{-\alpha}$: $[X \wedge x, y] = [X \wedge y, x] - [\Theta(x) \wedge y, \Theta(X)]$

The Cartan involution identifies $\mathfrak{s}_\alpha$ and $\mathfrak{s}_{-\alpha}$, which when identified with $\mathbb{R}^n$, $\mathfrak{m}$ acts on them as $\mathfrak{so}(n)$. In this case, the map $. \vee .$ can be seen as a bilinear symmetric map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathfrak{a}$, and when composed with α gives rise to an $\mathfrak{m}$-invariant scalar product $\langle ., . \rangle_0$ on $\mathbb{R}^n$. Moreover, by Lemma 4.4, for every $x, X \in \mathbb{R}^n$, $X \wedge x$ is the antisymmetric endomorphism of $\mathbb{R}^n$ defined by $X \wedge x(y) = \langle X, y \rangle_0 x - \langle x, y \rangle_0 X$.

Let $x, X \in \mathbb{R}^n$ and consider P the plane generated by x, X. Then $X \wedge x$ when seen as element of $\mathfrak{m} \cong \mathfrak{so}(n)$ is the infinitesimal generator of a one parameter group acting trivially on the orthogonal $P^\perp$ of P with respect to the scalar product $\langle .,. \rangle_0$. Hence $X \wedge x \in \mathfrak{so}(P)$. More generally:

Proposition 4.5. *Let E be a linear subspace of $\mathbb{R}^n$ and let $x \in E$. Consider $\mathfrak{c}$ the Lie subalgebra of $\mathfrak{so}(n)$ generated by $\{X \wedge x / X \in E\}$. Then $\mathfrak{c}$ equals the Lie algebra linearly generated by $\{X \wedge X' / X, X' \in E\}$, which in turn equals $\mathfrak{so}(E)$, the Lie algebra of orthogonal transformations preserving E and acting trivially on its orthogonal (with respect to $\langle .,. \rangle_0$).*

Proof. First we have $\mathfrak{c}(E) \subset E$ and hence $\mathfrak{c} \subset \mathfrak{so}(E)$. It is then sufficient to prove that $\mathfrak{c}$ and $\mathfrak{so}(E)$ have the same dimensions. For that let $\{x, X_2, ..., X_k\}$ be a basis of E. Note that $\{X_2 \wedge x, ..., X_k \wedge x, [X_i \wedge x, X_j \wedge x]$, for $2 \leq i < j \leq k\}$ are linearly independent. Thus $\mathfrak{c} = \mathfrak{so}(E)$. □

For every $x \neq 0 \in \mathfrak{s}_{-\alpha}$ consider:

$$Z_x = \{X \in \mathfrak{s}_\alpha, \text{ such that } [X, x] \in \mathfrak{h}\}.$$

Denote $\overline{\Theta(Z_x)}$ the projection of $\Theta(Z_x)$ in $\mathfrak{g}/\mathfrak{h}$. Then:

Proposition 4.6. *We have:*

$$\overline{\Theta(Z_x)} = \overline{\Theta(Z_y)} \Longleftrightarrow \overline{\Theta(Z_x)} \cap \overline{\Theta(Z_y)} \neq \{0\} \Longleftrightarrow x \in \Theta(Z_y) \Longleftrightarrow y \in \Theta(Z_x).$$

Proof. By Proposition 4.5 we have,

$$[Z_x, \Theta(Z_x)] = \mathfrak{a} \oplus \mathrm{alg}\left(\{X \wedge x / X \in Z_x\}\right) \subset \mathfrak{h}.$$

This implies that:

$$\Theta(Z_x) = \Theta(Z_y) \Longleftrightarrow x \in \Theta(Z_y) \Longleftrightarrow y \in \Theta(Z_x). \tag{5}$$

Now if $\bar{y} \in \overline{\Theta(Z_x)}$, then using (5) we can assume that $\bar{y} = \bar{x}$ and so $y + x \in \mathfrak{h}$. On the one hand $[\Theta(y), y + x] \in [\Theta(y), \mathfrak{h}] \subset \mathfrak{h}$. On the other hand, $[\Theta(y), y] \in \mathfrak{a} \subset \mathfrak{h}$. Thus $[\Theta(y), x] \in \mathfrak{h}$, which means exactly that $\Theta(y) \in Z_x$ or equivalently $y \in \Theta(Z_x)$. This together with (5) give us:

$$\overline{\Theta(Z_x)} = \overline{\Theta(Z_y)} \Longleftrightarrow \overline{\Theta(Z_x)} \cap \overline{\Theta(Z_y)} \neq \{0\} \Longleftrightarrow x \in \Theta(Z_y) \Longleftrightarrow y \in \Theta(Z_x). \quad \square$$

Next we prove:

Proposition 4.7. *The pseudo-Riemannian manifold M is conformally flat.*

Proof. We need to prove that the Weyl tensor W (or the Cotton tensor C if the dimension of M is 3) vanishes. Actually we will just make use of their conformal invariance property. Namely: if f is a conformal transformation of M then,

$$d_x f\, \mathrm{W}(X, Y, Z) = \mathrm{W}(d_x f(X), d_x f(Y), d_x f(Z)) \tag{6}$$

We denote by $\bar{x}$ the projection in $\mathfrak{g}/\mathfrak{h}$ of an element $x \in \mathfrak{g}$. A direct application of Equation (6) gives us:

$$\mathrm{W}(\bar{x}, \bar{y}, \bar{z}) = 0 \text{ for every x, y, z } \in \mathfrak{s}_{-\alpha} \tag{7}$$

$$\mathrm{W}(\bar{x},\bar{y},\overline{m}) = 0 \text{ for every x, y } \in \mathfrak{s}_{-\alpha} \text{ and every m } \in \mathfrak{m}; \tag{8}$$

$$[X, \mathrm{W}(\bar{x},\overline{m}_1,\overline{m}_2)] = \mathrm{W}([X,\bar{x}],\overline{m}_1,\overline{m}_2)\ \forall X \in \mathfrak{s}_\alpha, x \in \mathfrak{s}_{-\alpha} \text{ and } m_1, m_2 \in \mathfrak{m}. \tag{9}$$

Let $x \in \mathfrak{s}_{-\alpha}$, $m_1, m_2 \in \mathfrak{m}$. Then, from Equation (9) we obtain:

$$[\Theta(x), \mathrm{W}(\bar{x},\overline{m}_1,\overline{m}_2)] = \mathrm{W}([\Theta(x),\bar{x}],\overline{m}_1,\overline{m}_2) = 0.$$

In other words

$$\mathrm{W}(\bar{x},\overline{m}_1,\overline{m}_2) \in \overline{\Theta(Z_x)}.$$

Now let $x, y \in \mathfrak{s}_{-\alpha}$, $X \in \mathfrak{s}_\alpha$ and $m \in \mathfrak{m}$. Then again Equation (6) gives us:

$$\mathrm{W}(\bar{x},[X,\bar{y}],\overline{m}) + \mathrm{W}([X,\bar{x}],\bar{y},\overline{m}) = 0.$$

But $\mathrm{W}(\bar{x},[X,\bar{y}],\overline{m}) \in \overline{\Theta(Z_x)}$ and $\mathrm{W}([X,\bar{x}],\bar{y},\overline{m}) \in \overline{\Theta(Z_y)}$. Thus, Proposition 4.6 gives us:

(1) If $y \notin \Theta(Z_x)$ then $\mathrm{W}(\bar{x},[X,\bar{y}],\overline{m}) = 0$;
(2) In the case $y \in \Theta(Z_x)$ and $X \in Z_x$, we have $\mathrm{W}(\bar{x},[X,\bar{y}],\overline{m}) = 0$
(3) If $y \in \Theta(Z_x)$ and $X \notin Z_x$. Then,

$$\mathrm{W}(\bar{x},[X,\bar{y}],\overline{m}) = \mathrm{W}(\bar{x},\overline{[X,y]},\overline{m})$$

But $\overline{[X,y]} = \overline{X \wedge y} = \overline{\Theta(X \wedge y)} = \overline{\Theta(X \wedge y) + \Theta(X \vee y)} = \overline{\Theta([X,y])}$. Thus,

$$\mathrm{W}(\bar{x},[X,\bar{y}],\overline{m}) = -\,\mathrm{W}(\bar{x},\left[\Theta(y),\overline{\Theta(X)}\right],\overline{m}) = 0.$$

So in sum we get:

$$\mathrm{W}(\bar{x},[X,\bar{y}],\overline{m}) = 0 \text{ for every x, y } \in \mathfrak{s}_{-\alpha}, X \in \mathfrak{s}_\alpha \text{ and } m \in \mathfrak{m} \tag{10}$$

From Proposition 4.5, we know that $\mathfrak{s}_\alpha \wedge \mathfrak{s}_{-\alpha}$ generates $\mathfrak{m}$. Thus applying this to Equation (10) and Equation (9) gives us:

$$\mathrm{W}(\bar{x},\overline{m}_1,\overline{m}_2) = 0 \text{ for every } x \in \mathfrak{s}_{-\alpha} \text{ and } m_1, m_2 \in \mathfrak{m}. \tag{11}$$

$$\mathrm{W}(\overline{m}_1,\overline{m}_2,\overline{m}_3) = 0 \text{ for every } m_1, m_2, m_3 \in \mathfrak{m}. \tag{12}$$

By putting Equations (7), (8), (11), (12) together we get $\mathrm{W} = 0$. □

We finish this section by proving Proposition 4.2:

Proof of Proposition 4.2. For the sake of simplicity of notation, in what follows simple connectedness and identification of spaces are considered up to finite covers. First note that if $n = 1$ then $\mathfrak{m} = 0$. Thus we assume $n \geq 2$. So far we have seen that $M = \mathrm{SO}(1, n+1)/H$ is a conformally flat pseudo-Riemannian manifold of signature (n, n). Since the Lie algebra $\mathfrak{h}$ contains $\mathfrak{a}+\mathfrak{s}_\alpha$, the group H is cocompact in $\mathrm{SO}(1, n+1)$. Therefore $\mathrm{SO}(1, n+1)/H$ is connected and compact, with a connected isotropy and hence simply connected. As M is connected, it covers $\mathrm{SO}(1, n+1)/H$ and thus equals it.

On the one hand, the Einstein universe $\mathrm{Ein}^{n,n}$ is simply connected. Thus M is identified to $\mathrm{Ein}^{n,n}$. So $\mathrm{SO}(1, n+1)$ acts transitively on $\mathrm{Ein}^{n,n}$ with isotropy H. By Montgomery Theorem [18, Theorem A] any maximal compact subgroup in $\mathrm{SO}(1, n+1)$, e.g. $K_2 = \mathrm{SO}(n+1)$, acts transitively on $\mathbb{S}^n \times \mathbb{S}^n$ the two fold cover of $\mathrm{Ein}^{n,n}$.

On the other hand, the conformal group of $\mathrm{Ein}^{n,n}$ is $\mathrm{SO}(n+1, n+1)$. A maximal compact subgroup of it is $K_1 = \mathrm{SO}(n+1) \times \mathrm{SO}(n+1)$. Up to conjugacy, we can assume $K_2 \subset K_1$. Therefore, $K_2 = \mathrm{SO}(n+1)$ acts via a homomorphism $\rho = (\rho_1, \rho_2) : \mathrm{SO}(n+1) \to \mathrm{SO}(n+1) \times \mathrm{SO}(n+1)$.

If $\mathrm{SO}(n+1)$ is simple, then:

- either ρ_1 or ρ_2 is trivial and the other one is bijective, in which case $\rho(\mathrm{SO}(n+1))$ does not act transitively on $\mathbb{S}^n \times \mathbb{S}^n$,

- or both are bijective, and $\rho(\mathrm{SO}(n+1)$ is up to conjugacy in $\mathrm{SO}(n+1) \times \mathrm{SO}(n+1)$ the diagonal $\{(g, g)/g \in \mathrm{SO}(n+1)\}$. The latter, too, does not act transitively on $\mathbb{S}^n \times \mathbb{S}^n$.

Hence $\mathrm{SO}(n+1)$ must be non-simple which implies $n = 1$ or $n = 3$, but $n = 1$ was excluded, and then remains exactly the case $n = 3$, for which M is conformally equivalent to $\mathrm{Ein}^{3,3}$.

4.3. The general case

In this section we will show Theorem 1.1 in the general case. We suppose that $\mathfrak{g} = \mathfrak{s}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_\alpha \oplus \mathfrak{s}_c \oplus \mathfrak{r}_1$. Let us denote by $\mathfrak{m}_0 = \mathfrak{m} \cap \mathfrak{h}$ so that $\mathfrak{so}(1, n+1) \cap \mathfrak{h} = \mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \mathfrak{m}_0$. A priori the subalgebra $\mathfrak{m}_0$ could be of any dimension in $\mathfrak{m}$. Nevertheless the hypothesis $\mathfrak{m} \not\subset \mathfrak{h}$ restricts drastically the possibilities. So we have:

Proposition 4.8. *The subalgebra $\mathfrak{m}_0$ has codimension n in $\mathfrak{m}$.*

Proof. If $n = 2$ then $\mathfrak{m} = \mathfrak{so}(2)$. Hence $[p, \mathfrak{s}_\alpha] = \mathfrak{a} \oplus \mathfrak{m}$ for any non null $p \in \mathfrak{s}_{-\alpha}$. Recall that $\mathfrak{s}_{-\alpha}$ preserves the metric so by applying Equation (3) for $p = v \in \mathfrak{s}_{-\alpha}$, $u \in \mathfrak{s}_\alpha$ we get $\langle \mathfrak{s}_{-\alpha}, \mathfrak{m} \rangle = 0$. Thus $\mathfrak{m} \subset \mathfrak{h}$ which contradicts our hypothesis.

Assume that $n \geq 3$ and suppose that $\mathfrak{m}_0$ has codimension less than $n - 1$. Denote by M_0 the connected subgroup of $\mathrm{SO}(n)$ corresponding to $\mathfrak{m}_0$.

If the action of M_0 on $\mathfrak{s}_{-\alpha} \cong \mathbb{R}^n$ is reducible then M_0 preserves the splitting $\mathbb{R}^d \times \mathbb{R}^{n-d}$ and hence is contained in $\mathrm{SO}(d) \times \mathrm{SO}(n-d)$. Thus M_0 has codimension bigger than the codimension of $\mathrm{SO}(d) \times \mathrm{SO}(n-d)$ which in turn achieves its minimum if $d = 1$ or $n - d = 1$ and hence $M_0 = \mathrm{SO}(n-1)$. One can identify $\mathfrak{m}_0$ with $\mathfrak{so}(E)$ for some $n-1$ dimensional linear subspace E of $\mathfrak{s}_{-\alpha}$. Let then $e \in \mathfrak{s}_{-\alpha}$ such that $\mathfrak{s}_{-\alpha} = \mathbb{R}e \oplus E$. Fix a non zero element $x \in \Theta(E)$, we have $\langle \mathrm{ad}_x(e), X \rangle + \langle e, \mathrm{ad}_x X \rangle = 0$ for every $X \in \mathfrak{s}_{-\alpha}$ and so in particular $\langle e, \mathrm{ad}_x e \rangle = 0$. In addition by Proposition 4.5, $[E, \Theta(E)] = \mathfrak{a} \oplus \mathfrak{m}_0 \subset \mathfrak{h}$ thus $\langle \mathrm{ad}_x e, X \rangle = 0$ for every $X \in E$ and hence $\mathrm{ad}_x e$ is orthogonal to $\mathfrak{s}_{-\alpha}$. This implies that $x \wedge e \in \mathfrak{h} \cap \mathfrak{m} = \mathfrak{m}_0 = \mathfrak{so}(E)$ which contradicts the fact that $x \wedge e$ is the infinitesimal rotation of the plane $\mathbb{R}e \oplus \mathbb{R}x$.

The last case to consider is when M_0 acts irreducibly. Let $m \in \mathfrak{m}_0$, $X \in \mathfrak{s}_{-\alpha}$ and $y \in \mathfrak{s}_c \oplus \mathfrak{r}_1$ then $\langle \mathrm{ad}_m(X), y \rangle + \langle X, \mathrm{ad}_m y \rangle = 0$. But $\mathrm{ad}_m y = 0$ and hence $\mathfrak{s}_c \oplus \mathfrak{r}_1$ is orthogonal to $[\mathfrak{m}_0, \mathfrak{s}_{-\alpha}]$ which is equal to $\mathfrak{s}_{-\alpha}$ by irreducibility. Thus $\mathfrak{s}_c \oplus \mathfrak{r}_1 \subset \mathfrak{h}$ and we are in the non-compact semi-simple case. Therefore $n = 3$ and $\mathfrak{m} \cong \mathfrak{so}(3)$. Non trivial Sub-algebras of $\mathfrak{so}(3)$ have dimension one and are reducible. So the only left possibility is $\mathfrak{m}_0 = \mathfrak{m} \cong \mathfrak{so}(3)$ which show that $\mathfrak{m} \subset \mathfrak{h}$ and this is a contradiction. □

End of Proof of Theorem 1.1. By Proposition 4.8, $\mathfrak{m}_0$ is of codimension n in $\mathfrak{m}$. But $\mathfrak{s}_{-\alpha}$ is paired with $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_c \oplus (\mathfrak{r}_1 \cap \mathfrak{r}_0)$. Thus $\mathfrak{s}_c \oplus (\mathfrak{r}_1 \cap \mathfrak{r}_0) \subset \mathfrak{h}$ and we are also in the non-compact semi-simple case. Therefore $n = 3$ and M is conformally equivalent to $\mathrm{Ein}^{3,3}$.

Data availability

No data was used for the research described in the article.

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