An example of a 2-dimensional no leaf*

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Abstract

We improve the "no leaf construction" of Attie and Hurder to get a 2-dimensional example.

We construct a riemannian metric on the 2 dimensional disc, which is of bounded geometry but not quasi-isometric to a leaf of a Lipschitz foliation of a compact manifold.

Our exposition will be short since this is just obtained by adding some geometrical extra complexity to the combinatorial one in the 6 dimensional examples of O. Attie and S. Hurder (see the article of S. Hurder in these proceedings). In contrast the topology will be radically simplified since it is a 2-disc (of course this is topologically the simplest example because the line can not be one).

The example is achieved like those of Attie and Hurder by showing that the growth complexity function is super-exponential. This is the growth with respect to a diameter R of the number of quasi-isometric patterns needed to cover all the subsets with diameter less than R. Attie and Hurder show the super-exponential growth of this function is an obstruction for a simply connected open manifold to be a leaf of a Lipschitz foliation in a compact manifold. They further define the entropy, an invariant describing a strong way for this growth to be super-exponential.

*This paper is in final form and no version of it will be submitted elsewhere

Note that the examples of Attie and Hurder are, not homeomorphically quasi isometric to leaves, but our example here is even not "roughly" quasi isometric to a leaf, i.e. the image of a quasi isometric map (not a homeomorphism) from a leaf. Similar (6-dimensional) examples exist in the article of Hurder, but the obstruction there comes from the theory of coarse cohomology of coronas.

Denote by $S(r)$ the punctured euclidean 2-sphere of radius r. That is more precisely a sphere of radius r, from which we remove a ball of radius 1 around the south pole.

Our blocks in the construction will be punctured spheres of various radii $S(r), S(r^2)$ and $S(r^3)$, that we attach to balls in the hyperbolic plane.

Fact *A ball of radius R in the hyperbolic plane $\mathbf{H}^2$ contains an exponential number $[c^R]$ of disjoint balls of radius $R/3$.*

We will use this in the following weak fashion : A ball of radius r^{n+1} contains $[c^r]$ disjoint balls of radius $r^n/2$, that is, a subset of r^n -separated points, with cardinality $[c^r]$.

Now consider a ball $B(y, r^5)$ in $\mathbf{H}^2$ and select a subset $x_1, ..., x_k$ of r^4 -separated points ($d(x_i, x_j) \geq r^4$ for $i \neq j$), where $k = [c^r]$. Let $W^-(y, d, r^5)$ (or just $W^-(d, r^5)$) be the riemannian manifold of uniform bounded geometry obtained from $B(y, r^5)$ by gluing : d copies of $S(r)$ around $x_1, ...x_d$; and $k-d-1$ copies of $S(r^2)$ around $x_{d+1}...x_{k-1}$. Define $W^+(y, d, r^5)$ (or $W^+(d, r^5)$) by adding $S(r^3)$ around x_k.

Now, let $y_1, ...y_k$ be r^5 -separated points in $B(z, r^6)$. If σ is a map : $\{1, ..., k\} \to \{-, +\}$, let $N(z, \sigma, r^6)$ (or $N(\sigma, r^6)$) be obtained from $B(z, r^6)$ by putting $W^{\sigma(d)}(y_d, d, r^5)$ in place of $B(y_d, r^5)$.

Recall that for $(a, b, c) \in \mathbf{R}^3$, an (a, b, c) quasi-isometry $f : X \to Y$, is a map with c -dense image (in Y) and such that : $bd(x, y) - c \leq d(f(x), f(y)) \leq ad(x, y) + c$.

Usualy one uses that such maps preserve the asymptotic volume growth, i.e. for a raduis tending to infinity. But also for finite raduis, we have the following fact. Before stating it, let us say that two positive quantities are proportionnal if their ratio is bounded, i.e. lies in a compact of $]0, \infty[$, depending only on (a, b, c). For a subset A and a real ϵ, we denote by $\mathcal{O}_\epsilon(A)$ the neighbourhood of raduis ϵ of A, i.e. the union of balls of raduis ϵ centered at points of A.

Fact *Let f be an (a, b, c) quasi-isometry between two riemannian manifolds X and Y of bounded geometry. Fix a raduis ϵ, say $\epsilon = 1$. Then for any $A \subset X$, the volumes of $\mathcal{O}_1(A)$ and $\mathcal{O}_1 f(A)$ are proportionnal.*

Proof. By the boundness of the geometry, two neighbourhoods of given fixed radii have proportionnal volumes (because this is true for balls). Let $B = f(A)$ and $B' = \{b_1, ..., b_n\}$ be a maximal 2ϵ-separated subset of $\mathcal{O}_1 f(A)$. Thus balls centreted at the b_i and of raduis ϵ are disjoint, and those of raduis 2ϵ cover $\mathcal{O}_1 f(A)$. It then follows that $Vol(\mathcal{O}_1 f(A)$ is proportionnal to $card(B') = n$.

Choose $a_i \in f^{-1}\{b_i\} \cap A$. Since f is a quasi isometry, for ϵ big enough, the balls centered at the a_i and of raduis 1 are disjoint, and those of some raduis ϵ' proportionnal to ϵ, cover $\mathcal{O}_1(A)$. Hence, also the volume of this last subset is proportionnal to $card(B') = n$. This finishes the proof of the Fact.

Proposition *Given (a, b, c), if r is big enough, then two manifolds $N(\sigma, r^6)$ and $N(\sigma', r^6)$ are not (a, b, c) quasi isometric unless $\sigma = \sigma'$. In particular there are $2^{[c^r]}$ quasi isometry types of $N(\sigma, r^6)$.*

Proof. Let $f : N(\sigma, r^6) \to N(\sigma', r^6)$ be an (a, b, c) quasi-isometry. We shall prove that an attached sphere $S(r^3)$ or radius r^3 in $N(\sigma, r^6)$ is necessarily mapped "near" an attached sphere of the same type (i.e. $S(r^3)$) in $N(\sigma', r^6)$.

For this, let M be the neighbourhood of radius $b'r^3$, for $b' = max\{1, b\}$, of the union of all attached spheres of type $S(r^3)$ in $N(\sigma', r^6)$. Then there is a constant α, such that for y in $N(\sigma', r^6) - M$, the ball $B(y, br^3 - c)$ has a volume at least α^r. Indeed this ball contains at most one sphere of type $S(r)$ or $S(r^2)$ (and no $S(r^3)$). Thus it contains a hyperbolic ball of radius $br^3 - c - 2r^2$. Hence, it has an exponential volume α^r.

Let x be a north pole in an attached sphere $S(r^3)$ and $y = f(x)$. By the fact above, since f is an (a, b, c) quasi-isometry, the ball $B(y, br^3 - c)$ has a volume proportional to r^6 (like that of $S(r^3)$). By the above property of points in $N(\sigma', r^6) - M$, y must belong to M. Now, since attached spheres are at least r^4 -separated, M is a disjoint union of neighbourhoods (of the same radius $b'r^3$) of individual attached spheres (of type $S(r^3)$). So we have the same combinatory for attached spheres of type $S(r^3)$ in both $N(\sigma', r^6)$ and $N(\sigma, r^6)$. The same augument applied respectively to spheres of type

$S(r^2)$ and $S(r)$, and after to the $W^{\pm}(d, r^5)$, yields the proposition.

Now, glue to $\mathbf{H}^2$ all the possible $N(\sigma, r^6)$ for r integer. The number of quasi-isometry types needed to cover balls of radius R is at least $2^{[c^r]}$, where $R = r^6$. Hence it is super-exponential (in R). This can not be the case of a simply connected leaf of a Lipschitz foliation in a compact manifold.