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On discrete projective transformation groups of Riemannian manifolds

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ABSTRACT

We prove rigidity facts for groups acting on pseudo-Riemannian manifolds by preserving unparameterized geodesics.

RÉSUMÉ

Nous démontrons des résultats de rigidité pour les groupes agissant sur des variétés pseudo-riemanniennes en préservant leurs géodésiques non-paramétrées.

1. Introduction

1.0.1. The projective group of a connection

Two linear connections ∇ and ∇' on a manifold M are equal iff they have the same (parameterized) geodesics. They are called projectively equivalent if they have the same

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unparameterized geodesics. This is equivalent to that the difference $(2,1)$-tensor $T = \nabla - \nabla'$ being trace free in a natural sense [13].

The affine group $\mathsf{Aff}(M, \nabla)$ is that of transformations preserving ∇ and the projective one $\mathsf{Proj}(M, \nabla)$ is that of transformations f sending ∇ to a projectively equivalent one. So, elements of Aff are those preserving (parameterized) geodesics and those of Proj preserve unparameterized geodesics.

Obviously $\mathsf{Aff} \subset \mathsf{Proj}$; and it is natural to look for special connections for which this inclusion is proper, that is, when projective non-affine transformations exist.

1.0.2. Case of Levi-Civita connections

Let now g be a Riemannian metric on M and ∇ its Levi-Civita connection. The affine and projective groups $\mathsf{Aff}(M, g)$ and $\mathsf{Proj}(M, g)$ are those associated to ∇.

More generally, g and g' are projectively equivalent if so is the case for their associated Levi-Civita connections. This defines an equivalence relation on the space $\mathsf{Riem}(M)$ of Riemannian metrics on M. Let $\mathcal{P}(M, g)$ denote the class of g, i.e. the set of metrics shearing the same unparameterized geodesics with g. It contains $\mathbb{R}^+.g$, the set of constant multiples of g. Generically, $\mathcal{P}(M, g) = \mathbb{R}^+.g$.

One crucial fact here is that $\mathcal{P}(M, g)$ is always a finite dimensional manifold whose dimension is called the *degree of projective mobility* of g. (This contrasts with the case of projective equivalence classes of connections which are infinitely dimensional affine spaces. Similarly, conformal classes of metrics are identified to spaces of positive functions on the manifold.) It is actually one culminant fact of projective differential geometry to identify $\mathcal{P}(M, g)$ to an open subset of a finite dimensional linear sub-space $\mathcal{L}(M, g)$ of endomorphisms of TM (see §3). Being projectively equivalent for connections is a linear condition, but this is no longer linear for metrics (say because the correspondence $g \to$ its Levi-Civita connection, is far from being linear!). The trick is to perform a transform leading to a linear equation, see [6] for a nice exposition.

1.0.3. Philosophy

The idea behind our approach here is to let a diffeomorphism f on a differentiable manifold M act on the space $\mathsf{Riem}(M)$ of Riemannian metrics on M. That this action has a fixed point means exactly that f is an isometry for some Riemannian metric on M. One then naturally wonders what is the counterpart of the fact that the f-action preserves some (finite dimensional) manifold $V \subset \mathsf{Riem}(M)$. A classical similar idea is to let the isotopy class of a diffeomorphism on a surface act on its Teichmuller space [34]. Here, as it will be seen bellow, we are specially concerned with the case $\dim V = 2$.

1.0.4. More general pseudo-Riemannian framework

All this generalizes to the pseudo-Riemannian case. One fashion to unify all is to generalize all this to the wider framework of second order ordinary differential equations (e.g. Hamiltonian systems) on M, by letting their solutions play the role of (parameterized) geodesics.

1.0.5. Rigidity of the projective group

We are interested here in a (very) natural and classical problem in differential geometry: *Characterize pseudo-Riemannian manifolds* (M, g) *for which* $\mathsf{Proj}(M, g) \supsetneq \mathsf{Aff}(M, g)$*, that is,* M *admits an essential projective transformation.* Constructing upon a long research history by many people (see for instance [24,26]), we dare to formulate more precisely:

Projective Lichnerowicz conjecture 1.1. *Let* (M, g) *be a compact pseudo-Riemannian manifold. Then, unless* (M, g) *is a finite quotient of the standard Riemannian sphere,* $\mathsf{Proj}(M, g)/\mathsf{Aff}(M, g)$ *is finite.*

— Same question when compactness is replaced by completeness (this does not contain the first case since general pseudo-Riemannian non-Riemannian compact manifolds may be non-complete).

1.0.6. Gromov's vague conjecture

It states that rigid geometric structures having a large automorphism group, are classifiable [10,16]! This needs a precise experimental (realistic) formulation for each geometric structure. Our question above is an optimistic formulation in the case of metric projective connections (those which are of Levi-Civita type). The historical case which corresponds to Riemannian conformal structures with a precise formulation, as in the projective case above with the sphere playing a central role, is generally attributed to Lichnerowicz, and was solved by J. Ferrand [14,29]. In the general conformal pseudo-Riemannian case, there are many "Einstein universes", i.e. conformally flat examples with an essential conformal group. A Lichnerowicz type conjecture would be that all pseudo-Riemannian manifolds with an essential conformal group are conformally flat. However, this was recently invalidated by C. Frances [15]. In the projective case, there is no natural candidate of a compact pseudo-Riemannian (non-Riemannian) manifold playing the role Einstein universes; it becomes a natural challenge to prove that indeed $\mathsf{Proj}/\mathsf{Aff}$ is always finite in this situation.

In the vein of this vague conjecture, it is surely interesting to study automorphism groups of non-metric projective connections...

1.1. Results

This very classical subject of differential geometry was specially investigated by the Italian and next the Soviet schools. All famous names: Beltrami, Dini, Fubini, Levi-Civita are still involved in results on projective equivalence of metrics [4,11,22]. As for the "Soviet" side, let us quote [2,3,30,24,26,31,32], and as names Solodovnikov who "introduced" the projective group problem, and last V. Matveev, who handled many remarkable cases of it.

1.1.1. Killing fields variant

Actually, it was $\mathsf{Proj}^0(M,g)$, the identity component of $\mathsf{Proj}(M,g)$, that got real interest in the literature. Its elements are those belonging to flows of projective Killing fields. There is a prompt formulation of the Lichnerowicz conjecture here: *if* $\mathsf{Proj}^0(M,g) \supsetneq \mathsf{Aff}^0(M,g)$*, then* (M,g) *is covered by the standard sphere* (assuming M compact).

This identity component variant was proved by V. Matveev in the case of Riemannian manifolds [24], and remains open in the case of higher signature.

Local actions, i.e. projective Killing fields with flows defined only locally, were also considered, see for instance [8].

However, situations with no Killing fields involved, say for example when Proj is a discrete group, do not seem to be studied. We think it is worthwhile to consider them because the discrete part may have dynamics stronger than the connected one, as in the case of a flat torus $\mathbb{T}^n$, but in fact for its affine group whose discrete part is the beautiful arithmetic (the best!) group $\mathsf{SL}_n(\mathbb{Z})$.

1.1.2. Non-dynamical variant

Without actions, one may think of having big $\mathcal{P}(M,g)$ as an index of symmetry, and one naturally may ask when this happens. For this, as in the projective case, consider $\mathcal{A}(M,g)$, the set of metrics affinely equivalent to g (i.e. having the same Levi-Civita connection). Here, we have the following wonderful theorem:

Theorem 1.2 *(Kiosak, Matveev, Mounoud). (See [26].) Let (M,g) be a compact pseudo-Riemannian manifold. If $\dim \mathcal{P}(M,g) \geq 3$, then $\mathcal{P}(M,g) = \mathcal{A}(M,g)$, unless (M,g) is covered by the standard Riemannian sphere. In particular $\mathsf{Proj}(M,g) = \mathsf{Aff}(M,g)$ in this case.*

1.1.3. Rank 1 case?

In view of this, it remains to consider the case $\dim \mathcal{P}(M,g) = 2$ (the dimension 1 case is trivial). Actually, this case occupies a large part in proofs of Lichnerowicz conjecture in the Riemannian as well as Kählerian cases [24,27,23]. (We think our approach here, besides it treats the discrete case, also simplifies these existing proofs.) We are not surprised of the resistance of this case, reminiscent to a rank 1 phenomenon, vs the higher rank case. Assuming $\dim \mathcal{P}(M,g) \geq 3$ hides a symmetry abundance hypothesis!

Anyway, in all our proofs, we will assume $\dim \mathcal{P}(M,g) = 2$.

1.1.4. Aim

Our first objective here is to provide a proof of the above conjecture in case of compact Riemannian manifolds.

Theorem 1.3. *Let (M,g) be a compact Riemannian manifold. If M is not a Riemannian finite quotient of a standard sphere, and $\mathsf{Proj}(M,g) \supsetneq \mathsf{Aff}(M,g)$, then $\mathsf{Proj}(M,g)$ is a finite extension of $\mathsf{Aff}(M,g)$.*

More precisely, $\mathsf{Aff}(M, g) = \mathsf{Iso}(M, g)$*, and a subgroup* $\mathsf{Iso}'(M, g)$ *of index* ≤ 2*, is normal in* $\mathsf{Proj}(M, g)$*, and the quotient group* $\mathsf{Proj}(M, g)/\mathsf{Iso}'(M, g)$ *is either cyclic of order* $\leq \dim M$*, or dihedral of order* $\leq 2 \dim M$*.*

Examples. In order to illustrate the non-linear character of projective equivalence, let us recall the Dini's classical result: two metrics on a surface are projectively equivalent, iff, at a generic point, they have the following forms in some coordinate system:

$$g = (X(x) - Y(y)(dx^2 + dy^2), \quad \bar{g} = (\frac{1}{Y(y)} - \frac{1}{X(x)})(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)})$$

It follows that for the metric $g = (a(x) - \frac{1}{a(y)})(\sqrt{a(x)}dx^2 + \frac{1}{\sqrt{a(y)}}dy^2)$, the involution $(x, y) \to (y, x)$ is projective. This example given by V. Matveev [24], shows that non-affine projective transformations may exist (outside the case of spheres) but are not in the identity component, because of his result. Theorem 1.3 says that the "discrete projective transformation group" is always finite, but we do not know examples more complicated than the last involution.

Remark 1.4. Some of quoted results are also true in the complete non-compact case, but we consider here compact manifolds, only.

1.2. Kähler version

Let (M, g) be a Hermitian manifold. Let $V \subset M$ be a geodesic surface which is at the same time a holomorphic curve. If g is Kähler, then any (real) curve c in V satisfies that its complexified tangent direction is parallel; it is therefore called *h-planar*. It is very special that such V exists, but h-planer curves always exist. Two Kähler metrics are h-projectively equivalent if they share the same h-planer curves. A holomorphic diffeomorphism f is *h-projective* if f_*g is h-projectively equivalent to g. Their group is denoted $\mathsf{Proj}^{\mathsf{Hol}}(M, g)$. (There exist equivalent terminologies for h-projective, as holomorphic-projective or c-projective.)

This holomorphic side of the projective transformation problem was classically investigated by the Japanese school [17,19,36,37].

Finally, V. Matveev and S. Rosemann generalize all known Riemannian results (on the identity component) to the Kähler case [27]. That is, if $\mathsf{Proj}^{\mathsf{Hol}}(M, g)$ contains a one parameter group of non-affine transformations, then, up to a scaling, (M, g) is holomorphically isometric to $\mathbb{P}^d(\mathbb{C})$ endowed with its standard metric (where $d = \dim_{\mathbb{C}} M$).

Like in the Riemannian case, we are able here to handle the discrete part of $\mathsf{Proj}^{\mathsf{Hol}}$:

Theorem *(Rigidity of h-projective transformation groups). Let* (M^d, g) *be a compact Kähler manifold. If* $\mathsf{Aff}^{\mathsf{Hol}}(M, g)$ *has not finite index in* $\mathsf{Proj}^{\mathsf{Hol}}(M, g)$*, then, up to a scaling,* (M, g) *is holomorphically isometric to* $\mathbb{P}^d(\mathbb{C})$ *endowed with its Fubini–Study metric* g_{SF}*.*

About the proof. V. Matveev and S. Rosemann proved their Kähler identity component version by showing that all the differential geometric tools developed in the (usual Riemannian) projective case, may be adapted to the h-projective one, and enjoy all the needed properties, see [27] for details. Thanks to this, we will not give details of proof in the h-projective case, because it goes exactly as in the (usual) projective one. Instead, we investigate the following new aspects in the Kähler case, in particular in order to elucidate another use of the word "projective"!

Projective vs projective. Recall that a complex manifold M is called projective if it is holomorphic to a (closed regular) complex submanifold of some projective space $\mathbb{P}^N(\mathbb{C})$. Endowed with the restriction of g_{SF}, $(M, g_{SF|M})$ is a Kähler manifold. However, only few (other) Kähler metrics (M, g) admit holomorphic isometric embedding in a projective space (but, of course real analytic isometric embedding exists, by Nash Theorem). The dramatic example is that of an elliptic curve, that is a 2-torus with a complex structure. It admits a large space of holomorphic embedding in projective spaces of different dimensions, but the induced metric on them can never be flat! This is one case of a "Theorema Egregium" due to Calabi [9] which says that holomorphic isometric immersions in space forms of constant holomorphic sectional curvature, are absolutely rigid (see §8).

Theorem 1.5. *Let (M^d, g) be a complex submanifold of a projective space $\mathbb{P}^N(\mathbb{C})$ endowed with the induced metric (from the normalized Fubini–Study). Then the group $\mathsf{Proj}^{\mathsf{Hol}}(M, g)$ of holomorphic projective transformations is a finite extension of $\mathsf{Iso}^{\mathsf{Hol}}(M, g)$, its group of holomorphic isometries, unless (M, g) is holomorphically homothetic to $\mathbb{P}^d(\mathbb{C})$. More precisely, up to composition with $\mathsf{SU}(1+N)$, M is the image of a Veronese map: $v_k : \mathbb{P}^d(\mathbb{C}) \to \mathbb{P}^N(\mathbb{C})$ (which expands the metric by a factor k).*

Remark 1.6. There are submanifolds $M \subset \mathbb{P}^N(\mathbb{C})$ with a big "projective" group, say such that $G_M = \{g \in \mathsf{GL}_{N+1}(\mathbb{C}), g.M = M\}$ is non-compact and acts transitively on M. So, G_M does not act projectively with respect to the induced metric, unless M is a Veronese submanifold. The G_M-action preserves another kind of geometric structures. It is however remarkable that all the automorphism group of any Kähler manifold preserves a huge class of minimal submanifolds (in the sense of Riemannian geometry), namely, complex submanifolds!

1.3. Towards the general (indefinite) pseudo-Riemannian case

It was proved in [26] that the quotient space $\mathsf{Proj}^0(M, g)/\mathsf{Aff}^0(M, g)$ has always dimension ≤ 1. We have the following generalization to full groups.

Theorem 1.7. *Let (M, g) be a compact pseudo-Riemannian manifold having an essential projective group, that is, $\mathsf{Proj}(M, g)/\mathsf{Aff}(M, g)$ is infinite. Then, up to finite index:*

1) $\mathsf{Aff}(M, g) = \mathsf{Iso}(M, g)$ and it is a normal subgroup of $\mathsf{Proj}(M, g)$.

2) $\mathsf{Proj}(M,g)/\mathsf{Iso}(M,g)$ *is isomorphic to a subgroup of* $\mathbb{R}$*. More precisely, there is a representation* $\mathsf{Proj}(M,g) \to \mathsf{SL}_2(\mathbb{R})$ *whose kernel is* $\mathsf{Aff}(M,g)$ *and range contained in a non-elliptic 1-parameter group.*

1.3.1. Organization

We restrict ourselves here to compact manifolds, and from §4 to the case of metrics of projective mobility $\dim \mathcal{P}(M,g) = 2$.

Our proofs are mostly algebraic, somewhere dynamical but rarely geometrical!

1.3.2. Added to last version

Around one year and half after the publication of our present work (in ArXiv), V. Matveev [25] improved estimate in Theorem 1.3 of the (finite) index of $\mathsf{Aff}(M,g)$ in $\mathsf{Proj}(M,g)$. He shows that this index is ≤ 2 (of course when (M,g) is not homothetic to a quotient of the standard sphere). Matveev's proof consists in pursuing analysis of our elliptic case §5.0.2 (otherwise, he bases on our results here, especially in the hyperbolic case).

2. Actions, general considerations

M is here a compact smooth manifold.

2.0.1. Let $\mathcal{E}$ be the space of $(1,1)$-tensors T, i.e. sections of the linear bundle $End(TM) \to M$: for any x, T_x is a linear map $T_xM \to T_xM$. The space $\mathcal{E}$ has a natural structure of algebra with unit element I the identity of TM (over $\mathbb{R}$ as well as over $C^\infty(M)$ or $C^k(M)$).

$\mathsf{Diff}(M)$ acts naturally on $\mathcal{E}$ by $(f,T) \to \rho^E(f)T = f_*T$ defined naturally by $(f_*P)_x = D_{f^{-1}x}fT_{f^{-1}x}D_xf^{-1}$.

2.0.2. Let $\mathcal{G}$ be the space of pseudo-Riemannian metrics on M. Then, $\mathsf{Diff}(M)$ acts on $\mathcal{G}$ by taking direct image, $(f,g) \to \rho^G(f)g = f_*g$ defined by $(f_*g)_x(u,v) = g_{f^{-1}(x)}((D_xf)^{-1}u, (D_xf)^{-1}v)$.

2.0.3. Notation

We will sometimes use the usual notations f_*T and f_*g for $\rho^E(f)T$ and $\rho^G(f)g$, respectively.

2.0.4. Transfer

Given a metric g_0 on M, any other metric g can by written $g(.,.) = g_0(T.,.)$, where the transfer tensor $T = T(g,g_0)$ is a g_0-symmetric $(1,1)$-tensor (i.e. T_x is a symmetric endomorphism of $(T_xM, {g_0}_x)$).

In fact, a metric g defines a bundle isomorphism $TM \to T^*M$, and thus $T(g,g_0) = g_0^{-1}g$.

In other words, we have a map $C_{g_0} : g \in \mathcal{G} \to T(g, g_0) = g_0^{-1} g \in \mathcal{E}$. In particular $C_{g_0}(g_0) = I$ (the identity of TM).

2.0.5. Transfer action

The transfer of the natural Diff-action ρ^E on $\mathcal{G}$ to $\mathcal{E}$ by means of C_{g_0}, is by definition

$$(f, T) \to \rho^{GE}(f)(T) = C_{g_0}(\rho^G(f)(C_{g_0}^{-1} T))$$

It equals:

$${g_0}^{-1} \rho^G(f)(g_0 T) = g_0^{-1}(\rho^G(f) g_0)(\rho^E(f) T) = S_f \rho^E(f) T$$

where the g_0-strength of f is $S_f = g_0^{-1}(\rho^G(f) g_0)$.

2.0.6. A preserved functional

The following "norm-like" functional $Q(T) = \int_M \sqrt{|\mathsf{det} T|} dv_{g_0}$ is preserved by ρ^{GE}. Indeed,

$$Q(\rho^{GE}(f)T) = \int\limits_M \sqrt{|(\mathsf{det} S_f)\mathsf{det}(f_* T)|} dv_{g_0} = \int\limits_M \sqrt{|\mathsf{det} T|}(f^{-1}(x))\mathsf{Jac}_x f^{-1} dv_{g_0},$$

and this equals $Q(T)$.

2.0.7. Consider now the partially defined transform $\mathcal{F} : L \in \mathcal{E} \to T = \frac{L^{-1}}{\mathsf{det} L} \in \mathcal{E}$. Its inverse map is given by $\mathcal{F}^{-1}(T) = (\mathsf{det} T)^{\frac{1}{1+d}} T^{-1}$ ($d = \dim M$).

It is remarkable that $\mathcal{F}$ commutes with the Diff-action ρ^E on $\mathcal{E}$. The finite dimensional version of this for a linear space E is that $u \to \mathsf{End}(E) \to \frac{u^{-1}}{\mathsf{det} u} \in \mathsf{End}(E)$ commutes with the $\mathsf{GL}(E)$ action given by $(A, u) \in \mathsf{GL}(E) \times \mathsf{End}(E) \to AuA^{-1} \in \mathsf{End}(E)$.

2.0.8. Action in the L-representation

Consider now the map

$$g \in \mathcal{G} \to L = \mathcal{F}^{-1}(C_{g_0}(g)) \in \mathcal{E}$$

In other words, to a metric g, we associate the $(1, 1)$-tensor L such that $g(.,.) = \frac{1}{\mathsf{det} L} g_0(L^{-1}.,.)$.

The corresponding action ρ on $\mathcal{E}$ is given by:

$$\rho(f)L = (\rho^E(f)L)K_f$$

where K_f, the g_0-strength of f in the L-representation, is the $\mathcal{F}^{-1}$-transform of S_f, that is, K_f is defined by $\rho^G(f) g_0(.,.) = \frac{1}{\mathsf{det} K_f} g_0(K_f^{-1}.,.)$.

Corresponding to Q, ρ preserves the partially defined functional: $L \to N(L) = \int_M \frac{1}{\mathsf{det} L^{(1+d)/2}} dv_{g_0}$.

2.0.9. The chain rule for strength

$$K_{f^n} = (f_*^{n-1}K_f)(f_*^{n-2}K_f)\dots(f_*K_f)K_f$$

(of course $(f^k)_* = (f_*)^k$).

2.0.10. Summarizing:

Fact 2.1. *Let g_0 be a fixed metric on M. To any metric g, let L be the $(1,1)$ tensor defined by $g(.,.) = \frac{1}{\mathsf{det}L}g_0(L^{-1}.,.)$. The Diff-action on $(1,1)$ tensors deduced from the usual action on metrics by means of this map $g \to L$ is given by*

$$(f, L) \in \mathsf{Diff}(M) \times \mathcal{E} \to \rho(f)L = (f_*L)K_f$$

Here K_f is the L-tensor associated to f_*g_0, i.e. $f_*g_0 = \frac{1}{\mathsf{det}K_f}g_0(K_f^{-1}.,.)$, and f_*L denotes the usual action on $\mathcal{E}$.

- *f is an isometry of $g_0 \iff K_f = I$.*
- *f is a g_0-similarity (that is $f_*g_0 = bg_0$ for some constant b) $\iff K_f = bI$ for some b.*
- *ρ preserves the function $L \to N(L) = \int_M \frac{1}{\mathsf{det}L^{(1+d)/2}}dv_{g_0}$.*

3. Linearization, representation of $\mathsf{Proj}(M, g)$ in $\mathcal{L}(M, g)$

(We will henceforth mostly deal with only one metric and so we will denote it g instead of g_0.)

3.0.1. The space $\mathcal{L}(M, g)$

Recall that $\mathcal{P}(M, g)$ denotes the class of metrics projectively equivalent to g.

Let $\mathcal{L}^*(M, g)$ be the image of $\mathcal{P}(M, g)$ under the correspondence of Fact 2.1, and $\mathcal{L}(M, g)$ its linear hull:

$$\mathcal{L}(M,g) = \{L = \Sigma_i a_i L_i, a_i \in \mathbb{R}, \text{ such that } \frac{1}{\mathsf{det}L_i}g(L_i^{-1}.,.) \text{ is projectively equivalent to } g\}$$

Let us call $\mathcal{L}$-tensors the elements of this space.

3.0.2. Linearization

Theorem 3.1. *(See [6,30].) $L \in \mathcal{L}(M, g)$ iff L satisfies the linear equation:*

$$g((\nabla_u L)v, w) = \frac{1}{2}g(v, u)d\mathsf{trace}(L)(w) + \frac{1}{2}g(w, u)d\mathsf{trace}(L)(v)$$

where ∇ is the Levi-Civita connection of g.

Furthermore:

- $\mathcal{L}^*(M,g)$ *is an open subset of* $\mathcal{L}(M,g)$*: an element* $L \in \mathcal{L}(M,g)$ *belongs to* $\mathcal{L}^*(M,g)$ *iff it is an isomorphism of* TM*.*
- $\mathcal{L}(M,g)$ *has finite dimension (bounded by that corresponding to the projective space of same dimension).*
- $L \in \mathcal{L}^*(M,g)$ *is parallel iff the corresponding metric* $\frac{1}{\mathsf{det} L} g(L^{-1}.,.)$ *is affinely equivalent to* g*, iff* L *has constant eigenvalues.*

3.0.3. Linear representation of $\mathsf{Proj}(M,g)$

Fact 3.2. *We have a finite dimensional representation,*

$$f \in \mathsf{Proj}(M,g) \to \rho(f) \in \mathsf{GL}(\mathcal{L}(M,g))$$

where $\rho(f)(L) = f_*(L).K_f$*.*

- ρ *preserves the norm-like function* $N(L) = \int_M \frac{1}{\mathsf{det} L^{(1+d)/2}} dv_g$*.*
- *Let* $p : \mathsf{GL}(\mathcal{L}(M,g)) \to \mathsf{PGL}(\mathcal{L}(M,g))$ *be the canonical projection, then* p *is injective on* $\rho(\mathsf{Proj}(M,g))$*, or has at most a kernel* $\cong \mathbb{Z}/2\mathbb{Z}$*.*
- *Let* $\mathcal{D}$ *be the subset of degenerate tensors in* $\mathcal{L}(M,g)$*:*

  $$\mathcal{D} = \{L \in \mathcal{L}(M,g),\ L \textit{ not an isomorphism of } TM\}$$

  Then $\mathcal{D}$ *is a closed cone invariant under* ρ*.*

Proof.

- The first point is imported from Fact 2.1.
- For the second one, let aA and A be in $\mathsf{GL}(\mathcal{L}(M,g))$ such that both preserve N, then $N(aA(L)) = N(A(L)) = N(L)$, for any L. But $N(aL) = |a|^s N(L)$ with $s = -d(d+1)/2$, and hence $a = \pm 1$.
- To prove ρ-invariance of $\mathcal{D}$, observe that $L \in \mathcal{D}$ iff for some $x \in M$, $\mathsf{det} L(x) = 0$. But $\rho(f)L = f_* L K_f$, and hence $\mathsf{det}(\rho(f)L)(f(x)) = \mathsf{det} L(x) \mathsf{det} K_f(x) = 0$. $\square$

Remark 3.3. Actually $\mathcal{D}$ coincides essentially with the ∞-level of N.

4. The case $\dim \mathcal{L}(M,g) = 2$, a homography

4.0.1. Hypothesis

Henceforth, we will assume that $\dim \mathcal{L}(M,g) = 2$.

Fix f that is not homothetic, i.e. $K = K_f$ is not a multiple of I. Hence $\mathcal{L}(M,g)$ is spanned by K and I.

4.1. The degenerate set $\mathcal{D}$

Fact 4.1. *The subset of degenerate tensors (defined above) satisfies:*

$$\mathcal{D} = \{a(K - tI), a \in \mathbb{R}, \\ \text{and } t \text{ is a real spectral value of } K : \mathsf{det}(K(x) - tI) = 0 \text{ for some } x\}$$

In particular I and $K_f \notin \mathcal{D}$.

If the spectrum is real and described by d continuous eigenfunctions $x \to \lambda_1(x) \leq \ldots \leq \lambda_d(x)$ ($d = \dim M$), then $\mathcal{D} = \cup_{i=1}^{i=d}(C_i \cup -C_i)$, where

$$C_i = \{a(K - tI), a \in \mathbb{R}^+, \text{ and } \inf \lambda_i \leq t \leq \sup \lambda_i\}$$

Each C_i is a proper convex cone (sector).

Finally, unifying intersecting sectors, we get a minimal union: $\mathcal{D} = \cup_{i=1}^{i=k}(D_i \cup -D_i)$, where the D_i are disjoint sectors.

Proof. I (as well as K) does not belong to $\mathcal{D}$ and hence any element of this set has the form $a(K - tI)$. This belongs to $\mathcal{D}$ iff $\mathsf{det}(K(x) - tI) = 0$, for some x, that is $t \in \cup_i(Image(\lambda_i))$, and the cones C_i follow. □

4.2. Action by homography

4.2.1. Equation

By the 2-dimensional assumption, there exist α, β such that:

$$\rho(f)(K) = (f_*K)K = \alpha K + \beta I$$

Equivalently,

$$f_*K = \alpha I + \beta K^{-1}$$

Say somehow formally, $f_*K = \frac{\alpha K + \beta I}{K}$.

Since $f_*I = I$, in the basis $\{K, I\}$, $\rho(f) : L \to f_*(L).K_f$ has a matrix

$$B = B_f = \begin{pmatrix} \alpha & 1 \\ \beta & 0 \end{pmatrix}$$

4.2.2. The group $\mathsf{GL}_2(\mathbb{R})$ (more faithfully $\mathsf{PGL}_2(\mathbb{R})$) acts on the (projective) circle $\mathbb{S}^1 = \bar{\mathbb{R}} = \mathbb{R} \cup \infty$, by means of the law

$$z \to A \centerdot z = \frac{az + b}{cz + d} \text{ for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{GL}_2(\mathbb{R})$$

In fact, we can also let $\mathsf{GL}_2(\mathbb{R})$ act on the space of $(1,1)$-tensors by the same formula: $(A \centerdot X)(x) = (aX(x) + bI)(cX(x) + dI)^{-1}$. In other words, the action is fiberwise, and when a fiber $\mathsf{End}(T_xM)$ is identified to $\mathsf{Mat}_n(\mathbb{R})$, then $A \centerdot X = \frac{aX+b}{cX+d}$.

Now, the previous equation $f_*K = \frac{\alpha K+\beta I}{K}$ can be interpreted by that the linear f_*-action on K equals the homographic action $A\centerdot K$ where $A = \begin{pmatrix} \alpha & \beta \\ 1 & 0 \end{pmatrix}$ is the transpose of B.

4.2.3. Iteration

We have:

$$f_*{}^n K = A^n \centerdot K, \text{ for any } n \in \mathbb{Z} \tag{4.1}$$

This can be proved in a formal way. Let C be an endomorphism on an abstract algebra $\{1, x, x^{-1}, \ldots\}$, such that $C(x) = \alpha + \beta x^{-1}$ (with $C(1) = 1$ and $C(x^{-1}) = C(x)^{-1}$). Then, $C^n(x) = A^n \centerdot x$, where $A = \begin{pmatrix} \alpha & \beta \\ 1 & 0 \end{pmatrix}$.

4.2.4. Significance for eigen-functions

Let $x \in M$ and $y = f(x)$ and denote $T = D_xf : T_xM \to T_yM$.

The relation $f_*K = \alpha I + \beta K^{-1}$ means that $T^{-1}K_yT = \alpha + \beta K_x^{-1}$. This implies in particular that T maps an invariant subspace of (T_xM, K_x) to an invariant subspace of (T_yM, K_y). If $E_\lambda(x) \subset T_xM$ is the (generalized) K_x-eigenspace associated to λ, then T maps it to $E_{A^{-1}\centerdot\lambda}(y) \subset T_yM$.

Let $x \to Sp(x) \subset \mathbb{C}$ be the multivalued spectrum function of K, that is, $Sp(x) \subset \mathbb{C}$ is the set of eigenvalues of K_x. Then the image $A \centerdot Sp(f(x))$ (of the subset $Sp(f(x))$ under the homography $A^{-1}\centerdot$) equals $Sp(x)$, and so

$$Sp(f(x)) = A^{-1} \centerdot Sp(x)$$

Also, if $\lambda : M \to \mathbb{C}$ is a continuous K-eigen-function, that is $\lambda(x) \in Sp(x)$ for any $x \in M$ and λ is continuous, then

$$x \to \lambda'(x) = A^{-1} \centerdot \lambda(f^{-1}(x))$$

is another continuous K-eigen-function.

4.3. Classification of elements of $\mathsf{SL}_2(\mathbb{R})$

Recall that non-trivial elements A of $\mathsf{SL}_2(\mathbb{R})$ split into three classes:

(1) Elliptic: A is conjugate in $\mathsf{SL}_2(\mathbb{R})$ to a rotation, i.e. an element of $\mathsf{SO}(2)$. Its homographic action on $\bar{\mathbb{R}} = \mathbb{R} \cup \infty$, as well as on $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$ is conjugate to a rotation.

(2) Parabolic: A is unipotent, i.e. $(A-1)$ is nilpotent. Its homographic action on $\bar{\mathbb{R}}$ as well as on $\bar{\mathbb{C}}$ is conjugate to a translation. It has a unique fixed point $F_A \in \bar{\mathbb{R}}$. Up to conjugacy $F_A = \infty$, and $A \centerdot z \to z + a$, where $a \in \mathbb{R}$.
It follows that if $C \subset \mathbb{R}$ is a bounded $A\centerdot$-invariant set, then $C = \{F_A\}$ (and necessarily $F_A \neq \infty$).
(3) Hyperbolic: A has two fixed points F_A^- and F_A^+. Up to conjugacy, $F_A^- = 0$, $F_A^+ = \infty$, and $f(x) = ax$, with $0 < a < 1$.

Now, if $A \in \mathsf{GL}_2^+(\mathbb{R})$, its homographic action coincides with that of $\frac{A}{\mathsf{det}A}$, and the same classification applies.

5. Riemannian metrics: non-hyperbolic cases

(M, g) is here a compact Riemannian manifold, and f as in the previous section a chosen element such that $\{K_f, I\}$ generate $\mathcal{L}(M, g)$, and f is not affine.

In this Riemannian setting, all elements of $\mathcal{L}(M, g)$ are diagonalizable (since self-adjoint).

Let $G = \rho(\mathsf{Proj}(M, g))$, and $G^+ = G \cap \mathsf{GL}_2^+(\mathbb{R})$.

Let $A = \rho(h) \in G^+$ be non-trivial, then K_h is not collinear to I. Indeed, assume by contradiction that $K_h = aI$, then recall that $h_* g(.,.) = g(S_h.,.)$ and $S_h = \frac{K_h^{-1}}{\mathsf{det}K_h}$. It follows that S_h has the form $S_h = bI$, but equality of volumes of (M, g) and $(M, h_* g)$ implies $b = 1$ and hence also $K_h = I$ (that is $h \in \mathsf{Iso}(M, g)$). Thus, $\rho(h)I = I$.

On the other hand, by Fact 4.1, $\rho(h)$ preserves a finite set of lines, all different from $\mathbb{R}I$. Let l_1^+ and l_2^+ be the two nearest half lines to $\mathbb{R}^+I$. If $\rho(h) \neq Id$, then necessarily $\rho(h)l_1^+ = l_2^+$, but this implies $\rho(h)$ is a reflection which contradicts our hypothesis $\mathsf{det}\rho(h) > 0$.

5.0.1. G^+ can not contain parabolic elements

Assume by contradiction that $\rho(h)$ is parabolic with fixed point F_h. Then, F_h is the unique real spectral value of K_h (because there is no other bounded set of $\mathbb{R}$ invariant under the associated homography), and thus K_h is proportional to I (since it is diagonalizable), which we have just proved to be impossible.

5.0.2. Case where all elements of G^+ are elliptic

Recall that we have a union of $k \leq \dim M$ disjoints sectors D_i, such that $\mathcal{D} = \cup_{i=1}^{i=k}(D_i \cup -D_i)$ is G-invariant (Fact 4.1). If $k > 1$, then the stabilizer of $\mathcal{D}$ in $\mathsf{SL}_2(\mathbb{R})$ is compact, and we can assume G is a subgroup of $\mathsf{O}(2)$.

G^+ $(= G \cap \mathsf{GL}_2(\mathbb{R}))$ is a finite subgroup of $\mathsf{SO}(2)$ and hence cyclic. Now, if a rotation preserves a set of k-disjoint sectors, then it has order $\leq 2k$. We will observe in our case that this order is in fact $\leq k$.

However, in our case, we know that any $\rho(h) \in G^+$ is $\neq -Id$ (since otherwise $-I = \rho(h)I = h_*(I)K_h = K_h$, and hence $K_h = -I$ which we have already excluded). Say, in other words we can see the rotation acting on the projective space rather than the circle and get exactly k-sectors and deduce that actually, G^+ has order $\leq k$.

As for G (if strictly bigger than G^+), it is dihedral of order $\leq 2k$.

Finally, in the case $k = 1$, that is $\mathcal{D} = D_1 \cup -D_1$, its stabilizer in $\mathsf{SL}_2(\mathbb{R})$ contains $-Id$ together with a one parameter hyperbolic group. So, if we assume all elements of G^+ elliptic, we get $G^+ = \{1\}$. In this case, G itself reduces to a single reflection (if non-trivial).

5.0.3. About $\mathsf{Iso}(M, g)$

Observe first that if $h \in \mathsf{Aff}(M, g)$, then necessarily K_h is proportional to I since otherwise K_h will be a combination with constant coefficients of I and K_h, and thus has constant eigenvalues, and therefore $f \in \mathsf{Aff}(M, g)$ (see 3.1) contradicting our hypothesis. As observed previously $K_h = I$, that is $h \in \mathsf{Iso}(M, g)$ and so $\mathsf{Aff}(M, g) = \mathsf{Iso}(M, g)$.

On the other hand, if $h \in \mathsf{Iso}(M, g)$, and $\rho(h) \in G^+$, then $\rho(h) = Id$.

In general, if $\rho(h) \neq Id$, then it is a reflection since $\rho(h^2) = Id$.

Let $\mathsf{Iso}^{(2)}(M, g)$ be the normal subgroup of $\mathsf{Iso}(M, g)$ generated by squares $h^2, h \in \mathsf{Iso}(M, g)$. Then:

- either $\ker \rho = \mathsf{Iso}(M, g)$,
- or $\ker \rho = \mathsf{Iso}^{(2)}(M, g)$, and this has index 2 in $\mathsf{Iso}(M, g)$,
- in all cases, $\mathsf{Iso}(M, g)$ or $\mathsf{Iso}^{(2)}(M, g)$ is normal in $\mathsf{Proj}(M, g)$, and the corresponding quotient is cyclic of order $\leq \dim M$, or dihedral of order $\leq 2 \dim M$.

6. Riemannian metrics, hyperbolic case

In the present section, (M, g) is a compact Riemannian manifold with $\dim \mathcal{L}(M, g) = 2$ and $f \in \mathsf{Proj}(M, g)$ is such that $\rho(f)$ is hyperbolic.

The final goal (of the section) is to prove that (M, g) is projectively flat. This will be done by proving the vanishing of its Weyl projective tensor W (recalled below). For this, one iterates a vector $z = W(u, v)w$ by the Df-dynamics to get a sequence $z_n = Df^n z = W(Df^n u, Df^n v)Df^n w$, and shows that it has two different growth rates (when $n \to \pm\infty$) unless $z = 0$.

6.1. Size of the spectrum

Fact 6.1. *The homography $A.$ defined by $\rho(f)$ has two real finite fixed points $\lambda_- < \lambda_+$.*

$K = K_f$ has exactly one non-constant eigen-function λ. It has multiplicity 1 (at generic points), range the interval $[\lambda_-, \lambda_+]$, and satisfies the equivariance: $\lambda(f(x)) = A^{-1} . \lambda(x)$.

The full spectrum of K may be $\{\lambda_-, \lambda, \lambda_+\}$, $\{\lambda_-, \lambda\}$ or $\{\lambda, \lambda_+\}$. We denote the multiplicities of λ_- and λ_+ by d_- and d_+ respectively, and hence $\dim M = 1 + d_1 + d_+$.

Proof. Let $\mu_1(x) \leq \ldots \leq \mu_d(x)$ be the eigenfunctions (with multiplicity) of $K(x)$. From 4.2.4, the map $\mu_i' : x \to A^{-1} \centerdot \mu_i(f^{-1}(x))$ is another eigenfunction and hence equals some μ_j. Taking a power of f, we can assume $\mu_i' = \mu_i$, that is $\mu_i(f^{-1}(x)) = A \centerdot \mu_i(x)$. In other words, μ_i is an equivariant map between the two systems (M, f) and $(\mathbb{R}, A^{-1}\centerdot)$. Thus, *Image*$(\mu_i)$ is a bounded $A^{-1}\centerdot$-invariant interval. Hence $\lambda_\pm$ belong to $\mathbb{R}$ (rather than $\bar{\mathbb{R}}$) and the image of μ_i can be $\{\lambda_-\}$, $\{\lambda_+\}$ or $[\lambda_-, \lambda_+]$. The fact that only one μ_i has range $[\lambda_-, \lambda_+]$ follows from the following nice fact. $\square$

Theorem 6.2. *(See [28].) Let (M, g) be a complete Riemannian manifold and $L \in \mathcal{L}(M, g_0)$. Then two eigen-functions $\mu_i \leq \mu_j$ satisfy $\sup \mu_i \leq \inf \mu_j$ (that is not only $\mu_i(x) \leq \mu_j(x)$, but even $\mu_i(x) \leq \mu_j(y)$ for any $x, y \in M$).*

6.2. Dynamics of f

Define the singular sets $\mathcal{S}_\pm = \{x \in M, \lambda(x) = \lambda_\pm\}$.

6.2.1. Lyapunov splitting

On $M \setminus (\mathcal{S}_- \cup \mathcal{S}_+)$, corresponding to the eigenspace decomposition of $K = K_f$, we have a regular and orthogonal splitting $TM = E_- \oplus E_+ \oplus E_\lambda$.

Due to the relation, $f_* K = \alpha I + \beta K^{-1}$, f preserves this splitting.

Remark 6.3. Even in the linear situation of a matrix $A \in \mathsf{GL}_d(\mathbb{R})$, it is rare that A^*A and its conjugate $A^{-1}(A^*A)A$ have the same eigenspace decomposition!

6.2.2. Distortion

Recall the definition of the L-strength $f_* g(.,.) = \frac{1}{\mathsf{det} K} g(K^{-1}.,.)$ vs the ordinary strength $S = \frac{K^{-1}}{\mathsf{det} K}$.

If $y = f(x)$ and $u \in T_y M$ belongs to a $\mu = \mu^S(y)$-eigenspace of S, then $g_x(D_y f^{-1} u, D_y f^{-1} u) = \mu g_y(u, u)$. In our case, $D_x f$ sends S-eigenspaces at x to S-eigenspaces at y, by applying a similarity of ratio $\frac{1}{\sqrt{\mu^S(y)}}$.

In order to compute this by means of K-eigenvalues, observe that

$$\mathsf{det} K(x) = \lambda_-^{d_-} \lambda_+^{d_+} \lambda(x)$$

(where d_-, d_+ are the respective dimension of E_- and E_+).

Thus, for any x, $D_x f$ maps similarly $E_-(x)$ to $E_-(f(x))$ with similarity ratio $\zeta_-(x)$ such that

$$\zeta_-^2(x) = (\mathsf{det} K(f(x))\lambda_- = (\lambda_-^{d_-} \lambda_+^{d_+} \lambda(f(x))\lambda_-$$

As for $D_x f : E_+(x) \to E_+(f(x))$ and $D_x f : E_\lambda(x) \to E_\lambda(f(x))$, their respective distortions are:

$$\zeta_+^2(x) = (\lambda_-^{d_-} \lambda_+^{d_+} \lambda(f(x)))\lambda_+ \ \text{ and } \ \zeta_\lambda^2(x) = (\lambda_-^{d_-} \lambda_+^{d_+} \lambda(f(x)))\lambda(f(x))$$

6.2.3. Data for f^{-1}

Let $\lambda_1^*, \lambda_-^*, \lambda_+^*, \zeta_\lambda^*, \ldots$ be the analogous quantities corresponding to f^{-1}. Observe that f^{-1} preserves the same Lyapunov splitting and thus $\zeta_-^*(x)\zeta_-(f^{-1}(x)) = 1$. It follows that $\lambda_1^*(x) = \frac{1}{\lambda_1(f^{-1}(x))}$, $\lambda_-^* = \frac{1}{\lambda_+}$, and $\lambda_+^* = \frac{1}{\lambda_-}$.

6.2.4. Estimation of the Jacobian

From above we infer that:

$$(\mathsf{Jac}\ f_x)^2 = (\mathsf{det} D_x f)^2 = (\zeta_-(x)\zeta_\lambda(x)\zeta_+(x))^2 = (\lambda_-^{d_-}\lambda_+^{d_+}\lambda(f(x)))^{1+d}$$

Now, in general, $\mathsf{Jac}\ f_x^n = \mathsf{Jac}\ f_{f^{n-1}x} \ldots \mathsf{Jac}\ f_x$, and hence

$$(\mathsf{Jac}\ f_x^n)^2 = ((\lambda_-^{d_-}\lambda_+^{d_+})^n(\lambda(f^n(x)) \ldots \lambda(f(x))))^{1+d}$$

Fact 6.4. *Assume that $A^{-1}\centerdot$ is decreasing on $[\lambda_-, \lambda_+]$, that is, λ_+ is repulsing and λ_- is attracting, equivalently, λ_1 is decreasing along f-orbits: $\lambda_1(f(x)) \leq \lambda_1(x)$. Then, on compact sets $M \setminus \mathcal{S}_+$, $(\mathsf{Jac}\ f_x^n)^2$ is uniformly equivalent to $(\lambda_-^{d_-+1}\lambda_+^{d_+})^{n(1+d)}$, when $n \to +\infty$ (recall that $\mathcal{S}_+ = \{x \in M/\lambda_1(x) = \lambda_+\}$).*

The proof is based on the relation $\lambda(f^k(x)) = (A^{-1})^k \centerdot \lambda(x)$ and the next lemma.

Lemma 6.5. *Let C be a hyperbolic element of $\mathsf{SL}_2(\mathbb{R})$ with fixed points $\lambda_- < \lambda_+$, with λ_- attracting. The sequence*

$$\frac{(C^n \centerdot z)(C^{n-1} \centerdot z) \ldots (C \centerdot z)}{\lambda_-^n}$$

converges simply in $[\lambda_-\lambda_+[$ to a continuous function. The convergence is uniform in any compact subset of $[\lambda_-, \lambda_+[$.

Proof. In a small neighbourhood of λ_-, the $C\centerdot$-action is equivalent to a linear contraction fixing λ_-, $h : z \to \alpha(z - \lambda_-) + \lambda_-$, with $0 < \alpha < 1$. This equivalence is valid also on any compact interval $[\lambda_-, \lambda_+ - \epsilon]$. Thus $h^n z = \alpha^n(z - \lambda_-) + \lambda_-$. The above product is $(c\alpha^n + 1)(c\alpha^{n-1} + 1) \ldots (c\alpha + 1)(c + 1)$, where $c = \frac{z-\lambda_-}{\lambda_-}$. This product is convergent since it can be bounded by $\Pi_{i=0}^{i=n}(e^{|c|\alpha^i}) \leq e^{|c|(\Sigma\alpha^i)}$. $\square$

Corollary 6.6. *Keep the assumption $A^{-1}\centerdot$ decreasing. Then $(\lambda_-^{d_-}\lambda_+^{d_+})\lambda_- \leq 1$ and $(\lambda_-^{d_-}\lambda_+^{d_+})\lambda_+ \geq 1$. (In particular $\lambda_- < 1 < \lambda_+$.)*

Proof. By the fact above, if $\lambda_-^{d_-+1}\lambda_+^{d_+} > 1$, then $\int_{M\setminus\mathcal{S}_-} \mathsf{Jac}\ f_x^n \to \infty$ when $n \to +\infty$ contradicting that M has a finite volume.

The other inequality holds by applying the previous fact to f^{-1}. Observe for this, that indeed the eigen-function λ_1^* corresponding to f^{-1} verifies the same decreasing hypothesis. $\square$

6.2.5. Justification of the decreasing hypothesis for A^{-1}.

Let us see what happens if A^{-1}. was increasing in $[\lambda_-, \lambda_+]$. In this case, the volume estimate would give $(\lambda_-^{d_-}\lambda_+^{d_+})\lambda_+ \leq 1$ and $(\lambda_-^{d_-}\lambda_+^{d_+})\lambda_- \geq 1$, which leads to the contradiction $\lambda_+ \leq \lambda_-$.

6.3. The projective Weyl tensor

This is a $(3,1)$-tensor $W : TM \times TM \times TM \to TM$, that is invariant under $\mathsf{Proj}(M, g)$: $W(D_x fu, D_x fv, D_x fw) = D_x f(W(u, v, w))$, for any $u, v, w \in TM$ (and any $f \in \mathsf{Proj}(M, g)$). In dimension ≥ 3, its vanishing is the obstruction to projective flatness of (M, g), that is the fact that (M, g) has a constant sectional curvature.

Unlike the conformal case, the projective Weyl tensor is not a curvature tensor, that is, it does not satisfy all the usual symmetries of curvature tensors (see for instance [5,13] for more information). Its true definition is as follows. If u, v, w, z are four vectors in T_xM such that any two of them are either equal or orthogonal (that is, they are part of an orthonormal basis), then:

$$g_x(W(u, v, w), z) = g_x(R(u, v)w, z) - \frac{1}{n-1}(\delta_v^z Ric(w, u) - \delta_u^z Ric(w, v)) \tag{6.1}$$

where Ric is the Ricci tensor and δ is the Kronecker symbol.

6.3.1. Boundedness

By compactness, W is bounded by means of g, that is $\|W(u, v, w)\| \leq C\|u\|\|v\|\|w\|$, for some constant C, where $\|.\|$ is the norm associated to g.

Notation $\approx$. We will deal with sequences of positive functions $A(n, x)$ and numbers a_n. We will say they have the same growth rate for x in a compact set $K \subset M$, and write $A(n, x) \approx a_n$, if the ratio $\frac{A(n,x)}{a_n}$ belongs to an interval $[\alpha, \beta]$, $0 < \alpha \leq \beta < \infty$, when $x \in K$. In general, K is given by the context and hence omitted.

6.3.2. Asymptotic growth under Df

The goal of this paragraph is to estimate the asymptotic behaviour under the tangent dynamics Df of vectors in each of the Lyapunov spaces (§6.2.1).

Case of E_-: $D_x f^n$ maps similarly $E_-(x)$ to $E_-(f^n(x))$ with a contraction factor

$$\zeta_-(n, x) = \zeta_-(x)\zeta_-(f(x)) \dots \zeta_-(f^{n-1}(x))$$

More concretely,

$$\text{If } u \in E_-(x), \text{ then } \|D_x f^n u\| = \zeta_-(n, x)\|u\|$$

Recall that $\zeta_-(x) = ((\lambda_-^{d_-}\lambda_+^{d_+}\lambda(f(x))\lambda_-)^{1/2}$. If $x \in M \setminus \mathcal{S}_+$, i.e. $\lambda(x) < \lambda_+$, then, by Lemma 6.5, $\zeta_-(n,x)$ grows as

$$\zeta_-(n,x) \approx (\lambda_-^{d_-+2}\lambda_+^{d_+})^{n/2}, \ n \to +\infty$$

Case of E_+ and E_λ: One defines in the same way $\zeta_+(n,x)$ and $\zeta_\lambda(n,x)$ as distortion factors of $D_x f^n$ from $E_+(x)$ to $E_+(f^n(x))$ and from $E_\lambda(x)$ to $E_\lambda(f^n(x))$, respectively. One gets when $n \to \infty$

$$\zeta_+(n,x) \approx (\lambda_-^{d_-+1}\lambda_+^{d_++1})^{n/2}$$
$$\zeta_\lambda(n,x) \approx (\lambda_-^{d_-+2}\lambda_+^{d_+})^{n/2} \approx \zeta_-(n,x)$$

In sum, when $n \to +\infty$,

$$\begin{aligned} \text{If } u \in E_-(x), &\quad \|D_x f^n u\| = \zeta_-(n,x)\|u\| \approx (\lambda_-^{d_-+2}\lambda_+^{d_+})^{n/2}\|u\| \\ \text{If } u \in E_\lambda(x), &\quad \|D_x f^n u\| = \zeta_\lambda(n,x)\|u\| \approx (\lambda_-^{d_-+2}\lambda_+^{d_+})^{n/2}\|u\| \\ \text{If } u \in E_+(x), &\quad \|D_x f^n u\| = \zeta_+(n,x)\|u\| \approx (\lambda_-^{d_-+1}\lambda_+^{d_++1})^{n/2}\|u\| \end{aligned}$$

Observe that all these behaviours are uniform on compact sets of $M \setminus \mathcal{S}_+$. Also, observe that, surely, $\zeta_-(n,x)$ is exponentially decreasing. Indeed, if not, $\lambda_-^{d_-+2}\lambda_+^{d_+} \geq 1$, and since $\lambda_- < \lambda_+$, $\zeta_+(x,n)$ would be exponentially increasing. On the other hand, the Jacobian of Df^n is at least equivalent to a power of $\zeta_+(x,n)$ (since ζ_- and ζ_λ are bounded from below).

6.3.3. *The first vanishing*

Fact 6.7. *For any $x \in M \setminus (\mathcal{S}_- \cup \mathcal{S}_+)$, $W(u,v,w) = 0$ once all u, v, w belong to $E_-(x) \oplus E_\lambda(x)$, or all belong to $E_+(x) \oplus E_\lambda(x)$.*

Proof. To begin with, assume $u, v, w \in E_-(x)$, and then denote $z_n = W(D_x f^n u, D_x f^n v, D_x f^n w)$. By the two previous paragraphs,

$$\|z_n\| \leq C\|D_x f^n u\|\|D_x f^n v\|\|D_x f^n w\| =$$
$$\zeta_-(n,x)^3\|u\|\|v\|\|w\| \approx (\lambda_-^{d_-+2}\lambda_+^{d_+})^{3n/2}C\|u\|\|v\|\|w\|$$

That is,

$$\|z_n\| \lessapprox (\lambda_-^{d_-+2}\lambda_+^{d_+})^{3n/2}$$

(where $\lessapprox$ means that the ratio of the left hand by the right one is bounded independently of x).

On the other hand, by f-invariance of W, $z_n = W(D_xf^nu, D_xf^nv, D_xf^nw) = D_xf^nz$, for $z = W(u, v, w)$.

Decompose $z = z^- + z^+ + z^\lambda \in E_-(x) \oplus E_+(x) \oplus E_\lambda(x)$, and let $z_n = z_n^- + z_n^+ + z_n^\lambda$ accordingly, i.e. $z_n^- = D_xf^nz^- \dots$.

Thus, the norm of each of these parts is dominated by $(\lambda_-^{d_-+2}\lambda_+^{d_+})^{3n/2}$.

However, by §6.3.2 $z_n^- \approx (\lambda_-^{d_-+2}\lambda_+^{d_+})^{n/2}\|z^-\|$.

If c denotes $\left(\lambda_-^{d_-+2}\lambda_+^{d_+}\right)^{1/2}$, then recall it is < 1. So, we have at the same time $z_n^- \lessapprox c^{3n}$ and $z_n^- \approx c^n\|z^-\|$, which obviously implies $z^- = 0$.

Vanishing of z^+ and z^λ is easier to check. Indeed, they behave like $\zeta_+(n, x)\|z^+\|$ (and $\zeta_\lambda(n, x)\|z^\lambda\|$ respectively). By §6.3.2, these are at least equivalent to $c^n\|z^+\|$ (and $c^n\|z^+\|$ respectively). They can not be dominated by c^{3n}, unless $z^+ = z^\lambda = 0$.

Finally, the case where all u, v and w belong to $E_-(x) \oplus E_\lambda(x)$, or all belong to $E_+(x) \oplus E_\lambda(x)$, can be handled in the same way. □

6.3.4. Commutation

It was observed in particular in [20] that any $L \in \mathcal{L}(M, g)$ commutes with the Ricci curvature Ric (seen as an endomorphism of TM). From it we deduce that $Ric(u, v) = 0$ when u and v belong to two different eigenspaces of L (two commuting symmetric endomorphisms of a Euclidean space have a common eigen-decomposition, which is furthermore orthogonal). More precisely $Ric(u, v) = 0$ when $u \in E_\lambda(x)$ and $v \in E_\pm(x)$, or $u \in E_-(x)$ and $v \in E_+(x)$.

6.3.5. The second vanishing

Fact 6.8. *Let $u \in E_-(x)$ and $v, w \in E_+(x) \oplus E_\lambda(x)$, then $W(u, v, w) \in E_-(x)$.*

Proof. We have to prove that $g_x(W(u, v, w), z) = 0$, whenever $z \in E_+(x) \oplus E_\lambda(x)$.

Recall the defining formula (6.1) of the Weyl tensor. Observe that $\delta_u^z = 0$ and $Ric(w, u) = 0$ (because of §6.3.4), and thus $g_x(W(u, v, w), z) = g_x(R(u, v)w, z)$.

On the other hand $g_x(W(w, z, v), u) = g_x(R(w, z)v, u)$ because $\delta_w^u = \delta_z^u = 0$.

Hence $g_x(W(u, v, w), z) = g_x(W(w, z, v), u)$. But, we already proved that $W(w, z, v) = 0$ since all w, z, v belong to $E_+(x) \oplus E_\lambda(x)$. □

Fact 6.9. *In the same conditions, that is, $u \in E_-(x)$ and $v, w \in E_+(x) \oplus E_\lambda(x)$, we have: $W(u, v, w) = 0$.*

Proof. We know by above that $z = W(u, v, w)$ belongs to E_-.

By §6.3.2, we can write: $D_xf^nz = \zeta_-(n, x)\overline{z_n}$, $D_xf^nu = \zeta_-(n, x)\overline{u_n}$, where $\overline{z_n}$ and $\overline{u_n}$ belong to $E_-(f^n(x))$, and $\|\overline{z_n}\| = \|z\|$ and $\|\overline{u_n}\| = \|u\|$.

Thus $\overline{z_n} = W(\overline{u_n}, D_xf^nv, D_xf^nw)$. But D_xf^nv and D_xf^nw tend to 0 when $n \to -\infty$, hence $\|z\| = \|z_n\| \to 0$, that is $z = 0$. □

6.3.6. Full vanishing

Similar to the above situation in Fact 6.8, we consider the case where $u, v \in E_+(x) \oplus E_\lambda(x)$ and $w \in E_-(x)$ and prove that $W(u, v, w) \in E_-(x)$. For this goal, observe that $Ric(w, u) = Ric(w, v) = 0$, and hence $g_x(W(u, v, w), z) = g_x(R(u, v)w, z)$. On the other hand, $g_x(W(u, v, z), w) = g_x(R(u, v)z, w)$, since $\delta_u^w = \delta_v^w = 0$. Next, $u, v, z \in E_+(x) \oplus E_\lambda(x)$ and thus $W(u, v, z) = 0$, and consequently $g_x(W(u, v, w), z) = 0$ as claimed.

Next, the proof of Fact 6.9 applies here and yields that $W(u, v, w) = 0$.

So, by this and the previous facts, we get that $W(u, v, w) = 0$ whenever (at least) two of them are in $E_+(x) \oplus E_\lambda(x)$ (observe that $W(u, v, w) = -W(v, u, w)$).

Obviously, we can switch roles of E_+ and E_-, and so get that $W(u, v, w) = 0$ in all cases. $\square$

7. Riemannian case, proof of Theorem 1.3

As previously, (M, g) is a compact Riemannian manifold with $\mathsf{Proj}(M, g) \supsetneq \mathsf{Aff}(M, g)$. By [24], the degree of projective mobility $\dim \mathcal{L}(M, g)$ equals 2. Pick $f \in \mathsf{Proj}(M, g)$ as in §4.

If $\rho(f)$ is non-hyperbolic for any choice of such f, then, by §5, $\mathsf{Proj}(M, g)/\mathsf{Iso}(M, g)$ is finite as stated in Theorem 1.3.

If $\rho(f)$ is hyperbolic, then by §6, the projective Weyl tensor of (M, g) vanishes.

End of proof in higher dimension. Vanishing of the Weyl tensor in dimension ≥ 3 means that (M, g) has constant sectional curvature (see for instance [5,13]). The universal cover of M is necessarily the sphere since the Euclidean and hyperbolic spaces have no projective non-affine transformations.

Proof in dimension 2. The remaining part of the present section is devoted to the case $\dim M = 2$ and $\rho(f)$ hyperbolic. Our goal in the sequel is to show that in this case, too, (M, g) has constant curvature.

The spectrum of K_f consists of λ, and one constant, say λ_-.

7.0.1. Warped product structure

Let $\mathcal{F}_-$ and $\mathcal{F}_\lambda$ be two one dimensional foliations tangent to the eigenspaces E_- and E_λ. They are regular foliations on $M \setminus \mathcal{S}_-$ (recall that $\mathcal{S}_- = \{x/\lambda(x) = \lambda_-\}$).

Recall that Dini normal form says that two projectively equivalent metrics g and $\bar{g}$ on a surface have the following form:

$$g = (X(x) - Y(y)(dx^2 + dy^2), \quad \bar{g} = (\frac{1}{Y(y)} - \frac{1}{X(x)})(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)})$$

in some coordinates system (x, y) and near any point where the $(1, 1)$-tensor L defined by $\bar{g}(., .) = g(\frac{L^{-1}}{\mathsf{det} L}., .)$, has simple eigenvalues. In fact, $X(x)$ and $Y(y)$ are the eigenvalues of $L(x, y)$, and the coordinates are adapted to eigenspaces.

From this normal form one deduces that $(\mathcal{F}_\lambda, \mathcal{F}_-)$ determines a warped product (for g as well as $\bar{g}$), that is, there are coordinates (r, θ), where $\frac{1}{\partial r}$ (resp. $\frac{1}{\partial \theta}$) is tangent to $\mathcal{F}_\lambda$ (resp. $\mathcal{F}_-$), and the metric has the form $dr^2 + \delta(r)d\theta^2$ (see [38] for more information on warped products). Indeed here $Y(y) = \lambda_-$, and hence $g = (X(x) - \lambda_-)dx^2 + (X(x) - \lambda_-)dy^2$, and then change coordinates according to $dr^2 = (X(x) - \lambda_-)dx^2$, $\theta = y$.

We deduce in particular that the leaves of $\mathcal{F}_\lambda$ are geodesic in (M, g).

7.0.2. Topology

By 6.3.2, $D_x f$ is contracting away from $\mathcal{S}_+ = \{x/\lambda(x) = \lambda_+\}$.

Let $c \in \,]\lambda_-, \lambda_+[$ and $M_c = \{x/\lambda(x) \leq c\}$. It is a codimension 0 compact submanifold with boundary the level $\lambda^{-1}(c)$, for c generic. In particular, it has a finite number of connected components. Since λ is decreasing with f (§6.2), f preserves M_c: $f(M_c) \subset M_c$. Taking a power of f, we can assume it preserves each component of M_c. On such a component, say M_c^0, f contracts the Riemannian metric, and hence also its generated distance. It follows that f has a unique fixed point $x_0 \in M_c^0$. The M_c^0's, for c decreasing to λ_-, is a decreasing family converging to x_0. It follows that these M_c's are topological discs, their boundaries $\lambda^{-1}(c)$ are circles surrounding x_0, and finally x_0 is the unique point in M_c^0 with $\lambda(x) = \lambda_-$.

It is a general fact on tensors of $\mathcal{L}(M, g)$, an eigenfunction is constant along leaves tangent to the eigenspaces associated to the other eigen-functions. In our case, λ is constant on the $\mathcal{F}_-$-leaves, equivalently, leaves of $\mathcal{F}_-$ are levels of λ. So, these leaves are circles surrounding x_0. Those of $\mathcal{F}_\lambda$ are orthogonal to these circles, and hence they are nothing but the geodesics emanating from x_0. Thus, $\mathcal{F}_\lambda$ and $\mathcal{F}_-$ have x_0 as a unique singularity in M_c^0.

7.0.3. Geometry

We infer from the above analysis that the polar coordinates around x_0 give rise to a warped product structure, that is, the metric on these coordinates (r, θ), has the form: $dr^2 + \delta(r)d\theta^2$ (in the general case δ depends rather on (r, θ)).

At x_0, $D_{x_0} f$ is a similarity with coefficient λ_-.

Observe next that, from the form of the metric, rotations $\theta \to \theta + \theta_0$ are isometries. Composing f with a suitable rotation, we can assume f is a "pure homothety", i.e. it fixes individually each geodesic emanating form x_0. Thus f acts only at the r-level: $f(r, \theta) = f(r)$.

Now, the idea is to construct a higher dimensional example with the same ingredients, e.g. δ and f. Precisely, consider the metric $dr^2 + \delta(r)d\Omega^2$, where $d\Omega^2$ is the standard metric on a sphere $\mathbb{S}^N$. We will show in Lemma 7.1 below that this metric is indeed smooth.

We let f act by $f(r, \Omega) = f(r)$.

One verifies that f is projective. Indeed, $\mathsf{SO}(N + 1)$ acts isometrically and commutes with f, and any geodesic is contained in a copy of our initial surface (since, as in the case of $\mathbb{R}^{N+1}$, for any two points x and y, one may find a copy of the surface containing them).

This new f has the same dynamical behaviour as the former one, and one proves as in §6 that the projective Weyl tensor of this new metric vanishes and it has therefore a constant sectional curvature. The same is true for our initial surface. □

Lemma 7.1. *Consider a metric g of the form $dr^2 + \delta(r)d\Omega^2$, where δ is defined on an interval $[0, R[$, smooth on $]0, R[$, and $d\Omega^2$ is the metric of $\mathbb{S}^N$ (thus g is defined on a ball $B(0, R) \setminus \{0\}$ in $\mathbb{R}^{N+1} \setminus \{0\}$). Then g extends smoothly to 0 if and only if $\delta(r) = \zeta(r^2)$, where ζ is smooth as a function of r and $\zeta'(0) = 1$. In particular g is smooth for some dimension $N > 0$ iff it is smooth for any.*

Proof. Observe firstly that the condition on δ is equivalent to that the function $\eta(r) = \frac{\delta(r)-r^2}{r^4}$ equals $\kappa(r^2)$ where κ is smooth on r.

Consider the mapping $\Omega : z = (z_1, \ldots, z_{N+1}) \in \mathbb{R}^{N+1} \setminus \{0\} \to \Omega(z) = \frac{z}{\|z\|} = (\Omega_1, \ldots, \Omega_{N+1}) \in \mathbb{S}^N$. So $d\Omega = (d\Omega_1, \ldots, d\Omega_{N+1})$ is a vectorial 1-differential form on $\mathbb{R}^{N+1} \setminus \{0\}$ and $d\Omega^2 = \Sigma d\Omega_i^2$ is a field of quadratic forms (on $\mathbb{R}^{N+1} \setminus \{0\}$) whose restriction to $\mathbb{S}^N$ coincides with the induced metric.

Similarly, $r = \|z\|$, and hence $dr = \frac{1}{r}(\Sigma z_i dz_i)$. Thus $r^2 dr^2 = (\Sigma z_i dz_i)^2$.

On the other hand, it is known that $g_E = r^2 + r^2 d\Omega^2$ is the Euclidean metric Σdz_i^2. Therefore $r^4 d\Omega^2$ is smooth and equals exactly:

$$r^4 d\Omega^2 = r^2(g_E - dr^2) = (\Sigma z_i^2)(\Sigma dz_i^2) - (\Sigma z_i dz_i)^2 = \Sigma_{i \neq j} z_i^2 dz_j^2 - z_i z_j dz_i dz_j$$

Now, let $g = dr^2 + \delta(r)d\Omega^2$. Then

$$\begin{aligned} g - g_E &= (dr^2 + \delta(r)d\Omega^2) - (dr^2 + r^2 d\Omega^2) = (\delta(r) - r^2)d\Omega^2 = \frac{(\delta(r) - r^2)}{r^4}(r^4 d\Omega^2) \\ &= \eta(r)(r^4 d\Omega^2) \end{aligned}$$

We deduce in particular that a sufficient condition for g to be smooth (as a function of z) is that η is smooth (as a function of z). To see that this is also a necessary condition, we infer from the previous formula for $r^4 d\Omega^2$ that $\eta(r) r^4 d\Omega^2$ is smooth iff the functions $\eta(r) z_i z_j$ are smooth for any i, j. Then apply the next lemma:

Lemma 7.2. *Let $\eta(r)$ be a function such that all the functions $\eta(r) z_i z_j$ are smooth on z. Then $\eta(r)$ is smooth (as a function of z) and it equals $\kappa(r^2)$ where κ is smooth as a function of r.*

Proof. First, $\eta(r) z_i^2 \to 0$ when $r \to 0$. Indeed if not, this limit does not depend on i, and one can take the ratio $1 = \lim \frac{\eta(r) z_i^2}{\eta(r) z_j^2} = \lim \frac{z_i^2}{z_j^2}$, but the latter limit does not exist.

For the next step, to simplify notations, let us assume the dimension is 2 and note $x = z_1, y = z_2$ (the proof in higher dimension is identical).

By hypothesis $T(x, y) = \eta(r)(x^2 + y^2) = \eta(r) r^2$ is smooth. Its Taylor expansion allows one to write it, up to any order, as a sum of homogeneous polynomials on x and y. Since

T is $\mathsf{SO}(2)$-invariant, the same is true for these polynomials. Now, let P be such a polynomial of degree k. By homogeneity and $\mathsf{SO}(2)$-invariance $\frac{P}{(x^2+y^2)^{\frac{k}{2}}}$ is constant on $\mathbb{S}^1$ and hence constant, that is, P is proportional to $r^{\frac{k}{2}}$. This implies in particular that k is even. Therefore, the Taylor expansion is on the powers $r^2, r^4, r^6, \ldots$. We finally get that $\eta(r)$ has a Taylor expansion on $1, r^2, r^4, \ldots$, which ensures the existence of κ. □

Coming back to the proof of Lemma 7.1, assume now that $g = dr^2 + \delta(r)d\Omega^2$ is a smooth Riemannian metric on a neighbourhood of 0. For fixed Ω, the ray $r \to (r, \Omega)$ is an arc-length parameterized geodesic. It follows that $(r, \Omega) \to z = r\Omega$ are normal coordinates, i.e. the inverse of the exponential map at 0. Thus, as in the Euclidean case, the metric g is smooth with respect to z. Therefore, $\delta(r)$ satisfies the same conditions as above. □

Remark 7.3. As an alternative for all this proof in dimension 2, the referee suggests to mimic the higher dimensional proof by replacing the Weyl tensor by its 2-dimensional version, the Liouville tensor (as used in [8]).

8. The Kähler case: proof of Theorem 1.5

Let $F(N, b)$ denote the simply connected Hermitian space of dimension N and constant holomorphic sectional curvature b. Calabi proved (in his thesis) the following striking fact:

Theorem 8.1. *(See Calabi [9].) Let M be a Kähler manifold (not necessarily complete) and $f : M \to F(N, b)$ a holomorphic isometric immersion. Then, f is rigid in the sense that any other immersion f' is deduced from f by composing with an element of $\mathsf{Iso}(F(N, b))$ (this element is unique if the image of f in not contained in a totally geodesic proper subspace of $F(N, b)$). In particular, f is equivariant with respect to some faithful representation $\mathsf{Iso}(M) \to \mathsf{Iso}(F(N, b))$.*

As for holomorphic isometric immersions between space forms, one deduces (for more information, see for instance in [33,18,12]):

Theorem 8.2.

- *The Kähler Euclidean space $\mathbb{C}^d$ can not embed holomorphically isometrically in a projective space $\mathbb{P}^N(\mathbb{C})$ (a radially simple example of this is the situation of a holomorphic vector field; it does not act isometrically, in particular its orbits are not metrically homogeneous).*
- *Up to ambient isometry, the holomorphic homothetic embeddings between projective spaces are given by Veronese maps: $v_k : (\mathbb{P}^d(\mathbb{C}), g_{FS}) \to (\mathbb{P}^N(\mathbb{C}), \frac{1}{k} g_{FS})$,*

$N = \binom{d+k}{k} - 1$, $v_k : [X_0, \ldots, X_d] \to [\ldots X^I \ldots]$, where X^I ranges over all monomials of degree k in $X_0, \ldots, X_d$.

Proof of Theorem 1.5. This will follow from our rigidity theorem of the h-projective group of Kähler manifolds (see §1.2), together with the following fact.

Fact 8.3. *Let $(M^d, g_{SF|M})$ be a submanifold of $(\mathbb{P}^N(\mathbb{C}), g_{SF})$, then $\mathsf{Aff}(M^d, g_{SF|M})/\mathsf{Iso}(M^d, g_{SF|M})$ is finite (vaguely bounded by $\frac{n}{2}!$).*

Proof. This is a standard idea (see for instance [21,39]), the unique special fact we use here is that, by Calabi Theorem, the universal cover has no flat factor in its De Rham decomposition. Thus $\tilde{M}$ is a product $\tilde{M}_1 \times \ldots \times \tilde{M}_m$ of irreducible Kähler manifolds.

The holonomy group $Hol^{\tilde{M}}$ equals the product $Hol^{\tilde{M}_1} \times \ldots \times Hol^{\tilde{M}_m}$. An affine transformation $\tilde{f}$ commutes with Hol and hence preserves the De Rham splitting. Taking a power, we can assume that $\tilde{f}$ actually preserves each factor, and we will thus prove that $\tilde{f}$ is isometric. Since $\tilde{M}_i$ is irreducible, $\tilde{f}$ induces a homothety on it, say of distortion c. If $c \neq 1$, then f or f^{-1} is contracting with respect to the distance of $\tilde{M}_i$. In this case, $\tilde{f}$ will have a (unique) fixed point in $\tilde{M}_i$. However, $\tilde{f}$ preserves the Riemann curvature tensor $R(X,Y)Z$ of $\tilde{M}_i$. But being invariant by a contraction (or a dilation), this tensor must vanish, that is, $\tilde{M}_i$ is flat, contracting the fact that it is irreducible. Therefore, $c = 1$, that is, $\tilde{f}$ is isometric. $\square$

Remarks 8.4. 1. By equivariance, Segre maps $\mathbb{P}^m(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^{(m+1)(n+1)-1}(\mathbb{C})$ are homothetic. In particular, some $(\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}), \frac{1}{k}(g_{FS} \oplus g_{FS}))$ can be embedded in some $(P^N(\mathbb{C}), g_{FS})$. By composing Veronese and Segre maps, one can also realize some metrics $(\mathbb{P}^1(\mathbb{C}), \frac{1}{k}g_{FS}) \times (\mathbb{P}^1(\mathbb{C}), g_{FS}))$.

2. In fact, it turns out that for M a submanifold of $\mathbb{P}^N(\mathbb{C})$, De Rham decomposition applies to M itself; that is, the splitting of $\tilde{M}$ descends to a one of M. I am indebted to A.J. Di Scala for giving me a proof of that using Calabi rigidity. Indeed, this rigidity has the following amazing corollary: if M is holomorphically isometrically embedded in $\mathbb{P}^N(\mathbb{C})$, then neither a cover nor a quotient of it can be embedded so. Now, the De Rham splitting of $\tilde{M}$ gives an immersion into products of projective spaces. Segre map is isometric from this product to one big projective space, which gives us another holomorphic isometric immersion of $\tilde{M}$. But this must coincide with the immersion given by the universal cover $\tilde{M} \to M$. This implies that the De Rham decomposition is defined on M itself.

9. Facts on the indefinite pseudo-Riemannian case: proof of Theorem 1.7

Let (M,g) be a compact pseudo-Riemannian manifold with projective degree of mobility $\dim \mathcal{L}(M,g) = 2$, such that $\mathsf{Proj}(M,g)/\mathsf{Aff}(M,g)$ is infinite. Consider $\rho : \mathsf{Proj}(M,g) \to \mathsf{GL}_2(\mathbb{R})$.

Denote $G = \rho(\mathsf{Proj}(M, g))$. Theorem 1.7 says that, up to finite index, $\ker \rho = \mathsf{Iso}(M, g) = \mathsf{Aff}(M, g)$, and G lies in a non-elliptic one parameter group.

9.1. "Projective linear" action of $\mathsf{Proj}(M, g)$

So far, we singled out an element $f \in \mathsf{Proj}(M, g)$ and associate to it a homography $A_\bullet$ acting on $\bar{\mathbb{R}}$. It turns out that this $A_\bullet$ is nothing but that corresponding to the (projective) action of $\rho(f)$ on the projective space of $\mathcal{L}(M, g)$, identified to $\mathbb{P}^1(\mathbb{R})$, via the basis $\{K = K_f, I\}$.

Indeed, the choice of the basis $\{K, I\}$, say co-ordinates (k, i), allows one to identify $\mathbb{P}(\mathcal{L}(M, g))$ with $\mathbb{P}^1(\mathbb{R})$. In the affine chart $[k : i] \in \mathbb{P}^1(\mathbb{R}) \to z = \frac{k}{i}$, the projective action of $\rho(f)$ is $z \to \frac{\alpha z+\beta}{z}$, where α and β are defined by $\rho(f)K = \alpha K + \beta I$, as in §4.

Now, we let the whole group $\mathsf{Proj}(M, g)$ act by means of ρ on the projective space, and in fact the complex one. More precisely, let

$$\Phi : \mathsf{Proj}(M, g) \to \mathsf{PGL}(\mathcal{L}(M, g) \otimes \mathbb{C})$$

be the action associated to ρ on $\mathbb{P}(\mathcal{L}(M, g) \otimes \mathbb{C})$, the projective space of the complexification of $\mathcal{L}(M, g)$.

The degeneracy set $\mathcal{D}$ is complexified as

$$\mathcal{D}^{\mathbb{C}} = \{L \in \mathbb{P}^1(\mathcal{L}(M, g) \otimes \mathbb{C}),\ L \text{ not an isomorphism of } TM \otimes \mathbb{C}\}$$

The proof of the following fact is similar to that of Fact 4.1.

Fact 9.1. *Let f be any element of $\mathsf{Proj}(M, g)$ with $K_f \neq -\pm I$, then $\mathcal{D}^{\mathbb{C}}$ can be computed by means of K_f as follows. Under the identification of $\mathbb{P}(\mathcal{L}(M, g) \otimes \mathbb{C})$ with $\mathbb{P}^1(\mathbb{C})$ via the basis $\{K_f, I\}$, the set $\mathcal{D}^{\mathbb{C}}$ corresponds to the range of the spectrum mapping of K_f: $x \in M \to Sp^{K_f}(x) = Spectre\ of\ K_f(x) \subset \mathbb{C}$.*

The point is that this set is invariant under the G-action.

9.1.1. By Fact 3.2, the projection of G on its image in $\mathsf{PGL}_2(\mathbb{R})$ has finite index. In fact, since we are interested in objects up to finite index, for simplicity seek, we will argue as if G is contained in $\mathsf{PGL}_2(\mathbb{R})$, in fact in $\mathsf{SL}_2(\mathbb{R})$ to be more concrete.

9.1.2. The kernel of ρ

Letting $h \in \mathsf{Aff}(M, g)$, we will prove that $\rho(h) = 1$, up to index 2. Since h is affine, all K_h-eigenvalues are constant. It follows that K_h has the form aI, since otherwise it generates together with I the whole $\mathcal{L}(M, g)$, and hence all the K_f will have constant eigenvalues for any f, contradicting the fact that $\mathsf{Proj}(M, g) \supsetneq \mathsf{Aff}(M, g)$. By finiteness of the volume, $a = \pm 1$, say $a = 1$, i.e. $K_h = I$. Now, $\rho(h)L = h_* L K_h = h_* L$, and thus $\rho(h)I = I$. Therefore, if $\rho(h) \neq 1$, $\rho(h)$ will be parabolic with unique fixed point I

(in $\mathbb{P}^1(\mathbb{C})$). So, any closed $\rho(h)$-invariant set contains I. But this is not the case of the degeneracy set $\mathcal{D}^{\mathbb{C}}$ (since it corresponds to the spectrum).

9.2. Proof that G is contained in a one parameter group

As suggested by the referee, we will make use of Theorem 1.11 of [7]. It states that if for some x, $K_f(x)$ has a complex eigenvalue λ, then this is a constant eigenvalue, that is, λ is eigenvalue of $K_f(y)$ for any $y \in M$. So the proof will be essentially similar to the Riemannian case. More precisely, let $f \in \mathsf{Proj}(M, g)$ such that $\rho(f)$ is hyperbolic or parabolic, then K_f has everywhere a real spectrum. Indeed, otherwise, the homography associated to f will have a non-real fixed point in $\mathbb{P}^1(\mathbb{C})$ which is impossible (since this homography is real).

Furthermore, the range of the spectrum of K_f is a compact interval in $\mathbb{R}$ (non-reduced to a point since f is not affine). Now, a parabolic homography preserves no non-trivial compact interval, and so this case is impossible. In the hyperbolic case, the unique non-trivial invariant interval is that joining the two fixed points. It follows that $\mathcal{D}^{\mathbb{C}}$ is an interval in $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$. Therefore, the group G preserves a subset of two points consisting in the extremities of this interval. But the subgroup of $\mathsf{SL}_2(\mathbb{R})$ preserving two points in $\mathbb{P}^1(\mathbb{R})$ has a one parameter subgroup as a normal subgroup of index two (e.g. in the case of $\{0, \infty\}$, this group is generated by of $z \to az$, $a > 0$, and $z \to \frac{1}{z}$).

9.2.1. Elliptic case

It remains now to consider the case where all the elements of G are elliptic, the goal here is to prove that G is finite. Let $\bar{G}$ be the closure of G and $\bar{G}^0$ its identity component. Thus $\bar{G}^0$ is a connected subgroup of $\mathsf{SL}_2(\mathbb{R})$. It can not be $\mathsf{SL}_2(\mathbb{R})$ since the set of elliptic elements there is not dense. The 4 others possibilities for non-trivial connected subgroups are, up to conjugacy: the affine group $\mathsf{Aff}(\mathbb{R})$ (upper triangular elements of $\mathsf{SL}_2(\mathbb{R})$) or a one parameter of hyperbolic, parabolic or elliptic type. But, the set of elliptic elements is dense (actually just non-trivial) only in the case of an elliptic one parameter group. Hence, if non-trivial, $\bar{G}^0$ is conjugate to $\mathsf{SO}(2)$. The group G itself is contained in the normalizer of $\bar{G}^0$ which also equals $\mathsf{SO}(2)$. We will now find a contradiction leading to that this situation is impossible. Indeed, since G is dense in $\mathsf{SO}(2)$, its orbits in $\mathbb{P}^1(\mathbb{R})$ are dense, and hence any G-invariant closed set in $\mathbb{P}^1(\mathbb{R})$ equals $\mathbb{P}^1(\mathbb{R})$. This implies that $\mathcal{D}^{\mathbb{C}} \cap \mathbb{P}^1(\mathbb{R}) = \emptyset$, since the closed G-invariant subset $\mathcal{D}^{\mathbb{C}}$ does not contain ∞. In sum, the spectrum of K_f is nowhere real. As above, by [7], this implies K_f has a constant spectrum and hence f is affine, but we have already excluded this possibility.

Let us consider now the case where $\bar{G}^0 = 1$ which means that G is discrete. Any element $A \in G$ is elliptic and hence conjugate to an element of $\mathsf{SO}(2)$ which has a finite order (by discreetness). Apply Selberg Lemma (see for instance [1]), which says that a finitely generated subgroup of $\mathsf{GL}_n(\mathbb{R})$ has a torsion free finite index subgroup (i.e. with no elements of finite order). Let G' be a finitely generated subgroup of G. Since all elements of G' have finite order, Selberg Lemma implies that G' is finite. However, a finite non-trivial subgroup of $\mathsf{SL}_2(\mathbb{R})$ is conjugate to a unique one parameter elliptic subgroup (geometrically, it has a unique fixed point in the hyperbolic plane). Say, if an element $A \in G$, up to conjugacy belongs to $\mathsf{SO}(2)$, then, for any $B \in G$, the group G' generated by A and B must be contained in $\mathsf{SO}(2)$, and therefore $G \subset \mathsf{SO}(2)$. As above, G can not be dense in $\mathsf{SO}(2)$ and is hence finite.

We have thus proved that in all cases and after neglecting finite index objects, $\rho(\mathsf{Proj}(M, g))$ lies in a hyperbolic or parabolic one parameter group, which completes the proof of Theorem 1.7. $\square$

Remark 9.2. In higher dimensions, i.e. for subgroups of $\mathsf{SO}(1, n)$, $n > 2$, it is no longer true that having all its elements elliptic implies the subgroup is contained in a compact subgroup, see [35].

Remark 9.3. Let P be the one parameter group that contains G (up to finite things). Then G may be equal to P, or dense (and $\neq P$), or finally discrete and hence cyclic generated by a single element. The case $G = P$ means that M has a projective vector field. One may ask if the dense case may happen, that is, if G is dense, then necessarily $G = P$?

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