RIGIDITY OF PSEUDO-HERMITIAN HOMOGENEOUS SPACES OF FINITE VOLUME

OLIVER BAUES, WOLFGANG GLOBKE, AND ABDELGHANI ZEGHIB

ABSTRACT. Let M be a pseudo-Hermitian homogeneous space of finite volume. We show that M is compact and the identity component G of the group of holomorphic isometries of M is compact. If M is simply connected, then even the full group of holomorphic isometries is compact. These results stem from a careful analysis of the Tits fibration of M, which is shown to have a torus as its fiber. The proof builds on foundational results on the automorphisms groups of compact almost pseudo-Hermitian homogeneous spaces. It is known that a compact homogeneous pseudo-Kähler manifold splits as a product of a complex torus and a rational homogeneous variety, according to the Levi decomposition of G. Examples show that compact homogeneous pseudo-Hermitian manifolds in general do not split in this way.

Contents

1.	Introduction and main results	1
2.	<i>h</i> -structures on Lie algebras	6
3.	Algebraic dynamics arising from automorphism groups of geometric	
	structures	8
4.	Nil-invariant and quasi-invariant h -structures	14
5.	Application to almost pseudo-Hermitian and almost symplectic	
	homogeneous spaces	21
6.	Compact pseudo-Hermitian homogeneous spaces	23
Ap	Appendix A. Additional proofs	
Ref	References	

1. INTRODUCTION AND MAIN RESULTS

1.1. Some directions in Gromov's vague pseudo-Riemannian conjecture. Understanding when the isometry group of a pseudo-Riemannian structure acts non-properly, particularly when the isometry group $\text{Iso}(M, \mathbf{g})$ of a compact pseudo-Riemannian manifold (M, \mathbf{g}) is non-compact, is a paradigmatic case of Gromov's vague conjecture stipulating that rigid geometric structures with large automorphism groups are classifiable. It is also a fundamental question in geometric dynamics that has been investigated in many works with much progress, especially in the lower

Date: June 11, 2020.

²⁰¹⁰ Mathematics Subject Classification. Primary 53C50; Secondary 32M10, 53C55, 57S20.

index case, for example the Lorentzian index [1, 11, 27] or the authors' results on index two in [4]. Indeed, let us observe for instance that for any semisimple Lie group G with a uniform lattice Γ , the Killing form endows $M = G/\Gamma$ with a pseudo-Riemannian metric whose isometry group contains G, acting by left-translations, so that M is a homogeneous pseudo-Riemannian manifold. However, these are not the only homogeneous pseudo-Riemannian manifolds, which in general have the form G/H where H is not discrete. Herein lies the difficulty of the problem. So, even with an additional homogeneity hypothesis, the pseudo-Riemannian problem stays intractable.

1.1.1. Complex framework. Our approach here in testing Gromov's vague conjecture is to enrich the pseudo-Riemannian structure by assuming the existence of a compatible complex structure. In other words, we consider *pseudo-Hermitian* structures. More precisely, a *pseudo-Hermitian metric* **g** on a complex manifold (M, J) is a pseudo-Riemannian metric compatible with J: $g(J \cdot, J \cdot) = g(\cdot, \cdot)$. Here we are interested in the automorphism group of such a structure, which is the group of holomorphic transformations preserving the pseudo-Hermitian metric.

To illustrate the effect of the complex structure, let us observe that the previously introduced pseudo-Riemannian homogeneous examples G/Γ cannot be pseudo-Hermitian. Indeed, the induced complex structure and pseudo-Hermitian product on the Lie algebra \mathcal{G} of G are $\operatorname{Ad}(\Gamma)$ -invariant and hence $\operatorname{Ad}(G)$ -invariant by the Borel Density Theorem (here we assume G to be semisimple without compact factors). For any $x \in \mathcal{G}$, the adjoint operator $\operatorname{ad}(x)$ is skew with respect to the pseudo-Hermitian product, and $\operatorname{ad}(\operatorname{J} x) = \operatorname{Jad}(x)$. Now, the point is that the Lie algebra $\mathcal{U}(p,q)$ of the unitary group of a pseudo-Hermitian product of signature (p,q) on \mathbb{C}^n is totally real in $\operatorname{Mat}(n, \mathbb{C})$, that is, $\mathcal{U}(p,q) \cap i\mathcal{U}(p,q) = \mathbf{0}$.

The main result of the present article (see Subsection 1.2 below) is the classification of compact pseudo-Hermitian homogeneous spaces G/H by showing that G° must be compact (compare the discussion following Corollary D). In the initial example above, H was discrete and we used the Borel Density Theorem. All this fails dramatically in the general case, where we essentially have to deal with "savage" groups that are semidirect products of a compact group by a solvable one.

1.1.2. Pseudo-Kähler case. The fundamental two-form associated to **g** is the differential two-form defined by $\omega(\cdot, \cdot) = \mathbf{g}(\cdot, \mathbf{J} \cdot)$. The metric is said to be pseudo-Kähler if ω is closed. Conversely, from the symplectic geometry point of view, pseudo-Kähler structures are special symplectic structures given by a symplectic form ω calibrated by a complex structure J, that is $\omega(\mathbf{J} \cdot, \mathbf{J} \cdot) = \omega(\cdot, \cdot)$.

Dorfmeister and Guan [12] proved the compactness of the identity component of the holomorphic isometry group of a homogeneous compact pseudo-Kähler manifold, and that the Levi decomposition of this group induces a holomorphic and metric splitting of the manifold. The rigidity here comes essentially from the symplectic side. In fact, Dorfmeister and Guan's theorem is to a large part implied by Zwart and Boothby's [29] results on compact symplectic homogeneous spaces (compare Appendix A).

In the pseudo-Kähler case, Gromov's vague conjecture seems to be tractable. More generally, symplectic actions of finite-dimensional Lie algebras on compact symplectic manifolds are "easy" to handle. The basic idea is that the derived subalgebra consists of Hamiltonian vector fields, from which one concludes that it is a direct sum of a compact semisimple and an abelian Lie algebra (see for instance Guan [14, p. 3362], but also [15, Section 26], [18, Section 2]).

1.2. **Statement of the main results.** We begin our investigation by considering the more general situation of *almost* pseudo-Hermitian homogeneous manifolds. Such a manifold carries a field of complex structures that is not required to be integrable to a complex structure on the manifold. Our first main result is:

Theorem A. Any almost pseudo-Hermitian homogeneous space M of finite volume is compact. The maximal semisimple subgroup of the identity component of the almost complex isometry group of M is compact, and its solvable radical is nilpotent and at most two-step nilpotent.

It is therefore sufficient to henceforth consider compact spaces.

In Section 5 we find that the only almost pseudo-Hermitian compact homogeneous manifolds with a transitive action by a solvable group are complex tori.

Theorem B. Let M be a compact almost pseudo-Hermitian homogeneous space with a transitive effective action by a connected solvable Lie group G of almost complex isometries. Then G is abelian and compact. Furthermore, M is a flat complex pseudo-Hermitian torus $\mathbb{C}^{p,q}/\Gamma$, where Γ is a lattice in $\mathbb{C}^{p,q}$.

Upon making the assumption that M is a complex manifold, we can obtain much stronger statements. In Section 6, we make use of Tits's [25] result that any compact complex homogeneous manifold fibers holomorphically over a generalized flag manifold with complex parallelizable fibers. We eventually derive the following structure theorem for holomorphic isometry groups:

Theorem C. Let M be a compact complex homogeneous manifold, and let G be a closed Lie subgroup of the holomorphic automorphism group of M. Assume that G preserves a pseudo-Hermitian metric and acts transitively on M. Then the fiber of the Tits fibration of M is a compact complex torus and the group G is compact.

In particular, we obtain that the identity component of the holomorphic isometry group of a compact homogeneous pseudo-Hermitian manifold is compact. It is not clear if this result is still valid if we merely assume an *almost* pseudo-Hermitian structure on the compact homogeneous space.

Even though we show in Theorem C that the identity component of the holomorphic isometry group is compact, the full holomorphic isometry group is not necessarily so. The following example illustrates this. **Example.** Let $T = \mathbb{C}^{p,q}/\mathbb{Z}^{2p+2q}$ be a complex torus with a pseudo-Hermitian product \mathfrak{h} of signature (p,q). The identity component of the holomorphic isometry group is T itself. However, the full holomorphic isometry group contains $\Lambda(\mathfrak{h}) = U(\mathfrak{h}) \cap SL(2p+2q,\mathbb{Z})$, where $U(\mathfrak{h})$ denotes the unitary group of \mathfrak{h} . In the case of the standard pseudo-Hermitian product $\mathfrak{h}(z,w) = -z_1\overline{w}_1 - \ldots - z_p\overline{w}_p + z_{p+1}\overline{w}_{p+1} + \ldots + z_{p+q}\overline{w}_{p+q}$, this group Λ is a lattice in SU(\mathfrak{h}) and thus non-compact (for p,q > 0). This fact extends to all pseudo-Hermitian products with rational coefficients (by the Borel-Harish-Chandra Theorem [7]).

However, in the simply connected case, Theorem C extends to the full holomorphic isometry group.

Corollary D. The holomorphic isometry group of a simply connected compact homogeneous pseudo-Hermitian manifold is compact.

A class of examples illustrating Corollary D are the compact homogeneous pseudo-Hermitian manifolds with non-zero Euler characteristic. These are precisely the generalized flag manifolds (cf. Example 6.8).

Theorem C can be interpreted as the identity component of the holomorphic isometry group of a compact homogeneous pseudo-Hermitian manifold being inessential in the sense that it preserves a positive definite Hermitian structure. Thus, to classify these objects, start with a compact complex manifold homogeneous under the action of a compact group K, and take any pseudo-Hermitian product invariant under the isotropy of some point. By compactness of this isotropy, there always exists an invariant Hermitian product, but existence of higher signature ones depends on the isotropy representation. The compact group splits, up to finite index, as $G = T \times K$, where T is abelian and K is semisimple. In the pseudo-Kähler case, this splitting induces a holomorphic and metric splitting of the manifold as a product of a complex torus and a flag manifold (a projective homogeneous manifold), see Borel and Remmert [8], Dorfmeister and Guan [12]. Without the assumption that the fundamental two-form is closed, we do not have such a splitting. This is illustrated by the examples discussed at the end of Section 6.

In Appendix A, we review the results by Zwart and Boothby [29] on compact symplectic homogeneous spaces and by Dorfmeister and Guan [12] on compact homogeneous pseudo-Kähler manifolds. We give short proofs based on the techniques developed in this article. Moreover, in Theorem A.7, we prove a structure theorem for the intermediate case of a compact symplectic homogeneous space equipped with an invariant pseudo-Riemannian metric, but without the assumption of compatibility between the metric and the symplectic form.

1.3. Additional remarks beyond the homogeneous case. In order to illustrate how our current homogeneity hypothesis is fairly reasonable, let us show that any semisimple real Lie group G can act (non-transitively, but topologically transitive) by preserving a pseudo-Hermitian structure on a locally homogeneous

compact manifold. Let κ be the Killing form on the Lie algebra \mathcal{G} . On its complexification $\mathcal{G}^{\mathbb{C}} = \mathbb{C} \otimes \mathcal{G}$, consider the quadratic form $q(x+iy) = \kappa(x,x) + \kappa(y,y)$, where $x + iy \in \mathcal{G}^{\mathbb{C}}$. Polarization yields an $\operatorname{ad}(\mathcal{G})$ -invariant pseudo-Hermitian form. This allows us to define a left-invariant pseudo-Hermitian metric \mathfrak{h} on the group $\mathcal{G}^{\mathbb{C}}$, that equals q at the identity element. By the $\operatorname{ad}(\mathcal{G})$ -invariance of q, \mathfrak{h} is invariant under the right-action of \mathcal{G} on $\mathcal{G}^{\mathbb{C}}$. If Γ is a uniform lattice of $\mathcal{G}^{\mathbb{C}}$, then \mathfrak{h} projects to a well-defined \mathcal{G} -invariant pseudo-Hermitian metric on $M = \mathcal{G}^{\mathbb{C}}/\Gamma$. By Moore's Ergodicity Theorem (see Zimmer [28, Theorem 2.2.6]), the \mathcal{G} -action is ergodic and hence topologically transitive.

These examples are not pseudo-Kähler, because, as mentioned above, a Lie group preserving a symplectic structure on a compact manifold has a product of a compact group by an abelian group as its derived subgroup.

If $G = SL(2, \mathbb{R})$, then $G^{\mathbb{C}} = SL(2, \mathbb{C})$, which is locally isomorphic to SO(1,3), and so M is the frame bundle of the hyperbolic three-manifold (or orbifold) $\mathbb{H}^3/\Gamma =$ $SO(3) \setminus SO(1,3)/\Gamma$. The pseudo-Hermitian metric in this case is of Hermite-Lorentz type, and in particular has index two when viewed as a pseudo-Riemannian metric.

The whole discussion generalizes to the case where κ is replaced by any $\operatorname{ad}(\mathcal{G})$ invariant scalar product and G is any group admitting such a form, sometimes called a quadratic group (there is a huge family of such groups, see, for instance Kath and Olbrich [20]). If G is an oscillator group, then, as in the SL(2, \mathbb{R})-case, one gets a Hermite-Lorentz example. These beautiful and rich examples motivate the systematic study of isometric actions of Lie groups on pseudo-Riemannian homogeneous manifolds of index two as initiated by the authors in [4].

Notations and conventions. All Lie algebras \mathcal{G} under consideration are assumed to be finite-dimensional and defined over the reals (although most of our algebraic results are valid over any field of characteristic zero).

For direct products of Lie algebras \mathcal{G}_1 , \mathcal{G}_2 we write $\mathcal{G}_1 \times \mathcal{G}_2$, whereas $\mathcal{G}_1 + \mathcal{G}_2$ or $\mathcal{G}_1 \oplus \mathcal{G}_2$ refer to sums as vector spaces.

The solvable radical \mathcal{R} of a Lie algebra \mathcal{G} is the maximal solvable ideal of \mathcal{G} . Any semisimple Lie algebra \mathcal{F} over the real numbers is a direct product $\mathcal{F} = \mathcal{K} \times \mathcal{S}$, where \mathcal{K} is a semisimple Lie algebra of *compact type*, meaning its Killing form is definite, and where \mathcal{S} is semisimple without factors of compact type.

Any maximal semisimple Lie subalgebra $\mathcal{F} = \mathcal{K} \times S$ of \mathcal{G} is called a *Levi subalgebra* of \mathcal{G} . Let us further put $\mathcal{G}_s = \mathcal{S} \ltimes \mathcal{R}$. Then there is a Levi decomposition

$$\mathcal{G} = \mathcal{F} \ltimes \mathcal{R} = (\mathcal{K} \times \mathcal{S}) \ltimes \mathcal{R} = \mathcal{K} \ltimes \mathcal{G}_{s}.$$

For a Lie group G, we let G° denote the connected component of the identity. For a subgroup H of G, we write $\operatorname{Ad}_{\mathcal{G}}(H)$ for the adjoint representation of H on the Lie algebra \mathcal{G} of G, to distinguish it from the adjoint representation $\operatorname{Ad}(H)$ on its own Lie algebra \mathcal{H} . The closed subgroup H is called *uniform* if G/H is compact.

The center of a group G, or a Lie algebra \mathcal{G} , is denoted by Z(G), or $Z(\mathcal{G})$, respectively. Similarly, the centralizer of a subgroup H in G (or a subalgebra \mathcal{H}

in \mathcal{G}) is denoted by $Z_G(H)$ (or $Z_{\mathcal{G}}(\mathcal{H})$). The normalizer of H in G is denoted by $N_G(H)$, that of \mathcal{H} in \mathcal{G} by $\mathcal{N}_{\mathcal{G}}(\mathcal{H})$.

Acknowledgements. Wolfgang Globke was partially supported by the Australian Research Council grant DE150101647 and the Austrian Science Fund FWF grant I 3248.

2. h-structures on Lie Algebras

Here we work out the elementary linear algebra regarding the definition of almost h-algebras and h-structures on Lie algebras (compare Definitions 5.1, 2.1, and Section 2.2.). Such algebraic structures serve as local models for isometry groups of certain geometric manifolds.

2.1. *h*-structures.

2.1.1. Bilinear forms. Let β be any bilinear form on V. We shall denote orthogonality of vectors in V with respect to β by the symbol \perp_{β} . If β is a symmetric or skew symmetric bilinear form, the subspace

$$V^{\perp_{\beta}} = \{ v \in V \mid \beta(v, V) = \mathbf{0} \}$$

is called the *kernel* of β . Accordingly, β is called *non-degenerate* if $V^{\perp_{\beta}} = \mathbf{0}$.

Furthermore, we put

$$\mathcal{O}(\beta) = \{\varphi \in \mathcal{GL}(V) \mid \beta(\varphi(x), y) = -\beta(x, \varphi(y))\}$$

for the Lie algebra of linear transformations that are skew with respect to β . The form β will be called *invariant* by $\varphi \in \mathcal{GL}(V)$ if $\varphi \in \mathcal{O}(\beta)$.

2.1.2. *J*-structures. A linear map $J \in \text{End}(V)$ will be a called a *J*-structure on *V* if $J^2 \equiv -\text{id}_V \mod \ker J$. We let

$$\mathcal{GL}_{J}(V) = \{ \varphi \in \mathcal{GL}(V) \mid \varphi (\ker J) \subseteq \ker J \text{ and } J \circ \varphi \equiv \varphi \circ J \text{ mod } \ker J \}$$

denote the Lie algebra of J-linear maps of V. For any J-structure,

$$(2.1) V = \ker J \oplus \operatorname{im} J.$$

A *J*-structure is a complex structure if ker J = 0.

2.1.3. *h*-structures on vector spaces. Suppose that V is equipped with a J-structure. Let $\langle \cdot, \cdot \rangle$ be a J-invariant symmetric bilinear form with kernel V^{\perp} .

Definition 2.1. The data $(J, \langle \cdot, \cdot \rangle)$ will be called an *h*-structure on V if

 $\ker J \subseteq V^{\perp}.$

In this case, we also have

 $\langle Ju, Jv \rangle = \langle u, v \rangle$ for all $u, v \in V$.

The associated J-invariant skew-symmetric form

$$\omega(\cdot, \cdot) = \langle J \cdot, \cdot \rangle$$

is called the *fundamental two-form of the h-structure*. Note that

$$V^{\perp} = V^{\perp_{\omega}}.$$

We call V^{\perp} the *kernel of the h-structure*. Given an *h*-structure on *V*, the symbol \perp shall always denote orthogonality with respect to $\langle \cdot, \cdot \rangle$.

Remark. If the *h*-structure is non-degenerate (that is, if $\langle \cdot, \cdot \rangle$ has trivial kernel), then $(V, J, \langle \cdot, \cdot \rangle)$ is a *Hermitian vector space* with complex structure J. We do not require that $\langle \cdot, \cdot \rangle$ is definite in the definition of a Hermitian vector space.

2.1.4. Unitary Lie algebra of h-structures. For any h-structure on V we put

(2.2)
$$O(V) \coloneqq O(\langle \cdot, \cdot \rangle)$$
 and $S\mathcal{P}(V) \coloneqq O(\omega)$.

Furthermore

(2.3)
$$\mathcal{U}(V) \coloneqq \mathcal{O}(V) \cap \mathcal{GL}_J(V) \quad (= \mathcal{SP}(V) \cap \mathcal{GL}_J(V))$$

is called the *unitary Lie algebra* for the h-structure. Note that

$$(2.4) \qquad \qquad \mathcal{U}(V) \subseteq \mathcal{O}(V) \cap \mathcal{SP}(V),$$

and that equality holds in (2.4) if ker $J = V^{\perp}$.

2.1.5. Skew linear maps in Hermitian vector spaces. Let $(V, J, \langle \cdot, \cdot \rangle)$ be a Hermitian vector space.

Lemma 2.2. Let $\varphi \in O(V) \cap JO(V)$. Then we have:

Jφ + φJ = 0.
If φ and Jφ commute, then φ² = 0.

Proof. Let $\varphi^* \in \text{End}(V)$ denote the dual map of φ with respect to $\langle \cdot, \cdot \rangle$. Since $J, \varphi \in \mathcal{O}(V), (J\varphi)^* = \varphi J$. Since also $J\varphi \in \mathcal{O}(V), (J\varphi)^* = -J\varphi$. Hence, (1) holds. This implies $J\varphi^2 + \varphi J\varphi = 0$. If φ and $J\varphi$ commute we obtain $J\varphi^2 = 0$. Hence, $\varphi^2 = 0$. This shows (2).

A real subspace W of a vector space with complex structure (V, J) is called *totally real* if $W \cap JW = \mathbf{0}$. The unitary Lie algebra of a Hermitian vector space is totally real. More generally:

Lemma 2.3. $\mathcal{U}(V) \cap J\mathcal{O}(V) = \mathbf{0}$.

Proof. Indeed, let $\varphi \in \mathcal{U}(V)$ with $\varphi = J\psi$, where $\psi \in \mathcal{O}(V)$. By Lemma 2.2 (1), ψ is complex anti-linear. However, $\psi = -J\varphi \in \mathcal{GL}_J(V)$. Thus $\psi = 0$.

2.2. h-structures on Lie algebras. We now consider h-structures on Lie algebras. We start by introducing some notions related to bilinear forms and J-structures on Lie algebras.

2.2.1. Lie algebras with bilinear forms. Let G be a Lie algebra and β any bilinear form on G. The subalgebra of β -skew elements of G is

$$\mathcal{G}_{\beta} = \{ x \in \mathcal{G} \mid \beta([x, y], z) = -\beta(y, [x, z]) \text{ for all } y, z \in \mathcal{G} \}.$$

For a subalgebra \mathcal{H} of \mathcal{G} , we say that β is \mathcal{H} -invariant if $\mathcal{H} \subseteq \mathcal{G}_{\beta}$.

Lemma 2.4. Let $\mathcal{J} \subseteq \mathcal{G}_{\beta}$ be an ideal of \mathcal{G} . Then $[\mathcal{I}^{\perp_{\beta}}, \mathcal{I}] \subseteq \mathcal{I} \cap \mathcal{G}^{\perp_{\beta}}$.

In case β is symmetric or skew-symmetric, the pair (\mathcal{G},β) is called *effective* if the kernel $\mathcal{G}^{\perp_{\beta}}$ does not contain any non-trivial ideal of \mathcal{G} .

2.2.2. *J*-structures on Lie algebras. Given a *J*-structure on the Lie algebra \mathcal{G} we put $\mathcal{H} = \ker J$, which is a subspace of \mathcal{G} . Define the subalgebra

$$\mathcal{G}_J = \{x \in \mathcal{G} \mid \operatorname{ad}(x)\mathcal{H} \subseteq \mathcal{H} \text{ and } \operatorname{ad}(x)Jy = J\operatorname{ad}(x)y \operatorname{mod} \mathcal{H}\}.$$

We observe:

Lemma 2.5. If
$$x \in \mathcal{G}$$
, $y \in \mathcal{G}_J$ then $J(\operatorname{ad}(x)y) = \operatorname{ad}(Jx)y \mod \mathcal{H}$.

Proof.
$$J(\operatorname{ad}(x)y) = -J(\operatorname{ad}(y)x) = -\operatorname{ad}(y)Jx \mod \mathcal{H} = \operatorname{ad}(Jx)y \mod \mathcal{H}.$$

2.2.3. *h*-structures on Lie algebras. Let $(\langle \cdot, \cdot \rangle, J)$ be an *h*-structure on \mathcal{G} , and let ω denote its fundamental two-form. We call

$$\mathcal{G}^{\perp} \coloneqq \mathcal{G}^{\perp_{\langle \cdot, \cdot \rangle}} = \mathcal{G}^{\perp_{\omega}}$$

the kernel of the h-structure $(\langle \cdot, \cdot \rangle, J)$. For an h-structure on \mathcal{G} we define

$$\mathcal{G}_{J}^{h} = \{x \in \mathcal{G} \mid \operatorname{ad}(x)\mathcal{G}^{\perp} \subseteq \mathcal{G}^{\perp} \text{ and } \operatorname{ad}(x)Jy = J\operatorname{ad}(x)y \operatorname{mod} \mathcal{G}^{\perp}\}.$$

We then have:

Lemma 2.6. If $x \in \mathcal{G}$, $y \in \mathcal{G}_J^h$ then $J(\operatorname{ad}(x)y) = \operatorname{ad}(Jx)y \mod \mathcal{G}^{\perp}$.

We further write

$$\mathcal{G}_{h} = \mathcal{G}_{\omega} \cap \mathcal{G}_{\langle \cdot, \cdot \rangle} \text{ and } \mathcal{G}_{\langle \cdot, \cdot \rangle, J} = \mathcal{G}_{\langle \cdot, \cdot \rangle} \cap \mathcal{G}_{J}$$

Observe that

(2.5)
$$\mathcal{G}_{\langle\cdot,\cdot\rangle,J} \subseteq \mathcal{G}_h \subseteq \mathcal{G}_J^h .$$

From (2.4) we further infer

(2.6)
$$\mathcal{G}_h \subseteq \mathcal{G}_J = \mathcal{G}_J^h \text{ and } \mathcal{G}_{\langle\cdot,\cdot\rangle,J} = \mathcal{G}_h \text{ if } \ker J = \mathcal{G}^\perp$$

3. Algebraic dynamics arising from automorphism groups of geometric structures

In this section, we explore the dynamics arising from automorphism groups of multilinear forms on manifolds of finite volume.

3.1. Preliminaries: isotropic and split subgroups of algebraic groups.

3.1.1. Solvable groups. For a solvable linear algebraic group A defined over \mathbb{R} , let A_{spl} denote the maximal \mathbb{R} -split connected subgroup of A. This means that A_{spl} is the maximal connected subgroup trigonalizable over the reals. For the real algebraic group $A = A_{\mathbb{R}}$, its maximal trigonalizable subgroup is $A_{spl} = A \cap A_{spl}$. Let T be a torus defined over \mathbb{R} . Then T is called *anisotropic* if $T_{spl} = \{e\}$. Otherwise, T is called *isotropic*. Equivalently, T is anisotropic if its group of real points $T = T_{\mathbb{R}}$ is compact. Every torus defined over \mathbb{R} has a decomposition into subgroups $T = T_{spl} \cdot T_c$, where T_c is a maximal anisotropic torus defined over \mathbb{R} and $T_{spl} \cap T_c$ is finite. Moreover, if $T \subseteq A$ is a maximal torus defined over \mathbb{R} and U is the unipotent radical of A, then there is a semidirect product decomposition

$$A_{\rm spl}$$
 = $U \cdot T_{\rm spl}$.

Note also that the split part A_{spl} is preserved under \mathbb{R} -defined morphisms of algebraic groups. See Borel [5, §15] for more background.

3.1.2. Semisimple groups. A simple linear algebraic group L defined over \mathbb{R} is called *isotropic* if any maximal \mathbb{R} -defined torus is isotropic. A semisimple linear algebraic group L defined over \mathbb{R} will be called *isotropic* if all of its simple factors are isotropic.

3.1.3. The isotropic kernel. If G is a linear algebraic group defined over \mathbb{R} with solvable radical R and a maximal semisimple subgroup L, then we define the *isotropic* part of G to be the normal subgroup

$$(3.1) G_{\rm is} = \boldsymbol{S} \cdot \boldsymbol{R}_{\rm spl} \,.$$

where S is the maximal normal isotropic subgroup of L. Note that G_{is} is a normal subgroup and that there is an almost semidirect product decomposition

 $G = B \cdot G_{\mathrm{is}}$,

where B is a reductive anisotropic subgroup. The normal subgroup G_{is} will be called the *isotropic kernel* of G. Any \mathbb{R} -defined homomorphism of G to an anisotropic group contains G_{is} in its kernel, that is, it factors over G/G_{is} .

Lemma 3.1. Let G and H be \mathbb{R} -defined linear algebraic groups and $\psi : G \to H$ an \mathbb{R} -defined algebraic homomorphism. Then $\psi(G_{is}) \subseteq H_{is}$.

Proof. The split part of the solvable radical of G is mapped to the split part of the solvable radical of H (see Borel [5, 15.4 Theorem]). Moreover, ψ maps the isotropic simple subgroups of G to isotropic simple subgroups of H.

3.1.4. Real algebraic groups. Let $G = G_{\mathbb{R}}$ be the group of real points of G. Then $G_{is} = G \cap G_{is}$ is called the isotropic kernel of G and $B = B \cap G$ is a compact group. We note that

$$G = B \cdot G_{is}$$
 .

We conclude that G_{is} is a normal uniform subgroup of G, and, in fact, every morphism of real algebraic groups from G to a compact group factors over G/G_{is} . Recall that a closed subgroup H of G is called *uniform* if G/H is compact.

3.2. Dynamics of linear maps, invariant probability measures. Let V be a real vector space and $G \subseteq GL(V)$. Let $A = \overline{G}_{\mathbb{R}}^{\mathbb{Z}}$ denote the real Zariski closure of G.

3.2.1. Quasi-invariant vectors. For $v \in V$, put $G_v = \{g \in G \mid g \cdot v = v\}$. Recall that v is called *invariant* (by G) if $G = G_v$.

Definition 3.2. We call $v \in V$ quasi-invariant by G if the closure of the orbit $G \cdot v$ is compact. Moreover, v is called *strongly quasi-invariant* if $\overline{G}_{\mathbb{R}}^{z} \cdot v$ has compact closure in V.

Clearly, for any group G, strong quasi-invariance implies quasi-invariance. If $G \subseteq GL(V)$ is a real linear algebraic group, then quasi-invariance and strong quasi-invariance of $v \in V$ are equivalent.

Example 3.3. Let $T \subseteq GL(V)$ be a trigonalizable subgroup. Then $v \in V$ is quasiinvariant by T if and only if v is invariant by T.

Lemma 3.4. Let $G \subseteq GL(V)$ be a real linear algebraic group and $v \in V$. Put $H = G_v$. Suppose that v is quasi-invariant by G. Then H contains the isotropic kernel G_{is} of G. In particular, H is a uniform subgroup of G.

Proof. By Example 3.3, $H = G_v$ must contain every trigonalizable subgroup of G. Therefore, H contains G_{is} .

3.2.2. Invariant probability measures. We now consider G-invariant probability measures on V.

Example 3.5 (Invariant measures are supported on fixed points). Let $A \subseteq GL(V)$ be a one-dimensional trigonalizable group and μ an A-invariant probability measure. If μ is ergodic then, by Zimmer [28, Proposition 2.1.10], μ is supported on some orbit $A \cdot v$. Since the Haar-measure on A is infinite, we must have $A \cdot v = v$. By the ergodic decomposition theorem [26, Chapter 5], any invariant probability measure μ may be approximated by finite convex combinations of ergodic invariant measures, which (as we just remarked) are all supported on the fixed point set V^A . Therefore, any A-invariant probability measure μ is supported on V^A .

The following is a key observation in the context of invariant probability measures on representation spaces:

Proposition 3.6. Let $G \subseteq GL(V)$ and μ a *G*-invariant probability measure on *V*. Put $A = \overline{G}_{\mathbb{R}}^{\mathbb{Z}}$. Then the following hold:

- (1) μ is invariant by A.
- (2) The action of A on the support of μ factors over a compact group.
- (3) There exists a G-invariant subspace $W \subseteq V$ containing the support of μ , such that the action of G on W factors over a compact group.
- (4) If $A = A_{is}$ then μ is supported on V^A .

Proof. Embed V as an affine subspace into some real projective space $\mathbb{P}^n\mathbb{R}$. For any probability measure ν on $\mathbb{P}^n\mathbb{R}$, let $\mathrm{PGL}(n)_{\nu}$ denote the stabilizer of ν in the projective linear group. By [28, Theorem 3.2.4], $\mathrm{PGL}(n)_{\nu}$ is a real algebraic subgroup of $\mathrm{PGL}(n)$. Since μ pushes forward to a *G*-invariant measure on $\mathbb{P}^n\mathbb{R}$, this implies (1).

To prove (4), observe that A_{is} is generated by its one-dimensional split subgroups. In light of Example 3.5, we deduce that $\operatorname{supp} \mu$ is contained in the fixed point set of A_{is} . In particular, A_{is} acts trivially on the support of μ , so (2) follows. Let W be the smallest G-invariant subspace of V that contains $\operatorname{supp} \mu$. Then A_{is} acts trivially on W. Hence (3) holds.

Note that the action of G on W factors over a compact group and the elements of W are therefore strongly quasi-invariant by G.

3.3. The characteristic map of a geometric *G*-manifold. Let *M* be a differentiable manifold, and ϑ an *s*-multilinear form on *M*. The group of diffeomorphisms of *M* that preserve ϑ will be denoted by Aut(*M*, ϑ). Let

$$G \subseteq \operatorname{Aut}(M, \vartheta)$$

be a Lie group. We may identify the Lie algebra \mathcal{G} of G with the subalgebra of vector fields on M whose flows are one-parameter subgroups of diffeomorphisms contained in G. We let $L^{s}(\mathcal{G})$ denote the space of *s*-linear forms on \mathcal{G} , and consider the associated map

$$\Phi: M \to L^s(\mathcal{G}), \quad p \mapsto \Phi_p$$

where, for $X_1, \ldots, X_s \in \mathcal{G}$, the multilinear form Φ_p on \mathcal{G} is declared by

$$\Phi_p(X_1,\ldots,X_s) = \vartheta_p(X_{1,p},\ldots,X_{s,p}) .$$

The adjoint representation Ad of G on G induces a representation

$$\varrho: G \to \mathrm{GL}(L^s(\mathcal{G})) \ .$$

The map Φ is equivariant with respect to the action of G on M and the representation ϱ .

Theorem 3.7. Let M be a differentiable manifold and $G \subseteq Aut(M, \vartheta)$. Suppose that the Lie group G preserves a finite measure on M. Then the action of G on

$$\Phi(M) \subseteq L^s(\mathcal{G})$$

factors over a compact group. Moreover, the forms Φ_p , $p \in M$, are strongly quasiinvariant by G.

Proof. Put $V = L^s(\mathcal{G})$ and $A = \overline{\varrho(G)}_{\mathbb{R}}^{\mathbb{Z}}$. Since Φ is *G*-equivariant, the finite *G*-invariant measure μ on *M* pushes forward to a finite *G*-invariant measure μ' on *V*, which is supported on $\Phi(M)$. By Proposition 3.6 (3), there exists an *A*-invariant linear subspace *W* containing $\Phi(M)$ on which the action of *A* factors over a compact group. In particular, Φ_p is strongly quasi-invariant for all $p \in M$.

3.4. Nil-invariant and quasi-invariant bilinear forms on Lie algebras. Let G be a Lie algebra and consider its inner automorphism group

$$G = \operatorname{Inn}(G)$$
.

By definition, G is the connected Lie subgroup of $GL(\mathcal{G})$ that is tangent to the Lie algebra of linear transformations

$$\operatorname{ad}(\mathcal{G}) = \{\operatorname{ad}(x) \mid x \in \mathcal{G}\} \subseteq \mathcal{GL}(\mathcal{G}).$$

We also consider the real Zariski closure

$$\overline{\mathrm{Inn}(\mathcal{G})}^{\mathrm{z}} \subseteq \mathrm{Aut}(\mathcal{G}) \subseteq \mathrm{GL}(\mathcal{G})$$

of $Inn(\mathcal{G})$, which is an algebraic group of automorphisms of \mathcal{G} with Lie algebra

We are interested in the action of G on the space of bilinear forms on G. The following notions are fundamental (cf. Section 3.2.1):

Definition 3.8. Let β be a bilinear form on \mathcal{G} . Then β is called *invariant* if G is contained in the orthogonal group $O(\beta)$ of β . Likewise, β will be called *quasi-invariant* if

$$\overline{\mathrm{Inn}(\mathcal{G})}^{z} \cap \mathrm{O}(\beta) \subseteq \overline{\mathrm{Inn}(\mathcal{G})}^{z}$$

is the inclusion of a uniform subgroup.

According to Lemma 3.4, β is quasi-invariant if and only if

(3.2)
$$\overline{\operatorname{Inn}(\mathcal{G})}^{z} \cap \operatorname{O}(\beta) \supseteq \overline{\operatorname{Inn}(\mathcal{G})}_{\mathrm{is}}^{z}$$

That is, the stabilizer of β contains the maximal normal subgroup

$$\overline{\mathrm{Inn}(\mathcal{G})}_{\mathrm{is}}^{\mathbf{z}}$$

that is generated by linear maps in $\overline{\text{Inn}(G)}^{z}$ that are trigonalizable over \mathbb{R} .

3.5. The quasi-invariance condition. The condition (3.2) may be formulated in Lie algebra terms only:

Lemma 3.9. The bilinear form β is quasi-invariant by G if and only if the Lie algebra

$$\overline{\mathrm{ad}(\mathcal{G})}^{\mathrm{z}} \cap \mathcal{O}(\beta)$$

contains all elements of $\overline{\mathrm{ad}(\mathcal{G})}^{z}$ that are trigonalizable over \mathbb{R} .

The Lie algebra $\overline{\operatorname{ad}(\mathcal{G})}_{is}^{z}$ generated by all elements trigonalizable over \mathbb{R} is tangent to $\overline{\operatorname{Inn}(\mathcal{G})}_{is}^{z}$. In particular, $\overline{\operatorname{ad}(\mathcal{G})}_{is}^{z}$ is an ideal in $\overline{\operatorname{ad}(\mathcal{G})}^{z}$, since $\overline{\operatorname{Inn}(\mathcal{G})}_{is}^{z}$ is a normal subgroup of $\operatorname{Inn}(\mathcal{G})_{is}$. The following is a noteworthy consequence:

Lemma 3.10. Suppose that β is a quasi-invariant bilinear form. Then

$$\overline{\mathrm{Inn}(\mathcal{G})}_{\mathrm{is}}^{^{\mathrm{z}}}(\mathcal{G}) \subseteq \mathcal{G}_{\beta}$$

That is, for any $\varphi \in \overline{\mathrm{ad}(\mathcal{G})}_{\mathrm{is}}^{\mathbb{Z}}$ and $a \in \mathcal{G}$, we have $\varphi(a) \in \mathcal{G}_{\beta}$.

Proof. Since $\overline{\mathrm{ad}(\mathcal{G})}_{is}^{z}$ is an ideal of $\overline{\mathrm{ad}(\mathcal{G})}^{z}$, we have

$$[\overline{\mathrm{ad}(\mathcal{G})}_{\mathrm{is}}^{\mathrm{z}}, \mathrm{ad}(\mathcal{G})] \subseteq \overline{\mathrm{ad}(\mathcal{G})}_{\mathrm{is}}^{\mathrm{z}}.$$

But $\operatorname{ad}(\mathcal{G})$ is also an ideal in the Lie algebra $\operatorname{Der}(\mathcal{G})$ of all derivations of \mathcal{G} . This implies that, for any $\varphi \in \operatorname{ad}(\mathcal{G})_{is}^{z}$,

$$[\varphi, \mathrm{ad}(a)] = \mathrm{ad}(\varphi(a)) \in \mathrm{ad}(\mathcal{G}) \cap \overline{\mathrm{ad}(\mathcal{G})}_{\mathrm{is}}^{2}$$

If β is quasi-invariant, we have $\overline{\operatorname{ad}(\mathcal{G})}_{is}^{z} \subseteq \mathcal{O}(\beta)$. Therefore, for any $\varphi \in \overline{\operatorname{ad}(\mathcal{G})}_{is}^{z}$, $\operatorname{ad}(\varphi(a)) \in \overline{\operatorname{ad}(\mathcal{G})}_{is}^{z} \subseteq \mathcal{O}(\beta)$. In particular, $\varphi(a) \in \mathcal{G}_{\beta}$.

3.6. The nil-invariance condition. The following condition is somewhat weaker than quasi-invariance since it takes into account only the nilpotent elements in the endomorphism Lie algebra $\overline{\operatorname{ad}(\mathcal{G})}^{z}$:

Definition 3.11. A bilinear form β on \mathcal{G} is called *nil-invariant* if

$$\varphi \in \mathcal{O}(\beta)$$

for all nilpotent elements φ of the Lie algebra $\overline{\mathrm{ad}(\mathcal{G})}^{\mathtt{a}}$.

We fix a Levi decomposition

$$\mathcal{G} = (\mathcal{K} \times \mathcal{S}) \ltimes \mathcal{R},$$

where \mathcal{K} is semisimple of compact type, \mathcal{S} is semisimple without factors of compact type, and \mathcal{R} is the solvable radical of \mathcal{G} .

Lemma 3.12. If β is nil-invariant, then

$$(3.3) \qquad \qquad \mathcal{S} + \mathcal{N} \subseteq \mathcal{G}_{\beta}.$$

To see that S is contained in \mathcal{G}_{β} recall that the subalgebra S is generated by the set of elements $x \in S$ such that $\mathrm{ad}_{G}(x)$ is nilpotent (for example [4, Lemma 5.1]).

3.7. Quasi-invariant and nil-invariant symmetric bilinear forms. In the following, we recall important structural properties of Lie algebras with nil-invariant symmetric bilinear form.

Theorem 3.13 ([4, Theorem A, Corollary C]). Let \mathcal{G} be a finite-dimensional real Lie algebra with nil-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$. Let $\langle \cdot, \cdot \rangle_{\mathcal{G}_s}$ denote the restriction of $\langle \cdot, \cdot \rangle$ to $\mathcal{G}_s = \mathcal{S} \ltimes \mathcal{R}$. Then:

- (1) $\langle \cdot, \cdot \rangle_{\mathcal{G}_s}$ is invariant by the adjoint action of \mathcal{G} on \mathcal{G}_s .
- (2) $\langle \cdot, \cdot \rangle$ is invariant by the adjoint action of \mathcal{G}_{s} .

Furthermore, if $(\mathcal{G}, \langle \cdot, \cdot \rangle)$ is effective, then

(3) $\mathcal{G}^{\perp} \subseteq \mathcal{K} \ltimes \mathcal{Z}(\mathcal{G}_{s}) \text{ and } [\mathcal{G}^{\perp}, \mathcal{G}_{s}] \subseteq \mathcal{Z}(\mathcal{G}_{s}) \cap \mathcal{G}^{\perp}. \text{ If } \mathcal{G}^{\perp} \subseteq \mathcal{G}_{\langle \cdot, \cdot \rangle} \text{ then } [\mathcal{G}^{\perp}, \mathcal{G}_{s}] = \mathbf{0}.$

By definition, quasi-invariance always implies nil-invariance. It is somewhat surprising and non-trivial that for a *symmetric* bilinear form also the converse holds and an even stronger implication is satisfied in the solvable case: **Corollary 3.14.** Let β be a symmetric bilinear form on G. Then β is quasiinvariant if and only if β is nil-invariant. Moreover, if G is solvable then any nil-invariant symmetric bilinear form β is invariant.

Corollary 3.14 is directly implied by Theorem 3.13, which strengthens (3.3) for symmetric bilinear forms considerably.

3.8. Nil-invariant and quasi-invariant skew-symmetric bilinear forms. As noted above for symmetric bilinear forms on Lie algebras, nil-invariance already implies quasi-invariance. The following simple example illustrates that, in general, this is not the case for skew-symmetric bilinear forms:

Example 3.15. Let \mathcal{G} be the two-dimensional solvable Lie algebra with generators x, y, satisfying [x, y] = y. Consider the skew-symmetric form

$$\omega = \mathrm{d}x \wedge \mathrm{d}y$$

on \mathcal{G} (which satisfies $\omega(x, y) = 1$). It is easily verified that ω is invariant by $\operatorname{ad}(y)$, but not by $\operatorname{ad}(x)$. This means ω is nil-invariant without being quasi-invariant.

In the following Section 4.1, we will develop the structure theory of quasiinvariant and nil-invariant skew-symmetric forms in more detail.

4. Nil-invariant and quasi-invariant h-structures

This section is devoted to the study of quasi-invariant h-structures.

4.1. Lie algebras with skew-symmetric form. Let ω be a skew-symmetric bilinear form on the Lie algebra G.

Lemma 4.1. Let $x, z \in \mathcal{G}, y \in \mathcal{G}_{\omega}$. Then

(4.1)
$$\omega([x,y],z) = \omega([z,y],x) .$$

If also $z \in \mathcal{G}_{\omega}$ then

(4.2) $\omega(\mathrm{ad}(x)y,z) = \omega(y,\mathrm{ad}(x)z),$

In particular,

$$(4.3) \qquad \qquad \left[\mathcal{G}_{\omega},\mathcal{G}_{\omega}\right] \bot_{\omega} \mathcal{G}_{\omega}$$

Proof. Since $y \in \mathcal{G}_{\omega}$, $\omega([x, y], z) = -\omega(x, [z, y]) = \omega([z, y], x)$. Hence, if $z \in \mathcal{G}_{\omega}$ then

$$\omega([x,y],z) = \omega([z,x],y) = \omega(y,[x,z]).$$

Thus (4.2) holds. If furthermore $x \in \mathcal{G}_{\omega}$, then

 $\omega([x,y],z) = -\omega(y,[x,z]).$

So (4.2) implies (4.3).

Lemma 4.2. Let $\varphi \in \mathcal{O}(\omega)$ satisfy $\varphi(\mathcal{G}) \subseteq \mathcal{G}_{\omega}$. Then $\varphi([\mathcal{G}_{\omega}, \mathcal{G}_{\omega}]) \subseteq \mathcal{G}^{\perp_{\omega}}$.

Proof. Let $x, y \in \mathcal{G}_{\omega}$ and $a \in \mathcal{G}$. Then $\omega(\varphi([x, y]), a) = -\omega([x, y], \varphi(a))$. Since $\varphi(a) \in \mathcal{G}_{\omega}, (4.3)$ implies $\varphi([\mathcal{G}_{\omega}, \mathcal{G}_{\omega}]) \perp_{\omega} \mathcal{G}$.

4.1.1. Invariant skew-symmetric forms. Recall that the skew-symmetric form ω is called invariant if $\mathcal{G} = \mathcal{G}_{\omega}$, and that (\mathcal{G}, ω) is called *effective* if $\mathcal{G}^{\perp \omega}$ does not contain any non-trivial ideal of \mathcal{G} . We deduce from (4.3):

Proposition 4.3. Let ω be an invariant skew-symmetric form. Then we have

 $[\mathcal{G},\mathcal{G}] \subseteq \mathcal{G}^{\perp_{\omega}}.$

In particular, if (\mathcal{G}, ω) is effective, then \mathcal{G} is abelian and ω is non-degenerate.

4.1.2. Nil-invariant skew-symmetric forms. We now assume that ω is nil-invariant (in the sense of Definition 3.11). This implies in particular that the nilradical \mathcal{N} of \mathcal{G} , and also any semisimple subalgebra \mathcal{S} of non-compact type are contained in \mathcal{G}_{ω} . In fact, as we show now, any semisimple subalgebra \mathcal{S} of non-compact type is contained in the kernel of ω :

Proposition 4.4 (Levi subalgebra is of compact type). Let ω be a nil-invariant skew-symmetric form on \mathcal{G} . Let $\mathcal{I}(\mathcal{S})$ be an ideal of \mathcal{G} generated by a semisimple subalgebra \mathcal{S} of non-compact type. Then

 $\mathcal{I}(\mathcal{S}) \subseteq \mathcal{G}^{\perp_{\omega}}$.

In particular, if (G, ω) is effective, then every semisimple subalgebra of G is of compact type.

Proof. Recall that any derivation of the solvable radical \mathcal{R} of \mathcal{G} has image in \mathcal{N} ([19, Theorem III.7]). Since the semisimple Lie algebra \mathcal{S} acts reductively, we infer that $[\mathcal{S}, \mathcal{R}] = [\mathcal{S}, \mathcal{N}]$. We may assume that the image of \mathcal{S} is an ideal in \mathcal{G}/\mathcal{R} . Therefore

$$[\mathcal{S},\mathcal{G}] = \mathcal{S} + [\mathcal{S},\mathcal{N}] \subseteq [\mathcal{G}_{\omega},\mathcal{G}_{\omega}].$$

By Lemma 4.2, $\mathcal{S} + [\mathcal{S}, \mathcal{N}] \subseteq \mathcal{G}^{\perp_{\omega}}$. Then we have

$$[\mathcal{G}, [\mathcal{S}, \mathcal{N}]] = [\mathcal{S}, \mathcal{N}] + [\mathcal{N}, [\mathcal{S}, \mathcal{N}]].$$

By induction, the ideal \mathcal{B} of \mathcal{G} generated by $[\mathcal{S}, \mathcal{N}]$ equals the ideal of \mathcal{N} generated by $[\mathcal{S}, \mathcal{N}]$. Since $[\mathcal{N}, \mathcal{G}^{\perp_{\omega}}] \subseteq \mathcal{G}^{\perp_{\omega}}$, we deduce that $\mathcal{B} \subseteq \mathcal{G}^{\perp_{\omega}}$. Hence, $\mathcal{I}(\mathcal{S}) = \mathcal{S} + \mathcal{B} \subseteq \mathcal{G}^{\perp_{\omega}}$.

Let \mathcal{R} denote the solvable radical of \mathcal{G} . We write $\mathcal{R}_{\omega} = \mathcal{R} \cap \mathcal{G}_{\omega}$. Since $\mathcal{N} \subseteq \mathcal{R}_{\omega}$, \mathcal{R}_{ω} is an ideal of \mathcal{G} .

Lemma 4.5.

- (1) $[\mathcal{R}_{\omega}, [\mathcal{G}_{\omega}, \mathcal{G}_{\omega}]] \subseteq \mathcal{G}^{\perp_{\omega}}.$
- (2) $[\mathcal{N}, \mathcal{N}] \perp_{\omega} \mathcal{N}.$
- (3) $[[\mathcal{N}, \mathcal{N}]^{\perp_{\omega}}, [\mathcal{N}, \mathcal{N}]] \subseteq \mathcal{G}^{\perp_{\omega}}.$
- (4) $[\mathcal{G}, \mathcal{N}] \perp_{\omega} Z_{\mathcal{G}}(\mathcal{N}).$

Proof. Since $[\mathcal{R}_{\omega}, \mathcal{G}] \subseteq \mathcal{N} \subseteq \mathcal{G}_{\omega}$, Lemma 4.2 implies (1). Whereas (2) follows from (4.3), and (3) from Lemma 2.4. Since $\mathcal{N} \subseteq \mathcal{G}_{\omega}$, (4) is immediate.

It then follows:

Lemma 4.6. Let (\mathcal{G}, ω) be effective. Then

- (1) $\mathcal{R}_{\omega} = \mathcal{N}.$
- (2) \mathcal{N} is at most two-step nilpotent (that is, $[\mathcal{N}, \mathcal{N}] \subseteq \mathcal{Z}(\mathcal{N})$).

Proof. Since $[\mathcal{G}, \mathcal{R}_{\omega}] \subseteq \mathcal{N} \subseteq \mathcal{R}_{\omega}$, \mathcal{R}_{ω} is an ideal in \mathcal{G} . By Lemma 4.5, the ideal $[\mathcal{R}_{\omega}, [\mathcal{R}_{\omega}, \mathcal{R}_{\omega}]]$ is contained in $\mathcal{G}^{\perp_{\omega}}$. Therefore, $[\mathcal{R}_{\omega}, [\mathcal{R}_{\omega}, \mathcal{R}_{\omega}]] = \mathbf{0}$. Hence, \mathcal{R}_{ω} is a nilpotent ideal of \mathcal{G} , so that $\mathcal{R}_{\omega} \subseteq \mathcal{N}$. Hence (1) and (2) are satisfied.

Proposition 4.7. Let ω be a nil-invariant skew symmetric form, such that (\mathcal{G}, ω) is effective. If \mathcal{G} is nilpotent then \mathcal{G} is abelian.

Proof. Since $\mathcal{G} = \mathcal{N}$ is nilpotent, Lemma 4.5 (2) implies that $[\mathcal{N}, \mathcal{N}] \subseteq \mathcal{N}^{\perp_{\omega}}$ is an ideal of \mathcal{N} . Since (\mathcal{N}, ω) is effective this implies $[\mathcal{N}, \mathcal{N}] = \mathbf{0}$.

We summarize the above as follows:

Theorem 4.8. Let ω be a nil-invariant skew-symmetric form, such that (\mathcal{G}, ω) is effective. Then the following hold:

- (1) Any Levi subalgebra of \mathcal{G} is of compact type.
- (2) The nilradical \mathcal{N} of \mathcal{G} is at most two-step nilpotent.
- (3) If \mathcal{G} is nilpotent, then \mathcal{G} is abelian.

We remark that the nilradical of a Lie algebra with nil-invariant skew-symmetric form ω may be non-abelian, even for non-degenerate ω :

Example 4.9 (Solvable algebra with nilradical \mathcal{H}_n of Heisenberg type). Consider a 2*n*-dimensional symplectic vector space (W, ω_0) . We extend ω_0 to a 2n + 2dimensional symplectic vector space

$$(\mathcal{G}_n = \mathcal{A} \oplus W \oplus \mathcal{B}, \omega)$$

where $\mathcal{A} = \operatorname{span} a$ and $\mathcal{B} = \mathcal{A}^* = \operatorname{span} z$ are dually paired, one-dimensional vector spaces. For the definition of ω we require:

 $W \perp_{\omega} (\mathcal{A} \oplus \mathcal{B}), \quad \omega(a, z) = \langle a, z \rangle = 1 \quad \text{and} \quad \omega = \omega_0 \text{ on } W.$

Next, turn \mathcal{G} into a Lie algebra with the non-zero Lie brackets satisfying

 $[x,y] = \omega_0(x,y)z, \quad [a,x] = x, \quad [a,z] = 2z, \quad \text{where } x, y \in W.$

Thus \mathcal{G} is a split extension of the one-dimensional Lie algebra by a 2n+1 dimensional Heisenberg Lie algebra

$$\mathcal{H}_n = W \oplus \mathcal{B}.$$

The element $a \in \mathcal{G}$ acts on \mathcal{H}_n by the derivation $D: \mathcal{H}_n \to \mathcal{H}_n$, defined by Dx = x, $x \in W$, Dz = 2z. We observe further that ω is a nil-invariant *non-degenerate* skew-symmetric form on \mathcal{G} . For the nil-invariance property of (\mathcal{G}, ω) it is sufficient to show that, for all $v \in \mathcal{H}_n$, the operators $\operatorname{ad}(v): \mathcal{G} \to \mathcal{G}$ are skew with respect to ω . This follows mainly by the equations

$$\omega([a, x], y) = \omega(x, y) = \omega(a, [x, y]), \text{ for all } x, y \in W.$$

16

4.1.3. Quasi-invariant skew-symmetric forms. By Definition 3.8 and Lemma 3.9 a skew-symmetric form ω on the Lie algebra \mathcal{G} is quasi-invariant if and only if all trigonalizable elements in the smallest algebraic Lie algebra containing $\operatorname{ad}(\mathcal{G})$ are skew with respect to ω . Clearly, quasi-invariance of ω implies nil-invariance but the converse is not true, as already observed in Example 3.15.

Example 4.10. Any quasi-invariant skew-form ω on a solvable Lie algebra \mathcal{G} of real type (where the adjoint representation has only real eigenvalues) must be invariant. Thus Proposition 4.3 shows that for any effective pair (\mathcal{G}, ω) , where ω is quasi-invariant and \mathcal{G} solvable of real type, \mathcal{G} is abelian. In particular, the family of nil-invariant almost symplectic solvable Lie algebras (\mathcal{G}_n, ω) from Example 4.9 is not quasi-invariant.

The following property distinguishes quasi-invariant and nil-invariant skew-symmetric forms:

Proposition 4.11. Suppose that ω is quasi-invariant. Then

$$[\mathcal{G}, [\mathcal{N}, \mathcal{N}]] \subseteq [\mathcal{N}, \mathcal{N}] \cap \mathcal{G}^{\perp_{\omega}}.$$

Proof. Define $U = \mathcal{G}/[\mathcal{N}, \mathcal{N}]^{\perp_{\omega}}$ and $V = [\mathcal{N}, \mathcal{N}]/([\mathcal{N}, \mathcal{N}] \cap \mathcal{G}^{\perp_{\omega}})$. By (4.1) and (3) of Lemma 4.5, the map

 $(u, u', v) \mapsto f(u, u', v) \coloneqq \omega([u, v], u'), \quad u, u' \in \mathcal{G}, v \in [\mathcal{N}, \mathcal{N}]$

descends to an element $\overline{f} \in S^2U^* \otimes V^*$. Observe that ω induces a dual pairing of U and V, and thus also an isomorphism $U \to V^*$. Hence, we may view \overline{f} as an element of $S^2U^* \otimes U$.

For $u \in \mathcal{G}$, decompose $\operatorname{ad}(u) = \varphi_i + \varphi_s$ as a sum of commuting derivations of \mathcal{G} , where φ_i is semisimple with purely imaginary eigenvalues and φ_s is trigonalizable over the reals (see the remark following below). The assumption that ω is quasiinvariant implies that $\varphi_s \in \mathcal{O}(\omega)$. Moreover, by Lemma 3.10, $\varphi_s(\mathcal{G}) \subseteq \mathcal{G}_{\omega}$. Hence, Lemma 4.2 shows that

$$\varphi_s([\mathcal{N},\mathcal{N}]) \subseteq [\mathcal{N},\mathcal{N}] \cap \mathcal{G}^{\perp_{\omega}}$$

Therefore, $f(u, u', v) = \omega(\varphi_i(v), u')$.

We deduce that \overline{f} defines an element of the first prolongation of the orthogonal Lie algebra $\mathcal{O}(\langle \cdot, \cdot \rangle)$ for some suitable positive definite scalar product $\langle \cdot, \cdot \rangle$ on U. Since the orthogonal Lie algebras have trivial first prolongation (cf. [21, Chapter I, Example 2.5]), we deduce that $\overline{f} = 0$.

Remark (Jordan decomposition in algebraic Lie algebras). For any element $\varphi \in \mathcal{GL}(\mathcal{G})$, let $\varphi = \varphi_s + \varphi_n$ denote its Jordan decomposition, where $\varphi_s \in \mathcal{GL}(\mathcal{G})$ is semisimple, $\varphi_n \in \mathcal{GL}(\mathcal{G})$ is nilpotent, and

$$[\varphi_{\rm s},\varphi_{\rm n}]=0.$$

Suppose now that $\varphi \in \overline{\mathrm{ad}(\mathcal{G})}^{\mathbb{Z}}$. Since the Lie algebra $\overline{\mathrm{ad}(\mathcal{G})}^{\mathbb{Z}}$ is the Lie algebra of an algebraic group, it also contains the Jordan parts $\varphi_{\mathrm{s}}, \varphi_{\mathrm{n}}$ of φ . Moreover,

as follows for example from [5, §8.15], $\varphi_{\rm s} = \varphi_{\rm i} + \varphi_{\rm split}$ decomposes uniquely as a sum of commuting semisimple derivations both contained in $\overline{{\rm ad}(\mathcal{G})}^{\rm z}$ such that the eigenvalues of $\varphi_{\rm i}$ are imaginary and $\varphi_{\rm split}$ is diagonalizable over \mathbb{R} .

Proposition 4.12. Suppose that ω is quasi-invariant. If (\mathcal{G}, ω) is effective then the following hold:

- (1) $[\mathcal{N}, \mathcal{N}] \cap \mathcal{G}^{\perp_{\omega}} = \mathbf{0}.$
- (2) $[\mathcal{N}, \mathcal{N}] \subseteq Z(\mathcal{G}).$

Proof. By Proposition 4.11, $[\mathcal{G}, [\mathcal{N}, \mathcal{N}]] \subseteq [\mathcal{N}, \mathcal{N}] \cap \mathcal{G}^{\perp_{\omega}}$. In particular, $[\mathcal{N}, \mathcal{N}] \cap \mathcal{G}^{\perp_{\omega}}$ is an ideal in \mathcal{G} . By effectiveness of (\mathcal{G}, ω) , (1) and also (2) follow.

Lemma 4.13. Let ω be a nil-invariant skew-symmetric form on \mathcal{G} . Then, for all $a \in \mathcal{R}, n, n' \in \mathcal{N}$, we have:

$$\left[[a,n],n' \right] - \left[n, [a,n'] \right] \in \mathcal{G}^{\perp \omega}$$

Proof. Let $b \in G$. Note that $[a, b] \in \mathcal{N}$, since $a \in \mathcal{R}$. Therefore, by Lemma 4.5 (2),

(4.4)
$$\omega([[a,b],n],n') = 0$$

Now

$$\begin{split} & \omega([a,[b,n]],n') = \omega(a,[[b,n],n']) = \omega([n,[n',a]],b) \text{ and} \\ & \omega([b,[a,n]],n') = \omega(b,[[a,n],n']) . \end{split}$$

In the view of (4.4), we deduce $\omega(b, [n, [a, n']]) = \omega(b, [[a, n], n'])$.

Proposition 4.14. Suppose that ω is quasi-invariant. If (\mathcal{G}, ω) is effective then

$$[\mathcal{R},\mathcal{N}] \subseteq Z(\mathcal{N}).$$

Moreover, this implies $[\mathcal{N}, \mathcal{N}] \subseteq \mathcal{R}^{\perp_{\omega}}$ and $[\mathcal{R}, \mathcal{N}] \subseteq \mathcal{N}^{\perp_{\omega}}$.

Proof. Let $a \in \mathcal{R}$, $n, n' \in \mathcal{N}$. By Lemma 4.13, $[[a, n], n'] - [n, [a, n']] \in [\mathcal{N}, \mathcal{N}] \cap \mathcal{G}^{\perp_{\omega}}$. According to Proposition 4.12, $[\mathcal{N}, \mathcal{N}] \subseteq \mathcal{Z}(\mathcal{G})$. In particular, $[\mathcal{N}, \mathcal{N}] \cap \mathcal{G}^{\perp_{\omega}}$ is an ideal in \mathcal{G} . By effectiveness, $[\mathcal{N}, \mathcal{N}] \cap \mathcal{G}^{\perp_{\omega}} = \mathbf{0}$. Hence, [[a, n], n'] = [n, [a, n']].

On the other hand, $[\mathcal{N}, \mathcal{N}] \subseteq \mathcal{Z}(\mathcal{G})$ and the Jacobi identity imply [[a, n], n'] + [n, [a, n']] = 0. We deduce [[a, n], n'] = -[n, [a, n']]. Hence, [[a, n], n'] = 0.

Let $r \in \mathcal{R}$ and let $\operatorname{ad}(r) = \operatorname{ad}(r)_{s} + \operatorname{ad}(r)_{n}$ be its Jordan decomposition. To show that $r \perp_{\omega} [\mathcal{N}, \mathcal{N}]$, decompose $\mathcal{N} = W + \mathcal{Z}(\mathcal{N})$, with $\operatorname{ad}(r)_{s}W = \mathbf{0}$. In particular, $[\mathcal{N}, \mathcal{N}] = [W, W]$. Let $u, v \in W$. Then $\omega(r, [u, v]) = \omega([r, u], v) = \omega(\operatorname{ad}(r)_{n}u, v) = -\omega(u, \operatorname{ad}(r)_{n}v) = -\omega(u, [r, v]) = -\omega(r, [u, v])$. Therefore, $\omega(r, [u, v]) = 0$. \Box

Theorem 4.15. Let ω be a quasi-invariant skew-symmetric form on a Lie algebra

$$\mathcal{G} = \mathcal{K} \times \mathcal{R}$$

Assume further that (\mathcal{G}, ω) is effective. Then the following hold:

- (1) The nilradical \mathcal{N} of \mathcal{G} is abelian.
- (2) The solvable radical \mathcal{R} of \mathcal{G} is of imaginary type (meaning that the eigenvalues of the adjoint representation are imaginary).

Proof. By Proposition 4.14, $[\mathcal{N}, \mathcal{N}] \subseteq \mathcal{R}^{\perp_{\omega}}$. Since \mathcal{K} and \mathcal{N} commute, $[\mathcal{N}, \mathcal{N}] \perp \mathcal{K}$. This shows $[\mathcal{N}, \mathcal{N}] \subseteq \mathcal{G}^{\perp_{\omega}}$. Since (\mathcal{G}, ω) is effective, $[\mathcal{N}, \mathcal{N}] = \mathbf{0}$. Hence (1) holds.

Let $a \in \mathcal{R}$, and as above decompose the Jordan semisimple part of ad(a) as a canonical sum of commuting semisimple derivations: $ad(a)_s = ad(a)_i + ad(a)_{split}$. By quasi-invariance of ω , $\varphi = ad(a)_{split}$ is a skew derivation of (\mathcal{G}, ω) .

By Proposition 4.14, $[\mathcal{R}, \mathcal{N}] \subseteq \mathcal{Z}(\mathcal{N}) \cap \mathcal{N}^{\perp_{\omega}}$ and therefore $\varphi(\mathcal{N}) = \varphi(\mathcal{Z}(\mathcal{N}) \cap \mathcal{N}^{\perp_{\omega}})$. Using that φ is skew, we compute, for any $n \in \mathcal{Z}(\mathcal{N}) \cap \mathcal{N}^{\perp_{\omega}}$, $b \in \mathcal{R}$, that $\omega(\varphi(n), b) = -\omega(n, \varphi(b)) = 0$, since $\varphi(b) \in \mathcal{N}^{\perp_{\omega}}$. Therefore, $\varphi(\mathcal{N}) = \varphi(\mathcal{Z}(\mathcal{N}) \cap \mathcal{N}^{\perp_{\omega}}) \subseteq \mathcal{R}^{\perp_{\omega}}$. From $[\mathcal{K}, \mathcal{R}] = \mathbf{0}$, we infer $\varphi(\mathcal{K}) = \mathbf{0}$, which implies $\mathcal{K} \perp_{\omega} \varphi(\mathcal{G})$ and $\varphi(\mathcal{N}) \subseteq \mathcal{G}^{\perp_{\omega}}$.

Observe that $[b, \varphi(n)] = \varphi([b, n]) + [ad(b), \varphi](n)$. Since \mathcal{N} is abelian, the restrictions of ad(a) and ad(b) to \mathcal{N} commute. Hence, the Jordan part $ad(a)_s$ also commutes with ad(b). Therefore, φ commutes with ad(b). This implies $[ad(b), \varphi](n) = 0$, showing $[b, \varphi(\mathcal{N})] \subseteq \varphi(\mathcal{N})$. Hence, $\varphi(\mathcal{N})$ is an ideal in \mathcal{G} . Since (\mathcal{G}, ω) is effective, we deduce that $\varphi(\mathcal{N}) = \mathbf{0}$. Hence, $\varphi = 0$ and $ad(a)_s = ad(a)_i$, showing (2).

Corollary 4.16. Let \mathcal{R} be a solvable Lie algebra with a quasi-invariant non-degenerate skew-symmetric form ω . Then \mathcal{R} is abelian.

Proof. We may decompose $\mathcal{N} = W \oplus (\mathcal{N} \cap \mathcal{N}^{\perp \omega})$ and $\mathcal{G} = \mathcal{A} \oplus \mathcal{N}$, such that $W^{\perp \omega} = \mathcal{A} \oplus (\mathcal{N} \cap \mathcal{N}^{\perp \omega})$. Since $\mathcal{N} \cap \mathcal{N}^{\perp \omega} \perp_{\omega} W$ and ω is non-degenerate, dim $\mathcal{A} \ge \dim \mathcal{N} \cap \mathcal{N}^{\perp \omega}$. On the other hand, \mathcal{A} is faithfully represented on $\mathcal{N} \cap \mathcal{N}^{\perp \omega}$ by an abelian Lie algebra of derivations with imaginary eigenvalues only. This shows dim $\mathcal{N} \cap \mathcal{N}^{\perp \omega} > \dim \mathcal{A}$, unless dim $\mathcal{A} = 0$. We conclude $\mathcal{A} = \mathcal{N} \cap \mathcal{N}^{\perp \omega} = \mathbf{0}$. Therefore, $\mathcal{G} = W$ is abelian. \Box

4.2. Lie algebras with nil-invariant *h*-structure. Let $(\langle \cdot, \cdot \rangle, J)$ be an *h*-structure for the Lie algebra \mathcal{G} . The *h*-structure

 $(\mathcal{G}, \langle \cdot, \cdot \rangle, J)$

is called *nil-invariant* if *both* the bilinear form $\langle \cdot, \cdot \rangle$ and the fundamental two-form ω are nil-invariant. Similarly, the *h*-structure is called *quasi-invariant* if both forms are quasi-invariant forms under the adjoint action of \mathcal{G} .

4.2.1. Invariant h-structures. The h-structure $(\langle \cdot, \cdot \rangle, J)$ on \mathcal{G} will be called an *invariant h*-structure if $\mathcal{G} = \mathcal{G}_h$. This of course is the case if and only if $\langle \cdot, \cdot \rangle$ and ω are invariant by \mathcal{G} . In particular, for an invariant h-structure (2.5) implies that $\mathcal{G} = \mathcal{G}_J^h$. Moreover, \mathcal{G}^{\perp} is an ideal in \mathcal{G} . The following is a direct consequence of Proposition 4.3:

Proposition 4.17. Let $(\langle \cdot, \cdot \rangle, \mathsf{J})$ be an invariant h-structure for \mathcal{G} . Then

$$(4.5) \qquad \qquad \left[\mathcal{G},\mathcal{G}\right] \subseteq \mathcal{G}^{\perp}$$

In particular, if $(\mathcal{G}, \langle \cdot, \cdot \rangle)$ is effective, then \mathcal{G} is abelian.

4.2.2. Nil-invariant h-structures. Let $(\mathcal{G}, \langle \cdot, \cdot \rangle, J)$ be a nil-invariant h-structure for \mathcal{G} . Let \mathcal{N} and \mathcal{R} denote the nilpotent and solvable radicals of \mathcal{G} . Nil-invariance implies that $\mathcal{N} \subseteq \mathcal{G}_{\omega}$, and also $\mathcal{R} \subseteq \mathcal{G}_{\langle \cdot, \cdot \rangle}$ by Theorem 3.13. Together with (2.6) we thus have inclusions

$$\mathcal{N} \subseteq \mathcal{R}_{\omega} = \mathcal{R}_{\langle \cdot, \cdot \rangle, \omega} \subseteq \mathcal{G}_h \subseteq \mathcal{G}_J^h.$$

Since $\mathcal{N} \subseteq \mathcal{G}_J^h$, Lemma 2.6 implies:

Lemma 4.18. $J[\mathcal{G}, \mathcal{N}] = [J\mathcal{G}, \mathcal{N}] \mod \mathcal{G}^{\perp}$ and $J\mathcal{N} = \mathcal{N} \mod \mathcal{G}^{\perp}$.

We then obtain:

Lemma 4.19. $[\mathcal{G}_{J}^{h}, [\mathcal{G}, \mathcal{N}]] \subseteq [\mathcal{G}, \mathcal{N}]^{\perp}$.

Proof. By Lemma 4.18, J maps the subspace $[\mathcal{G}, \mathcal{N}] + \mathcal{G}^{\perp}$ of \mathcal{G} to itself. In fact, since ker $J \subseteq \mathcal{G}^{\perp}$ and \mathcal{G}^{\perp} is stable by J, J induces a complex structure J' on $[\mathcal{G}, \mathcal{N}]/([\mathcal{G}, \mathcal{N}] \cap \mathcal{G}^{\perp})$ that is compatible with the symmetric bilinear form induced by $\langle \cdot, \cdot \rangle$. Thus $V = [\mathcal{G}, \mathcal{N}]/([\mathcal{G}, \mathcal{N}] \cap [\mathcal{G}, \mathcal{N}]^{\perp})$ is a Hermitian vector space. As $\langle \cdot, \cdot \rangle$ is nil-invariant, Theorem 3.13 states that, for any $x \in \mathcal{G}$, $\operatorname{ad}(x)$ induces a skew linear map $[\mathcal{G}, \mathcal{N}] \to [\mathcal{G}, \mathcal{N}]$. In particular, $\operatorname{ad}(x)$ descends to a linear map $\varphi_x \in \mathcal{O}(V)$. Moreover, since $[\mathcal{G}, \mathcal{N}] \subseteq \mathcal{N} \subseteq \mathcal{G}_J^h$, Lemma 2.6 implies $\varphi_{Jx} = J'\varphi_x$. In particular, $J'\varphi_x \in \mathcal{O}(V)$ is skew as well, which implies that φ_x is a complex anti-linear map (see Lemma 2.2 (1)). If $x \in \mathcal{G}_J^h$ then φ_x is also complex linear, so that $\varphi_x = 0$. \Box

Remark. One can prove in the same way the following stronger statement: $[\mathcal{G}_J^h, \mathcal{N}] \subseteq \mathcal{N}^{\perp}$.

Let $a \in \mathcal{G}_{\langle \cdot, \cdot \rangle}$. Choose a Fitting decomposition for the linear map $\operatorname{ad}(a)$ of \mathcal{G} . This is the unique $\operatorname{ad}(a)$ -invariant decomposition $\mathcal{G} = E_0(a) \oplus E_1(a)$, where $E_0(a) = \operatorname{ker} \operatorname{ad}(a)^n$, $n = \dim \mathcal{G}$, and $\operatorname{ad}(a) : E_1(a) \to E_1(a)$ is a vector space isomorphism. Furthermore, $E_0(a) \perp E_1(a)$, since $\operatorname{ad}(a) \in \mathcal{O}(\mathcal{G})$. This implies $E_1(a)^{\perp} \cap E_1(a) = \mathcal{G}^{\perp} \cap E_1(a)$.

Lemma 4.20. Let $a \in \mathcal{R}$. Then $E_1(a)$ is contained in $[\mathcal{G}, \mathcal{N}] \cap \mathcal{G}^{\perp}$.

Proof. Let $\varphi_a \in \mathcal{O}(V)$ denote the map induced by $\operatorname{ad}(a)$, where V is as in the proof of Lemma 4.19. Since $\mathcal{N} \subseteq \mathcal{G}_J^h$, \mathcal{N} acts trivially on V by Lemma 4.19. Since $[\mathcal{G}, \mathcal{R}] \subseteq \mathcal{N}$, this implies that φ_a commutes with φ_x for all $x \in \mathcal{G}$. In particular, φ_a commutes with $J\varphi_a = \varphi_{Ja}$. Thus, as remarked in (2) of Lemma 2.2, $\varphi_a^2 = 0$. Since $\operatorname{ad}(a)$ is an isomorphism on $E_1(a)$, we infer that $E_1(a) \subseteq [\mathcal{G}, \mathcal{N}] \cap [\mathcal{G}, \mathcal{N}]^{\perp}$. In particular, $E_1(a) \subseteq [\mathcal{G}, \mathcal{N}]^{\perp} \subseteq E_1(a)^{\perp}$. By the above remark on the Fitting decomposition of $\operatorname{ad}(a)$, this shows $E_1(a) \subseteq [\mathcal{G}, \mathcal{N}] \cap \mathcal{G}^{\perp}$.

Our main result on nil-invariant almost h-structures is now:

Theorem 4.21. Let \mathcal{G} be a Lie algebra with a nil-invariant and effective almost h-structure. Then $\mathcal{G} = \mathcal{K} \ltimes \mathcal{N}$, where \mathcal{K} is compact semisimple. Moreover, the nilradical \mathcal{N} is at most two-step nilpotent.

Proof. By Theorem 4.8, we have S = 0. Now let $a \in \mathcal{R}$. Lemma 4.20 states that $E_1(a) \subseteq \mathcal{N} \cap \mathcal{G}^{\perp}$. By Theorem 3.13 (3), $\mathcal{N} \cap \mathcal{G}^{\perp}$ is contained in $\mathcal{Z}(\mathcal{R})$. Since $a \in \mathcal{R}$, this implies $E_1(a) = 0$. Hence, ad(a) is nilpotent. This shows that $\mathcal{R} = \mathcal{N}$. By Lemma 4.6, \mathcal{N} is at most two-step nilpotent.

Remark. Note that, under the assumption of Theorem 4.21, also the various properties of Lemma 4.5 are satisfied for (\mathcal{G}, ω)

Corollary 4.22. Let G be a solvable Lie algebra with a nil-invariant and effective almost h-structure. Then G is abelian and the almost h-structure is non-degenerate.

Proof. By Theorem 4.21, \mathcal{G} is in fact nilpotent. Therefore the *h*-structure is invariant. Thus \mathcal{G} is abelian by Proposition 4.17

5. Application to almost pseudo-Hermitian and almost symplectic homogeneous spaces

Given a differentiable manifold M with almost complex structure J_M , a *pseudo-Hermitian metric* is a pseudo-Riemannian metric g_M satisfying the compatibility condition

$$g_M(J_M X, J_M Y) = g_M(X, Y)$$

for all vector fields X, Y on M. We call $(M, \mathsf{J}_M, \mathsf{g}_M)$ an almost pseudo-Hermitian manifold. If J_M is a complex structure, M is called a pseudo-Hermitian manifold. Associated to g_M and J_M is the fundamental two-form ω_M which is given by

$$\omega_M(X,Y) = \mathsf{g}_M(\mathsf{J}_M X,Y)$$

5.1. Lie algebra models for pseudo-Hermitian homogeneous spaces. Consider a homogeneous manifold

M = G/H

that admits a G-invariant almost complex structure J_M . Let \mathcal{G} denote the Lie algebra of G and \mathcal{H} the Lie subalgebra of \mathcal{G} corresponding to H. Then J_M can be described by (the choice of) an endomorphism J on the Lie algebra \mathcal{G} with the following properties:

- (J1) Jx = 0 for $x \in \mathcal{H}$.
- (J2) $J^2 x = -x \mod \mathcal{H}$ for all $x \in \mathcal{G}$.
- (J3) $JAd(h)x = Ad(h)Jx \mod \mathcal{H}$ for all $h \in H, x \in \mathcal{G}$.

Moreover, J_M is a complex structure if and only if (see e.g. Koszul [22, Section 2])

 $(J4) [Jx, Jy] - [x, y] - J[x, Jy] - J[Jx, y] \in \mathcal{H} \text{ for all } x, y \in \mathcal{G}.$

If in addition (M, J_M) has a *G*-invariant pseudo-Hermitian metric g_M , then g_M induces a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on *G* that satisfies

(H) $\langle x, y \rangle = \langle Jx, Jy \rangle$ for all $x, y \in \mathcal{G}$.

Similarly, ω_M induces a skew-symmetric bilinear form ω on \mathcal{G} , satisfying

$$\omega(x,y) = \langle Jx,y \rangle.$$

Remark. Note that since $\langle \cdot, \cdot \rangle$ is induced by g_M , we have

(5.1)
$$\mathcal{H} = \mathcal{G}^{\perp} = \{ x \in \mathcal{G} \mid \langle x, y \rangle = 0 \text{ for all } y \in \mathcal{G} \}$$

$$= \{ x \in \mathcal{G} \mid \omega(x, y) = 0 \text{ for all } y \in \mathcal{G} \}$$

5.1.1. The h-algebra model of a pseudo-Hermitian homogeneous space. By conditions (J1), (J2) and (H), the data

 $(\mathcal{G}, \mathsf{J}, \langle \cdot, \cdot \rangle)$

associated with the almost pseudo-Hermitian homogeneous space

$$M = (G/H, \mathsf{J}_M, \mathsf{g}_M)$$

form an *almost h-algebra* with kernel \mathcal{H} (see Definition 5.1). In view of (J4), $(\mathcal{G}, \mathsf{J}, \langle \cdot, \cdot \rangle)$ is an *h*-algebra if and only if M is a pseudo-Hermitian manifold.

Recall that $\mathcal{H} = \mathcal{G}^{\perp}$ is called the kernel of the *h*-structure.

Definition 5.1. An *h*-structure $(\mathcal{G}, J, \langle \cdot, \cdot \rangle)$ is called an *almost h-algebra* with kernel \mathcal{H} if the following conditions are satisfied:

- (1) $\mathcal{H} = \ker J = \mathcal{G}^{\perp}$ is a subalgebra of \mathcal{G} .
- (2) \mathcal{H} is contained in $\mathcal{G}_h = \mathcal{G}_{\langle \cdot, \cdot \rangle, \mathsf{J}}$.

If J is also *integrable*, that is, if

(3) $[Jx, Jy] - [x, y] - J[x, Jy] - J[Jx, y] \in \mathcal{H}$ for all $x, y \in \mathcal{G}$,

then we call $(\mathcal{G}, J, \langle \cdot, \cdot \rangle)$ an *h*-algebra.

Remark. It follows that (almost) h-algebras are precisely the local models of (almost) pseudo-Hermitian homogeneous spaces, compare Section 5.1.

5.1.2. The h-algebra models of Hermitian homogeneous spaces of finite volume. We note the following application of Theorem 3.7:

Corollary 5.2. Let M = G/H be an almost pseudo-Hermitian homogeneous space with finite Riemannian volume. Then the forms $\langle \cdot, \cdot \rangle$, ω induced on \mathcal{G} by \mathbf{g}_M , ω_M are quasi-invariant (with respect to the adjoint representation of G).

This states that the almost *h*-algebra model $(\mathcal{G}, J, \langle \cdot, \cdot \rangle)$ for *M* constitutes a quasi-invariant *h*-structure in the sense of Section 4.2.

5.2. Almost pseudo-Hermitian homogeneous spaces of finite volume. We now derive some global consequences of the algebraic results on *h*-structures and quasi-invariant skew-symmetric forms on Lie algebras as developed in Section 4. The *almost complex isometry group* of an almost pseudo-Hermitian manifold (M, J, g)consists of the diffeomorphisms of M preserving both J and g.

Theorem A. Any almost pseudo-Hermitian homogeneous space M of finite volume is compact. The maximal semisimple subgroup of the identity component of the almost complex isometry group of M is compact, and its solvable radical is nilpotent and at most two-step nilpotent. *Proof.* Since the almost complex isometry group G of M acts effectively and transitively, the almost h-algebra model $(\mathcal{G}, J, \langle \cdot, \rangle)$ for M is quasi-invariant and effective.

Thus Theorem 4.21 applies and it follows that G = KR, where K is compact semisimple and the solvable radical R is nilpotent and at most two-step nilpotent. As a consequence of Mostow's result [24, Theorem 6.2], the finite volume space M = (KR)/H is in fact compact.

Theorem B. Let M be a compact almost pseudo-Hermitian homogeneous space with a transitive effective action by a connected solvable Lie group G of almost complex isometries. Then G is abelian and compact. Furthermore, M is a flat complex pseudo-Hermitian torus $\mathbb{C}^{p,q}/\Gamma$, where Γ is a lattice in $\mathbb{C}^{p,q}$.

Proof. As proved in [3, Theorem A], since G is solvable, the stabilizer subgroup H is a discrete group. Therefore, in the almost h-algebra model $(\mathcal{G}, J, \langle \cdot, \rangle)$ the scalar product $\langle \cdot, \cdot \rangle$ on \mathcal{G} is non-degenerate. Then Corollary 4.22 asserts that G is abelian. Effectivity of the G-action implies that the stabilizer is trivial and that G is compact.

The Lie algebra \mathcal{G} being abelian also implies that the almost complex structure on M is integrable, since (J4) is clearly satisfied. Similarly, any left-invariant pseudo-Riemannian connection on an abelian Lie group is flat. It follows that M is a flat complex pseudo-Hermitian torus.

It is interesting to compare the pseudo-Hermitian case with related situations of considerably weaker assumptions. In this regard we mention that the existence of a non-degenerate invariant skew-symmetric form on M (e.g. the fundamental two-form in the pseudo-Hermitian case) already exhibits a considerable rigidity:

Corollary 5.3. Let M be a compact manifold equipped with a non-degenerate twoform ω_M invariant by a locally simply transitive action of a connected solvable Lie group G on M. Then G is abelian and compact and M is a flat symplectic twotorus, that is, ω arises from a parallel symplectic two-form on \mathbb{R}^{2n} .

Proof. The condition that G acts locally simply transitively means that the stabilizer group H is discrete, that is, $M = G/\Gamma$ is a lattice quotient of G. The associated Lie algebra model (G, ω) for M is therefore given by a quasi-invariant non-degenerate skew-symmetric form on G. By application of Corollary 4.16 we infer that G is abelian. Any left-invariant non-degenerate skew-symmetric two-form on the abelian group G is closed, hence symplectic. It is also parallel with respect to the canonical connection.

6. Compact pseudo-Hermitian homogeneous spaces

Let M be a compact complex manifold. By a classical result of Bochner and Montgomery the group of biholomorphic maps Hol(M) is a complex Lie group and acts holomorphically on M. We shall consider real connected Lie subgroups G of Hol(M) and our interest is to study the properties of G-invariant geometric structures on M. The group G itself is not necessarily complex, but its action on M extends to that of a complex group $G^{\mathbb{C}}$ contained in $\operatorname{Hol}(M)$. If G acts transitively on M, then $G^{\mathbb{C}}$ acts transitively on M, so that $M = G^{\mathbb{C}}/H^{\mathbb{C}}$ becomes a complex homogeneous manifold.

Note on terminology. By a complex Lie group $G^{\mathbb{C}}$ we mean a complex analytic group. A complex Lie subgroup $H^{\mathbb{C}}$ of $G^{\mathbb{C}}$ is a complex analytic submanifold of $G^{\mathbb{C}}$ with an induced structure of complex Lie group. A complex homogeneous space is a quotient space $G^{\mathbb{C}}/H^{\mathbb{C}}$, where $G^{\mathbb{C}}$ is a complex Lie group and $H^{\mathbb{C}}$ a closed complex Lie subgroup. We let $M = G^{\mathbb{C}}/H^{\mathbb{C}}$ denote the underlying complex manifold. Every complex Lie group $G^{\mathbb{C}}$ is also a real Lie group in the usual sense, and, if not stated otherwise, a Lie subgroup of $G^{\mathbb{C}}$ refers to a Lie subgroup with respect to the real Lie group structure of $G^{\mathbb{C}}$. A complex Lie group $A^{\mathbb{C}}$ isomorphic to $(\mathbb{C}^*)^k$ is customarily called a *torus* and we will stick to this convention. A compact abelian complex Lie group $T^{\mathbb{C}} = \mathbb{C}^k / \Lambda$ is, as a real Lie group, isomorphic to a compact real torus $T^{2k} = (S^1)^{2k}$. The underlying complex manifold of $T^{\mathbb{C}}$ is called a (compact) complex torus. A complex Lie group is called reductive if it is finitely covered by a product of a torus and a semisimple complex Lie group. In the context of complex Lie groups, any maximal reductive complex subgroup of $G^{\mathbb{C}}$ is called a Levi subgroup of $G^{\mathbb{C}}$, whereas in the real case, traditionally, a Levi subgroup denotes a maximal semisimple Lie subgroup. In the complex Lie group $G^{\mathbb{C}}$, any maximal semisimple Lie subgroup $S^{\mathbb{C}}$ is a complex subgroup as well.

6.1. The Tits fibration. According to Tits [25], any compact complex homogeneous manifold $M = G^{\mathbb{C}}/H^{\mathbb{C}}$ fibers holomorphically over a generalized flag manifold with complex parallelizable fibers. This means the following: The normalizer of $(H^{\mathbb{C}})^{\circ}$ in $G^{\mathbb{C}}$ is a parabolic subgroup $Q^{\mathbb{C}}$, meaning it contains a maximal solvable connected subgroup $B^{\mathbb{C}}$ of $G^{\mathbb{C}}$. Then $X = G^{\mathbb{C}}/Q^{\mathbb{C}}$ is a homogeneous complex projective manifold, and it is called a *generalized flag manifold*. In particular, X is compact and simply connected. There is an induced holomorphic fibration $F \to M \to X$ with fiber $F = Q^{\mathbb{C}}/H^{\mathbb{C}}$. The fiber F is a compact homogeneous complex manifold, and $F = F^{\mathbb{C}}/\Lambda$ is the quotient of a connected complex Lie group $F^{\mathbb{C}} = Q^{\mathbb{C}}/(H^{\mathbb{C}})^{\circ}$ by the uniform lattice $\Lambda = H^{\mathbb{C}}/(H^{\mathbb{C}})^{\circ}$ contained in $F^{\mathbb{C}}$. Such a quotient manifold F is called a *complex parallelizable* manifold. The Tits fibration is characterized as the unique holomorphic fibration of the complex manifold Mover a flag manifold with parallizable fibers.

6.1.1. Nilmanifold fiber F. If the complex Lie group $F^{\mathbb{C}}$ is nilpotent then the fibers of the Tits fibration are compact complex nilmanifolds. This assumption has strong consequences for the structure of M and $G^{\mathbb{C}}$. In the following we assume that $G^{\mathbb{C}} \subseteq \operatorname{Hol}(M)$ is a connected complex Lie group that acts effectively and transitively on M. Let $N^{\mathbb{C}}$ denote the nilradical of $G^{\mathbb{C}}$ and let $S^{\mathbb{C}}$ be a Levi subgroup.

Lemma 6.1. Suppose that the Tits fibration of M has nilmanifold fiber F. Then: (1) $G^{\mathbb{C}} = N^{\mathbb{C}} \cdot S^{\mathbb{C}}$.

- (2) $S^{\mathbb{C}}$ commutes with $N^{\mathbb{C}}$.
- (3) $N^{\mathbb{C}} \cap (H^{\mathbb{C}})^{\circ} = \{1\}.$

Proof. Let $G^{\mathbb{C}} = R^{\mathbb{C}} \cdot S^{\mathbb{C}}$ be a Levi decomposition, where $R^{\mathbb{C}}$ is the solvable radical of $G^{\mathbb{C}}$. Then $P^{\mathbb{C}} = S^{\mathbb{C}} \cap Q^{\mathbb{C}}$ is parabolic and contains a Borel subgroup $B^{\mathbb{C}}$ of $S^{\mathbb{C}}$. Let $A^{\mathbb{C}}$ denote a maximal torus contained in $B^{\mathbb{C}}$. Observe that $Q^{\mathbb{C}}$ acts holomorphically by complex automorphisms on $F^{\mathbb{C}} = Q^{\mathbb{C}}/(H^{\mathbb{C}})^{\circ}$. Since $A^{\mathbb{C}}$ acts reductively for every complex representation and the group $F^{\mathbb{C}}$ is nilpotent, the image of $A^{\mathbb{C}}$ in $F^{\mathbb{C}}$ must be central. This means that $A^{\mathbb{C}}$ acts trivially on $F^{\mathbb{C}}$. As $A^{\mathbb{C}}$ is a Cartan subgroup of $B^{\mathbb{C}}$, the action of the Borel subgroup $B^{\mathbb{C}}$ on $F^{\mathbb{C}}$ is also trivial. In particular, $B^{\mathbb{C}}$ acts trivially on the subgroup $R^{\mathbb{C}}/(R^{\mathbb{C}} \cap (H^{\mathbb{C}})^{\circ})$. Since $R^{\mathbb{C}} \cap (H^{\mathbb{C}})^{\circ}$ is normalized by $B^{\mathbb{C}}$, every non-trivial subrepresentation of the action of $S^{\mathbb{C}}$ on the Lie algebra of $R^{\mathbb{C}}$ is contained in the Lie algebra of $R^{\mathbb{C}} \cap (H^{\mathbb{C}})^{\circ}$, cf. Lemma 6.2. It follows that $R^{\mathbb{C}} \cap (H^{\mathbb{C}})^{\circ}$ is normalized by $S^{\mathbb{C}}$, and $S^{\mathbb{C}}$ acts trivially on the factor group. In particular, $R^{\mathbb{C}} \cap (H^{\mathbb{C}})^{\circ}$ is a normal subgroup of $G^{\mathbb{C}}$. Since we assume that $G^{\mathbb{C}}$ acts effectively on M, this implies $R^{\mathbb{C}} \cap (H^{\mathbb{C}})^{\circ} = \{1\}$. Thus $R^{\mathbb{C}}$ embeds as a subgroup into $F^{\mathbb{C}}$, showing that $R^{\mathbb{C}} = N^{\mathbb{C}}$ is nilpotent. □

Lemma 6.2. Let S be a complex semisimple Lie algebra and \mathcal{B} a Borel subalgebra of S. If V is a finite-dimensional S-module, then $SV = \mathcal{B}V$.

Proof. For a certain choice of positive simple roots, let \mathcal{N}^+ , \mathcal{N}^- denote the nilpotent subalgebras spanned by the positive and negative root spaces, respectively. Then we may assume $\mathcal{B} = \mathcal{A} \ltimes \mathcal{N}^+$ for some Cartan subalgebra \mathcal{A} of \mathcal{S} . Also we may assume that V is irreducible and non-trivial. Let $\lambda_- \neq 0$ be a *minimal weight* for the action of \mathcal{A} on V. Then $\mathcal{A}V_{\lambda_-} = V_{\lambda_-}$ and $V = V_{\lambda_-} + \mathcal{N}^+V_{\lambda_-}$. In particular, $V = \mathcal{B}V$.

6.1.2. Transitive Lie subgroups of $G^{\mathbb{C}}$. Let G be a connected Lie subgroup of $G^{\mathbb{C}}$ that acts transitively on M. Then M = G/H with $H = G \cap H^{\mathbb{C}}$. Moreover, $L = G \cap Q^{\mathbb{C}}$ contains H and X = G/L. It also follows that L is connected, since X is simply connected. The fiber of $M \to X$ can be identified with L/H and therefore the Lie group L/H° is a covering group of $F^{\mathbb{C}} = L/(L \cap (H^{\mathbb{C}})^{\circ})$. In particular, L/H° attains the structure of a complex Lie group locally isomorphic to $F^{\mathbb{C}}$.

Proposition 6.3. Assume that $G^{\mathbb{C}}$ admits a transitive Lie subgroup G with a Levi decomposition of the form $G = N \cdot K$, where N is the nilradical of G and K is compact semisimple. Then the Tits fibration of M has nilmanifold fiber. In particular, K centralizes N.

The proof of the proposition builds on the following lemma:

Lemma 6.4. Let $E^{\mathbb{C}}$ be a complex Lie group such that $E^{\mathbb{C}} = V \cdot T$ is an extension of a compact Lie group T by some connected nilpotent normal Lie subgroup V of $E^{\mathbb{C}}$. Then $E^{\mathbb{C}}$ is nilpotent. In particular, the compact subgroup T is contained in the center of $E^{\mathbb{C}}$.

Proof. By assumption, V is contained in the nilradical $N^{\mathbb{C}}$ of $E^{\mathbb{C}}$. It also follows by assumption that the Levi subgroup $S^{\mathbb{C}}$ of $E^{\mathbb{C}}$ is compact. Since $S^{\mathbb{C}}$ is a complex semisimple Lie group, this implies that $S^{\mathbb{C}}$ is trivial. Hence, $E^{\mathbb{C}}$ is solvable and $T_1^{\mathbb{C}} = E^{\mathbb{C}}/N^{\mathbb{C}}$ is an abelian complex Lie group and compact. Note that $T_1^{\mathbb{C}}$ acts by automorphisms on the complex abelian Lie group $H^1(N^{\mathbb{C}}) = N^{\mathbb{C}}/[N^{\mathbb{C}}, N^{\mathbb{C}}]$. Since every complex linear representation of $T_1^{\mathbb{C}}$ is trivial, $T_1^{\mathbb{C}}$ acts trivially on $H^1(N^{\mathbb{C}})$. Therefore, the compact group T acts trivially on $H^1(N^{\mathbb{C}})$ and (thus) also on $N^{\mathbb{C}}$. Hence, $E^{\mathbb{C}} = N^{\mathbb{C}}T$ is nilpotent.

Proof of Proposition 6.3. First note that N is contained in $Q^{\mathbb{C}}$, since every solvable Lie group of complex automorphisms of the projective variety X has a fixed point. Let $L = G \cap Q^{\mathbb{C}} = N \cdot C$, where $C = K \cap Q^{\mathbb{C}}$ is compact. By the above remarks, L/H° is a complex Lie group covering $F^{\mathbb{C}}$ and it satisfies the assumption of Lemma 6.4. Hence, L/H° is nilpotent. So $F^{\mathbb{C}} = Q^{\mathbb{C}}/(H^{\mathbb{C}})^{\circ}$ is nilpotent, which means that the Tits fibration for M has nilmanifold fiber. By Lemma 6.1, $G^{\mathbb{C}} = N^{\mathbb{C}}S^{\mathbb{C}}$, where $S^{\mathbb{C}}$ centralizes $N^{\mathbb{C}}$. We may assume that N is contained in $N^{\mathbb{C}}$ and K in $S^{\mathbb{C}}$. It follows that K centralizes N.

Remark. Since N is contained in L, X = K/C for the compact subgroup $C = L \cap K$. Since the parabolic subgroup $P^{\mathbb{C}} = Q^{\mathbb{C}} \cap S^{\mathbb{C}}$ of $S^{\mathbb{C}}$ is the centralizer of a torus [9, 4.15 Théorème], $C = K \cap P^{\mathbb{C}}$ is the centralizer of a torus T_0 in K. In particular, C contains a maximal torus T of K. We may write $C = T_0C_0$, where C_0 is compact semisimple. Since L/H° is nilpotent, C_0 is contained in H° . Thus $H^\circ \cap K = T_1C_0$, where T_1 is contained in T_0 . Choose a compact torus T_2 such that $T_0 = T_1T_2$. Then $L/H^\circ = N\overline{T}_2$, where \overline{T}_2 is a compact torus that centralizes N.

We infer the following addendum to Lemma 6.1:

Lemma 6.5. Suppose that M has nilmanifold fiber. Then $F^{\mathbb{C}}/N^{\mathbb{C}}$ is compact and $N^{\mathbb{C}}$ has compact center.

Proof. Let K be a compact real form of $S^{\mathbb{C}}$. Since K is a maximal compact subgroup, $S^{\mathbb{C}} = K \cdot B^{\mathbb{C}}$ by the Iwasawa decomposition. Hence, K acts transitively on X. Thus the Lie subgroup $G = N^{\mathbb{C}}K$ acts transitively on M. As the above remark shows, $F^{\mathbb{C}} = N^{\mathbb{C}}\overline{T}_2$, where \overline{T}_2 is a compact torus. Since $F = F^{\mathbb{C}}/\Lambda$, where $\Lambda = H^{\mathbb{C}}/(H^{\mathbb{C}})^{\circ}$, is compact, Λ intersects $N^{\mathbb{C}}$ in a uniform lattice. Since $N^{\mathbb{C}}$ is nilpotent, it follows that Λ also intersects the center Z of $N^{\mathbb{C}}$ in a uniform lattice. However, since Z is also central in $G^{\mathbb{C}}$ by Proposition 6.3, $H^{\mathbb{C}} \cap Z$ is trivial. We deduce that Z is compact.

6.1.3. *G*-invariant pseudo-Hermitian metric. We say that the Tits fibration for M has torus fiber if $F^{\mathbb{C}}$ is abelian. This means that the fiber $F = F^{\mathbb{C}}/\Lambda$ of the Tits fibration is a compact abelian complex Lie group. In fact, as we assume that the action of $G^{\mathbb{C}}$ on M is effective, $F^{\mathbb{C}}$ is a compact complex Lie group due to Lemma 6.5 above. It follows:

Lemma 6.6. If the compact complex homogeneous space $G^{\mathbb{C}}/H^{\mathbb{C}}$ has torus fiber then the solvable radical of $G^{\mathbb{C}}$ is a compact complex abelian group. The following theorem characterizes compact complex homogeneous manifolds with an underlying homogeneous pseudo-Hermitian structure:

Theorem C. Let M be a compact complex homogeneous manifold, and let G be a closed Lie subgroup of Hol(M). Assume that G preserves a pseudo-Hermitian metric and acts transitively on M. Then the Tits fibration of M has torus fiber and the group G is compact.

Proof. Since G acts holomorphically by isometries of a pseudo-Hermitian metric on M, it also preserves its associated fundamental two-form ω on M. By Theorem 4.21, the existence of the invariant non-degenerate two-form ω implies $G = N \cdot K$, where N is a connected nilpotent normal subgroup and K is compact semisimple. Since G is a subgroup of $G^{\mathbb{C}}$ that acts transitively on M, Proposition 6.3 asserts that K centralizes N. Theorem 4.21 also states that the commutator subgroup [N,N] is contained in the center of N. Thus, [N,N] is contained in the center of G. In view of the remark following Theorem 4.21, this implies that [N,N] is contained in H° , showing that the normal subgroup [N,N] of G is trivial, since G acts effectively. Therefore, N is abelian. By the remark following the proof of Proposition 6.3, $L/H^{\circ} = N\overline{T}_2$ is abelian. Therefore, $F^{\mathbb{C}}$ is abelian and the Tits fibration of M has torus fiber. Lemma 6.6 implies that $N^{\mathbb{C}}K$ is compact. Hence, the closed subgroup G is compact. \square

Corollary 6.7. A compact complex homogeneous manifold M admits a homogeneous pseudo-Hermitian structure if and only if its Tits fibration has torus fiber. Moreover, every such manifold admits a homogeneous positive definite Hermitian metric.

Proof. If the Tits fibration of $M = G^{\mathbb{C}}/H^{\mathbb{C}}$ has torus fiber we may always choose an underlying homogeneous Hermitian structure on M: In fact, by Lemma 6.6, we have $G^{\mathbb{C}} = T^{\mathbb{C}}S^{\mathbb{C}}$, where $T^{\mathbb{C}}$ is compact and central in $G^{\mathbb{C}}$ and $S^{\mathbb{C}}$ is a semisimple complex Lie group. Let K be a compact real form of $S^{\mathbb{C}}$. As noted in the proof of Lemma 6.5, $G = T^{\mathbb{C}}K$ acts transitively on M. Invariant integration on M = G/Hfurnishes a G-invariant (definite) Hermitian structure.

Remark. Hano and Kobayashi studied the implications of the existence of *G*-invariant volume forms on complex homogeneous spaces. The assertion of Theorem C for a definite Hermitian metric is actually contained in [16, Theorem B].

6.2. Applications of Theorem C. Let (M, g, J) be a compact homogeneous pseudo-Hermitian manifold. Taking $G = \operatorname{Aut}(M, g, J)^{\circ}$ in Theorem C we obtain that the identity component of the holomorphic isometry group of a compact homogeneous pseudo-Hermitian manifold is compact. In the simply connected case, this can be strengthened to:

Corollary D. The holomorphic isometry group of a simply connected compact homogeneous pseudo-Hermitian manifold is compact.

Proof. Since M is simply connected, a maximal compact semisimple subgroup K of G already acts transitively on M (cf. [23]). Therefore we may identify $T_x M$ at $x \in M$ with $\mathcal{K}/(\mathcal{H} \cap \mathcal{K})$, where \mathcal{K} is the Lie algebra of K. Now G = KA is an almost direct product of a compact semisimple Lie group K and a compact abelian group A. In particular, K is characteristic in G and the isotropy representation of the stabilizer $\operatorname{Aut}(M, \mathbf{g}, \mathsf{J})_x$ factorizes over a closed subgroup of the automorphism group of K. As this latter group is compact, the isotropy representation has compact closure in $\operatorname{GL}(T_x M)$. If follows that there exists a Riemannian metric on M that is preserved by $\operatorname{Aut}(M, \mathbf{g}, \mathsf{J})$. Hence $\operatorname{Aut}(M, \mathbf{g}, \mathsf{J})$ is compact. \Box

A particular interesting case for Corollary D arises if in the Tits fibration the fiber is trivial, that is, if M is a generalized flag manifold. In fact, generalized flag manifolds are always simply connected and have non-zero Euler characteristic, conversely any compact complex homogeneous space with non-zero Euler characteristic is a generalized flag manifold (see [13, 25]).

Example 6.8. Let M be a compact homogeneous pseudo-Hermitian manifold with non-zero Euler characteristic. Then M is biholomorphic to a generalized flag manifold. By Corollary D, the holomorphic isometry group G of M is compact. Since the Euler characteristic of M is non-zero, the stabilizer H contains a maximal torus of G. Hence G = K is a compact semisimple Lie group. Moreover, M is a product of pseudo-Hermitian irreducible flag manifolds.

Proof. The only thing that remains to be proved is the last statement. Let $\mathcal{G}_1, \mathcal{G}_2$ be two simple factors of \mathcal{G} , and let $x_1 \in \mathcal{G}_1, x_2 \in \mathcal{G}_2$. Since \mathcal{H} contains a maximal torus \mathcal{T} of \mathcal{G}, x_1 can be written $x_1 = [t, y_1]$ for some $t \in \mathcal{T} \cap \mathcal{G}_1$ and $y_1 \in \mathcal{G}_1$. Recalling that the induced symmetric bilinear form $\langle \cdot, \cdot \rangle$ on \mathcal{G} is \mathcal{H} -invariant, we obtain

$$\langle x_1, x_2 \rangle = \langle [t, y_1], y_2 \rangle = -\langle y_1, [t, y_2] \rangle = 0.$$

It follows that the simple factors of \mathcal{G} are mutually orthogonal with respect to $\langle \cdot, \cdot \rangle$. This implies that the pseudo-Hermitian metric **g** on M is a product of the metrics induced on the irreducible factors of M.

6.2.1. *Examples.* In general, the decomposition of $\operatorname{Aut}(M, \mathsf{g}, \mathsf{J})^\circ$ into a compact semisimple factor and an abelian factor does not translate into a splitting of M into corresponding complex factors. This distinguishes the case of arbitrary pseudo-Hermitian metrics from that of pseudo-Kähler metrics (discussed in Appendix A).

Example 6.9 (Complex structures on the unitary group U(2)). Let Λ be a discrete uniform subgroup in $\mathbb{R}^{>0} \leq \mathbb{C}^*$ and let

$$M = (\mathbb{C}^2 - \mathbf{0}) / \Lambda$$

be a two-dimensional Hopf manifold, which is a homogeneous space for $GL(2, \mathbb{C})$. The complex homogeneous manifold

$$\widetilde{M} = \operatorname{GL}(2, \mathbb{C}) / \operatorname{Aff}(\mathbb{C}) = \mathbb{C}^2 - \mathbf{0}$$

is a covering manifold of M, where $\operatorname{Aff}(\mathbb{C}) = \{ \begin{pmatrix} \alpha & u \\ 0 & 1 \end{pmatrix} \mid \alpha \neq 0 \}$. Observe that \widetilde{M} has a holomorphic fibration

$$\mathbb{C}^* \longrightarrow \widetilde{M} \longrightarrow \mathrm{P}^1 \mathbb{C} = \mathrm{SL}(2, \mathbb{C})/B$$
,

where B is the Borel subgroup of $SL(2, \mathbb{C})$ and \mathbb{C}^* acts as the center of $GL(2, \mathbb{C})$. We then put

$$G^{\mathbb{C}} = \operatorname{GL}(2,\mathbb{C})/\Lambda$$
 and $M = G^{\mathbb{C}}/\operatorname{Aff}(\mathbb{C}).$

Note that $G^{\mathbb{C}}$ has maximal compact subgroup $\mathrm{SU}(2) \cdot T^1_{\mathbb{C}}$, where $T^1_{\mathbb{C}} = \mathbb{C}^* / \Lambda$. Let $N = \mathbb{R}^{>0} / \Lambda$ and observe that the real compact Lie subgroup

$$G = \mathrm{SU}(2) \cdot N$$

acts simply transitively on M. Since G is isomorphic to U(2), M is biholomorphic to the compact Lie group U(2) endowed with a left-invariant complex structure. Clearly, M (being homeomorphic to $S^1 \times S^3$) cannot be Kähler, and neither is it a product of complex manifolds. Consider the compact torus $S^1 = SU(2) \cap B$. Then the Tits fibration of M (induced by $T^1_{\mathbb{C}}$) takes the form

$$NS^1 \longrightarrow M = G \longrightarrow P^1 \mathbb{C} = S^3 / S^1$$
.

Remark. Left-invariant pseudo-Hermitian metrics on the unitary group U(2) are computed in [2, Theorem 4.6]). These metrics are *locally conformally pseudo-Kähler*. In fact, [17, Theorem 1] shows that the Tits fibration of a complex compact homogeneous manifold with a (non-Kähler) locally conformally Kähler metric has one-dimensional fiber.

The Hopf manifolds are not simply connected. It is a famous observation due to Calabi and Eckmann [10] that there are non-Kähler simply connected compact homogeneous complex manifolds.

Example 6.10 (Calabi-Eckmann manifolds). Let $\Lambda \cong \mathbb{C}$ act on $\mathbb{C}^p \times \mathbb{C}^q$ via $(u, v) \mapsto$ (e^tu, e^{αt}v), where $\alpha \in \mathbb{C} - \mathbb{R}$, and consider

$$M = \left(\left(\mathbb{C}^p - \mathbf{0} \right) \times \left(\mathbb{C}^q - \mathbf{0} \right) \right) / \Lambda, \quad p, q > 1.$$

Then M is a compact complex homogeneous manifold for $\operatorname{GL}(p, \mathbb{C}) \times \operatorname{GL}(q, \mathbb{C})$. Its Tits fibration is of the form

$$\mathbb{C}/(\mathbb{Z} + \alpha \mathbb{Z}) \longrightarrow M \longrightarrow \mathrm{P}^{p-1}\mathbb{C} \times \mathrm{P}^{q-1}\mathbb{C}$$

and has torus fiber (an elliptic curve). In particular, M admits invariant (pseudo-) Hermitian metrics and is diffeomorphic to $S^{2p-1} \times S^{2q-1}$.

APPENDIX A. ADDITIONAL PROOFS

In this appendix we derive that any compact homogeneous pseudo-Kähler manifold is a pseudo-Kählerian product of homogeneous spaces for a compact semisimple group and for a compact abelian group. Besides the compactness of the identity component of the holomorphic isometry group, this is the essential content of Dorfmeister and Guan's theorem [12]. The splitting is distinctive for the Kähler and symplectic case. For the proof we use our Theorem C on compact pseudo-Hermitian homogeneous spaces and specialize it to the case that the fundamental two-form is closed. Note that proofs for Dorfmeister and Guan's theorem using the properties of Hamiltonian group actions have been given by Huckleberry [18] and Guan [14].

A.1. Splitting of compact homogeneous symplectic manifolds. We first review the important splitting result on homogeneous symplectic manifolds given by Zwart and Boothby [29, Theorem 5.11, Corollary 6.10]:

Theorem A.1 (Zwart & Boothby). Let M be a compact symplectic homogeneous manifold, with $G \subseteq \operatorname{Aut}(M, \omega_M)$ a connected Lie group acting transitively on M, so that M = G/H for some closed subgroup H of G. Then:

- (1) G is a direct product G = KR, where K is a compact semisimple subgroup with trivial center and R is the solvable radical of G.
- (2) H is a direct product of $H_K = H \cap K$ and $H_R = H \cap R$. Moreover, H_K is connected and is the centralizer of a torus.
- (3) $M = M_K \times M_R$ splits as a product of compact symplectic homogeneous spaces $M_K = K/H_K$ and $M_R = R/H_R$.
- (4) $R = A \ltimes N$ with an abelian subgroup A and abelian nilradical N, where A acts on N by rotations.

A.1.1. Nil-invariant closed skew-symmetric forms. Let ω be a nil-invariant closed skew-symmetric form on the Lie algebra G. A skew-symmetric form on G is called closed, if for all $x, y, z \in G$,

(A.1)
$$\omega([x,y],z) + \omega([y,z],x) + \omega([z,x],y) = 0.$$

Since ω is nil-invariant, we may assume that $\mathcal{G} = \mathcal{K} \ltimes \mathcal{R}$, where \mathcal{K} is a compact semisimple Lie subalgebra and \mathcal{R} is the solvable radical of \mathcal{G} . Furthermore, let $\mathcal{N} \subseteq \mathcal{R}$ denote the nilradical of \mathcal{G} . The following two results can then be seen as an addendum to Section 4.1.2:

Lemma A.2 (Nil-invariant closed skew-symmetric forms).

- (1) $[\mathcal{G}, \mathcal{G}] \perp_{\omega} \mathcal{N}$ and $\mathcal{K} \perp_{\omega} \mathcal{R}$.
- $\begin{array}{ll} (2) \ \left[\mathcal{K}, \mathcal{N} \right] \bot_{\omega} \ \mathcal{G} \ and \ \left[\mathcal{N}, \mathcal{N} \right] \bot_{\omega} \ \mathcal{G}. \\ (3) \ \mathcal{G}^{\bot_{\omega}} = \left(\mathcal{G}^{\bot_{\omega}} \cap \mathcal{K} \right) + \left(\mathcal{G}^{\bot_{\omega}} \cap \mathcal{R} \right). \end{array}$

Proof. Using (A.1) and $\mathcal{N} \subseteq \mathcal{G}_{\omega}$, for all $x \in \mathcal{N}, k_1, k_2 \in \mathcal{G}$, we compute

$$\omega([k_1,k_2],x) = -\omega([k_2,x],k_1) - \omega([x,k_1],k_2) = \omega(k_2,[k_1,x]) - \omega([x,k_1],k_2) = 0.$$

Hence, $[\mathcal{G}, \mathcal{G}] \perp_{\omega} \mathcal{N}$. Recall that $\mathcal{K} = [\mathcal{K}, \mathcal{K}]$ and $[\mathcal{K}, \mathcal{R}] \subseteq \mathcal{N}$. Thus, for any $k = [k_1, k_2]$, and $r \in \mathcal{R}$, $\omega(k, r) = -\omega([k_2, r], k_1) - \omega([r, k_1], k_2) = 0$. Hence, $\mathcal{K}_{\perp \omega} \mathcal{R}$. Finally, $\omega([k, x], r) = -\omega([x, r], k) - \omega([r, k], x) = 0$, for $x \in \mathcal{N}$. This shows $[\mathcal{K}, \mathcal{N}] \perp_{\omega} \mathcal{R}$. Since $[\mathcal{K}, \mathcal{N}] \subseteq \mathcal{N} \perp_{\omega} \mathcal{K}$, this implies $[\mathcal{K}, \mathcal{N}] \perp_{\omega} \mathcal{G}$. Similarly, $[\mathcal{N}, \mathcal{N}] \perp_{\omega} \mathcal{G}$. Hence (1) and (2) hold and (3) follows.

30

Proposition A.3. Suppose that (\mathcal{G}, ω) is effective. Then:

- (1) \mathcal{N} is abelian.
- (2) $\mathcal{G} = \mathcal{K} \times \mathcal{R}$ is a direct product.

Proof. By the previous lemma, $[\mathcal{N}, \mathcal{N}] \subseteq \mathcal{G}^{\perp_{\omega}}$. Part (1) follows. Since \mathcal{N} is abelian, $[\mathcal{K}, \mathcal{N}]$ is an ideal in \mathcal{G} and contained in $\mathcal{G}^{\perp_{\omega}}$. Thus $[\mathcal{K}, \mathcal{N}] = \mathbf{0}$.

A.1.2. Proof of the symplectic splitting theorem. In the algebraic model (\mathcal{G}, ω) for the homogeneous symplectic manifold ω is a quasi-invariant *closed* skew-symmetric form and $\mathcal{G}^{\perp \omega} = \mathcal{H}$ is the Lie algebra of H. Moreover, (\mathcal{G}, ω) is effective.

According to Proposition 4.4, $\mathcal{G} = \mathcal{K} \ltimes \mathcal{R}$, where \mathcal{K} compact semisimple and \mathcal{R} is solvable. By Proposition A.3, \mathcal{K} and \mathcal{R} commute. Hence $G = K \cdot R$ is an almost direct product, meaning that both K and R are closed normal subgroups of G, and that their intersection is finite and central in G.

Moreover, $\operatorname{Ad}(k)|_{\mathcal{R}} = \operatorname{id}_{\mathcal{R}}$ and $\operatorname{Ad}(r)|_{\mathcal{K}} = \operatorname{id}_{\mathcal{K}}$ for all $k \in K$, $r \in R$. Let G_{ω} denote the subgroup formed by all elements $g \in G$ such that $\operatorname{Ad}(g) \in O(\omega)$. Since $\mathcal{K}_{\perp \omega} \mathcal{R}$, by Lemma A.2, it follows that $g \in G_{\omega}$ if and only if g = kr with $k \in K \cap G_{\omega}$ and $r \in R \cap G_{\omega}$. Furthermore, by construction, we have that $H \subseteq G_{\omega}$.

By Borel [6, Theorem 1], K and $K_{\omega} = K \cap G_{\omega}$ are connected, K_{ω} is the centralizer of a torus, and $K_{\omega} \subseteq H$. Therefore $H = (H \cap K)(H \cap R)$ (see [29, 5.10, 5,11]). Note that $K \cap R$ is central in K and therefore trivial. In the view of Lemma A.2 (3), we have now proved (1) to (3) of Theorem A.1.

A.2. Compact homogeneous pseudo-Kähler manifolds. Let M be a compact homogeneous pseudo-Kähler manifold. That is, M is pseudo-Hermitian manifold where the metric \mathbf{g}_M is a pseudo-Kähler metric on M, which means that the fundamental two-form ω_M is closed.

Theorem A.4 (Dorfmeister & Guan). Let M be a compact homogeneous pseudo-Kähler manifold, $G = Aut(M, \mathbf{g}_M, \mathbf{J}_M)^\circ$, and H a closed subgroup of G such that M = G/H. Then:

- (1) G is compact and $H \subseteq K$.
- (2) $M = T \times M_K$, where T is a complex torus and $M_K = K/H$ is a rational homogeneous variety.
- (3) H is connected, has trivial center and is the centralizer of a torus in K.
- (4) $M_K = M_{K_1} \times \cdots \times M_{K_m}$ is a product of pseudo-Kähler manifolds, where the K_i are the simple factors of K and $M_{K_i} = K_i/(H \cap K_i)$.

From Theorem C we know that the Lie algebra \mathcal{G} of G is a product $\mathcal{K} \times \mathcal{R}$, where \mathcal{K} is semisimple of compact type and the solvable radical \mathcal{R} is abelian. Recall that $\mathcal{G}^{\perp} = \mathcal{G}^{\perp \omega} = \mathcal{H}$ is the Lie algebra of H, where ω is the fundamental two-form of the effective *h*-structure $(\mathcal{G}, J, \langle \cdot, \cdot \rangle)$, which describes the homogeneous pseudo-Kähler manifold, see Section 5.1.

Lemma A.5. $\mathcal{K} \perp_{\omega} \mathcal{R}$ and $\mathcal{G}^{\perp} = \mathcal{K}^{\perp_{\omega}} \cap \mathcal{K}$.

Proof. This is a direct consequence of Lemma A.2, taking into account that, since \mathcal{R} is the center of \mathcal{G} , we have $\mathcal{G}^{\perp} \cap \mathcal{R} = \mathbf{0}$.

Let $\mathcal{K}_1, \ldots, \mathcal{K}_m$ denote the simple ideals of \mathcal{K} and $\mathcal{H}_i = \mathcal{G}^{\perp} \cap \mathcal{K}_i$ for $i = 1, \ldots, m$.

Lemma A.6. $\mathcal{K} \perp_{\langle \cdot, \cdot \rangle} \mathcal{K}$, and $\mathcal{K}_i \perp_{\langle \cdot, \cdot \rangle} \mathcal{K}_j$ for all $i \neq j$. In particular, $\mathcal{G}^{\perp} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_m$.

Proof. Since \mathcal{K} is semisimple and ω is closed, there exists a linear form $\lambda \in \mathcal{K}^*$ such that $\omega(x, y) = \lambda([x, y])$ for all $x, y \in \mathcal{K}$, and since the Killing form κ of \mathcal{K} is non-degenerate, there exists $a \in \mathcal{K}$ such that $\lambda = \kappa(a, \cdot)$. By invariance of κ , $\mathcal{K}^{\perp_{\omega}} \cap \mathcal{K} = \mathbb{Z}_{\mathcal{K}}(a)$. Combined with Lemma A.5,

$$\mathcal{G}^{\perp} = \mathcal{K}^{\perp_{\omega}} \cap \mathcal{K} = Z_{\mathcal{K}}(a).$$

As $a \in \mathcal{K}$ is contained in some maximal torus of \mathcal{K} which in turn is contained in $\mathcal{Z}_{\mathcal{K}}(a)$, it follows that \mathcal{G}^{\perp} is its own normalizer in \mathcal{K} . By (J3) (cf. Section 5.1), $[\mathsf{J}\mathcal{R}, \mathcal{G}^{\perp}] \subseteq \mathcal{G}^{\perp}$, so $\mathsf{J}\mathcal{R}$ normalizes \mathcal{G}^{\perp} and thus $\mathsf{J}\mathcal{R} \subseteq \mathcal{R} + \mathcal{G}^{\perp}$. It follows that for all $r \in \mathcal{R}, x \in \mathcal{K}$,

$$\langle x, r \rangle = \omega(x, \mathsf{J} r) = 0,$$

which proves $\mathcal{K} \perp_{\langle ... \rangle} \mathcal{R}$.

Suppose $x_i \in \mathcal{K}_i, y_j \in \mathcal{K}_j$ for $i \neq j$. We may assume that $x_i = [x'_i, x''_i]$ for some $x'_i, x''_i \in \mathcal{K}_i$. By (A.1),

$$\omega(x_i, y_j) = \omega([x'_i, x''_i], y_j) = -\omega([x''_i, y_j], x'_i) - \omega([y_j, x'_i], x''_i) = 0.$$

It follows that $\mathcal{K}_i \perp_{\omega} \mathcal{K}_j$. This immediately implies

$$\mathcal{G}^{\perp} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_m.$$

Similar to above we can now use (J3) to find that $\mathcal{K}_j \cap \mathsf{J} \mathcal{K}_i$ normalizes \mathcal{G}^{\perp} and is thus contained in $\mathcal{K}_j \cap (\mathcal{R} + \mathcal{G}^{\perp}) = \mathcal{H}_j$. So

$$\langle x_i, y_j \rangle = \omega(\mathsf{J} x_i, y_j) = 0,$$

which concludes the proof.

Now we prove Dorfmeister and Guan's theorem.

Proof of Theorem A.4. As remarked in A.1.2, $H = (K \cap G_{\omega})(H \cap R)$. Since G is compact by Theorem C, $H \cap R$ is central in G and hence trivial, so $H \subseteq K$.

By Lemmas A.5 and A.6, $\mathcal{K} \perp \mathcal{R}$ with respect to both ω and $\langle \cdot, \cdot \rangle$. Hence $J \mathcal{K} \subseteq \mathcal{K}$ and $J \mathcal{R} \subseteq \mathcal{R} + \mathcal{G}^{\perp}$. We may thus assume that $J \mathcal{R} \subseteq \mathcal{R}$ without changing the complex structure on M. It follows that $M_K = K/H$ and R are both compact pseudo-Kähler manifolds, where R is a compact torus.

The claim (3) and the symplectic splitting in (4) is the content of the aforementioned theorem of Borel [6, Theorem 1]. Moreover, $M_K = M_{K_1} \times \cdots \times M_{K_m}$ is a product of pseudo-Kähler manifolds by Lemma A.6.

32

A.3. Metric compact symplectic homogeneous spaces. In this section, we adopt a slightly different point of view and study compact homogeneous manifolds M = G/H equipped with an invariant symplectic form ω_M and an invariant pseudo-Riemannian metric \mathbf{g}_M , but with no assumption of compatibility between the two.

Theorem A.7. Let M be a compact symplectic homogeneous manifold, with $G \subseteq Aut(M, \omega_M)$ a connected Lie group acting transitively on M, so that M = G/H for some closed subgroup H of G. In addition, suppose M is equipped with a G-invariant pseudo-Riemannian metric \mathbf{g}_M . Then:

- (1) G is an almost direct product G = KR, where K is a compact semisimple subgroup and the solvable radical R of G is a compact abelian subgroup.
- (2) $H \subseteq K$ is connected and contains the center of K.
- (3) $M = (K/H) \times R$ is a product of symplectic manifolds.

Proof. The metric \mathbf{g}_M induces a nil-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ on \mathcal{G} , and, as in (5.1), $\mathcal{G}^{\perp \omega} = \mathcal{G}^{\perp \langle \cdot, \cdot \rangle}$ is the Lie algebra of H, and we simply write \mathcal{G}^{\perp} for it.

The subalgebra \mathcal{G}^{\perp} contains no non-trivial ideal of \mathcal{G} , and by the splitting in Theorem A.1, also

$$\mathcal{K}^{\perp_{\omega}} \cap \mathcal{K} = \mathcal{G}^{\perp} \cap \mathcal{K}$$

contains no non-trivial ideal of ${\mathcal K}$ and

$$\mathcal{R}^{\perp_{\omega}} \cap \mathcal{R} = \mathcal{G}^{\perp} \cap \mathcal{R}$$

contains no non-trivial ideal of \mathcal{R} .

By Theorem 3.13, the restriction of $\langle \cdot, \cdot \rangle$ to \mathcal{R} is an invariant bilinear form. Hence $\mathcal{G}^{\perp_{\langle \cdot, \cdot \rangle}} \cap \mathcal{R}$ is an ideal in \mathcal{R} , and since $[\mathcal{K}, \mathcal{R}] = \mathbf{0}$, it is an ideal in \mathcal{G} . Hence

$$\mathcal{R}^{\perp_{\omega}} \cap \mathcal{R} = \mathcal{G}^{\perp} \cap \mathcal{R} = \mathcal{G}^{\perp_{\langle \cdot, \cdot \rangle}} \cap \mathcal{R} = \mathbf{0}.$$

This means ω is non-degenerate on \mathcal{R} . From the splitting $\mathcal{G}^{\perp} = (\mathcal{G}^{\perp} \cap \mathcal{K}) \times (\mathcal{G}^{\perp} \cap \mathcal{R})$ it follows that $\mathcal{G}^{\perp} \subseteq \mathcal{K}$. By Corollary 5.3, R is abelian. Then H_R is a normal subgroup of G, and since G acts effectively, H_R is trivial. So R is compact.

Example A.8. Let $M_K = K/H$ be any symplectic homogeneous space for a compact semisimple Lie group K and a suitable closed subgroup H. Consider some vector space decomposition $\mathcal{K} = \mathcal{H} \oplus W$ of the Lie algebra \mathcal{K} of K. Let $d = \dim W$ and equip \mathbb{R}^d with the canonical symplectic form. On the Lie algebra $\mathcal{K} \times \mathbb{R}^d$, define a symmetric bilinear form $\langle \cdot, \cdot \rangle$ by $\langle \cdot, \cdot \rangle|_{\mathcal{K} \times \mathcal{K}} = 0$, $\langle \cdot, \cdot \rangle|_{\mathbb{R}^d \times \mathbb{R}^d} = 0$ and such that W and \mathbb{R}^d are dually paired. This $\langle \cdot, \cdot \rangle$ extends to a symmetric bilinear form on the Lie group $G = K \times \mathbb{T}^d$, and $\langle \cdot, \cdot \rangle$ is trivially nil-invariant. Then M = G/H is a compact symplectic homogeneous space with a pseudo-Riemannian metric. Furthermore, M splits as a product of symplectic manifolds K and \mathbb{T}^d , but not as a product of pseudo-Riemannian manifolds.

References

 S. Adams, G. Stuck, The isometry group of a compact Lorentz manifold I, Inventiones Mathematicae 129, 1997, 239-261

- [2] D.V. Alekseevsky, V. Cortés, K. Hasegawa, Y. Kamishima, Homogeneous locally conformal Kähler and Sasaki manifolds, International Journal of Mathematics 26, 2015 (6), 1541001
- [3] O. Baues, W. Globke, Rigidity of compact pseudo-Riemannian homogeneous spaces for solvable Lie groups, International Mathematics Research Notices 2018 (1), 3199-3223
- [4] O. Baues, W. Globke, A. Zeghib, Isometry Lie algebras of indefinite homogeneous spaces of finite volume, Proceedings of the London Mathematical Society 119, 2019 (4), 1115-1148
- [5] A. Borel, Linear Algebraic Groups, second edition, GTM 126, Springer, 1991
- [6] A. Borel, Kählerian coset spaces of semi-simple Lie groups, Proceedings of the National Academy of Sciences of the United States of America 40, 1954, 1147-1151
- [7] A. Borel, Harish-Chandra, Arithmetic Subgroups of Algebraic Groups, Annals of Mathematics 75, 1962 (3), 485-535
- [8] A. Borel, R. Remmert, Über kompakte homogene Kählersche Mannigfaltigkeiten, Mathematische Annalen 145, 1962, 429-439
- [9] A. Borel, J. Tits, Groupes réductifs, Publications Mathématiques de l'I.H.E.S. 27, 1965, 55-151
- [10] E. Calabi, B. Eckmann, A class of compact complex manifolds which are not algebraic, Annals of Mathematics, 58, 1953 (3), 494-500
- [11] G. D'Ambra, M. Gromov, Lectures on Transformation Groups: Geometry and Dynamics, Surveys in Differential Geometry 1, 1991, 19-111
- [12] J. Dorfmeister, Z.D. Guan, Classification of compact homogeneous pseudo-Kähler manifolds, Commentarii Mathematici Helvetici 67, 1992, 499-513
- [13] M. Goto, On Algebraic Homogeneous Spaces, American Journal of Mathematics 76, 1954 (4), 811-818
- [14] D. Guan, On compact symplectic manifolds with Lie group symmetries, Transactions of the American Mathematical Society 357, 2005 (8), 3359-3373
- [15] V. Guillemin, S. Sternberg, Symplectic techniques in physics, Cambridge University Press, 1984
- [16] J. Hano, S. Kobayashi, A fibering of a class of homogeneous complex manifolds, Transactions of the American Mathematical Society 94, 1960, 233-243
- [17] K. Hasegawa, Y. Kamishima, Compact homogeneous locally conformally Kähler manifolds, Osaka Journal of Mathematics 53, 2016 (3), 683-703
- [18] A.T. Huckleberry, Homogeneous pseudo-Kählerian manifolds: A Hamiltonian viewpoint, Note di Matematica X, 1990 (2), 337-342
- [19] N. Jacobson, Lie Algebras, John Wiley & Sons, 1962
- [20] I. Kath, M. Olbrich, Metric Lie algebras and Quadratic extensions, Transformation Groups 11, 2006 (1), 87-131
- [21] S. Kobayashi, Transformation Groups in Differential Geometry, Springer, 1972
- [22] J.L. Koszul, Sur la forme Hermitienne canonique des espaces homogènes complexes, Canadian Journal of Mathematics 7, 1955, 562-576
- [23] D. Montgomery, Simply connected homogeneous spaces, Proceedings of the American Mathematical Society 1, 1950, 467-469
- [24] G.D. Mostow, Homogeneous Spaces with Finite Invariant Measure, Annals of Mathematics 75, 1962 (1), 17-37
- [25] J. Tits, Espaces homogènes complexes compacts, Commentarii Mathematici Helvetici 37, 1962, 111-120
- [26] M. Viana, K. Oliveira, Foundations of Ergodic Theory, Cambridge Studies in Advanced Mathematics 151, Cambridge University Press, 2016
- [27] A. Zeghib, *Sur les espaces-temps homogènes*, Geometry and Topology Monographs 1: The Epstein Birthday Schrift, 1998, 551-576
- [28] R.J. Zimmer, Ergodic Theory and Semisimple Groups, Birkhäuser, 1984
- [29] P.B. Zwart, W.M. Boothby, On compact homogeneous symplectic manifolds, Annales de l'Institute Fourier 30, 1980 (1), 129-157

OLIVER BAUES, DEPARTMENT OF MATHEMATICS, CHEMIN DU MUSÉE 23, UNIVERSITY OF FRI-BOURG, CH-1700 FRIBOURG, SWITZERLAND *E-mail address*: oliver.baues@unifr.ch

Wolfgang Globke, Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria

 $E\text{-}mail\ address:\ \texttt{wolfgang.globke@univie.ac.at}$

Abdelghani Zeghib, École Normale Supérieure de Lyon, Unité de Mathématiques Pures et Appliquées, 46 Allée d'Italie, 69364 Lyon, France

 $E\text{-}mail\ address: \texttt{abdelghani.zeghib@ens-lyon.fr}$