Rigidity of degenerate Riemannian metrics

Abdelghani Zeghib

UMPA, ENS-Lyon
http://www.umpa.ens-lyon.fr/~zeghib/
(in collaboration with E. Bekkara, S. Bekkara et C. Frances)

Préambule: La quatrième géométrie

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Object and Subject

E a vector space,

b a light(-like) scalar product,

$b : E \times E \to \mathbb{R}$ positive, and $\ker b$ has dimension $= 1$

Example, on $\mathbb{R}^{n+1}$, the quadratic form: $x_1^2 + \dots x_n^2$

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Example, on $\mathbb{R}^{n+1}$, the quadratic form: $x_1^2 + \ldots x_n^2$

M a manifold

Definition: a lightlike metric is a tensor g such that g_x is lightlike scalar on T_xM, $\forall x \in M$.

$f : (M, g) \longrightarrow (N, h)$ isometry, if $f^*h = g$:

$g_x(u_x, v_x) = h_{f(x)}(D_x f(u_x), D_x f(v_x))$

Goal: describe $\mathrm{Iso(M, g)}$, the isometry group of (M, g).

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Questions:
Is $\mathrm{Iso}(\mathrm{M}, \mathrm{g})$ a Lie group (why, why-not, and how)?

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Goal: describe $\mathrm{Iso(M, g)}$, the isometry group of (M, g).

Questions:

Is $\mathrm{Iso(M, g)}$ a Lie group (why, why-not, and how)?

Essentially,

– Is there a kind of connection associated naturally to (M, g)?

– Is there of family of curves (like geodesics) naturally associated to (M, g)?

More useful definitions

Definition of local isometry: f a local isometry of (M, g): means
$f : U \to V$, U, V open subsets of M,
$f : (U, g_{|U}) \to (V, g_{|V})$ isometry,

- The collection of local isometries is a pseudo-group ($\cong$ a groupoid...)

(M, g) a lightlike manifold,

$N = \ker g \subset TN$ a line fiber bundle

$\mathcal{N}$ its tangent foliation,

Terminology: characteristic (or normal, null, radical...) foliation,

Local expression

$\dim M = n + 1$

Local coordinate system $\rightarrow$ the characteristic foliation corresponds to $\frac{\partial}{\partial r}$

$(x_1, \ldots, x_n, r) = (x, r)$

$$g = \sum_{i,j \leq n} g_{ij}(x, r) dx^i dx^j$$

A one parameter family of Riemannian metric of the local quotient space $M/\mathcal{N}$, but not privileged parameter!

Similar structures, Lightlike vs sub-Riemannian

$E \to M$ a vector bundle,

• A sub-Riemannian metric on E consists in giving a Riemannian metric on a codimension 1 sub-bundle $D \subset E$

• A lightlike metric on E consists in giving $N \subset E$ a sub-bundle of rank (i.e. dimension) 1, together with a Riemannian metric on E/N.

Fact

A lightlike metric on E is equivalent to a sub-Riemannian metric on E^*. In particular:
lightlike metric on M (i.e. on TM) $\iff$ sub-Riemannian metric on T^*M.
subriemannian metric on M (i.e. on TM) $\iff$ lightlike metric on

Proof:

• if $h : E \to \mathbb{R}$ lightlike, $N \subset E$ its characteristic line sub-bundle,
let $D_x = \{\alpha_x \in E^*, \alpha_x(n_x) = 0\}$
$D \subset E^*$ has codimension 1,
α_x is a form on E_x/N_x,
$h^* : D \to \mathbb{R}$ defined by: $h^*(\alpha_x) =$ square norm of α_x (as an element of the euclidean space $(E_x/N_x)^*$)

• If $h : D \subset E \to \mathbb{R}$ is a sub-Riemannian metric, $h^* : E^* \to \mathbb{R}$, the hamiltonian (in the case $E = TM$)
$h^*(\alpha_x) =$ the square of the norm of $\alpha_{x|D_x}$.
This is lightlike with Kernel D^*

Contents

Some motivations

Induced pseudo-Riemannian metrics

Principle Start with the linear situation and then generalize straightforwardly to the manifold situation

Pseudo-euclidean scalar products: $\mathbb{R}^{p,q} = \mathbb{R}^{p+q}$ endowed with $Q = -x_1^2 - \ldots - x_p^2 + y_1^2 + \ldots + y_q^2$

Essential Difficulty: $E \subset \mathbb{R}^{p+q}$, subspace, $Q_{|E}$ is not necessarily a pseudo-euclidean product!

The induced Lorentz case

Lorentz: $p = 1$: $\mathbb{R}^{1,n} : Q_0 = -t^2 + x_1^2 + \ldots + x_n^2$

$\cong$

$Q_1 = x_0 x_1 + (x_1^2 + \ldots x_n^2)$

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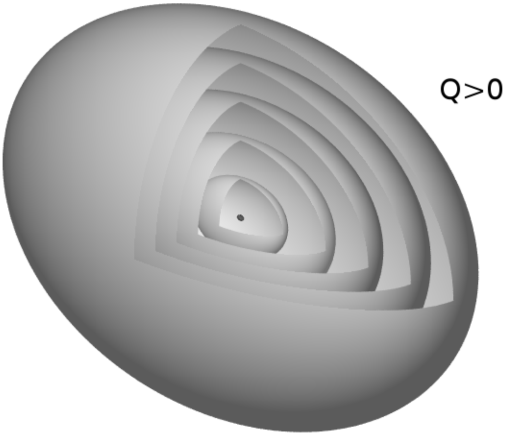
Recall causal characters:

- Spacelike $Q_{|E}$ positive definite
- timelike $Q_{|E}$ of Lorentz type
- Lightlike $Q_{|E}$ degenerate

Figures



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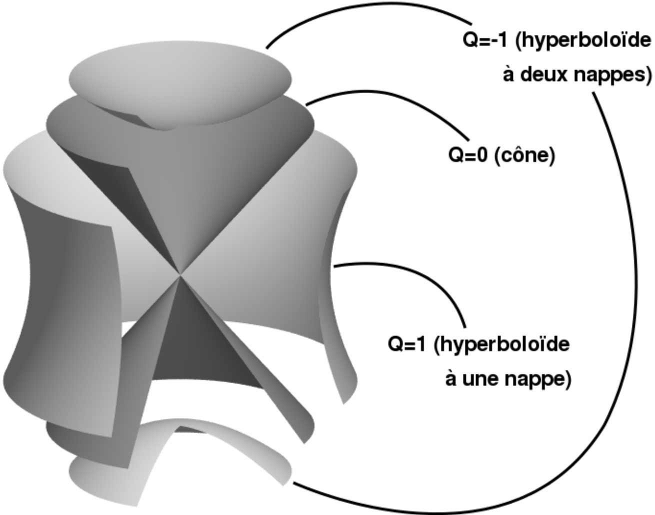
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à deux nappes)
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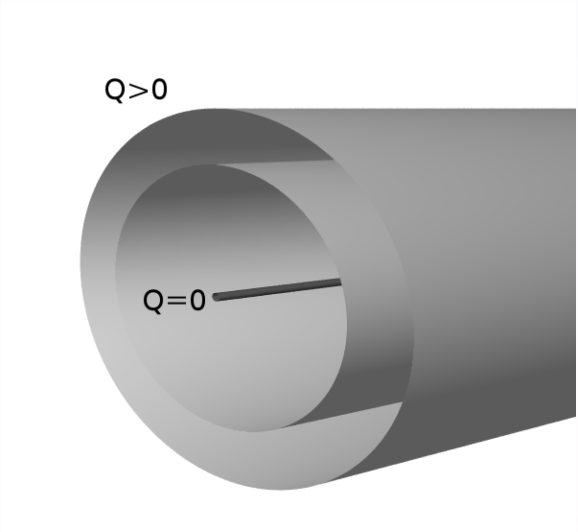

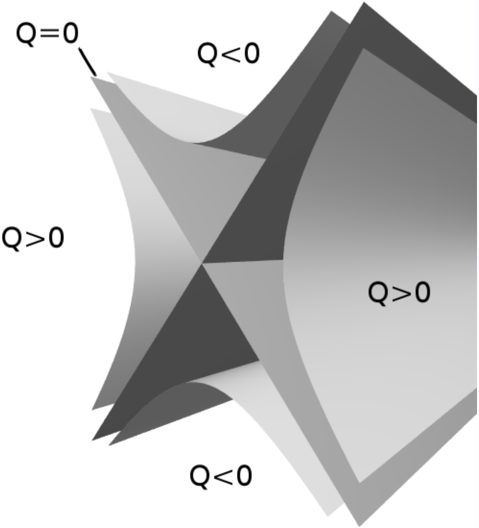
Q=0
Q<0
Q>0
Q>0
Q<0

More terminology, sub-Lorentz metrics

A tensor g on M is "**sub-Lorentz**" metric, if at each x, g_x is a Riemannian, Lorentzian or a lightlike scalar product.

Fact If (N, h) is a lorentz manifold, and $f : M \to N$ is a differentiable immersion, then, $g = f^*h$ is sub-Lorentz.

Example: the Euclidean sphere in the Minkowski space.

Philosophy

– Sub-Lorentz metrics are the abstraction of metrics induced form immersions in Lorentz manifolds,

– Study them intrinsically (and maybe, then, ask an isometric immersion problem?)

Lightlike metrics appear as regular sub-Lorentzian metrics (they have a constant type)!

In the sequel: natural situations of lightlike manifolds (usually immersed)

Black holes

(L, h) Lorentz (chronologically-oriented)

A (spacelike) hypersurface S has:
$D(S)$ its domain of dependence

$H(S) = \partial D(S)$ its horizon of S:
The horizon is lightlike (by maximality)

Black hole: the horizon of infinity!

Difficulty: horizons are generally non-regular (non-smooth submanifolds)

Characteristic surfaces of wave equation

$$\Box = \frac{\partial^2}{\partial t^2} - (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})$$

Point of view: This is the Laplacian for $\mathbb{R}^{1,3}$, the Minkowski space $\mathbb{R}^4$ endowed with $-t^2 + x^2 + y^2 + z^2$

general situation:
Algebra: $Q(x) = \Sigma g_{ij} x^i x^j$ (symbol)
Geometry: $g = g_{ij} dx^i dx^j$ (metric)
Analysis: $P_g = \Sigma g_{ij} \frac{\partial^2}{\partial x^i \partial x^j}$ (opertor)
$P_g u =$ div grad(u)....
Q and g classical data $\to P_g$ (quantum)

Characteristic surfaces of wave equation

(L, h) a Lorentz space,
$\Box_h u =$ div grad(u), D'Alembertian (Wave operator)

Example, for $\mathbb{R}^{1,3}$, the Minkowski space $\mathbb{R}^4$ endowed with $-t^2 + x^2 + y^2 + z^2$:

$$\Box = \frac{\partial^2}{\partial t^2} - (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})$$

(up to a sign)

Def: $M^n \subset N$ Characteristic hypersurface for $\Box_g$
$\iff$ $h_{|M}$ is degenerate $\iff$ M lightlike

Significance from the point of view of P_g: the Cauchy problem can not be solved for data given on M.

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See however, Hormander (Nicolas...): M is (locally) a limit of Cauchy surfaces, and somehow stronger problem can be solved ...

Orbits of pseudo-Riemannian actions

(N, h) a pseudo-Riemannian manifold,
$G \times N \to N$ acts isometrically on N
Any orbit $G.p$ is a homogeneous space, but not necessarily a pseudo-Riemannian homogeneous space.

Definition: a homogeneous space G/H is a pseudo-Riemannian homogeneous space of type (p, q) if the left G action preserves some pseudo-Riemannian metric of type (p, q).

Similar definition of lightlike homogeneous spaces (and also other structures like sub-Riemannian ...)

Problem: Classify homogeneous spaces of a given type (under aditionnal natural hypotheses)?

Related questions

M a lighlike submanifold in the Lorentz space (N, h).

There is no well submanifold theory for M, no "unit " normal", no second fundamental form, no curvature...

The "rigidity " suggests to associate higher order differential objects as substitution!

The two paradigmatic examples

Riemannian flows

I has dimension 1 (an interval or S^1) a lightlike metric is $(I, 0)$

• (M, g) is said transversally Riemannian, if it is locally isometric to $(N, h) \times (I, 0)$, where (N, g) is Riemannian.

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• Classical approach, transversally Riemannian flow (Molino, Carrière, Hector, Ghys, ...): a 1-dimensional foliation $\mathcal{N}$ of M endowed with a holonomy invariant (bundle-like) Riemannian metric,

$\Longleftrightarrow$

(M, g) lightlike, such that any flow X parametrizing $\mathcal{N}$ has flow ϕ^t which preserves g

A transvection $f : M \to M \iff, \forall x,\ f(\mathcal{N}_x) = \mathcal{N}_x$
(M, g) transversally Riemannian $\iff$ any transvection is an isometry

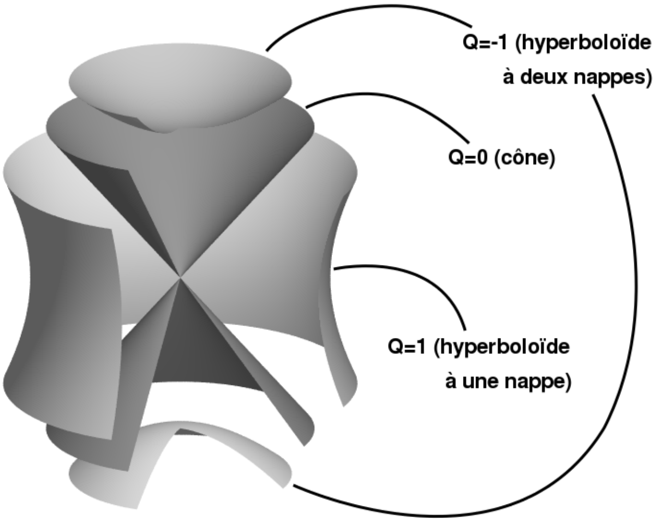
In particular, in this case, $\dim \mathrm{Iso}(\mathrm{M}, \mathrm{g}) = \infty$
example $(M, g) = (N, h) \times (S^1, 0)$, $f(x, y) = (x, t(x, y))$ is isometric

The light Minkowski cone

$\mathbb{R}^{1,n} : Q(x) = -x_0^2 + x_1^2 + \ldots + x_n^2$
$Co^n = Q^{-1}(0) - 0$ endowed with the induced metric,
- The characteristic foliation: radial lines,

$Co^n \cong (\mathbb{R}^n - \{0\}, d\Omega^2)$
(the Euclidean metric is $r^2 + d\Omega^2$)

$O^+(1, n)$ acts on $\mathbb{R}^{1,n}$ preserving Co^n,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The surprise!

Theorem

For $n \geq 4$, any local isometry of Co^n is the restriction of an element of $O^+(1,n)$. In particular $\mathrm{Iso}(\mathrm{Co}^n) = \mathrm{O}^+(1,\mathrm{n})$ is a Lie group (of finite dimension).

Similar to classical statements:

$\mathrm{Iso}^{\mathrm{loc}}(\mathbb{R}^n)$ are composition of translations and rotation,

$\mathrm{Iso}^{\mathrm{loc}}(\mathbf{S}^n)$...

$\mathrm{Conf}(\mathbf{S}^n)$ (Liouville) composition of similarities and inversions...

Proof

$Co^n \cong \mathbb{R} \times \mathbf{S}^{n-1}$

Metric $0 \oplus e^{2t} h_{\mathbf{S}^{n-1}}$

Let f be an isometry,

$\mathbf{S}^{n-1}$ is the space null leaves and so f acts on it

f has the form: $(t, x) \mapsto (\lambda(t, x), \phi(x))$.

Isometry $\Longrightarrow$

$$\phi^* g_{\mathbf{S}^{n-1}} = e^{2(t-\lambda(t,x))} g_{\mathbf{S}^{n-1}}$$

Thus,

$f : (t, x) \mapsto (t - \mu(x), \phi(x))$,

ϕ a conformal transformation of the sphere, $\phi^* g_{\mathbf{S}^{n-1}} = e^{2\mu} g_{\mathbf{S}^{n-1}}$.

– Apply Liouville Theorem for Conf($\mathbf{S}^n$)

(in particular no freedom for μ)

Lighlike geometry contains conformal Riemannian geometry

(N, h), $h = h_{ij}(x)dx^i dx^j$ Riemannian,
$M = N \times \mathbb{R}$
(x, r)
$g_{x,r)} = c(r)h_x$,
Assume: $\partial c/\partial r \neq 0$, then

local isometry for M $\iff$ local conformal transformation for N

Global rigidity

Homogeneous lightlike manifolds?

The problem: classify lightlike homogeneous spaces

Homogeneous lightlike manifolds?

The problem: classify lightlike homogeneous spaces

Remarks:

1) This looks like a linear algebra problem, but turns out to be a quadratic one!

2) A natural general problem about geometric structures: homogeneous spaces are like polynomials among general functions! (or exact solutions...)

Hierarchy

Dynamical condition: G/H such that H non-compact,

Otherwise, G preserves a Riemannian metric on G/H, a simpler structure,

In the case of non-transitive actions, the hypothesis is the action is non-proper

Theorem

Let $M = G/H$ be a homogeneous lightlike manifold, such that,
– H non-compact
– G is semi-simple with no factor locally isomorphic to $SL_2(\mathbb{R})$
Then, up to a cyclic cover, $M \cong Co^n$

Case of non-transitive actions

Assuming the action non-proper, there are orbits $\cong Co^n$ (up to a cover).
M itself is a kind of amalgamed product of cones by a Riemannian manifold.

What is a semi-simple Lie group?

G has no abelian normal subgroup,

Practical definition:

$SL_n(\mathbb{R})$

$O(p, q)$

$Sp(n, \mathbb{R})$

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Remark: their structure is encoded in combinatorial data: a root system...(important tool in many areas of mathematics and physics with no apparent connection)!

Isometry groups of left invariant lightlike metrics on Lie groups

An example: $G = SO(3)$

A left invariant lightlike metric g on $SO(3) \iff$ a lightlike scalar product q on $so(3)$

The null direction is a left-invariant direction field.... Hopf fibration

$S(q) =$ stabilizer of q in the adjoint action

$S(q) \cong SO(2)$ or $\cong 1$

Theorem (Bekkara-Oussalah)

If $S(q) = 1$, then full isometry group coincides with $SO(3)$ (no extra isometry).
(In the other case, the lightlike metric is transversally Riemannian, and has infinitely dimensional isometry group)

Local Rigidity (as a geometric structure)

The concept

Rigidity vs flexibility of geometric structures?

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Examples:

Rigid (Solid): Riemannian metric — Pseudo-Riemannian metric — Connection — Conformal structure on dimension > 2....

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Non-rigid (Fluid): Symplectic structure — Complex structure — Contact structure — ...

The concept

Rigidity vs flexibility of geometric structures?

Examples:

Rigid (Solid): Riemannian metric — Pseudo-Riemannian metric — Connection — Conformal structure on dimension > 2....

Non-rigid (Fluid): Symplectic structure — Complex structure — Contact structure — ...

• Rigid $\cong$ the pseudo-group of local isometries is a LIE GROUP (of finite dimension).

Generalized geodesics

Rigid:

– there is a kind of naturally associated connection,

– there is a naturally associated family of curve, generalized like geodesics,

Rigid at order k $\longrightarrow$ a differential equation of order $k+1$

Example: a connection (order 1)

$\ddot{x}^i = \Sigma \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k$

$(x = (x^1, \ldots, x^n)$ a coordinate system)

Remark, case of homogeneous spaces (at least if $M = G/H$, $H = 1$): Γ^i_{jk} do not depend on x,

Yet, these equations are very complicated (they can generate chaos...)

Another example, structure of order 2 (equations of order 3) conformal structure (in dim ≥ 3) $\rightarrow$ circles

Classic: H-structures

$H \subset GL_n(\mathbb{R})$ a subgroup (this is not an abstract group)
$\dim M = n$
The frame bundle $P(M) \to M$, with structural group $GL_n(\mathbb{R})$
An H-structure on M: reduction of the structural group to H.
$\iff$ a section of the bundle $P(M)/H \to H$

Explanation

$E = TM$

Transitions between trivialisation charts $U \times \mathbb{R}^n \to U \times \mathbb{R}^n$

$(x, u) \to (x, A(x)u)$

$x \in U \to A(x) \in \mathrm{GL_n}(\mathbb{R})$

Reduction to $H \subset \mathrm{GL_n}(\mathbb{R})$

$\iff$ an atlas with all the A's valued in H.

Examples

- $H = \{1\}$: Parallelism (Framing)

- $H = O(n)$ a Riemannian metric
- The groups $O(n_-, n_0, n_+)$, the orthogonal group of a quadratic form of type (n_-, n_0, n_+)
- Case $n_0 = 0$: pseudo-Riemannian metric
- Case of $O(0, 1, n)$: lightlike metric

- The groups $D(k,n)$ (the stabilizer of $\mathbb{R}^k$ in $\mathrm{GL_n}(\mathbb{R})$): a field of k- planes
- $OD(k,n)$ subgroup elements of $D(k,n)$ preserving the Euclidean product on $\mathbb{R}^k$.

Groups

$O(0, 1, n)$

$$\left\{ \begin{pmatrix} \lambda & a_1 & \dots & a_n \\ 0 & & & \\ . & & A & \\ . & & & \\ 0 & & & \end{pmatrix} A \in O(n),\ \lambda, a_i \in \mathbb{R} \right\}$$

$OD(n-1,n)$:

$$\begin{pmatrix} A & \overrightarrow{u} \\ 0 & b \end{pmatrix}$$

The automorphic mapping $B \to B^{-1*}$ sends $OD(n-1,n)$ bijectively on $O(1,0,n)$.

(Duality between lightlike and sub-Riemannian)

Finite type, Elie Cartan

$\mathcal{H} \subset Mat_n(\mathbb{R})$ the Lie algebra of H

$\mathcal{H}_k = Pro_k(\mathcal{H})$, the space of k-prolongations of $\mathcal{H}$

$A \in \mathcal{H}_k$ means:

$A : \mathbb{R}^n \times \ldots \mathbb{R}^n = (\mathbb{R}^n)^{k+1} \longrightarrow \mathbb{R}^n$,

- A is symmetric multilinear
- $\forall u = (v_1, \ldots v_k) \in (\mathbb{R}^n)^k$ fixed, the map

$$A_u : v \longrightarrow A(v, v_1, \ldots, v_k)$$

belongs to $\mathcal{H}$.

Explanation

$\mathbb{R}^n$, $P(\mathbb{R}^n) = \mathbb{R}^n \times GL_n(\mathbb{R})$

$H \subset GL_n(\mathbb{R})$

The (globally) flat H-structure on $\mathbb{R}^n$: defined by any translation invariant framing of $\mathbb{R}^n$.

$f : \mathbb{R}^n \to \mathbb{R}^n$ preserves the H-structure (isometry of the structure)

$\iff$

$\forall x, \quad D_x f \in H,$

An H-Killing field X = its flow preserves the H-structure $\iff$ $\forall x \in \mathbb{R}^n$: $x \to D_x X \in \mathcal{H}$ (the Lie algebra of H)

(example: $H = O(n)$, an Euclidean vector field, infinitesimal isometry $\iff$ $D_x X$ skew-symmetric)

$A = D_{x_0}^{k+1} X$ (the higher derivative of order $(k+1)$ at x_0)

$A \in \mathcal{H}_k$

Examples

• Finite type

$H = O(n)$: $\mathcal{H}_1 = 0$: the second derivative of an infinitesimal Euclidean isometry is 0...

$H = \mathbb{R} \times O(n)$ (conformal structure) $\mathcal{H}_2 = 0$ (if $n > 2$)

• Infinite type

$H = sp(n, \mathbb{R})$ (symplectic) (e.g. $SL_2(\mathbb{R})$): $\mathcal{H}_i \neq 0$, $\forall i$

$H = O(0, 1, n)$: (e.g. consider transvections on $\mathbb{R}^{n+1}$ endowed with $x_1^2 + \ldots + x_n^2$)

Synthesis

– The word "rigidity" doesn't exist in Cartan appraoch...

– Finite type $\Longrightarrow$ The isometry group is a Lie group

This uses "Cartan connections"

– Finite type $\iff$ there is an associate parallelism of some frame bundle

– Central Fact (remark): by definition, being of finite order depends only on H (e.g. all Riemannian metrics have order 2, not only the flat one)

Associated higher order parrallelism

Example 1: case of a connection (this is not a usual H-structure but rather an H-structure of order 2)

Conformal case:

M, $[g_0]$ a conformal class

$X_0 \to M$: the (trivial) line bundle whose (local) sections are Riemmaian metric in this conformal class: $g = e^\sigma g_0$

$X_1 \to M$: bundle of 1-jets of sections of X_0: $\cong T^*M$ ($d_x\sigma$)

Fact

Given $x_1 \in X_1$, i.e. a section $g_1 = e^\sigma g_0$ up to order 1, equivalently, they are given: $m \in M, \sigma(m), d_m\sigma$
($x_0 = e^{\sigma(m)} g_0(m)$, and $x_1 \cong d_m\sigma$).
Then,
– There exists $g = e^\sigma g_0$ such that $Ric_m(g) = 0$
– The 2-jet of g is unique (well determined) at m

Thus, the derivative of $g : M \to X_1$ is well defined
Its image is a horizontal space H_{x_1}

Equip H_{x_1} with the metric $x_0 = e^{\sigma}(m)g_0(m)$

Anything on the vertical space...

Gromov's approach

What is a geometric structure?

A chart $c = (U, x)$, $c : U \to \mathbb{R}^n$

Germs of charts

$\mathcal{C}(M) = \{(U, x)\}$

$D = Diff(\mathbb{R}^n, 0)$ acts by composition on the target

A geometric structure is a map: $\phi : \mathcal{C}(M) \to Z$ obeying to some law: it is equivariant with respect to an action of D on Z

Example: Riemannian metric: to $c = (U, x)$ associate its local expression $g_{ij}(x)dx^i dx^j$

$\phi : c = (U, x) \to (g_{ij}(x)) \in Sym_n(\mathbb{R})$

Action of D:

$f \in D = Diff(\mathbb{R}^n, 0) \to D_0 f \in GL_n(\mathbb{R})$

D acts on $Sym_n(\mathbb{R})$ via $GL_n(\mathbb{R})$

A connection: $c = (U, x) \to (\Gamma^k_{ij}(x))$ Christofel symbols

Structure of Order k : equivalence relation: $c \sim c'$ if they have the same k-jet at x

$\mathcal{C}^k(M)$ the quotient space

$D \to D^k$

Z a manifold with a D^k-action
$\phi : \mathcal{C}^k(M) \to Z$ equivariant,

Rigidity

$\mathrm{Iso}_x^{\mathrm{Loc}} = \{\mathrm{f}$ local isometry defined around x such that $f(x) = x\}$
$\Phi_x^k : f \in \mathrm{Iso}_x^{\mathrm{Loc}} \to \mathrm{jet}_x^k(\mathrm{f}) \in \ldots$ (some complicated algebraic space)

"Local rigidity at order k": Φ_x^k is injective

Example: A Riemannian isometry f: $f(x) = x$, and $D_x f = 1$, then $f = id$.

Infinitesimal rigidity

True definition, stronger than the local rigidity:
$\Psi_x^k : \mathrm{Iso}_x^{k+1} \longrightarrow \mathrm{Iso}_x^k$ is injective.

• Definition: f is an isometry up to order j, if $f^*g - g$ vanishes up to order j at x (The Taylor development vanishes up to order j, in some, and hence any, chart)

Examples

— Riemannian metrics, $\mathrm{Iso}^2 \to \mathrm{Iso}^1$ is injective

(A step in the proof of the existence of the Levi-Civita connection)

— Conformal case: $\mathrm{Iso}^3 \to \mathrm{Iso}^2$ is injective

Liouville Theorem $\to$ the explicit form of local conformal transformations on $\mathbb{R}^n$, $n \geq 3$

System of equations in the Riemannian and conformal cases

Rigidity vs finiteness of type for H-structures

Theorem

For H-structures:
Rigidity (in Gromov sense)$\cong$ finite type (in the Cartan sense)

this depends only on H (as a subgroup of $GL_n(\mathbb{R})$): if there is one example of a rigid H-structure, then, any H-structure is rigid!

In particular:

Corollary

– lightlike and sub-Riemannian metrics are never rigid (since there are examples with infinite dimensional groups)

Contradiction!?

The sub-Riemannian case, rigidity flavor

(M^{2n+1}, h) a contact sub-Riemannian structure:
$[\omega]$ a contact structure, h a metric on $D = \ker[\omega]$

The geodesic flow $\cong$ the flow associated to the hamiltonian $h^* : T^*M \to \mathbb{R}$...
Hence, geodesics...

- $(h, [\omega]) \to g = Rie(h, [\omega]) = Rie(h)$ a true Riemannian metric

Indeed, $h \to \omega_0 \in [\omega]$ a contact form
Then, define g, by decreeing the Reeb flow of ω_0 is unitary and orthogonal to D

• Construction of ω_0, by the condition
$d{\omega_0}^n_{|D}$ = volume form of h (on D).

If $\omega_1 = c\omega_0$, $d\omega_1 = dc \wedge \omega_0 + cd\omega_0$
e.g.$n = 1$, $dc \wedge \omega_0$ vanishes on $D \implies c = 1$.

Contradiction with the fact that sub-Riemannian metrics are not of finite type!

Or, finiteness of type is not the exact counterpart of the rigidity concept...!?

A Rigidity aspect in the lightlike case

Remember the transversally conformal case (e.g. the lightcone case):
$g_{(x,r)} = c(r)h$: local conformal diffeomorphisms for $h \iff$ local isometries for h,
with the genericty hypothesis (reminiscent to the contact condition): $L_X g \neq 0$, where X is (any) null vector field.

Contradiction with the fact that the type is infinite, equivalently no rigidity in the Gromov sense!

Lightlike case, infinitesimal fact

Theorem ("Sub-rigidity")

(M^n, g) a generic lightlike structure: $L_N g$ has maximal rank $= n - 1$, where N is tangent to the characteristic foliation.

If $f \in \mathrm{Iso}^3$, $Jet^1 f = 1$ $(= Jet^1(Id))$, then $Jet^2 f = 1$.

Similar statement for sub-Riemannian contact structures.

See "Singular Riemannian metrics, sub-rigidity vs rigidity" in
`http://www.umpa.ens-lyon.fr/~zeghib/pubs.html`

Definition

There are natural maps $\mathrm{Iso}^{\mathrm{i+j}} \longrightarrow \mathrm{Iso}^{\mathrm{j}}$

Definition

The geometric structure is sub-rigid of order (k, δ) if

$$Image(Is^{k+\delta}) \subset Is^{k+1} \longrightarrow \mathrm{Iso}^{\mathrm{k}}$$

is injective

f isometric up to order $k + \delta$

f trivial up to order k

Then f is trivial up to order $k + 1$

Explanation

The definition is du to D. Fisher, J. Bennvensite (under the name "quasi-rigid")

Their Prototype: singular metrics:
$g = fh$, h a Riemannian metric, f a non-negative function,
$f(x_0) = 0$, x_0 isolated zero...

Or: a (singular) framing which vanish up to order δ at an isolated point

Their motivation: a "geometrization conjecture" du to Gromov-Zimmer: whenever a lattice Γ in a higher rank semisimple Lie group (e.g. $SL_n(\mathbb{Z})$) acts smoothly on a compact manifold, there is a partition into pieces on which Γ preserves a rigid geometric structure.

The test:
The blow up of $\mathbb{R}^n$ (at 0) $\to X$
$p : X \to \mathbb{R}^n$, bijective above $\mathbb{R}^n - 0$, but $p^{-1}(0) = \mathbb{R}P^{n-1}$
$X = \{(x, d) \in \mathbb{R}^n \times \mathbb{R}P^{n-1}$ such that $x \in d\}$

The "inverse image" of the flat connection is defined on X
It is sub-rigid...
$GL_n(\mathbb{R})$ preserves everything

Their Natural questions:

– Is "sub-rigid" essentially similar to "rigid":

– Do sub-rigid structures give rise to singular framing...

Here: generic lightlike and contact -Riemannian metrics are "conter-examples": they can be homogeneous:

– there is no localization of their singularity

– they are everywhere sub-rigid non-rigid,

One default of sub-rigidity

Fact

Let M be simply connected and endowed with an analytic rigid geometric structure. Let V be a Killing field of the structure (its local flow preserves the structure) defined on an open subset $U \subset M$.
Then, V extends to M.

This is not true in the sub-rigid case

$Aff(\mathbb{R}^n) = GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$ acts $\mathbb{R}^n$, but not on its blow up...

Generic lightlike and contact sub-Riemannian metrics are less than rigid, but better than sub-rigid?!