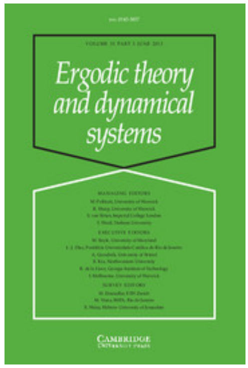

Actions of discrete groups on stationary Lorentz manifolds

PAOLO PICCIONE and ABDELGHANI ZEGHIB

Ergodic Theory and Dynamical Systems / *FirstView* Article / June 2013, pp 1 - 34
DOI: 10.1017/etds.2013.17, Published online: 05 June 2013

Ergod. Th. & Dynam. Sys. (First published online 2013), page 1 of 34*
doi:10.1017/etds.2013.17
*Provisional—final page numbers to be inserted when paper edition is published

Actions of discrete groups on stationary Lorentz manifolds

PAOLO PICCIONE† and ABDELGHANI ZEGHIB‡

† *Departamento de Matemática, Universidade de São Paulo, Rua do Matão 1010, 05508-900, São Paulo, SP, Brazil*
(e-mail: piccione@ime.usp.br)
‡ *Unité de Mathématiques Pures et Appliquées, École Normale Supérieure de Lyon, 46, Allée d'Italie, 69364 Lyon Cedex 07, France*
(e-mail: abdelghani.zeghib@ens-lyon.fr)

(*Received* 29 *November* 2012 *and accepted in revised form* 28 *February* 2013)

Abstract. We study the geometry of compact Lorentzian manifolds that admit a somewhere timelike Killing vector field, and whose isometry group has infinitely many connected components. Up to a finite cover, such manifolds are products (or amalgamated products) of a flat Lorentzian torus and a compact Riemannian (respectively, lightlike) manifold.

Contents

1. *Introduction*

1.1. *Paradigmatic example.* We will deal with dynamics and geometry of the following flavor. Let q be a Lorentz form on $\mathbb{R}^k$; this induces a (flat) Lorentz metric on the torus $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$. The linear isometry group of $\mathbb{T}^k$ is $\mathrm{O}(q, \mathbb{Z}) = \mathrm{GL}(k, \mathbb{Z}) \cap \mathrm{O}(q)$, and its full isometry group is the semi-direct product $\mathrm{O}(q, \mathbb{Z}) \ltimes \mathbb{T}^k$.

The global and individual structure of $\mathrm{O}(q, \mathbb{Z})$ involves interesting geometric, arithmetic and dynamical interactions. For generic q, $\mathrm{O}(q, \mathbb{Z})$ is trivial. Nonetheless, if q is rational, i.e. if $q(x) = \sum a_{ij} x_i x_j$, where a_{ij} are rational numbers, then $\mathrm{O}(q, \mathbb{Z})$ is *big* in $\mathrm{O}(q)$; more precisely, by the Harish-Chandra–Borel theorem, it is a lattice in $\mathrm{O}(q)$. When q is not rational, many intermediate situations are possible. It is a finite volume non-co-compact lattice in the case of the standard form $q_0 = -x_1^2 + x_2^2 + \cdots + x_k^2$, but can be co-compact for other forms. On the other hand, a given element $A \in \mathrm{O}(q_0, \mathbb{Z})$ could have complicated dynamics. For instance, if A is hyperbolic, then it has a leading simple eigenvalue. If furthermore A is irreducible, that is, it preserves no non-trivial sub-torus, then this eigenvalue is a Salem number. Conversely, any Salem number is the eigenvalue of such a hyperbolic $A \in \mathrm{O}(q, \mathbb{Z})$, for some rational Lorentz form q, see more details in Appendix A (in fact, essentially, one can increase the dimension and get an integer orthogonal matrix A for the standard Lorentz form, i.e. $A \in \mathrm{O}(1, n)(\mathbb{Z}) = \mathrm{O}(q_0, \mathbb{Z})$, where q_0 is the standard Lorentz form in dimension $1 + n$).

1.2. *Lorentz geometry and dynamics.* The global geometry of compact manifolds endowed with a non-positive definite metric (pseudo-Riemannian manifolds) can be quite different from the geometry of Riemannian manifolds. For instance, compact pseudo-Riemannian manifolds may fail to be geodesically complete or geodesically connected; moreover, the isometry group of a compact pseudo-Riemannian manifold fails to be compact in general. The main goal of this paper is to investigate the geometric structure of Lorentz manifolds *essentially* non-Riemannian, i.e. with non-compact isometry group.

Lorentzian manifolds, i.e. manifolds endowed with metric tensors of index 1, play a special role in pseudo-Riemannian geometry, due to their relations with general relativity. The lack of compactness of the isometry group is due to the fact that, unlike the Riemannian case, Lorentzian isometries need not be equicontinuous, and may generate chaotic dynamics on the manifold. For instance, the dynamics of Lorentz isometries can be of Anosov type, evocative of the fact that in general relativity one can have contractions of time and expansion in space. A celebrated result of D'Ambra (see [**7**]) states that the isometry group of a *real analytic* simply-connected compact Lorentzian manifold is compact. It is not known whether this results holds in the C^∞ case. In the last decade several authors have studied isometric group actions on Lorentz manifolds. Most notably, a complete classification of (connected) Lie groups that act locally faithfully and isometrically on compact Lorentzian manifolds has been obtained independently by Adams and Stuck (see [**1, 2**]) and the second author (see [**16**]). Roughly speaking, the identity component G_0 of the isometry group of a compact Lorentz manifold is the direct product of an abelian group, a compact semi-simple group, and, possibly, a third factor which is locally isomorphic to either $\mathrm{SL}(2, \mathbb{R})$ or to an oscillator group, or else to a

Heisenberg group. The geometric structure of a compact Lorentz manifold that admits a faithful isometric action of a group G isomorphic to $\mathrm{SL}(2, \mathbb{R})$ or to an oscillator group is well understood; such manifolds can be described using right quotients G/Γ, where Γ is a co-compact lattice of G, and warped products, see §4 for more details. Observe that such constructions produce Lorentz manifolds on which the G_0-action has some timelike orbit. Recall that a Lorentz manifold is said to be *stationary* if it admits an everywhere timelike Killing vector field. Our first result (Theorem 4.1) is that when the identity component of the isometry group is non-compact and it has some timelike orbit, then it must contain a non-trivial factor locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$ or to an oscillator group.

Thus, the next natural question is to study the geometry of manifolds whose isometry group is non-compact for having an infinite number of connected components.

1.3. *Results.* We will show in this paper that compact Lorentz manifolds with a *large* isometry group are essentially constructed from flat tori. In order to define the appropriate notion of the lack of compactness of the isometry group of a Lorentzian manifold, let us give the following definition.

Definition. Let $\rho : \Gamma \to \mathrm{GL}(\mathcal{E})$ be a representation of the group Γ on the vector space $\mathcal{E}$. Then, ρ is said to be *of Riemannian type* if it preserves some positive definite inner product on $\mathcal{E}$. We say that ρ is *of post-Riemannian type* if it preserves a positive semi-definite inner product on $\mathcal{E}$ having kernel of dimension equal to 1.

Observe that ρ is of Riemannian type if and only if it is precompact, i.e. $\rho(\Gamma)$ is precompact in $\mathrm{GL}(\mathcal{E})$.

Given a Lorentzian manifold $(M, \mathbf{g})$, we will denote by $\mathrm{Iso}(M, \mathbf{g})$ its isometry group, and by $\mathrm{Iso}_0(M, \mathbf{g})$ the identity connected component of $\mathrm{Iso}(M, \mathbf{g})$. The Lie algebra of $\mathrm{Iso}(M, \mathbf{g})$ will be denoted by $\mathfrak{Iso}(M, \mathbf{g})$. By a large isometry group we mean in particular that $\mathrm{Iso}(M, \mathbf{g})$ is non-compact. The lack of compactness may occur in one of the following situations:

(1) the strongest situation where the identity connected component $\mathrm{Iso}_0(M, \mathbf{g})$ is non-compact; this case was studied and essentially understood in [**1, 2, 16**];
(2) the weakest case where $\mathrm{Iso}_0(M, \mathbf{g})$ is trivial, and the discrete factor $\Gamma = \mathrm{Iso}(M, \mathbf{g})/\mathrm{Iso}_0(M, \mathbf{g})$ is infinite;
(3) an intermediate situation, where both $\mathrm{Iso}_0(M, \mathbf{g})$ and Γ are non-trivial, that is, $\mathrm{Iso}_0(M, \mathbf{g})$ is compact and the action of $\mathrm{Iso}(M, \mathbf{g})$ on the Lie algebra $\mathfrak{Iso}(M, \mathbf{g})$ is not post-Riemannian (in particular, $\mathrm{Iso}_0(M, \mathbf{g})$ is non-trivial).

We are dealing here with such an intermediate case. The main result of the paper is that compact Lorentz manifolds that belong to this intermediate category are essentially built up by tori. More precisely, we prove the following structure theorem.

THEOREM 1. *Let* $(M, \mathbf{g})$ *be a compact Lorentz manifold, and assume that the action of* $\mathrm{Iso}(M, \mathbf{g})$ *on the Lie algebra of* $\mathrm{Iso}_0(M, \mathbf{g})$ *is not post-Riemannian, and that* $\Gamma = \mathrm{Iso}(M, \mathbf{g})/\mathrm{Iso}_0(M, \mathbf{g})$ *is infinite. Then,* $\mathrm{Iso}_0(M, \mathbf{g})$ *contains a torus* $\mathbb{T} = \mathbb{T}^d$*, endowed with a Lorentz form* q*, such that* Γ *is a subgroup of* $\mathrm{O}(q, \mathbb{Z})$.

Up to a finite cover, there is a new Lorentz metric $\mathbf{g}^{\text{new}}$ *on* M *having a larger isometry group than* $\mathbf{g}$*, such that the discrete factor is* $\Gamma^{\text{new}} = \mathrm{O}(q, \mathbb{Z})$*, where* $\Gamma^{\text{new}} = \mathrm{Iso}(M, \mathbf{g}^{\text{new}})/\mathrm{Iso}_0(M, \mathbf{g}^{\text{new}})$. *Geometrically,* M *is the metric direct product* $\mathbb{T} \times N$*, where* N *is a compact Riemannian manifold, or* M *is an amalgamated metric product* $\mathbb{T} \times_{\mathbb{S}^1} L$*, where* L *is a lightlike manifold with an isometric* $\mathbb{S}^1$*-action. The last possibility holds when* Γ *is a parabolic subgroup of* $\mathrm{O}(q)$.

A more precise description of the original metric $\mathbf{g}$ is given in §11 for the hyperbolic case and in §12 for the parabolic case. We will in fact prove Theorem 1 under an assumption weaker than the non-post-Riemannian hypothesis for the conjugacy action of Γ. The more general statement proved here is the following theorem.

THEOREM 2. *Assume that* Γ *is infinite and that* $\mathrm{Iso}_0(M, \mathbf{g})$ *has a somewhere timelike orbit. Then the conclusion of Theorem 1 holds.*

Theorem 1 will follow from Theorem 2 once we show that, under the assumption that the action of $\mathrm{Iso}(M, \mathfrak{g})$ on the Lie algebra of $\mathrm{Iso}_0(M, \mathfrak{g})$ is not of post-Riemannian type, then the connected component of the identity of the isometry group must have some timelike orbits, see §3.1.

A first consequence of our main result is the following.

COROLLARY 3. *Assume that* $(M, \mathbf{g})$ *is a compact Lorentzian manifold with infinite discrete factor* Γ. *If* $(M, \mathbf{g})$ *has a* somewhere *timelike Killing vector field, then* $(M, \mathbf{g})$ *has an* everywhere *timelike Killing vector field.*

We will also prove (Theorem 2.1, part (2)) that, when $(M, \mathbf{g})$ has a Killing vector field which is timelike somewhere, then the two situations (a) and (b) below are mutually exclusive:

(a) the connected component of the identity $\mathrm{Iso}_0(M, \mathbf{g})$ of $\mathrm{Iso}(M, \mathbf{g})$ is non-compact;

(b) $\mathrm{Iso}(M, \mathbf{g})$ has infinitely many connected components, as in the case of the flat Lorentzian torus.

The point here is that, in a compact Lorentzian manifold, the flow of a Killing vector field which is timelike somewhere generates a non-trivial precompact group in the (connected component of the identity of the) isometry group. Thus, by continuity, the Lie algebra of the isometry group of such manifolds must contain a non-empty open cone of vectors generating precompact one-parameter subgroups in the isometry group. The proof of Theorem 2.1 is obtained by ruling out the existence of a non-compact abelian or nilpotent factor in the connected component of the isometry group. The argument is based on an algebraic precompactness criterion for one-parameter subgroups of Lie groups proved in Proposition 3.2.

Moreover, using Theorem 2 and previous classification results by the second author, we prove the following partial extension of D'Ambra's result to the C^∞-realm.

THEOREM 4. *The isometry group of a simply connected compact Lorentzian manifold that admits a Killing vector field which is somewhere timelike is compact.*

2. *A guide: steps of proofs*

Let us discuss the sequence of steps of our proofs.

2.1. *Notation.* $(M, \mathbf{g})$ will denote a compact Lorentz manifold, $\mathrm{Iso}(M, \mathbf{g})$ its isometry group and $\mathrm{Iso}_0(M, g)$ its identity component (the connected component of the identity). Its Lie algebra is denoted $\mathfrak{Iso}(M, g)$.

For G a subgroup of $\mathrm{Iso}(M, \mathbf{g})$, we denote by G_0 its identity component and $\mathfrak{g}$ its Lie algebra. In fact, G will generally designate the full group $\mathrm{Iso}(M, \mathbf{g})$ itself.

Let $\mathrm{Aut}(G_0)$ be the group of automorphisms of G_0, and $\mathrm{Inn}(G)$ its subgroup of inner automorphisms. Since G_0 is normal in G, G acts by conjugation on G_0, and we have a homomorphism $G \to \mathrm{Aut}(G_0)$, and composing with the quotient map $\mathrm{Aut}(G_0) \to \mathrm{Aut}(G_0)/\mathrm{Inn}(G_0)$, we have a homomorphism:

$$\rho : G \longrightarrow \mathrm{Out}(G_0), \tag{2.1}$$

where $\mathrm{Out}(G_0) = \mathrm{Aut}(G_0)/\mathrm{Inn}(G_0)$.

2.2. *The identity component of weakly stationary manifolds.* The following theorem was the initial motivation of the present work. It completes the previously quoted works on actions of non-compact connected groups [**1, 2, 16**], with the hypothesis that the orbits are somewhere timelike.

THEOREM 2.1. *Let $(M, \mathbf{g})$ be a compact Lorentz manifold that admits a Killing vector field which is timelike at some point. Then:*

(1) *the identity component of its isometry group is compact, unless it contains a group locally isomorphic to* $\mathrm{SL}(2, \mathbb{R})$ *or to an oscillator group;*
(2) *if* $\mathrm{Iso}(M, \mathbf{g})$ *has infinitely many connected components, then* $\mathrm{Iso}_0(M, \mathbf{g})$ *is compact;*
(3) *in this last case, the homomorphism* $\mathrm{Iso}(M, \mathbf{g}) \to \mathrm{Out}(\mathrm{Iso}_0(M, \mathbf{g}))$ *is quasi-injective in the sense that its kernel is compact.*

The proof will be given in §§4–6. As for §3, it is to an extent independent and devoted to the proof of Theorem 1 from Theorem 2.

2.3. *A fixed point theorem in linear dynamics.* The are many results in dynamical systems stating that, for some classes of systems, recurrence occurs only in a 'trivial' manner. We have for instance Rosenlicht's theorem about algebraic actions of algebraic groups, Furstenberg's lemma on projective dynamics, and also Bendixson–Poincaré's theorem on the dynamics of general flows on the 2-sphere [**13, 14, 18**]. In §7, we will prove the following variant.

THEOREM 2.2. *Let Γ be a group and $\rho : \Gamma \to \mathrm{GL}(\mathcal{E})$ a representation in a vector space $\mathcal{E}$. Let $F = \mathrm{Sym}(\mathcal{E}^*)$ be the space of quadratic forms of $\mathcal{E}$, and ρ^F the associated representation.*

Assume that $\rho(\Gamma)$ is non-precompact, and furthermore that some Lorentz form (on $\mathcal{E}$) has a bounded orbit under the ρ^F-action on F. Then, (some finite index subgroup of) $\rho(\Gamma)$ preserves some Lorentz form on $\mathcal{E}$.

This will be applied essentially as follows. With the notation above, let $\mathcal{E} = \mathfrak{g}$ be the Lie algebra of G. We have a Gauss map $\mathcal{G} : M \ni x \mapsto q_x \in \mathrm{Sym}(\mathcal{E})$, where q_x is the quadratic form on $\mathfrak{g}$ defined as the pullback of the induced metric on the orbit Gx by the projection $\mathfrak{g} \to T_x(Gx)$ (see §3.1). This map is equivariant with respect to the given action of G on M and the action associated with the G-adjoint action on $\mathfrak{g}$. The theorem applies once one assumes there exists x_0 such that q_{x_0} is of Lorentz type (of course, assuming M to be compact).

When it applies, the theorem says that G embeds into $\mathrm{SO}(1, n-1)$, $n = \dim M$. Results of this kind are sometimes called *embedding theorems*: if G acts on M by preserving an H-structure, then G has an injective homomorphism in H. However, such results are generally proved for actions of semi-simple Lie groups; see [**4**] as a recent account on the subject.

2.4. *A reduction.* In §8, under the assumptions of Theorem 2, and applying the previous steps, we show that we can assume that the identity component $\mathrm{Iso}_0(M, \mathbf{g})$ is a torus $\mathbb{T}^k$, and that ρ is an almost faithful representation $\Gamma = G/G_0 \to \mathrm{O}(k, \mathbb{Z}) = \mathrm{GL}(k, \mathbb{Z}) \cap \mathrm{O}(q)$, where q is a rational Lorentz form on $\mathbb{R}^k$.

2.5. *Lorentz dynamics: weakly stationary implies stationary.* From this last reduction we see how the algebraic structure of the isometry group G (essentially $\Gamma = G/G_0$) is now related to the isometry group of a Lorentz flat torus $(\mathbb{T}^k, q)$. In §9, we pursue the analogy at a dynamical level; we pick $f \in G$, with $\rho(f) \in \mathrm{GL}(k, \mathbb{Z})$ of infinite order and compare the two Lorentz dynamical systems (M, f) and $(\mathbb{T}^k, \rho(f))$. Our principal ingredient will be the fact that non-equicontinuous Lorentz isometries possess approximately stable foliations, see [**17**]. They are codimension-one lightlike geodesic foliations; for instance, in the case where $\rho(f)$ is partially hyperbolic, the approximately stable foliation of $(\mathbb{T}^k, \rho(f))$ coincides with its weakly stable one (that is, the sum of the stable and the central ones). Comparison of the dynamics of f and of $\rho(f)$ allows us to prove the following theorem.

THEOREM 2.3. *Let $f \in \mathrm{Iso}(M, \mathbf{g})$ be such that $\rho(f)$ is an element of infinite order (in* $\mathrm{Out}(\mathrm{Iso}_0(M, \mathbf{g}))$*). Then, there is a minimal timelike $\rho(f)$-invariant torus $\mathbb{T}^d \subset \mathrm{Iso}_0(M, \mathbf{g})$ of dimension $d = 3$ or $d \geq 2$ according to whether $\rho(f)$ is parabolic or hyperbolic, respectively. The action of $\mathbb{T}^d$ on M is (everywhere) free and timelike.*

The proof of Theorem 2.3 is presented in §9.

The other ingredient, besides the theory of approximately stable foliation, is the fact that non-spacelike Killing fields are singularity free [**5**]. This will be used in the deduction of Theorem 1 from Theorem 2 in §3.

2.6. *Geometric structure: dynamics forces integrability.* The $\mathbb{T}^d$-action determines a regular timelike foliation $\mathcal{G}$. Hence, on one hand we get a quotient space N which is a Riemannian orbifold together with a pseudo-Riemannian (Seifert) $\mathbb{T}^d$-principal fibration $M \to N$. On the other hand, we have a spacelike orthogonal bundle $\mathcal{N}$. The obstruction to its integrability is encoded in a Levi form $(X, Y) \mapsto l(X, Y) \in \mathcal{N}^{\perp} = \mathcal{G}$, where X and Y are vector fields tangent to $\mathcal{N}$, and $l(X, Y)$ is the orthogonal projection of the bracket $[X, Y]$. All structures are preserved by the isometry f (specified in the previous step).

One may then expect that the conflict between the Riemannian dynamics of f on $\mathcal{N}$ and its Lorentzian dynamics on $\mathcal{G}$ leads to the vanishing of l. This is what really happens, in fact, when $\rho(f)$ is hyperbolic. In the same vein, one proves that the leaves of $\mathcal{N}$ are compact. In other words, if $\mathcal{N}$ is seen as a connection on the $\mathbb{T}^d$ bundle $M \to N$, then this connection is flat and has a finite holonomy: Theorem 11.4 in §11.

In the case where $\rho(f)$ is parabolic, it is an augmentation $\mathcal{L}$ of $\mathcal{N}$ which is integrable, and enjoys the same properties as $\mathcal{N}$ in the previous case: Theorem 12.3 in §12. It is at this point that amalgamated structures show up.

3. *Precompactness. Proof of Theorem 1 from Theorem 2*

3.1. *A Gauss map.* Let $(M, \mathbf{g})$ be a compact Lorentzian manifold, let $\mathrm{Iso}(M, \mathbf{g})$ denote its isometry group, which is a Lie group (see for instance [**10**]), and denote by $\mathrm{Iso}_0(M, \mathbf{g})$ the connected component of the identity of $\mathrm{Iso}(M, \mathbf{g})$. The Lie algebra of $\mathrm{Iso}(M, \mathbf{g})$ will be denoted by $\mathfrak{Iso}(M, \mathbf{g})$; let us recall that there is a Lie algebra anti-isomorphism from $\mathfrak{Iso}(M, \mathbf{g})$ to the space of Killing vector fields $\mathrm{Kill}(M, \mathbf{g})$ obtained by mapping a vector $\mathfrak{v} \in \mathfrak{Iso}(M, \mathbf{g})$ to the Killing field $K^{\mathfrak{v}}$ which is the infinitesimal generator of the one-parameter group of isometries $\mathbb{R} \ni t \mapsto \exp(t\mathfrak{v}) \in \mathrm{Iso}(M, \mathbf{g})$. If Φ is a diffeomorphism of M and K is a vector field on M, we will denote by $\Phi_*(K)$ the *push-forward* of K by Φ, which is the vector field given by $\Phi_*(K)(p) = \mathrm{d}\Phi(\Phi^{-1}(p))K(\Phi^{-1}(p))$ for all $p \in M$. If Φ is an isometry and K is Killing, then $\Phi_*(K)$ is Killing.

If $\Phi \in \mathrm{Iso}(M, \mathbf{g})$, then

$$\Phi_*(K^{\mathfrak{v}}) = K^{\mathrm{Ad}_\Phi(\mathfrak{v})} \quad \text{for all } \mathfrak{v} \in \mathfrak{Iso}(M, \mathbf{g}). \tag{3.1}$$

It will be useful to introduce the following map. Let $\mathrm{Sym}(\mathfrak{g})$ denote the vector space of symmetric bilinear forms on $\mathfrak{g}$. The *Gauss map* $\mathcal{G} : M \to \mathrm{Sym}(\mathfrak{g})$ is the map defined by

$$\mathcal{G}_p(\mathfrak{v}, \mathfrak{w}) = \mathbf{g}_p(K^{\mathfrak{v}}(p), K^{\mathfrak{w}}(p)), \tag{3.2}$$

for $p \in M$ and $\mathfrak{v}, \mathfrak{w} \in \mathfrak{g}$. The following identity is immediate:

$$\mathcal{G}_{\Phi(p)} = \mathcal{G}_p(\mathrm{Ad}_\Phi \cdot, \mathrm{Ad}_\Phi \cdot), \tag{3.3}$$

for all $\Phi \in \mathrm{Iso}(M, \mathbf{g})$. In this paper we will be interested in the case where (M, g) admits a Killing vector field which is timelike somewhere. In this situation, the image of the Gauss map contains a Lorentzian (non-degenerate) symmetric bilinear form on $\mathfrak{g}$ (in fact, a non-empty open subset consisting of Lorentzian forms; this will be used in Lemma 8.8).

We now have the necessary ingredients to show how the proof of Theorem 1 is obtained from Theorem 2.

Proof of Theorem 1 from Theorem 2. Let us assume that the action of $\mathrm{Iso}(M, g)$ on $\mathfrak{Iso}(M, \mathbf{g})$ is not of post-Riemannian type; we will show by contradiction that $\mathrm{Iso}_0(M, \mathbf{g})$ has a somewhere timelike orbit. Let κ be the quadratic form on $\mathfrak{Iso}(M, \mathbf{g})$ defined by

$$\kappa(\mathfrak{v}, \mathfrak{w}) = \int_M \mathcal{G}_p(\mathfrak{v}, \mathfrak{w}) \,\mathrm{d}M(p),$$

the integral being taken relative to the volume element of the Lorentzian metric $\mathbf{g}$. By (3.3), κ is invariant by the conjugacy action. If $\mathrm{Iso}_0(M, \mathbf{g})$ has no timelike orbit, then κ is positive semi-definite. The proof will be concluded if we show that the kernel $\mathcal{K} = \mathrm{Ker}(\kappa)$ has dimension less than or equal to one. Assume that $\mathcal{K}$ is not trivial, i.e. that κ is positive semi-definite. If $\mathfrak{v} \in \mathcal{K}$, then for all $\mathfrak{w} \in \mathfrak{Iso}(M, \mathbf{g})$ and all $p \in M$, $\mathbf{g}_p(K^{\mathfrak{v}}_p, K^{\mathfrak{w}}_p) = 0$; in particular, $\mathbf{g}_p(K^{\mathfrak{v}}_p, K^{\mathfrak{v}}_p) = 0$, i.e. $K^{\mathfrak{v}}$ is an everywhere isotropic† Killing vector field on M. Non-trivial isotropic Killing vector fields are never vanishing, see for instance [**5**, Lemma 3.2]; this implies that the map $\mathcal{K} \ni \mathfrak{v} \mapsto K^{\mathfrak{v}}_p \in T_pM$ is an injective vector space homomorphism for all $p \in M$. On the other hand, its image has dimension one, because an isotropic subspace of a Lorentz form has dimension at most one, hence $\mathcal{K}$ has dimension one.

3.2. *Precompactness of one-parameter subgroups.* It will be useful to recall that there is a natural smooth left action of $\mathrm{Iso}(M, \mathbf{g})$ on the principal bundle $\mathcal{F}(M)$ of all linear frames of TM, defined as follows. If $b = (v_1, \dots, v_n)$ is a linear basis of T_pM, then for $\Phi \in \mathrm{Iso}(M, \mathbf{g})$ set $\overline{\Phi}(b) = (\mathrm{d}\Phi_p(v_1), \dots, \mathrm{d}\Phi_p(v_n))$, which is a basis of $T_{\Phi(p)}M$. The action of $\mathrm{Iso}(M, \mathbf{g})$ on $\mathcal{F}(M)$ is defined by $\mathrm{Iso}(M, \mathbf{g}) \times \mathcal{F}(M) \ni (\Phi, b) \longmapsto \overline{\Phi}(b) \in \mathcal{F}(M)$. Given any frame $b \in \mathcal{F}(M)$, the map $\mathrm{Iso}(M, \mathbf{g}) \ni \Phi \mapsto \overline{\Phi}(b) \in \mathcal{F}(M)$ is a proper embedding of $\mathrm{Iso}(M, \mathbf{g})$ onto a closed submanifold of $\mathcal{F}(M)$ (see [**10**, Theorems 1.2, 1.3]), and thus the topology and the differentiable structure of $\mathrm{Iso}(M, \mathbf{g})$ can be studied by looking at one of its orbits in the frame bundle. In particular, the following will be used at several points.

Precompactness criterion. If $H \subset \mathrm{Iso}(M, \mathbf{g})$ is a subgroup that has one orbit in $\mathcal{F}(M)$ which is contained in a compact subset of $\mathcal{F}(M)$, then H is precompact. For instance, if H preserves some Riemannian metric on M and it leaves a non-empty compact subset of M invariant, then H is precompact.

LEMMA 3.1. *Let $(M, \mathbf{g})$ be a compact Lorentzian manifold and K be a Killing vector field on M. If K is timelike at some point, then it generates a precompact one-parameter subgroup of isometries in* $\mathrm{Iso}_0(M, \mathbf{g})$.

Proof. Let $p \in M$ be such that $\mathbf{g}(K(p), K(p)) < 0$. Consider the compact subsets of TM given by

$$V = \{K(q) : q \in M \text{ such that } \mathbf{g}(K(q), K(q)) = \mathbf{g}(K(p), K(p))\},$$

and

$$V^{\perp} = \{v \in K(q)^{\perp} : q \in M \text{ such that } \mathbf{g}(K(q), K(q)) = \mathbf{g}(K(p), K(p)), \mathbf{g}(v, v) = 1\}.$$

Consider an orthogonal basis $b = (v_1, \dots, v_n)$ of T_pM with $v_1 = K(p)$ and $\mathbf{g}(v_i, v_j) = \delta_{ij}$ for $i, j \in \{2, \dots, n\}$. The one-parameter subgroup generated by K in $\mathrm{Iso}(M, \mathbf{g})$ can be identified with the $\mathbb{R}$-orbit of the basis b by the action of the flow of K on the frame bundle $\mathcal{F}(M)$. Every vector of a basis of the orbit belongs to the compact subset $V \cup V^{\perp}$, and this implies that the orbit of b is precompact in the frame bundle $\mathcal{F}(M)$. □

† Here we use the following terminology: a vector $v \in TM$ is *isotropic* if $\mathbf{g}(v, v) = 0$, and it is *lightlike* if it is isotropic and non-zero.

Lemma 3.1 has been used in [**8**] to prove the existence of periodic timelike geodesics in compact weakly stationary Lorentz manifolds.

3.3. *An algebraic criterion for precompactness.* Now observe that if a compact manifold $(M, \mathbf{g})$ admits a Killing vector field which is timelike somewhere, then, by continuity, sufficiently close Killing fields are also timelike somewhere. Thus, if one wants to study the (connected component of the) isometry group of a Lorentz manifold that has a Killing vector field which is timelike at some point, it is natural to ask which (connected) Lie groups have open sets of precompact one-parameter subgroups. The problem is better cast in terms of the Lie algebra; we will settle this question in our next proposition.

PROPOSITION 3.2. *Let G be a connected Lie group, $K \subset G$ be a maximal compact subgroup, and $\mathfrak{k} \subset \mathfrak{g}$ be their Lie algebras. Let $\mathfrak{m}$ be an Ad_K-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then, $\mathfrak{g}$ has a non-empty open cone of vectors that generate precompact one-parameter subgroups of G if and only if there exists $\mathfrak{v} \in \mathfrak{k}$ such that the restriction $\mathrm{ad}_{\mathfrak{v}} : \mathfrak{m} \to \mathfrak{m}$ is an isomorphism.*

Proof. Let $\mathfrak{C} \subset \mathfrak{g}$ be the cone of vectors that generate precompact one-parameter subgroups of G; we want to know when $\mathfrak{C}$ has non-empty interior. Clearly $\mathfrak{C}$ contains $\mathfrak{k}$, and every element of $\mathfrak{C}$ is contained in the Lie algebra $\mathfrak{k}'$ of some maximal compact subgroup K' of G. Since all maximal compact subgroups of G are conjugate (see, for instance, [**9**]), it follows that $\mathfrak{C} = \mathrm{Ad}_G(\mathfrak{k})$, i.e. $\mathfrak{C}$ is the image of the map $F : G \times \mathfrak{k} \to \mathfrak{g}$ given by $F(g, \mathfrak{v}) = \mathrm{Ad}_g(\mathfrak{v})$. We claim that $\mathfrak{C}$ has non-empty interior if and only if the differential $\mathrm{d}F$ has maximal rank at some point $(g, \mathfrak{v}) \in G \times \mathfrak{k}$. The condition is clearly sufficient, and by Sard's theorem is also necessary; namely, if $\mathrm{d}F$ never has maximal rank then all the values of F are critical, and they must form a set with empty interior. The second claim is that it suffices to look at the rank of $\mathrm{d}F$ at the points $(e, \mathfrak{v})$, where e is the identity of G. This follows easily observing that the function is G-equivariant in the first variable. Now, the differential of F at $(e, \mathfrak{v})$ is easily computed as

$$\mathrm{d}F_{(e,\mathfrak{v})}(\mathfrak{g}, \mathfrak{k}) = [\mathfrak{g}, \mathfrak{v}] + \mathfrak{k} = [\mathfrak{m}, \mathfrak{v}] + \mathfrak{k}.$$

Thus, $\mathrm{d}F_{(e,\mathfrak{v})}$ is surjective if and only if there exists $\mathfrak{v} \in \mathfrak{k}$ such that $[\mathfrak{m}, \mathfrak{v}] = \mathfrak{m}$, which concludes the proof. □

4. *Case when the identity connected component is non-compact. Proof of part* (1) *of Theorem 2.1*

The geometric structure of compact Lorentz manifolds whose isometry group contains a group which is locally isomorphic to an *oscillator group* or to $\mathrm{SL}(2, \mathbb{R})$ is well known. Let us recall (see [**16**, §1.6]) that a compact Lorentz manifold that admits a faithful isometric action of a group locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$ has universal cover which is given by a warped product of the universal cover of $\mathrm{SL}(2, \mathbb{R})$, endowed with the bi-invariant Lorentz metric given by its Killing form, and a Riemannian manifold. Every such manifold admits everywhere a timelike Killing vector field, corresponding to the timelike vectors of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$.

Oscillator groups are characterized as the only simply-connected solvable and non-commutative Lie groups that admit a bi-invariant Lorentz metric and possess lattices, i.e. co-compact discrete subgroups (see [**11**]). More precisely, an oscillator group G is (the universal cover of) a semi-direct product $\mathbb{S}^1 \ltimes \mathrm{Heis}$, where Heis is a Heisenberg group (of some dimension $2d+1$). There are positivity conditions on the eigenvalues of the automorphic $\mathbb{S}^1$ action on the Lie algebra $\mathfrak{heis}$ (ensuring the existence of a bi-invariant Lorentz metric), and arithmetic conditions on them (ensuring the existence of a lattice).

It is interesting and useful to consider oscillator groups as objects completely similar to $\mathrm{SL}(2, \mathbb{R})$, from a Lorentz geometry viewpoint. In particular, regarding our arguments in the present paper, both cases are perfectly parallel. however, let us notice some differences (but with no relevance to our investigation here). First, of course, the bi-invariant Lorentz metrics on an oscillator group do not correspond to its Killing form, since this latter is degenerate (because the group is solvable). Another fact is the non-uniqueness of these bi-invariant metrics, but, surprisingly, their uniqueness up to automorphisms. In the $\mathrm{SL}(2, \mathbb{R})$-case, we have uniqueness up to a multiplicative constant. Also, we have essential uniqueness of lattices in an oscillator group, versus their abundance in $\mathrm{SL}(2, \mathbb{R})$.

Let us describe briefly the construction of Lorentz manifolds endowed with a faithful isometric G-action, where G is either $\mathrm{SL}(2, \mathbb{R})$ or an oscillator group. The construction starts by considering right quotients G/Γ, where Γ is a lattice of G. The G-left action is isometric exactly because the metric is bi-invariant. A slight generalization is obtained by considering a Riemannian manifold $(\widetilde{N}, \widetilde{\mathbf{g}})$ and quotients of the direct metric product $X = \widetilde{N} \times G$ by a discrete subgroup Γ of $\mathrm{Iso}(\widetilde{N}, \widetilde{\mathbf{g}}) \times G$. Observe here that since the isometry group of the Lorentz manifold X is $\mathrm{Iso}(\widetilde{N}, \widetilde{\mathbf{g}}) \times (G \times G)$, it is possible to take a quotient by a subgroup Γ contained in this full group. The point is that we assumed that G acts (on the left) on the quotient, and hence, G normalizes Γ; but since G is connected, it centralizes Γ. Therefore, only the right G factor in the full group remains (since the centralizer of the left action is exactly the right factor). Observe, however, that Γ does not necessarily split. Indeed, there are examples where Γ is discrete co-compact in $\mathrm{Iso}(\widetilde{N}, \widetilde{\mathbf{g}}) \times G$, but its projection on each factor is dense!

Next, warped products yield a more general construction. Rather than a direct product metric $\widetilde{\mathbf{g}} \oplus \kappa$, one endows $\widetilde{N} \times G$ with a metric of the form $\widetilde{\mathbf{g}} \oplus w\kappa$, where w is a positive function on $\widetilde{N}$, and κ is the bi-invariant Lorentz metric on G. Here, there is one difference between the case of $\mathrm{SL}(2, \mathbb{R})$ and the oscillator case. For $\mathrm{SL}(2, \mathbb{R})$ this is the more general construction, but in the oscillator case, some 'mixing' between G and $\widetilde{N}$ and a mixing of their metrics is also possible, see [**16**, §1.2].

Let us study now the situation when the isometry group does not contain any group which is locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$ or to an oscillatory group.

THEOREM 4.1. *Let* $(M, \mathbf{g})$ *be a compact Lorentz manifold that admits a Killing vector field which is timelike at some point. Then, the identity component of its isometry group is compact, unless it contains a group locally isomorphic to* $\mathrm{SL}(2, \mathbb{R})$ *or to an oscillator group.*

Proof. By the classification result in [**1, 2, 16**], if $\mathrm{Iso}_0(M, \mathbf{g})$ does not contain a group locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$ or to an oscillator group, then $\mathfrak{Iso}(M, \mathbf{g})$ can be written as a Lie algebra direct sum $\mathfrak{h} + \mathfrak{a} + \mathfrak{c}$, where $\mathfrak{h}$ is a Heisenberg algebra, $\mathfrak{a}$ is abelian, and $\mathfrak{c}$ is semi-simple and compact. Our aim is to show that the Heisenberg summand $\mathfrak{h}$ in fact does not occur in the decomposition, and that the abelian group A corresponding to the summand $\mathfrak{a}$ is compact. By the assumption that $(M, \mathbf{g})$ has a Killing vector field which is timelike at some point, $\mathfrak{Iso}(M, \mathbf{g})$ must contain a non-empty open cone of vectors that generate a precompact one-parameter subgroup of $\mathrm{Iso}_0(M, \mathbf{g})$ (Lemma 3.1). The first observation is that, since $\mathfrak{c}$ is compact, if $\mathfrak{h} + \mathfrak{a} + \mathfrak{c}$ has an open cone of vectors that generate precompact one-parameter subgroups, than so does the subalgebra $\mathfrak{h} + \mathfrak{a}$. Moreover, by the same compactness argument, we can also assume that the abelian Lie subgroup A is simply connected. The proof of our result will be concluded once we show that any Lie group G with Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{a}$, $\mathfrak{h}$ and $\mathfrak{a}$ as above, does not have an open set of precompact one-parameter subgroups. To this aim, write $\mathfrak{h} + \mathfrak{a} = \mathfrak{m} + \mathfrak{k}$, with $\mathfrak{k}$ the Lie algebra of a maximal compact subgroup K of G and $\mathfrak{m}$ a $\mathfrak{k}$-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$. If either $\mathfrak{h}$ or $\mathfrak{a}$ is not zero, then also $\mathfrak{m}$ is non-zero. Since $\mathfrak{h} + \mathfrak{a}$ is nilpotent, for no $\mathfrak{x} \in \mathfrak{k}$ is the map $\mathrm{ad}_{\mathfrak{x}} : \mathfrak{m} \to \mathfrak{m}$ injective, and by Proposition 3.2, G does not have an open set of precompact one-parameter subgroups. □

5. *Case when the identity component is compact. Proof of part* (3) *of Theorem 2.1*

Let us start with a general result on group actions having orbits of the same dimension.

LEMMA 5.1. *Let G be a Lie group acting on a manifold R, and let G_0 be a compact normal subgroup of G all of whose orbits in R have the same dimension. Then:*

(1) *the distribution Δ tangent to the G_0-orbits is smooth, and it is preserved by G;*

(2) *there exists a Riemannian metric $\mathbf{h}_0$ on Δ which is preserved by G_0 and by the centralizer* $\mathrm{Centr}(G_0)$ *of G_0 in G.*

Proof. Introduce the following notation: for $x \in R$, let $\beta_x : G \to R$ be the map $\beta_x(g) = g \cdot x$, and let $L_x : \mathfrak{g} \to T_x M$ be its differential at the identity. Here $\mathfrak{g}$ is the Lie algebra of G. The map $M \times \mathfrak{g} \ni (x, \mathfrak{v}) \mapsto L_x(\mathfrak{v}) \in TM$ is a smooth vector bundle morphism from the trivial bundle $M \times \mathfrak{g}$ to TM. The distribution Δ is the image of the sub-bundle $M \times \mathfrak{g}_0$, where $\mathfrak{g}_0 \subset \mathfrak{g}$ is the Lie algebra of G_0. Since the orbits of G_0 have the same dimension, then the image of $M \times \mathfrak{g}_0$ is a smooth sub-bundle of TM (recall that the image of a vector bundle morphism is a smooth sub-bundle if it has constant rank). The action of G preserves Δ because G_0 is normal, which concludes the proof of the first assertion.

The construction of $\mathbf{h}_0$ goes as follows. Choose a positive definite inner product B on $\mathfrak{g}_0$ which is Ad_{G_0}-invariant; the existence of such B follows from the compactness of G_0. For all $x \in R$, the restriction to $\mathfrak{g}_0$ of L_x gives a surjection $L_x|_{\mathfrak{g}_0} : \mathfrak{g}_0 \to \Delta_x$; denote by V_x the B-orthogonal complement of the kernel of this map, given by $\mathrm{Ker}(L_x|_{\mathfrak{g}_0}) = \mathrm{Ker}(L_x) \cap \mathfrak{g}_0$. The value of $\mathbf{h}_0$ on Δ_x is defined to be the push-forward via the map L_x of the restriction of B to V_x. In order to see that such a metric is invariant by the action of G_0 and of its centralizer, for $g \in G$ denote by $\mathcal{I}_g : G_0 \to G_0$ the conjugation by g (recall that G_0 is normal) and by $\gamma_g : M \to M$ the diffeomorphism $x \mapsto g \cdot x$; for fixed $x \in M$ we have a commutative diagram.

$$\begin{tikzcd} G_0 \arrow[r, "\mathcal{I}_g"] \arrow[d, "\beta_x"'] & G_0 \arrow[d, "\beta_{gx}"] \\ M \arrow[r, "\gamma_g"'] & M \end{tikzcd}$$

Differentiating at the identity the diagram above we get the following diagram.

$$\begin{tikzcd} \mathfrak{g}_0 \arrow[r, "\mathrm{Ad}_g"] \arrow[d, "L_x"'] & \mathfrak{g}_0 \arrow[d, "L_{g\cdot x}"] \\ \Delta_x \arrow[r, "\mathrm{d}\gamma_g(x)"'] & \Delta_{g\cdot x} \end{tikzcd}$$

Now assume that g is such that Ad_g preserves B; this holds by assumption when $g \in G_0$, and clearly also for g in the centralizer of G_0 (in which case Ad_g is the identity!). For such a g, since $\mathrm{Ad}_g(\mathrm{Ker}(L_x) \cap \mathfrak{g}_0) = \mathrm{Ker}(L_{g\cdot x} \cap \mathfrak{g}_0)$, then also $\mathrm{Ad}_g(V_x) = V_{g\cdot x}$, and thus we have a commutative diagram

$$\begin{tikzcd} V_x \arrow[r, "\mathrm{Ad}_g"] \arrow[d, "L_x"'] & V_{g\cdot x} \arrow[d, "L_{g\cdot x}"] \\ \Delta_x \arrow[r, "\mathrm{d}\gamma_g(x)"'] & \Delta_{g\cdot x} \end{tikzcd}$$

from which it follows that $\mathrm{d}\gamma_g(x)$ preserves the metric $\mathbf{h}_0$, proving the last statement in the thesis. □

As an application of Lemma 5.1, we have the following proposition which implies part (3) of Theorem 2.1.

PROPOSITION 5.2. *Let* $(M, \mathbf{g})$ *be a compact Lorentz manifold that admits a Killing vector field* K *which is timelike somewhere,* $G = \mathrm{Iso}(M, \mathbf{g})$*,* $G_0 = \mathrm{Iso}_0(M, \mathbf{g})$ *and* $\rho : G \to \mathrm{Out}(G_0)$*. If* G_0 *is compact, then* $G_1 = \mathrm{Ker}(\rho) = G_0 \cdot \mathrm{Centr}(G_0)$ *is also compact.*

Proof. Let R be the (non-empty) open subset of M consisting of all points x whose G_0-orbit $\mathcal{O}(x)$ has maximal dimension (among all G_0-orbits), and such that $\mathcal{O}(x)$ is timelike, i.e. the restriction of $\mathbf{g}$ to $\mathcal{O}(x)$ is Lorentzian. Recall that the set of points whose G_0-orbit has maximal dimension is open and dense, and R is the intersection of this dense open subset with the open subset of M where K is timelike. We claim that there exists a Riemannian metric $\mathbf{h}$ on R which is preserved by G_1. Such a metric $\mathbf{h}$ is constructed as follows: on the distribution Δ tangent to the G_0-orbits it is given by the metric $\mathbf{h}_0$ as in Lemma 5.1, on the $\mathbf{g}$-orthogonal distribution Δ' it is the (positive definite) restriction of $\mathbf{g}$, furthermore Δ and Δ' are declared to be $\mathbf{h}$-orthogonal. Note that the distributions Δ and Δ' are preserved by G, the metric $\mathbf{g}$ on Δ' is preserved by all elements of G, and the metric $\mathbf{h}_0$ on Δ is preserved by all elements of G_1, which proves our claim.

Observe that G_1 is closed in G. In order to conclude the proof, we will use the precompactness criterion of §3.2, by showing that G_1 leaves some compact subset of R

invariant. More precisely, we will show the existence of a continuous G_1-invariant function $v : M \to R$, whose set T of maximum points is compact and contained in R. Clearly, such a set T is compact and G_1-invariant. The function v, which is a sort of *parametric orbit volume function*, is constructed in Lemma 5.3 below. □

5.1. *A generalized volume function for orbits of isometric actions.* The generalized orbit volume function mentioned in the proof of Proposition 5.2 is based on the notion of *parametric volume* for submanifolds of pseudo-Riemannian manifolds, defined as follows. Let K be a compact Lie group acting by isometries on a pseudo-Riemannian manifold $(P, \mathbf{g}_P)$; for $x \in P$, let Kx denote the orbit of x under the action of K, and consider the submersion $\mathfrak{p}_x : K \to Kx$ given by $\mathfrak{p}_x(k) = kx$. Note that, by compactness, the K-action on P is proper, hence all the orbits are embedded compact submanifolds of P. Assume that Kx is a non-degenerate submanifold of P, i.e. that the restriction of $\mathbf{g}_P$ to Kx is non-degenerate somewhere (hence, everywhere). Assume also that K is endowed, say, with a bi-invariant Riemannian metric $\mathbf{g}_K$ having unit volume. For $k \in K$, let α_k be the volume form on $\mathrm{Ker}(\mathrm{d}\mathfrak{p}_x(k))$ induced by the restriction of $\mathbf{g}_K$ to $\mathfrak{p}_x^{-1}(x)$. Moreover, denote by β_k the volume form on the $\mathbf{g}_K$-orthogonal complement of $\mathrm{Ker}(\mathrm{d}\mathfrak{p}_x(k))$ obtained as pull-back by $\mathrm{d}\mathfrak{p}_x(k)$ of the volume form of Kx, induced by the restriction of $\mathbf{g}_P$. Then, the wedge product $\alpha_k \wedge \beta_k$ defines a volume form on K. The *parametric volume* of the orbit Kx, denoted by $\underline{\mathrm{vol}}_x$, is defined to be the integral $\int_K \alpha_k \wedge \beta_k$. If K_x denotes the stabilizer of x, using Fubini's theorem it is easy to show that $\underline{\mathrm{vol}}_x$ equals the product $\mathrm{vol}(Kx) \cdot \mathrm{vol}(K_x)$. Here, $\mathrm{vol}(K_x)$ is computed relative to the restriction of the bi-invariant metric of K, while $\mathrm{vol}(Kx)$ is computed using the (non-degenerate) restriction of $\mathbf{g}_P$ to the orbit Kx.

Let us state and prove the following result, which has some interest in its own right.

LEMMA 5.3. *Let K be a compact Lie group acting isometrically on a pseudo-Riemannian manifold $(P, \mathbf{g}_P)$, and let $d > 0$ be the dimension of the principal orbits of K in P. Define a function $v : M \to \mathbb{R}$ by setting $v(x) = \underline{\mathrm{vol}}_x$ if $\dim(Kx) = d$ and Kx is non-degenerate, otherwise set $v(x) = 0$. Then, v is a continuous function.*

Proof. Let us call *regular* an orbit of dimension d (possibly exceptional, and also possibly degenerate). The set of points with regular orbit is open and dense. Continuity of v at such points is obtained from the following argument. The volume form $\alpha_k \wedge \beta_k$ is left-invariant in K, and thus its norm at every point, computed using the (unit volume) bi-invariant metric on K, is equal to its integral. Thus, the continuity of $\underline{\mathrm{vol}}_x$ follows from that of $\mathrm{Ker}(\mathfrak{p}_x(k))$, for $k \in K$ fixed and x varying in the set of points with regular orbits. For the continuity at a non-regular orbit, assume that x_n is a sequence of regular points tending to a singular point x_∞. Denote by $\mathfrak{k}_{x_n}$ the Lie algebra of K_{x_n}; then, $\mathfrak{k}_{x_n}$ tends to a strict subspace of $\mathfrak{k}_{x_\infty}$. Thus, we can find a sequence u_n of unit vectors in $\mathfrak{k}_{x_n}^\perp$ that converges to some unit vector in $\mathfrak{k}_\infty$. Therefore, the image of u_n by $\mathrm{d}\mathfrak{p}_{x_n}$ is small, hence the pull-back of the volume on $\mathfrak{k}_{x_n}^\perp$ is small. In particular, $v(x_n)$ tends to 0, and v is continuous. □

Lemma 5.3 is applied in Proposition 5.2 to $K = G_0$ and $(P, \mathbf{g}_P) = (M, \mathbf{g})$. Note that the corresponding continuous function $v : M \to \mathbb{R}$ is actually G-invariant, since G_0 is normal in G.

6. *When the isometry group has infinitely many connected components. Proof of part* (2) *of Theorem 2.1*

Let us now study the situation when the isometry group of a Lorentzian manifold with a timelike Killing vector field has infinitely many connected components.

To begin, let us formulate the following generalization of Lemma 5.1 (and Proposition 5.2), which can proved in the same manner.

LEMMA 6.1. *Let G_0 be a connected normal subgroup of* $\mathrm{Iso}(M, \mathbf{g})$ *contained in* $\mathrm{Iso}_0(M, \mathbf{g})$ *such that all the G_0-orbits are timelike and have the same dimension and let* $r : \mathrm{Iso}(M, \mathbf{g}) \to \mathrm{Aut}(\mathfrak{g}_0)$ *be the action by conjugacy on the Lie algebra of G_0. Let L be a subgroup of* $\mathrm{Iso}(M, \mathbf{g})$ *such that $r(L)$ is precompact.*

Then, L preserves a Riemannian metric on M. In particular, if L is closed in $\mathrm{Iso}(M, \mathbf{g})$, *then L is compact.*

Proof. Totally analogous to the proof of Lemma 5.1. One only has to start with an inner product B which is $r(L)$-invariant. The precompactness hypothesis ensures its existence. □

COROLLARY 6.2. *Let G_0 be as above, and assume there exists a compact subgroup S of* $\mathrm{Aut}(\mathfrak{g}_\mathrm{o})$ *such that* $r(\mathrm{Iso}(M, g))$ *is contained in the subgroup of* $\mathrm{Aut}(\mathfrak{g}_\mathrm{o})$ *generated by S and* $\mathrm{Inn}(G_0)$ *(inner automorphisms). Then,* $\mathrm{Iso}(M, \mathbf{g})/G_0$ *is compact. In particular,* $\mathrm{Iso}(M, \mathbf{g})$ *has a finite number of connected components (since G_0 is connected).*

Proof. Let $L = \rho^{-1}(S)$. It is a closed subgroup of G. It projects surjectively on $\Gamma = \mathrm{Iso}(M, \mathfrak{g})/G_0$. Indeed, let $f \in \mathrm{Iso}(M, \mathfrak{g})$, then $r(f)$ belongs to the group generated by S and $\mathrm{Inn}(G_0)$, and hence has the form $r(f) = sr(f')$, where $s \in S$ and $f' \in G_0$, and therefore $ff'^{-1} \in L$. By Lemma 6.1 above, L is compact, and so is also its factor $\mathrm{Iso}(M, \mathfrak{g})/G_0$. □

Proof of part (2) *of Theorem 2.1.* Recall that $(M, \mathbf{g})$ is assumed to have a non-compact isometry group with somewhere timelike orbit. By part (1) of Theorem 2.1, $\mathrm{Iso}_0(M, \mathbf{g})$ contains a subgroup G_0 which is locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$ or to an oscillator group. In fact G_0 is normal in $\mathrm{Iso}(M, \mathbf{g})$, see [**16**] for a proof of this fact†.

Our goal now is to apply Corollary 6.2 by showing that $r(\mathrm{Iso}(M, \mathbf{g}))$ is contained in a compact extension of $\mathrm{Inn}(\mathfrak{g}_0)$.

In the case where G_0 is locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$, up to a finite index, all automorphisms are inner, and the claim is obvious.

In the other case where G_0 is an oscillator group, $\mathrm{Aut}(\mathfrak{g}_0)$ is 'large' with respect to $\mathrm{Inn}(\mathfrak{g}_0)$. However, the image of $r : \mathrm{Iso}(M, \mathbf{g}) \to \mathrm{Aut}(\mathfrak{g}_0)$ lies in fact inside $H = \mathrm{Aut}(\mathfrak{g}_0) \cap \mathrm{SO}(q_0)$, where q_0 is some bi-invariant Lorentz form on $\mathfrak{g}_\mathrm{o}$ (so H is the group of q_0-orthogonal automorphisms of $\mathfrak{g}_0$). Indeed, as we have seen in §3.1, we know that the action by conjugacy of $\mathrm{Iso}(M, \mathbf{g})$ on its Lie algebra preserves some Lorentz form, and by [**16**], the induced form q_0 on the Lie sub-algebra $\mathfrak{g}_\mathrm{o}$ is also of Lorentz type.

† When $\mathrm{Iso}_0(M, \mathbf{g})$ contains a subgroup G_0 which is locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$ or to an oscillator group, the geometric structure of $(M, \mathbf{g})$ is in fact well understood (see [**16**, Theorems 1.13 and 1.14]). This could be used to prove our theorem; we will rather present in what follows an 'algebraic proof' based on Corollary 6.2 above.

The following lemma allows us to apply Corollary 6.2, and hence it completes the proof of part (2) of Theorem 2.1. □

LEMMA 6.3. *For an oscillator group G_0, $\mathrm{Inn}(G_0)$ is co-compact in the orthogonal automorphism group H; more precisely, H is a semi-direct product of the form $H = S \ltimes N$, where $N \subset \mathrm{Inn}(G_0)$ and S is a compact group.*

Proof. Any automorphism preserves the center $\mathfrak{c}$ of the Lie algebra $\mathfrak{g}_0$ of G_0. This center is lightlike for (any bi-invariant) q_0. So $\mathrm{Aut}(\mathfrak{g}_0) \cap \mathrm{SO}(q_0)$ is contained in the parabolic group P corresponding to $\mathfrak{c}$ (i.e. the stablizer of $\mathfrak{c}$ in the orthogonal group of q_0). Such a group P is amenable, and it has a semi-direct product structure $P = (D \times \mathrm{O}(n-1)) \ltimes N$, where D is diagonal and has dimension one, and N is the unipotent radical of P.

On one hand, by an easy look at oscillator groups, one first observes that the action by conjugacy of its Heisenberg subgroup gives exactly N as a subgroup of $\mathrm{Inn}(\mathfrak{g}_0)$. In particular, N is contained in the orthogonal automorphism group H.

Let us prove that, on the other hand, H is contained in $\mathrm{O}(n-1) \ltimes N$, that is, any element $h \in H$ has a trivial D-part, i.e. h cannot be hyperbolic. Indeed, a generator Z of the center would be a non-trivial λ-eigenvector for h. There is a unique λ^{-1}-eigenvector T. The orthogonal $Z^{\perp}$ is the Heisenberg sub-algebra of $\mathfrak{g}_0$, and T lies outside of it. Let $X \in Z^{\perp}$, then the bracket $[T, X]$ belongs to $Z^{\perp}$, since $Z^{\perp}$ is an ideal†. Apply h to $[T, X]$: $h^n[T, X] = [h^n T, h^n X] = \lambda^{-n}[T, h^n X]$. If we see all this mod $\mathbb{R}Z$, that is, we consider their projections on $Z^{\perp}/\mathbb{R}Z$, then h acts as a Euclidean isometry, i.e. it is conjugate to an element of $\mathrm{O}(n-1)$. Hence, for some sequence $n_i \to \infty$, $h^{n_i} \to 1$, but $h^{n_i}[T, X] = \lambda^{-n_i}[T, h^{n_i} X]$ which equals approximately $\lambda^{-n}[T, X]$. This contradicts the fact $h^{n_i} \to 1$, unless $[T, X] = 0$ in $Z^{\perp}/\mathbb{R}Z$, that is, $[T, X] \in \mathbb{R}Z$. However, for the oscillator group, any T outside the Heisenberg ideal is such that ad_T is skew-symmetric with no kernel on $Z^{\perp}/\mathbb{R}Z$.

We conclude that H is contained in $\mathrm{O}(n-1) \ltimes N$, and since it contains N, it has the form $S \ltimes N$, where S is the closed subgroup of $\mathrm{O}(n-1)$ consisting of elements that act by isomorphisms on the oscillator algebra, i.e. that preserve Lie brackets. □

7. *Linear dynamics. Proof of Theorem 2.2*

Gauss maps (and variants) have the advantage to transform the dynamics on M into linear dynamics, i.e. an action of the group in question on a linear space, or on an associated projective space, via a linear representation. We will prove in the following a stability result: if a linear group 'almost-preserves' some Lorentz form, then it (fully) preserves another one. We start with the individual case, i.e. with actions of the infinite cyclic group $\mathbb{Z}$, and then we will consider general groups.

7.1. *Individual dynamics.* Let $\mathcal{E}$ be a vector space, and $A \in \mathrm{GL}(\mathcal{E})$. It has a Jordan decomposition $A = EHU$, where U is unipotent (i.e. $U - 1$ is nilpotent), H hyperbolic (i.e. diagonalizable over $\mathbb{R}$ with positive eigenvalues), and E is elliptic (i.e. diagonalizable over $\mathbb{C}$, and all its eigenvalues have norm equal to 1).

† $Z^{\perp}$ is the Heisenberg algebra, which is an ideal of the oscillator algebra.

If F is a space obtained from $\mathcal{E}$ by functorial constructions, e.g. $F = \mathrm{Sym}(\mathcal{E}^*)$ the space of quadratic forms on $\mathcal{E}$, or $F = \mathrm{Gr}^d(\mathcal{E})$ the Grassmannian of d-dimensional subspaces of $\mathcal{E}$, the associated A-action on F will be denoted by A^F. Naturally, when F is a vector space, we have $A^F = E^F H^F U^F$.

A point $p \in \mathcal{E}$ is *A-recurrent* if there is $n_i \in \mathbb{N}$, $n_i \to \infty$, such that $A^{n_i}(p) \to p$ as $i \to \infty$. A point p is *A-escaping* if for any compact subset $K \subset \mathcal{E}$ there is N such that $A^n(p) \notin K$, for $n > N$. So, p is non-escaping if there is $n_i \to \infty$, such that $A^{n_i} p$ stay in some compact set $K \subset \mathcal{E}$.

Let us prove the following lemma.

LEMMA 7.1. *Let $p \in F$ be a point recurrent under the A^F-action. Then p is fixed by H^F and U^F. If F is a vector space, p is A^F-non-escaping if and only if p is fixed by H^F and U^F.*

Proof. Any Grassmannian space is a subspace of a suitable projective space†. For the first statement of the lemma, we will consider the case where F is the projective space associated with $\mathcal{E}$. Consider the decomposition $\mathcal{E} = \bigoplus \mathcal{E}_i$ into eigenspaces of H, say $H|_{\mathcal{E}_i} = \lambda_i \,\mathrm{Id}_{\mathcal{E}_i}$, and choose $\lambda_1 > \lambda_2 \cdots$. This decomposition is invariant under H, U and E. Since U is unipotent, there is an endomorphism u nilpotent such that $U = \exp u$. For t integer, and $x_i \in \mathcal{E}_i$, we have $A^t(x_i) = {\lambda_i}^t(E^t(x_i) + tu\, E^t(x_i) + (t^2/2)u^2\, E^t(x_i) + \cdots (t^k/k!)u^k\, E^t(x_i))$, where $k = \dim \mathcal{E}_i$.

Let $x = \Sigma x_i$, with $x_1 \neq 0$. Clearly, the direction of $A^t x$ converges to a direction in $\mathcal{E}_1$. In particular if x is recurrent, then $x \in \mathcal{E}_1$. The same argument yields, in general, that if the direction of x is recurrent, then it belongs to some $\mathcal{E}_i$. We can then assume $x = x_1$. Since E is elliptic, the norms of the $u^j(E^t(x_1))$ are bounded, and there exists $t_n \to \infty$, such that $E^{t_n} \to 1$. Assume $u^k(x_1) \neq 0$. Then, the direction of $A^{t_n} x$ converges to $u^k(x_1)$. If x_1 is recurrent, then x_1 is an eigenvector of u^k, and hence $u^k(x_1) = 0$, since u is nilpotent. The same argument yields $u^{k-1}(x_1) = \cdots = u(x_1) = 0$. In conclusion, we have then proved that $H(x) = \lambda x$, and $U(x) = x$.

Let us now prove the second statement. Consider the case that F is a vector space. If x is non-escaping, then $A^t(x)$ is bounded, and so also is $A^t(x_i)$ for any i, when $t \to \pm\infty$. For $x_i \neq 0$, this can happen only if $\lambda_i = \pm 1$, and $U(x_i) = x_i$. Therefore $U(x) = x$ and $H(x) = \pm x$. Since H has only positive eigenvalues, $H(x) = x$. □

7.2. *Recalls on the classification of elements of* $\mathrm{SO}(1, n)$. Let q be a Lorentz form on $\mathcal{E}$. (For many purposes here, we can assume q is the standard Lorentz form $-x_1^2 + x_2^2 + \cdots + x_k^2$ on $\mathbb{R}^k$.)

A vector $u \in \mathcal{E}$ is non-spacelike if it is timelike ($q(u, u) < 0$) or lightlike ($q(u, u) = 0$ and $u \neq 0$). The space of non-spacelike vectors consists of two disjoint convex cones. A time orientation of $(\mathcal{E}, q)$ means a choice of one of them, call it C^+. For the sake of simplicity, we will denote by $\mathrm{SO}(q)$ (instead of the more accurate but heavy $\mathrm{SO}^+(q)$) the identity connected component of $\mathrm{SO}(q)$, which consists of orthogonal transformations that preserve both space and time orientation.

† The Grassmannian space of d-planes of a vector space V is a subset of the projective space of $\wedge^d V$.

Any element $A \in \mathrm{SO}(q)$ fixes a direction in C^+. If A fixes a timelike direction (and hence a timelike vector, since A preserves q), it will be called elliptic. In the other cases, A may fix exactly one lightlike direction and will be called parabolic or exactly two distinct lightlike directions, in which case A will be called hyperbolic. (The meaning of this classification is that an element A which preserves three distinct lightlike directions preserves in fact a timelike one.)

The classification of subgroups of $\mathrm{SO}(q)$ is much more complicated. An easy case is when Γ is an elementary subgroup, which means that Γ preserves a direction in C^+.

7.3. *Normal forms.* One can prove that A is hyperbolic if its spectrum consists of real simple eigenvalues $\{\lambda, \lambda^{-1}\}$, $\lambda > 1$, and eigenvalues in $\mathbb{S}^1$; A is parabolic if it is not diagonalizable over $\mathbb{C}$. In other words, A is hyperbolic if it is conjugate in $\mathrm{GL}(k, \mathbb{R})$ to a matrix of the form

$$\begin{pmatrix} \lambda & 0 & \\ 0 & \lambda^{-1} & \mathbf{0} \\ & \mathbf{0} & R \end{pmatrix}, \tag{7.1}$$

where $\lambda \in \mathbb{R}$, $\lambda > 1$, and $R \in \mathrm{SO}(k-2)$. Note that this matrix belongs to the orthogonal group of the quadratic form $x_1x_2 + x_3^2 + \cdots + x_k^2$.

Similarly, A is parabolic if it has the normal form

$$\begin{pmatrix} 1 & t & -t^2/2 & \\ 0 & 1 & t & \mathbf{0} \\ 0 & 0 & 1 & \\ & & \mathbf{0} & R \end{pmatrix} \tag{7.2}$$

with $t \in \mathbb{R}$ and $R \in \mathrm{SO}(k-3)$. Note that this matrix belongs to the orthogonal group of the quadratic form $x_1x_3 + x_2^2 + x_4^2 + \cdots + x_k^2$.

A unipotent element of $\mathrm{SO}(q)$ is parabolic with a trivial rotational part R. Similarly, an $\mathbb{R}$-diagonal element of $\mathrm{SO}(q)$ is hyperbolic with a trivial R-part.

7.4. *Intersection of orthogonal groups*

LEMMA 7.2. *Let $A \in \mathrm{SO}(q_0)$ be non-elliptic (i.e. parabolic or hyperbolic). If A preserves another Lorentz form q, then it keeps its nature as parabolic or hyperbolic as an element of $\mathrm{SO}(q)$, with the same characteristics: lightlike eigendirections as well as with their orthogonal hyperplanes. In particular, on these hyperplanes, any q is positive semi-definite, with kernel the corresponding eigendirection.*

Proof. Let us consider the parabolic case since the hyperbolic one is easier. Let A have the normal form (7.1). For q_0, its lightlike eigendirection is $\mathbb{R}e_1$, whose q_0-orthogonal is the hyperplane $e_1^\perp = \{x \in \mathbb{R}^k : x_3 \neq 0\}$. Let us show that e_1 can be characterized topologically (i.e. in a form that does not involve q_0). Indeed, the direction of e_1 is the unique attractor for the A-action, i.e. there is an open set of directions converging to $\mathbb{R}e_1$ under the A-action. Indeed, for any $x \notin e_1^\perp$, the direction of $A^n(x)$ tends to $\mathbb{R}e_1$. Therefore, there is no other attracting direction since its basin would intersect $e_1^\perp$. As for the hyperplane $e_1^\perp$, it is the unique attractor for the A-action on the dual space of $\mathbb{R}^k$. □

7.5. *Parabolic case*

COROLLARY 7.3. *Let $A \in \mathrm{SO}(q_0)$ be parabolic. Any combination (with positive weights) of Lorentz forms preserved by A, is a Lorentz form (of course preserved by A). The same holds, more generally, for any average of such forms by means of any positive measure.*

Proof. As previously, let A have the normal form (7.2). Let q_i be A-invariant Lorentz forms and $q = \Sigma_i \lambda_i q_i$ with $\lambda_i > 0$. By Lemma 7.2, q is positive semi-definite on $e_1^\perp = \{x \in \mathbb{R}^k,\ x_3 \neq 0\}$ with kernel $\mathbb{R}e_1$. Let us prove that q is non-degenerate. If not, let u be a null vector for q. If $u \notin e_1^\perp$, then also e_1 is null for q, and hence the kernel of q is a 2-plane P intersecting $e_1^\perp$ along $\mathbb{R}e_1$. However, there is no such A-invariant plane. Indeed such a plane is timelike for q_0, and A must be diagonal on it, which is not possible.

Suppose now that the kernel of q is $\mathbb{R}e_1$. Then, A preserves a non-degenerate form q' on $\mathbb{R}^k/\mathbb{R}e_1$. On $e_1^\perp/\mathbb{R}e_1$, q' is positive definite, and therefore, q' is either positive definite or of Lorentz type. Now, a quick look at the expression of the reduction of A on $\mathbb{R}^k/\mathbb{R}e_1$ shows that it can preserve neither a Euclidean nor a Lorentzian form (for instance, in the latter case, such a reduction must be parabolic, which is far from being the case).

Finally, since q is positive definite on any supplementary space of $\mathbb{R}e_1$ in $e_1^\perp$, it has a signature $(++), (+\cdots+)$ or $(-+), (+\cdots+)$, that is, q is either positive definite or Lorentzian. But A cannot preserve a positive definite form, and hence q is Lorentzian.

The same proof applies for any average $q = \int Q \,\mathrm{d}\mu(Q)$, where μ is a positive finite measure on the set of A-invariant Lorentz forms. □

7.6. *Hyperbolic case*

COROLLARY 7.4. *Let $A \in \mathrm{SO}(q_0)$ be hyperbolic. Let P be the 2-plane generated by the two lightlike eigendirections of A, and $P^\perp$ its orthogonal with respect to q_0. Then, any Lorentz A-invariant form has an orthogonal decomposition $q = \lambda q^P + q^{P^\perp}$, where $q^{P^\perp}$ is a positive definite form on $P^\perp$ and q^P is any Lorentz form on P with isotropic directions the lightlike eigendirections of A. In particular, all the q^P forms are proportional. It follows, in particular, that if an average q of A-invariant Lorentz forms is not a Lorentz form, then it is degenerate with kernel P.*

Proof. Let A have the hyperbolic form above. Its lightlike eigendirections are $\mathbb{R}e_1$ and $\mathbb{R}e_2$, and their q_0-orthogonals are $e_1^\perp = \{x_2 \neq 0\}$ and $e_2^\perp = \{x_1 \neq 0\}$. Now, P is $\mathrm{Span}(e_1, e_2)$, and $P^\perp = e_1^\perp \cap e_2^\perp$.

By Lemma 7.2, these directions and their orthogonal hyperplanes are the same for any A-preserved Lorentz form q. In particular, q is positive definite on $P^\perp$ and $q_{|P}$ is proportional to $q_{0|P}$ (two Lorentz forms on a 2-vector space are proportional if and only if they have the same isotropic directions). □

Let us here observe that $P^\perp$ has an easy topological interpretation. It is the stable space for A, that is, the subspace of vectors $u \in \mathbb{R}^k$ with a bounded A-orbit $\{A^n u : n \in \mathbb{Z}\}$.

7.7. *Group dynamics. Proof of Theorem 2.2.* We consider now a group Γ acting on $\mathcal{E}$ via a representation $\rho : \Gamma \to \mathrm{GL}(\mathcal{E})$. We are going to prove the following proposition which implies Theorem 2.2.

PROPOSITION 7.5. *Let $\rho : \Gamma \to \mathrm{GL}(\mathcal{E})$ be such that $\rho(\Gamma)$ is non-precompact. Let $F = \mathrm{Sym}(\mathcal{E})$, and assume that some Lorentz form has a bounded orbit under the associated action ρ^F. Then, up to a finite index, $\rho(\Gamma)$ preserves some Lorentz form.*

The proof of Proposition 7.5 will be given in the remainder of this section.

LEMMA 7.6. *Let Γ be a subgroup of $\mathrm{GL}(F)$, and $B \subset F$ the set of points with bounded Γ-orbit. Then, B is a linear subspace on which the Zariski closure Γ^{Zar} acts via a homomorphism $\rho' : \Gamma^{\mathrm{Zar}} \to \mathrm{GL}(B)$ having compact image.*

Proof. It is easy to see that B is a Γ-invariant vector subspace. It is thus also invariant under Γ^{Zar}. The action of Γ on B factorizes through a homomorphism $\alpha : \Gamma \to K \subset \mathrm{GL}(B)$, with K a compact subgroup. Indeed, in any basis of B, the matrices representing the elements of Γ (acting on B) have bounded entries. Therefore, they constitute a bounded subset of $\mathrm{GL}(B)$, whose closure is a compact subgroup K of $\mathrm{GL}(B)$.

Now, the group G of elements g of $\mathrm{GL}(F)$ preserving B and whose restriction $g|_B \in K$ belongs to K is an algebraic group (since, as a compact group, K is algebraic in $\mathrm{GL}(B)$). Therefore, $G \supset \Gamma^{\mathrm{Zar}}$. □

For Proposition 7.5, we take $F = \mathrm{Sym}(\mathcal{E}^*)$ the space of quadratic forms on $\mathcal{E}$, and Γ acts via the associated representation ρ^F. The subspace B is that of quadratic forms with bounded $\rho^F(\Gamma)$-orbit. By hypothesis, B contains (at least) one form of Lorentz type q_1.

Let H be the Zariski closure $\rho^F(\Gamma)$. It acts on B via $\rho' : H \to \mathrm{GL}(B)$, with image a compact group K and a kernel L, say. By definition and since $\rho(\Gamma)$ is non-precompact (and so is $\rho^F(\Gamma)$), L is a non-compact (algebraic) subgroup of $\mathrm{SO}(q_1)$. If μ is a Haar measure on K, then any average $q_2 = \int_K k \cdot q_1 \, \mathrm{d}\mu$ is a K-invariant element of B, and hence Γ-invariant. Since L is non-compact, then it contains an element $A \in \mathrm{SO}(q_0)$ which is either parabolic or hyperbolic.

Assume that A is parabolic. By Corollary 7.3, q_2 is a Lorentz form.

In the case A is hyperbolic, by Corollary 7.4, then q_2 is Lorentzian, unless its kernel is the 2-plane P generated by the lightlike eigendirections of A. Hence P is H-invariant. Any other hyperbolic element C of L shares with A the same characteristics P and $P^{\perp}$. Indeed, any $C \in \mathrm{SO}(q_1)$ preserving P preserves its two isotropic directions and its orthogonal $P^{\perp}$. By uniqueness of lightlike eigendirections, C and A have the same characteristics. Summarizing, all the hyperbolic elements of L preserve $P^{\perp}$, P and the isotropic directions within it. But, these characteristics have a topological characterization, and hence if $C = hAh^{-1}$, then h preserves these characteristics. This applies, in particular, to any $h \in H$. It follows that H preserves the conformal class of the form q_3 which vanishes on $P^{\perp}$ and equals q_1 on P. Since a co-compact subgroup L of H preserves this form, H itself preserves it (not only up to a factor). The sum $q_2 + q_3$ is an H-invariant (non-degenerate) Lorentz form.

8. *Reduction of G: the toral factor*

The content of the present section can be summarized in the following proposition.

PROPOSITION 8.1. *Under the hypotheses of Theorem 2, there is a subgroup G of $\mathrm{Iso}(M, \mathbf{g})$ with a compact abelian identity component $G_0 = \mathbb{T}^k$, say, having somewhere*

timelike orbits, and such that the discrete factor $\Gamma = G/G_0$ *is infinite. Furthermore:*
(1) *the* $\mathbb{T}^k$*-action is locally free on an open dense set;*
(2) *the* ρ*-representation* $G \to \mathrm{Out}(G_0)$ *becomes a representation* $\rho : \Gamma \to \mathrm{GL}(k, \mathbb{Z})$*;*
(3) ρ *has a finite kernel;*
(4) ρ *preserves some flat Lorentz metric on* $\mathbb{T}^k$*, say given by a Lorentz form* q *on* $\mathbb{R}^k$*.*

8.1. *Reduction of* $\mathbf{G_0}$. Recall the homomorphism $\rho : G \to \mathrm{Out}(G_0)$ in (2.1); by Proposition 5.2, $G_1 = \mathrm{Ker}(\rho)$ is compact, and, in particular, $\rho(G)$ is non-compact.

If G_0 has an almost decomposition $\mathbb{T}^k \times K$, where K is semi-simple, then the image of ρ is contained in $\mathrm{Out}(\mathbb{T}^k) = \mathrm{GL}(k, \mathbb{Z})$.

Since K is normal, we have a representation $r : G \to \mathrm{Aut}(K)$; let G' be its kernel, which is the centralizer of K in G. One can see that $G/G' = K/\mathrm{Center}(K)$. Indeed, since $\mathrm{Aut}(K) = \mathrm{Inn}(K)$, for any $f \in G$, there exists $k \in K$ such that $r(f) = r(k)$, that is, $fk^{-1} \in G'$. Therefore, G/G' is a quotient of K. This quotient is easily identified with $K/\mathrm{Center}(K)$.

In some sense, going from G to G' allows one to kill the semi-simple factor, that is, to assume that the identity component is a torus, and that the discrete factor has not changed, i.e. $G/G^0 = G'/G'_0$. More precisely, let us now describe how to 'forget' the semi-simple factor K keeping the identity component with somewhere timelike orbits. Let X be a somewhere timelike Killing field. The closure of its flow is a product (possibly trivial) of two tori, $K_1 \times K_2$, where K_1 (respectively, K_2) is a subgroup of $\mathbb{T}^k$ (respectively, of K). Since G' centralizes K_2, we have a direct product group $G' \times K_2$.

Summarizing, we have proven the following lemma.

LEMMA 8.2. *There is a subgroup* G *of* $\mathrm{Iso}(M, \mathbf{g})$ *having an abelian identity component* $G_0 = \mathbb{T}^k$ *which has a timelike orbit, and, moreover, it is such that* $G/G_0 = \mathrm{Iso}(M, \mathbf{g})/\mathrm{Iso}_0(M, \mathbf{g})$. *In other words, keeping the hypotheses of Theorem 2, we can and do assume that* $\mathrm{Iso}_0(M, \mathbf{g})$ *is a torus* $\mathbb{T}^k$.

With such a reduction of the group G, we can now consider the action of G on $G_0 \cong \mathbb{T}^k$ given by the representation $\rho : G \to \mathrm{Out}(\mathbb{T}^k) = \mathrm{GL}(k, \mathbb{Z})$; in order to distinguish the action of G on M and on $\mathbb{T}^k$, we will call the latter the ρ*-action*.

COROLLARY 8.3. *Up to a finite index reduction, the quotient group* $\Gamma = G/G_0$ *is torsion free, i.e. all its non-trivial elements have infinite order.*

Proof. Choose any torsion free finite index subgroup H of $\mathrm{GL}(k, \mathbb{Z})$ (it exists by Selberg's lemma [**3**]), and set $G' = \rho^{-1}(H)$. Its projection on G/G_0 is a finite index torsion-free subgroup of Γ, by part (3) of Theorem 2.1. □

8.2. *Generalities on toral actions.* Our aim here is to determine the freeness of isometric toral actions on manifolds. The key fact is that the set $S(\mathbb{T}^d)$ of all closed subgroups of the d-torus $\mathbb{T}^d$ is countable, and it satisfies a uniform discreteness property.

LEMMA 8.4. *Let* X *be a locally compact metric space, and let* $\phi : X \to S(\mathbb{T}^d)$ *be a* semi-continuous *map, that is, if* $x_n \to x$*, then any limit of* $\phi(x_n)$ *is contained in* $\phi(x)$*. Then, there exists* $A \in S(\mathbb{T}^d)$ *such that* $\phi^{-1}(A)$ *has non-empty interior.*

Proof. For $A \in S(T)$, set $F_A = \{x \in X : \text{ such that } A \subset \phi(x)\}$. By the semi-continuity, the closure $\overline{\phi^{-1}(A)} \subset F_A$ for all $A \in S(\mathbb{T}^d)$. Clearly, $X = \bigcup_{A \in S(\mathbb{T}^d)} \phi^{-1}(A)$. By Baire's theorem, the interior $\mathrm{int}(\overline{\phi^{-1}(A)})$ of some $\overline{\phi^{-1}(A)}$ must be non-empty. Thus, the intersection $\mathrm{int}(\overline{\phi^{-1}(A)}) \cap \phi^{-1}(A)$ is non-empty. Let x be a point of such intersection, so that $A = \phi(x)$, and there is a neighborhood V of x such that $\phi(y) \supset \phi(x)$ for all $y \in V$. By semi-continuity, we must have equality $\phi(y) = \phi(x)$ for y in some neighborhood $V' \subset V$. This follows from the fact that A is an isolated point of the set

$$S(\mathbb{T}^d;\, A) = \{B \in S(\mathbb{T}^d) : A \subset B\},$$

see Lemma 8.5. Hence, $\phi^{-1}(A)$ has non-empty interior. □

LEMMA 8.5. *Every $A \in S(\mathbb{T}^d)$ is isolated in $S(\mathbb{T}^d;\, A)$.*

Proof. Let us consider the case that A is the trivial subgroup. To prove that $A = \{1\}$ is isolated in $S(\mathbb{T}^d)$ it suffices to observe that there exist two disjoint closed subsets $C_1, C_2 \subset \mathbb{T}^d$ such that C_1 is a neighborhood of 1, $C_1 \cap C_2 = \emptyset$, and with the property that if $B \in S(\mathbb{T}^d)$ is such that $B \cap C_1 \neq \{1\}$, then $B \cap C_2 \neq \emptyset$. For instance, one can take C_1 to be the closed ball around one of radius $r > 0$ small, and $C_2 = \{p \in \mathbb{T}^d : 2r \leq \mathrm{dist}(p, 1) \leq 3r\}$. Here we are considering the distance on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ induced by the Euclidean metric of $\mathbb{R}^d$. The number r can be chosen in such a way that $C_1 \cap C_2 = \emptyset$. Every non-trivial element in C_1 has some power in C_2, which proves that C_1 and C_2 have the required properties. Thus, if $A_n \in S(\mathbb{T}^d)$ is any sequence which is not eventually equal to A, then some limit point of A_n must be contained in C_2, and therefore $\lim A_n \neq A$.

If one replaces $\mathbb{T}^d$ by a finite quotient of $\mathbb{T}^d$, then one gets to the same conclusion by essentially the same proof. The case of an arbitrary $A \in S(\mathbb{T}^d)$ is obtained by considering the quotient $\mathbb{T}^d/A$, which is equal to a finite quotient of a torus, and applying the first part of the proof. □

COROLLARY 8.6. *If X is a locally compact metric space and $\phi : X \to S(\mathbb{T}^d)$ is semi-continuous, then there is a dense open subset $U \subset X$, where ϕ is locally constant, i.e. any $x \in U$ has a neighborhood V where ϕ is constant.*

Proof. Let U be the open subset of X given by the union of the interiors of the sets $\phi^{-1}(A)$, with A running in $S(\mathbb{T}^d)$. This is the largest open subset of X where ϕ is locally constant. If U were not dense, then there would exist a non-empty open subset $V \subset X$ with $V \cap U = \emptyset$. The restriction $\widetilde{\phi}$ of ϕ to V is a semi-continuous map, with the property that $\widetilde{\phi}^{-1}(A)$ has empty interior for all $A \in S(\mathbb{T}^d)$. By Lemma 8.4, this is impossible, hence U is dense. □

COROLLARY 8.7. *Any faithful isometric action of a torus $\mathbb{T}^d$ on some pseudo-Riemannian manifold $(M, \mathbf{g})$ is free on a dense open subset of M.*

Proof. Apply Corollary 8.6 to the map $\phi : M \to S(\mathbb{T}^d)$, which associates each $p \in M$ with its stabilizer. Such a map is obviously semi-continuous. Thus, on a dense open subset U of M, the stabilizer of the isometric action is locally constant. No non-trivial isometry of a pseudo-Riemannian manifold fixes all points of a non-empty open subset†, and this implies that the stabilizer of each point of U is trivial. □

8.3. *Preliminary properties of the* $\mathbb{T}^k$*-action.* From Corollary 8.7, the $\mathbb{T}^k$-action on M is free on a dense open set.

LEMMA 8.8. *After replacing G by a finite index subgroup, the ρ-action on $\mathbb{T}^k$ preserves some Lorentz metric. In particular, one can see $\rho(G)$ as lying in* $\mathrm{GL}(k,\mathbb{Z})\cap\mathrm{SO}(q)$*, where q is a Lorentz form on $\mathbb{R}^k$. Furthermore, q can be chosen to be rational.*

Proof. The ρ-action of G on $G_0\cong\mathbb{T}^k$ by conjugation induces an action of G on the space $\mathrm{Sym}(\mathbb{R}^k)$ of symmetric bilinear forms on $\mathbb{R}^k$, the Lie algebra of $\mathbb{T}^k$. By (3.3), the compact subset given by the image of the Gauss map $\mathcal{G}$ is invariant by this action. Such a compact subset contains a non-empty open subset consisting of Lorentz forms, because G_0 has timelike orbits in M. By Theorem 2.2, the ρ-action preserves some Lorentz form q. It remains to check that one can choose q rational. For this, let $B\subset\mathrm{Sym}(\mathbb{R}^k)$ be the space of $\rho(G)$-invariant forms. This linear space is defined by rational equations $A\cdot q=q$, $A\in\rho(G)$. Therefore, the rational forms in B are dense. In particular, since it contains a Lorentz form, it contains a rational Lorentz form as well. □

9. *Actions of almost cyclic groups. Proof of Theorem 2.3*

Choose $f\in G$, $f\notin G_0$; then, $\rho(f)\in\mathrm{GL}(k,\mathbb{Z})$ has infinite order by Corollary 8.3. Consider the group $G=G_f$ generated by f and $\mathbb{T}^k$. Up to a compact normal subgroup, G_f is cyclic, which justifies the name *almost cyclic*. One can prove the following lemma.

LEMMA 9.1. *$G=G_f$ is a closed subgroup of* $\mathrm{Iso}(M,\mathbf{g})$ *with compact identity connected component having a timelike orbit. It is isomorphic to a semi-direct product $\mathbb{Z}\ltimes\mathbb{T}^k$.*

9.1. *Normal forms over the rationals*

LEMMA 9.2. *If $A\in\mathrm{GL}(k,\mathbb{Z})\cap\mathrm{SO}(q)$ is parabolic, with q a rational Lorentz form, then some power of A is* rationally equivalent‡ *to*

$$\begin{pmatrix} 1 & t & -t^2/2 & \\ 0 & 1 & t & \mathbf{0} \\ 0 & 0 & 1 & \\ & \mathbf{0} & & \mathrm{Id}_{k-3} \end{pmatrix}.$$

In particular, there is an A-invariant rational 3-space on whose orthogonal, which is not necessarily rational, the A-action is trivial.

Proof. The proof is quite standard. Let A have a normal form as in (7.2) in a basis $\{e_1,\ldots,e_k\}$. Let $\mathcal{E}$ be the kernel of $(A-1)^3$. It is a rational subspace, and it contains $\mathcal{E}_0=\mathrm{Span}\{e_1,e_2,e_3\}$. On $\mathcal{E}/\mathcal{E}_0$, A is elliptic and it satisfies $(A-1)^3=0$, and hence it is trivial.

† Semi-Riemannian isometries are uniquely determined by the value and the derivative at one point, see for instance [**12**, Ch. 3, Proposition 62].

‡ This means that the subspaces $\{e_1\}$, $\{e_1,e_2\}$, $\{e_1,e_2,e_3\}$ and $\{e_4,\ldots,e_k\}$ are rational.

Since $\mathcal{E}$ is rational, A determines an integer matrix in $\mathrm{GL}(\mathbb{R}^k/\mathcal{E})$, which is furthermore elliptic. So, all its eigenvalues are roots of unity, and therefore, after passing to a power, we can assume that one is the unique eigenvalue; more precisely, we can assume that some power of A is trivial.

All this shows that the elliptic part R of some power of A is trivial. It remains to check the rationality of the involved spaces. Consider $A - 1$ and $(A - 1)^2$. Their images are, respectively, $\mathrm{Span}\{e_1, e_3\}$ and $\mathbb{R}e_1$. These two subspaces are thus rational. The space $\mathrm{Span}\{e_1, e_4, \ldots, e_k\}$ is rational since it equals the 1-eigenspace of A. We can modify the basis in order that $e_4, \ldots, e_k$ become rational. It remains to show that e_3 too can be chosen rational. This can be done by taking any rational vector that does not belong to $\mathrm{Span}\{e_1, e_2, e_4, \ldots, e_k\}$. □

In the hyperbolic case, we have the following result, the proof of which is standard.

LEMMA 9.3. *Let $A \in \mathrm{GL}(k, \mathbb{Z}) \cap \mathrm{SO}(q)$ be hyperbolic, with q a rational Lorentz form. There exists a unique rational minimal timelike A-invariant subspace E of $(\mathbb{R}^k, q)$, which contains the two lightlike eigendirections of A. We have a rational A-invariant q-orthogonal splitting $\mathbb{R}^k = E \oplus E^\perp$. The projection of E in $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$ is a closed torus $\mathbb{T}$, on which A induces a hyperbolic Lorentz transformation. In fact, $\mathbb{T}$ is nothing but the closure of the projection in $\mathbb{T}^k$ of the plane P generated by the two lightlike eigendirections of A. Finally, $A|_\mathbb{T}$ is irreducible in the sense that it preserves no non-trivial sub-torus.*

9.2. *Structure theorem*

THEOREM 9.4. *Let $f \in \mathrm{Iso}(M, \mathbf{g})$ act non-periodically on $\mathrm{Iso}_0(M, \mathbf{g})$ (i.e. as an element of $\mathrm{Out}(\mathrm{Iso}_0(M, \mathbf{g}))$). Then, there is a minimal timelike $\rho(f)$-invariant torus $\mathbb{T}^d \subset \mathrm{Iso}_0(M, \mathbf{g})$ of dimension $d = 3$ or $d \geq 2$ according to whether $\rho(f)$ is parabolic or hyperbolic, respectively. The action of $\mathbb{T}^d$ on M is (everywhere) locally free and timelike.*

We will present the proof of the theorem in the parabolic case; the hyperbolic case is analogous, in fact, easier. So, let f be such that $\rho(f)$ is parabolic. The 3-torus $\mathbb{T}^d = \mathbb{T}^3$ in question is the one corresponding to the rational 3-space associated with A in Lemma 9.2. The normal form of $\rho(f)$ on this rational 3-space is

$$\rho(f) \cong \begin{pmatrix} 1 & t & -t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad t \neq 0. \tag{9.1}$$

We need to show that this torus acts freely with timelike orbits on M, and the idea is to relate the dynamics of f on M and the dynamics of $\rho(f)$ on the toral factor. Towards this goal, we will use the approximately stable foliation of a Lorentz isometry, introduced in [**17**].

9.3. *Recalls on approximate stability.* Let ϕ be a diffeomorphism of a compact manifold M. A vector $v \in T_xM$ is called *approximately stable* if there is a sequence $v_n \in T_xM$, $v_n \to v$ such that the sequence $D_x\phi^n v_n$ is bounded in TM. The vector v is called *strongly approximately stable* if $D_x\phi^n v_n \to 0$. The set of approximately stable vectors in T_xM is denoted $\mathrm{AS}(x, \phi)$, or sometimes $\mathrm{AS}(x, \phi, M)$. Their union over M is denoted $\mathrm{AS}(\phi)$, or $\mathrm{AS}(\phi, M)$. Similarly, $\mathrm{SAS}(x, \phi)$ will denote the set of strongly approximately stable vectors in T_xM, and $\mathrm{SAS}(\phi) = \bigcup_{x\in M} \mathrm{SAS}(x, \phi)$.

The structure of $\mathrm{AS}(\phi)$ when ϕ is a Lorentzian isometry has been studied in [**17**].

THEOREM 9.5. (Zeghib [**17**]) *Let ϕ be an isometry of a compact Lorentz manifold $(M, \mathbf{g})$ such that the powers $\{\phi^n\}_{n\in\mathbb{N}}$ of ϕ form an unbounded set (i.e. non-precompact in* $\mathrm{Iso}(M, \mathbf{g})$*). Then:*

(1) $\mathrm{AS}(\phi)$ *is a Lipschitz codimension-one vector subbundle of TM which is tangent to a codimension-one foliation of M by geodesic lightlike hypersurfaces;*
(2) $\mathrm{SAS}(\phi)$ *is a Lipschitz one-dimensional subbundle of TM contained in* $\mathrm{AS}(\phi)$ *and everywhere lightlike.*

9.4. *The action on M versus the toral action.* Denote by $\mathcal{T}$ the Lie algebra of $\mathbb{T}^3$, and by $\rho_0(f)$ the linear representation associated with $\rho(f)$. More explicitly, $\rho_0(f)$ is the push-forward by f of Killing vector fields, see (3.1).

LEMMA 9.6. *Let $X \in \mathcal{T}$ be a Killing field which is approximately stable for $\rho(f)$ at $1 \in \mathbb{T}^3$. Then, for all $x \in M$, $X(x) \in T_xM$ is approximately stable. In other words, if $X \in \mathrm{AS}(0, \rho_0(f), \mathcal{T})$, then $X(x) \in \mathrm{AS}(x, f, M)$ for any $x \in M$.*

A totally analogous statement holds for the strong approximate stability.

Proof. Let X_n be a sequence of Killing fields in $\mathcal{T}$ such that $X_n \to X$ and with $Y_n = f_*^n X_n$ bounded. Clearly $X_n(x) \to X(x)$ for all $x \in M$; moreover, by assumption, the Y_n are bounded vector fields, and so $D_x f^n X_n(x) = Y_n(f^n x)$ is bounded, that is, $X(x) \in \mathrm{AS}(f)$. □

LEMMA 9.7. *Assume $\rho(f)$ parabolic. Then, there is a Killing field $Z \in \mathcal{T}$ such that*

(a) *Z defines a periodic flow ϕ^t;*
(b) *f preserves Z, i.e. f commutes with the one-parameter group of isometries ϕ^t generated by Z;*
(c) *Z generates the strong approximate stable one-dimensional bundle of f;*
(d) *Z is everywhere isotropic;*
(e) *Z is non-singular, hence Z is everywhere lightlike.*

Proof. Let Z be a 1-eigenvector of $\rho(f)$; since $\rho_0(f)Z = f_*Z = Z$, then f preserves Z. In the normal form (9.1) of $\rho(f)$, the vector Z corresponds to the first element of the basis.

The Z-direction is rational, since it is the unique 1-eigendirection of $\rho_0(f)$. Thus Z defines a periodic flow.

One verifies that Z is strongly approximately stable for the $\rho_0(f)$-action at $0 \in \mathcal{T}$. Therefore, at any x where it does not vanish, $Z(x)$ determines the strongly stable one-dimensional bundle of f. In particular, $Z(x)$ is isotropic for all $x \in M$. But non-trivial isotropic Killing fields cannot have singularities. □

9.5. *Proof of Theorem 9.4 (parabolic case)*

LEMMA 9.8. *The torus $\mathbb{T}^3$ preserves the approximate stable foliation $\mathcal{F}$ of f.*

Proof. The group G_f generated by $\mathbb{T}^3$ and f is amenable (it is an extension of the abelian $\mathbb{T}^3$ by the abelian $\mathbb{Z}$). The statement then follows from [**17**, Theorems 2.4, 2.6]. □

LEMMA 9.9. *The $\mathbb{T}^3$-action is locally free.*

Proof. Let Σ be the set of points x having a stabilizer S_x of positive dimension. We claim that if Σ is non-empty, then there must be some point of Σ whose stabilizer contains the flow ϕ^t of the vector field Z given in Lemma 9.7. This is clearly a contradiction, because such a Z has no singularity.

In order to prove the claim, consider the set $\Sigma^2 = \{x \in M : \dim(S_x) = 2\}$. This is a closed subset of M, because two is the highest possible dimension of the stabilizers of the $\mathbb{T}^3$-action. If Σ^2 is non-empty, then there exists an f-invariant measure on Σ^2, and by Poincaré's recurrence theorem there is at least one recurrent point $x_0 \in \Sigma^2$. The Lie algebra $\mathfrak{s}_{x_0}$ of S_{x_0} is then $\rho(f)$-recurrent, and since $\rho(f)$ is parabolic, by Lemma 7.1 (applied to the $\rho(f)$-action on the Grassmannian of 2-planes in $\mathcal{T}$), then $\mathfrak{s}_{x_0}$ is fixed by $\rho(f)$. There is only one 2-plane fixed by $\rho(f)$ in $\mathcal{T}$ (the one spanned by the first two vectors of the basis that puts $\rho(f)$ in normal form), and such a plane contains Z.

Similarly, if Σ^2 is empty, then $\Sigma^1 = \{x \in M : \dim(S_x) = 1\}$ is closed in M. As above, there must be a recurrent point x_0 in Σ^1, and $\mathfrak{s}_{x_0}$ is fixed by $\rho(f)$. This implies that $\mathfrak{s}_{x_0}$ contains Z.

The proof is concluded. □

LEMMA 9.10. *The $\mathbb{T}^3$-action is everywhere timelike.*

Proof. If not, there exists $x \in M$ such that the restriction $\mathbf{g}_x$ of the metric $\mathbf{g}$ to $T_x\mathbb{T}^3x \cong \mathcal{T}$ is lightlike, i.e. positive semi-definite (note that Z is a lightlike vector of such restriction, which cannot be positive definite). Consider the f-invariant compact subset $M_+ = \{x \in M : \mathbf{g}_x \text{ is positive semi-definite}\}$; it has an f-invariant measure, and by Poincaré's recurrence theorem there is a recurrent point $x_0 \in M_+$ for f. Also the metric $\mathbf{g}_{x_0}$ on $\mathcal{T}$ is $\rho(f)$-recurrent, and by Lemma 7.1 (applied to the $\rho(f)$-action on the space of quadratic forms on $\mathcal{T}$), $\mathbf{g}_{x_0}$ is fixed by $\rho(f)$. But there exists no non-zero $\rho(f)$-invariant quadratic form on $\mathcal{T}$ whose kernel is Z. This is proved with an elementary computation using the normal form (9.1) of $\rho(f)$. □

10. *A general covering lemma*

Let us now go back to the general case where $\rho(f)$ is either parabolic or hyperbolic, and proceed with the study of the geometrical structure of M. The product structure of (a finite covering of) M will be established using a general covering result.

PROPOSITION 10.1. *Let M be a compact manifold, and let X be a non-singular vector field on M generating an equicontinuous flow ϕ^t (i.e. ϕ^t preserves some Riemannian* *metric). Assume there exists a codimension-one foliation $\mathcal{N}$ such that:*

(1) *$\mathcal{N}$ is everywhere transverse to X;*

(2) *$\mathcal{N}$ is preserved by ϕ^t.*

Then, X and $\mathcal{N}$ define a global product structure in the universal cover $\tilde{M}$. More precisely, let $x_0 \in M$ and let N_0 be its $\mathcal{N}$-leaf. Then, the map $p : \mathbb{R} \times N_0 \to M$ defined by $p(t, x) = \phi^t x$ is a covering.

A generalization is available for some group actions. Namely, consider an action of a compact Lie group K on a compact manifold M such that:

(1) *the action is locally free (in particular, all orbits have the same dimension);*

(2) *K preserves a foliation $\mathcal{N}$ transverse to its orbits (with a complementary dimension).*

Then, for all $x_0 \in M$, denoting by N_0 the leaf of $\mathcal{N}$ through x_0, the map $p : K \times N_0 \to M$ defined by $p(g, x) = gx$ is an equivariant† covering.

Proof. In order to prove the first statement, consider the class of Riemannian metrics for which X and $\mathcal{N}$ are orthogonal, and X has norm equal to one. The equicontinuity assumption implies that the ϕ^t generate a precompact subgroup Φ of the diffeomorphisms group of M. By averaging over the compact group $\overline{\Phi}$, one obtains a metric $\mathbf{g}_*$ on M (in our specified class of metrics) which is preserved by ϕ^t. Now, endow $\mathbb{R} \times N_0$ with the product metric, where $\mathbb{R}$ is endowed with the Euclidean metric $\mathrm{d}t^2$ and N_0 has the induced metric from $\mathbf{g}_*$. Observe that the induced metric on N_0 is complete (leaves of foliations in compact manifolds have *bounded geometry*, i.e. they are complete, have bounded curvature and injectivity radius bounded from below). One then observes that p is a local isometry; namely, $\mathbb{R} \times N_0$ is complete, and therefore p is a covering.

For the second statement, one can choose a left-invariant Riemannian metric $\mathbf{h}$ on K, and taking an average on K one obtains a K-invariant Riemannian metric $\mathbf{g}_*$ on M such that:

(a) the K-orbits and the leaves of $\mathcal{N}$ are everywhere orthogonal;

(b) the map $K \ni k \mapsto kx_0 \in Kx_0$ is a local isometry when the orbit Kx_0 is endowed with the Riemannian metric induced by $\mathbf{g}_*$.

As above, with such a choice the equivariant map $p : K \times N_0 \to M$ defined by $p(k, x) = kx$ is a local isometry, and since $K \times N_0$ is complete, p is a covering map. □

11. *On the product structure: the hyperbolic case*

Let us now assume that $\rho(f)$ is hyperbolic; in this section we will denote by $\mathbb{T}$ the torus $\mathbb{T}^d \subset G_0$ given in Theorem 9.4.

LEMMA 11.1. *The orthogonal distribution $\mathcal{N}$ to the $\mathbb{T}$-foliation is integrable.*

Proof. Let N be the quotient of M by the $\mathbb{T}$-action, and $\pi : M \to N$ the projection. It is a compact Riemannian orbifold. The f-action induces an isometry g of N. Consider the *Levi form* (i.e. the integrability tensor of the distribution $\mathcal{N}$) $l : \mathcal{N} \times \mathcal{N} \to \mathcal{N}^\perp$. Observe that $\mathcal{N}^\perp$ is the tangent bundle of the $\mathbb{T}$-foliation.

† The action of K on $K \times N_0$ is the left multiplication on the first factor.

Let X and Y be two vector fields on N. Suppose they are g-invariant: $g_*X = X$ and $g_*Y = Y$. Let $\bar{X}$ and $\bar{Y}$ be their horizontal lifts on M. Then, $f_*\bar{X} = \bar{X}$ and $f_*\bar{Y} = \bar{Y}$. Hence, $l(\bar{X}, \bar{Y})$ is an f-invariant vector field tangent to the $\mathbb{T}$-foliation. However, by definition of the minimal torus $\mathbb{T}$, the $\rho(f)$-action on it has no invariant vector field. This means $l(\bar{X}, \bar{Y}) = 0$.

This proof will be finished thanks to the following fact. □

PROPOSITION 11.2. *Let g be an isometry of a Riemannian manifold N. There is an open dense set U such that for any $x \in U$, any vector $u \in T_xU$ can be extended to a g-invariant vector field.*

Proof. Either the group $\{g^n, n \in \mathbb{Z}\}$ acts properly on N, or its closure in the isometry group of N is a compact group with a torus S as identity connected component. The proof in the proper case is straightforward, so let us consider the case where the closure is compact, and, to simplify matters, assume it to be connected and thus to coincide with the torus S. Apply Corollary 8.7 to conclude that, for the associated isometric action on N, the isotropy group is trivial on an open dense set U. Given $x \in U$ and $u \in T_xN$, extend first u to an arbitrary smooth vector field having compact support on the slice through x of the S-action, then extend to an open neighborhood of x using the S-action, and extend to zero outside such a neighborhood. □

We shall now prove the compactness of the leaves of $\mathcal{N}$.

LEMMA 11.3. *Let N_0 be a leaf of $\mathcal{N}$. Then N_0 is compact.*

Proof. The distribution $\mathcal{N}$ can be seen as a connection on the $\mathbb{T}$-principal bundle $M \to N$. We have just proved that this connection is flat, i.e. $\mathcal{N}$ is integrable, which is equivalent to the fact that its holonomy group is discrete. The leaves will be compact if we prove that the holonomy group is indeed finite; for $x \in N$, we will denote by $\mathbb{T}_x$ the fiber at x of the principal bundle $M \to N$. Recall that if c is a loop at $x \in N$, then the holonomy map $H(c) : \mathbb{T}_x \to \mathbb{T}_x$ is obtained by means of horizontal lifts of c. It commutes with the $\mathbb{T}$-action and therefore it is a translation itself. In fact, $H(c)$ can be seen as an element of the acting torus $\mathbb{T}$ (and so, it is independent of the base point x). We have a holonomy map $H : \pi_1(N, x) \to \mathbb{T}$. In fact, since $\mathbb{T}$ is commutative, we have canonical identification of holonomy maps defined on different base points. In other words $H(c) = H(c')$, whence c and c' are freely homotopic curves.

Up to replacing f by some power, we can assume that the basic Riemannian isometry $g : N \to N$ is in the identity component of $\mathrm{Iso}(N)$ (since this group is compact). Therefore, any loop c is freely homotopic to $g(c)$, and hence $H(c) = H(g(c))$.

Now, f preserves all the structure, and thus if $\tilde{c}$ is a horizontal lift of c, then $f \circ \tilde{c}$ is a horizontal lift of $g(c)$. So, $fH(c)f^{-1} = H(g(c))$. If $g(c)$ is freely homotopic to c, then $H(c)$ is a fixed point of $\rho(f)$. But we know that $\rho(f)$ has only finitely many fixed points (by the definition of $\mathbb{T}$). Therefore, the holonomy group is finite. □

Now apply Proposition 10.1 to deduce that M is covered by a product $\mathbb{T} \times N_0 \to M$. The covering is finite because N_0 is compact.

Observe that we can assume that the leaf N_0 is f-invariant. Indeed, the leaf N_0 meets all the fibers $\mathbb{T}_x$, and, say, it contains $\tilde{x} \in \mathbb{T}_x$. So after composing with a suitable translation $t \in \mathbb{T}$, i.e. replacing f by $t \circ f$, we can assume that $f(\tilde{x}) \in N_0$. This implies that $f(N_0) = N_0$. Summarizing, all things (the $\mathbb{T}$-action and f) can be lifted to the finite cover $T \times N_0$.

11.1. *The metric.* The Lorentz metric $\mathbf{g}$ is not necessarily a product of the Riemannian metric on N_0 and that of $\mathbb{T}$. It is true that $\mathbb{T}$ and N are $\mathbf{g}$-orthogonal. Also, two leaves $\{t\} \times N$ and $\{t'\} \times N$ are isometric, via the $\mathbb{T}$-action. However, the metric induced on each $\mathbb{T} \times \{n\}$ may vary with n. Observe here that one can choose the same metric for all these toral orbits, and get a new metric $\mathbf{g}^{\mathrm{new}}$ on M, of course keeping the same initial group acting isometrically. Remember, however, that we broke symmetries of this initial metric in Proposition 8.1, where we reduced to the case that $\mathrm{Iso}_0(M, \mathbf{g})$ was a torus. In other words, we eliminated its semi-simple part K (see §8.1). In order to prove that $\mathbf{g}^{\mathrm{new}}$ inherits all the isometries of the initial metric $\mathbf{g}$, we only need to show that K preserves $\mathbf{g}^{\mathrm{new}}$. This follows from the fact that K commutes with $\mathbb{T}$, and it preserves all the structures involved in our construction, in particular foliations. In fact K acts on N_0, and hence it acts isometrically for $\mathbf{g}^{\mathrm{new}}$ since the metric of N_0 has not been changed.

11.2. *The non-elementary case.* Let Γ be a discrete subgroup of $\mathrm{SO}(1, k-1)$ and L_Γ be its limit set in the sphere (boundary at infinity of the hyperbolic space $\mathbb{H}^{k-1}$). The group Γ is *elementary parabolic* if L_Γ has cardinality equal to one, and *elementary hyperbolic* if L_Γ has cardinality equal to two. It is known that if Γ is not elementary, then L_Γ is infinite, and the action of Γ on L_Γ is *minimal*, i.e. every orbit is dense, see [**15**].

If Γ is elementary hyperbolic, then Γ is virtually a cyclic group, i.e. up to a finite index, it consists of powers A^n of the same hyperbolic element A. If Γ is elementary parabolic, then it is virtually a free abelian group of rank $d \leq k-2$, i.e. it has a finite index subgroup isomorphic to $\mathbb{Z}^d$.

One fact about non-elementary groups is that they contain hyperbolic elements. More precisely, the set of fixed points of hyperbolic elements in L_Γ is dense in L_Γ.

All the previous considerations in the case of a hyperbolic isometry f extend to the case of a non-elementary group. We get the following theorem.

THEOREM 11.4. *Let* $(M, \mathbf{g})$ *be a compact Lorentz manifold with* $\mathrm{Iso}(M, \mathbf{g})$ *non-compact, but* $\mathrm{Iso}_0(M, \mathbf{g})$ *compact, and let* $\Gamma = \mathrm{Iso}(M, \mathbf{g})/\mathrm{Iso}_0(M, \mathbf{g})$ *be the discrete factor. Assume that* $\mathrm{Iso}_0(M, \mathbf{g})$ *has some timelike orbit. Then, there is a torus* $\mathbb{T}^k$ *contained in* $\mathrm{Iso}_0(M, \mathbf{g})$, *invariant under the action by conjugacy of* Γ, *and such that the* $\mathbb{T}^k$*-action is everywhere locally free and timelike.*

The Γ*-action on* $\mathbb{T}^k$ *preserves some Lorentz metric on* $\mathbb{T}^k$, *which allows one to identify* Γ *with a discrete subgroup of* $\mathrm{SO}(1, k-1)$, *as well as a subgroup of* $\mathrm{GL}(k, \mathbb{Z})$.

If Γ *is not elementary parabolic, then, up to a finite covering,* M *splits as a topological product* $\mathbb{T}^k \times N$, *where* N *is a compact Riemannian manifold. One can modify the original metric* $\mathbf{g}$ *along the* $\mathbb{T}^k$ *orbits, and get a new metric* $\mathbf{g}^{\mathrm{new}}$ *with a larger isometry group,* $\mathrm{Iso}(M, \mathbf{g}^{\mathrm{new}}) \supset \mathrm{Iso}(M, \mathbf{g})$, *such that* $(M, \mathbf{g}^{\mathrm{new}})$ *is a pseudo-Riemannian direct product* $\mathbb{T}^k \times N$.

12. *On the product structure: the parabolic case*

As above, we have a $\mathbb{T}^3$-principal fibration $M \to N$ over a Riemannian orbifold N, and $\mathcal{N}$ is seen as a connection. Let Z be the Killing field (as defined in Lemma 9.7), that is, the unique vector field which commutes with f.

Consider the codimension-two bundle $\mathcal{L} = \mathcal{N} \oplus \mathbb{R}Z$.

LEMMA 12.1. *$\mathcal{L}$ is integrable.*

Proof. Let X and Y be two vector fields tangent to $\mathcal{N}$. As in the proof above in the hyperbolic case, we can choose X and Y to be f-invariant, see Proposition 11.2. Therefore, $l(X, Y)$ is also f-invariant, where l is the Levi form of $\mathcal{N}$ (not of $\mathcal{L}$!). But Z is the unique $\rho(f)$-invariant vector. Thus, $l(X, Y)$ is tangent to $\mathbb{R}Z$.

Now, consider the Lie bracket $[X, Z]$. The $\mathbb{T}^3$-action preserves $\mathcal{N}$, and, in particular, $[Z, X]$ is tangent to $\mathcal{N}$, for any X tangent to $\mathcal{N}$. Consequently, $\mathcal{L}$ is integrable. □

As in the hyperbolic case, one can prove the following lemma.

LEMMA 12.2. *The leaves of $\mathcal{L}$ are compact.*

One can then apply Proposition 10.1 to $\mathcal{L}$ and any two-dimensional torus $\mathbb{T}^2$ transverse to Z ($\mathcal{L}$ is invariant by $\mathbb{T}^3$ and so also by any such $\mathbb{T}^2$). One obtains that M is finitely covered by $\mathbb{T}^2 \times L_0$.

The essential difference from the hyperbolic case is that, since $\mathbb{T}^2$ is not f-invariant, this product is not compatible with f. Thus, we have to analyze this situation slightly more deeply in order to carry out an f-invariant 'decomposition' of M.

Observe first that, as in the hyperbolic case, we can choose L_0 invariant by f. The metric structure of L_0 is that of a lightlike manifold, that is, L_0 is endowed with a positive semi-definite (degenerate) metric with a one-dimensional null space. Here, the null space corresponds to the foliation defined by the $\mathbb{S}^1$-action given by the flow of the vector field Z. The circle $\mathbb{S}^1$ acts isometrically on the lightlike L_0, as well as on the Lorentz $\mathbb{T}^3$. One then shows that, up to a finite cover, M is constructed by means of these ingredients, as an amalgamated product, i.e. M is a 'metric' quotient $(\mathbb{T}^3 \times L_0)/\mathbb{S}^1$, see §12.1 for details. This structure is compatible with f.

We have proven the following theorem.

THEOREM 12.3. *Let f be an isometry of a compact Lorentz manifold $(M, \mathbf{g})$ such that the action $\rho(f)$ on the toral component of* $\mathrm{Iso}_0(M, \mathbf{g})$ *is parabolic. Then, there is a new metric on M having a larger isometry group such that M is the amalgamated product of a Lorentz torus $\mathbb{T}^3$, and a lightlike manifold L_0. Both have an isometric $\mathbb{S}^1$-action. The isometry f is obtained by means of an isometry h of L_0 commuting with the $\mathbb{S}^1$-action, and a linear isometry on the Lorentz $\mathbb{T}^3$.*

The same statement is valid if, instead of a single parabolic f, we have an elementary parabolic group Γ of rank d. In this case, the torus has dimension $2 + d$.

In this last higher rank case, $\rho(\Gamma)$ has a normal form as in Lemma 9.2, with t running over a lattice (isomorphic to $\mathbb{Z}^d$) in $\mathbb{R}^d$ (where now t^2 denotes the square of its norm). Here we get a torus of dimension $d + 2$ playing the role of our $\mathbb{T}^3$ in the case of a single parabolic f. All the steps of the proofs are adapted to the higher rank case. One observes, in particular, that the $\mathbb{S}^1$-action is common for all Γ.

12.1. *Amalgamated products.* Given any two manifolds X and Y carrying free (left) actions of the circle $\mathbb{S}^1$, one can consider the diagonal action of $\mathbb{S}^1$ on the product $X \times Y$: $g(x, y) = (gx, gy)$ for all $g \in \mathbb{S}^1$, $x \in X$ and $y \in Y$. Let Z be the quotient $(X \times Y)/\mathbb{S}^1$ of this diagonal action. Assume that X is Lorentzian, Y is Riemannian, and the action of $\mathbb{S}^1$ in each manifold is isometric; one can define a natural Lorentzian structure on Z as follows (see below about the case where Y is lightlike). Let $A \in \mathfrak{X}(X)$ and $B \in \mathfrak{X}(Y)$ be smooth vector fields tangent to the fibers of the $\mathbb{S}^1$-action on X and on Y, respectively; for $(x_0, y_0) \in X \times Y$, denote by $[(x_0, y_0)] \in Z$ the $\mathbb{S}^1$-orbit $\{(gx_0, gy_0) : g \in \mathbb{S}^1\}$. The subspace $T_{x_0}X \oplus B_{y_0}^\perp$ is complementary to the one-dimensional subspace spanned by (A_{x_0}, B_{y_0}) in $T_{x_0}X \oplus T_{y_0}Y$. If we denote by $\mathfrak{q} : X \times Y \to Z$ the projection, then the linear map $\mathrm{d}\mathfrak{q}_{(x_0,y_0)} : T_{x_0}X \oplus T_{y_0}Y \to T_{[(x_0,y_0)]}Z \cong (T_{x_0}X \oplus T_{y_0}Y)/\mathbb{R} \cdot (A_{x_0}, B_{y_0})$ restricts to an isomorphism:

$$\mathrm{d}\mathfrak{q}_{(x_0,y_0)} : T_{x_0}X \oplus B_{y_0}^\perp \xrightarrow{\cong} T_{[(x_0,y_0)]}Z.$$

A Lorentzian metric can be defined on Z by requiring that such isomorphism be isometric; in order to see that this is well defined, we need to show that this definition is independent of the choice of (x_0, y_0) in the orbit $[(x_0, y_0)]$. For $g \in \mathbb{S}^1$, denote by $\phi_g : X \to X$ and $\psi_g : Y \to Y$ the isometries given by the action of g on X and on Y, respectively. By differentiating at (x_0, y_0) the commutative diagram

$$\begin{tikzcd} X \times Y \arrow[rr, "(\phi_g,\psi_g)"] \arrow[dr, "\mathfrak{q}"'] & & X \times Y \arrow[dl, "\mathfrak{q}"] \\ & Z & \end{tikzcd}$$

we get a commutative diagram

$$\begin{tikzcd} T_{x_0}X \oplus T_{y_0}Y \arrow[rr, "((\mathrm{d}\phi_g)_{x_0},(\mathrm{d}\psi_g)_{y_0})"] \arrow[dr, "\mathrm{d}\mathfrak{q}_{(x_0,y_0)}"'] & & T_{gx_0}X \oplus T_{gy_0}Y \arrow[dl, "\mathrm{d}\mathfrak{q}_{(gx_0,gy_0)}"] \\ & T_{[(x_0,y_0)]}Z & \end{tikzcd}$$

As $((\mathrm{d}\phi_g)_{x_0}, (\mathrm{d}\psi_g)_{y_0})$ carries $T_{x_0}X \oplus B_{y_0}^\perp$ onto $T_{gx_0}X \oplus B_{gy_0}^\perp$, we get the following commutative diagram of isomorphisms:

$$\begin{tikzcd} T_{x_0}X \oplus B_{y_0}^\perp \arrow[rr, "((\mathrm{d}\phi_g)_{x_0},(\mathrm{d}\psi_g)_{y_0})", "\cong"'] \arrow[dr, "\cong", "\mathrm{d}\mathfrak{q}_{(x_0,y_0)}"'] & & T_{gx_0}X \oplus B_{gy_0}^\perp \arrow[dl, "\cong"', "\mathrm{d}\mathfrak{q}_{(gx_0,gy_0)}"] \\ & T_{[(x_0,y_0)]}Z & \end{tikzcd}$$

Since $((\mathrm{d}\phi_g)_{x_0}, (\mathrm{d}\psi_g)_{y_0})$ is an isometry, the above diagram shows that the metric induced by $\mathrm{d}\mathfrak{q}_{(x_0,y_0)}$ coincides with the metric induced by $\mathrm{d}\mathfrak{q}_{(gx_0,gy_0)}$. This shows that the Lorentzian metric tensor on Z is well defined.

Observe now that an analogous construction can be carried out naturally when, instead of a Riemannian metric, Y is endowed with a lightlike metric which is invariant under the $\mathbb{S}^1$-action and having its direction as null space. In this case, the quotient $(T_{x_0}X \oplus T_{y_0}Y)/\mathbb{R} \cdot (A_{x_0}, B_{y_0})$ is canonically identified with $T_{x_0}X \oplus (T_{y_0}Y/\mathbb{R}B_{y_0})$ which has a natural Lorentz product since by definition $T_{y_0}Y/\mathbb{R}B_{y_0}$ has positive definite inner product.

As to the topology of Z, we have the following lemma.

LEMMA 12.4. *If X and Y are simply connected, then Z is simply connected. If the product of the fundamental groups $\pi_1(X) \times \pi_1(Y)$ is not a cyclic group, then Z is* not *simply connected.*

Proof. The diagonal $\mathbb{S}^1$-action on $X \times Y$ is free (and proper), and therefore the quotient map $\mathsf{q} : X \times Y \to Z$ is a smooth fibration. The thesis follows from an immediate analysis of the long exact homotopy sequence of the fibration, that reads

$$\mathbb{Z} \cong \pi_1(\mathbb{S}^1) \longrightarrow \pi_1(X) \times \pi_1(Y) \longrightarrow \pi_1(Z) \longrightarrow \pi_0(\mathbb{S}^1) \cong \{1\}. \qquad \square$$

13. *Proof of Corollary 3 and Theorem 4*

Proof of Corollary 3. This is one of the steps of the proof of our structure result, see Lemma 9.10.

Proof of Theorem 4. By the structure result of [**16**], compact Lorentzian manifolds admitting an isometric action of (some covering of) $\mathrm{SL}(2, \mathbb{R})$ or of an oscillator group are not simply connected. Thus, if M is simply connected, by Theorem 4.1 $\mathrm{Iso}_0(M, \mathbf{g})$ is compact. Now, if $\mathrm{Iso}(M, \mathbf{g})$ has infinitely many connected components, then (a finite covering of) M is not simply connected. When $\Gamma = \mathrm{Iso}(M, \mathbf{g})/\mathrm{Iso}_0(M, \mathbf{g})$ is not elementary parabolic, this follows directly from Theorem 11.4. When Γ is elementary parabolic, this follows from Theorem 12.3 and the second statement of Lemma 12.4. Hence, $\mathrm{Iso}(M, \mathbf{g})$ is compact.

Acknowledgements. The authors would like to thank François Brunault for his help on questions related to Salem numbers.

The first author was partially sponsored by Fundación Séneca grant 09708/IV2/08, Spain. Part of this work was carried out during the visit of the first author to the École Normale Supérieure de Lyon in November 2009. The authors gratefully acknowledge the financial support for this visit provided by *ANR Geodycos*.

A. *Appendix. Salem Numbers and leading eigenvalues of Lorentz hyperbolic transformations*

As mentioned in the introduction, our problem on Lorentz isometry groups leads naturally to hyperbolic arithmetic lattices of type $\mathrm{O}(q, \mathbb{Z})$, where q is a rational Lorentz form on $\mathbb{R}^k$. We give in the present appendix more details on this arithmetic side of our problem since they seem absent in the literature.

If $A \in \mathrm{O}(q, \mathbb{Z}) = \mathrm{GL}(k, \mathbb{Z}) \cap \mathrm{O}(q)$, then it preserves the canonical lattice $\mathbb{Z}^k \subset \mathbb{R}^k$. By definition, A is irreducible (over the rationals) if A preserves no non-trivial subgroup of rank $\leq k$ in $\mathbb{Z}^k$, or, equivalently, no non-trivial $\mathbb{Q}$-subspace. In this case, its characteristic polynomial is $\mathbb{Q}$-irreducible.

Let A be irreducible and $k \geq 4$. If an eigenvalue λ of A is greater than 1, then A is hyperbolic, and λ is a Salem number. Indeed, by definition [**6**] a real $\lambda > 1$ is a Salem number if it is an algebraic integer (say of degree k) such that all the solutions x of its minimal polynomial (with integer coefficients) P belong to the unit disk $\{|x| \leq 1\}$, with at least one on the unit circle $\{|x_0| = 1\}$. Observe that, in dimension two, λ is a quadratic integer, which is also the case in dimension three (A cannot be irreducible in odd dimensions).

A.1. *The converse.* The existence of such a root x_0 implies that P is reciprocal: $z^k P(1/\bar{z}) = P(z)$. Indeed, $z^k P(1/\bar{z})$ is another minimal polynomial for x_0. We infer from this that $1/\lambda$ is another root of P and all the others are in the unit circle. Hence, any matrix A having P as a characteristic polynomial is $\mathbb{C}$-diagonalizable with simple real spectrum $\{\lambda, \lambda^{-1}\}$, and a complex spectrum in the unit circle. It then follows that A is conjugate to the normal form §7.3 of a hyperbolic element in the orthogonal group $\mathrm{O}(q_1)$ of some Lorentz form q_1.

Observe now that, since P has integer entries, the companion matrix A of P belongs to $\mathrm{GL}(k, \mathbb{Z})$. Recall that if $P(z) = c_0 + c_1 z + \cdots + c_n z^n$, then its companion matrix is

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{k-1} \end{pmatrix},$$

say in a compact form

$$A = \begin{pmatrix} 0 & \\ \mathrm{Id}_{k-1} & -\vec{c} \end{pmatrix},$$

where

$$\vec{c} = \begin{pmatrix} c_0 \\ \vdots \\ c_{k-1} \end{pmatrix}.$$

Summarizing, there exists $A \in \mathrm{GL}(k, \mathbb{Z}) \cap \mathrm{O}(q_1)$, with λ as a leading eigenvalue of A.

Let us prove that we can choose q_1 to be a rational form, i.e. with rational coefficients in the canonical basis of $\mathbb{R}^k$. For this, consider $F = \mathrm{Sym}(\mathbb{R}^k)$ the space of quadratic forms on $\mathbb{R}^k$. Let E be the space of A-invariant forms. This linear space is defined by a rational equation $A \cdot q = q$. Therefore, the rational forms in E are dense, and, in particular, since $q_1 \in E$ is Lorentzian, there exists a rational Lorentz form q_2 in E.

A.2. *Isometric embedding in the standard form.* It is now natural to ask if any Salem number is a leading eigenvalue of some matrix A of $\mathrm{O}(1, n)(\mathbb{Z}) = \mathrm{O}(1, n) \cap \mathrm{GL}(n + 1, \mathbb{Z}) = \mathrm{O}(q_0)$, where q_0 is the standard form $-x_0^2 + x_1^2 + \cdots + x_n^2$. One allows here n

to be larger than k (the degree of transcendency of λ). We will give here a partial answer to this question.

The first observation towards this is that, for any rational Lorentz form q on $\mathbb{R}^k$ (up to a constant), there is a rational isometric embedding in $(\mathbb{R}^{n+1}, q_0)$ for some n. To do this, recall that any rational form is diagonalizable, that is, up to applying an element of $\mathrm{GL}(k, \mathbb{Q})$, we can assume that q_2 has the form $q_2 = a_1 x_1^2 + \cdots + a_k x_k^2$, with $a_i \in \mathbb{Q}$. Up to multiplication, we can assume $a_i \in \mathbb{Z}$.

Now, the one-dimensional space $(\mathbb{R}, ax^2)$, with a a positive integer can be embedded in $(\mathbb{R}^a, x_1^2 + \cdots + x_a^2)$, by means of the diagonal map $x \to (x, \ldots, x)$. In the case $a < 0$, write $ax^2 = -(b)^2 x^2 + cx^2$, with b and c non-negative integers. Then $x \to (bx, x, \ldots, x)$ yields an isometric embedding in $(\mathbb{R}, -x^2) \times (\mathbb{R}^c, x_1^2 + \cdots + x_c^2)$. From all this, one deduces the existence of a rational isometric embedding of $(\mathbb{R}^k, q_2)$ into a standard Lorentz space.

The image $E \subset \mathbb{R}^n$ of such an embedding is a rational subspace. An element of $\mathrm{O}(q_{0|E})$ can be extended trivially on $E^\perp$ to give an element of $\mathrm{O}(q_0)$. Because of rationality, this gives an embedding $\mathrm{O}(q_2, \mathbb{Q}) = \mathrm{O}(q_2) \cap \mathrm{GL}(k, \mathbb{Q})$ into $\mathrm{O}(q_0, \mathbb{Q}) = \mathrm{O}(q_0) \cap \mathrm{GL}(n + 1, \mathbb{Q})$. This embedding is natural, and it preserves eigenvalues. From all this, one obtains an embedding of a finite index subgroup Γ of $\mathrm{O}(q_2, \mathbb{Z})$ in $\mathrm{O}(q_0, \mathbb{Z})$. Hence, for any Salem number λ, there is a power λ^l which is a leading eigenvalue of some $A \in \mathrm{O}(1, n)(\mathbb{Z})$, for some n. It is natural to wonder whether, by any means, l may be chosen to be equal to one.

References

[1] S. Adams and G. Stuck. The isometry group of a compact Lorentz manifold. I. *Invent. Math.* **129**(2) (1997), 239–261.

[2] S. Adams and G. Stuck. The isometry group of a compact Lorentz manifold. II. *Invent. Math.* **129**(2) (1997), 263–287.

[3] R. C. Alperin. An elementary account of Selberg's lemma. *Enseign. Math.* (2) **33**(3–4) (1987), 269–273.

[4] U. Bader, C. Frances and K. Melnick. An embedding theorem for automorphism groups of Cartan geometries. *Geom. Funct. Anal.* **19**(2) (2009), 333–355.

[5] J. K. Beem, P. E. Ehrlich and S. Markvorsen. Timelike isometries and Killing fields. *Geom. Dedicata* **26**(3) (1988), 247–258.

[6] M. J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse and J. P. Schreiber. *Pisot and Salem Numbers*. Birkhäuser, Basel, 1992.

[7] G. D'Ambra. Isometry groups of Lorentz manifolds. *Invent. Math.* **92**(3) (1988), 555–565.

[8] J. L. Flores, M. A. Javaloyes and P. Piccione. Periodic geodesics and geometry of compact Lorentzian manifolds with a Killing vector field. *Math. Z.* **267**(1–2) (2011), 221–233.

[9] G. Hochschild. *The Structure of Lie Groups*. Holden-Day, San Francisco, 1965.

[10] S. Kobayashi. *Transformation Groups in Differential Geometry (Classics in Mathematics)*. Springer, Berlin, 1995, reprint of the 1972 edition.

[11] A. Medina and P. Revoy. Les groupes oscillateurs et leurs réseaux. *Manuscripta Math.* **52**(1–3) (1985), 81–95.

[12] B. O'Neill. *Semi-Riemannian Geometry with Applications to Relativity*. Academic Press, New York, 1983.

[13] J. Palis and W. de Melo. *Geometric Theory of Dynamical Systems*. Springer, New York, 1982.

[14] M. Rosenlicht. A remark on quotient spaces. *An. Acad. Brasil. Ciênc.* **35** (1963), 487–489.

[15] W. Thurston. *Three-dimensional Geometry and Topology (Princeton Mathematical Series, 35)*. Princeton University Press, Princeton, NJ, 1997.

[16] A. Zeghib. Sur les espaces-temps homogènes. *The Epstein Birthday Schrift (Geometry & Topology Monographs, 1)*. Geometry and Topology Publishing, Coventry, 1998, pp. 551–576; electronic.

[17] A. Zeghib. Isometry groups and geodesic foliations of Lorentz manifolds, Part I: Foundations of Lorentz dynamics. *Geom. Funct. Anal.* **9** (1999), 775–822.

[18] R. J. Zimmer. *Ergodic Theory and Semisimple Groups (Monographs in Mathematics, 81)*. Birkhäuser, Basel, 1984.