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SUBSYSTEMS OF ANOSOV SYSTEMS

By A. ZEGHIB

1. Introduction. Let (M, ϕ) be an Anosov system (flow or diffeomorphism). Thus the tangent bundle TM splits into $E^s \bigoplus E^u \bigoplus E^0$, where E^s (resp. E^u) is the contracting (resp. expanding) subbundle and E^0 is trivial in the case of diffeomorphisms and is the one-dimensional tangent bundle in the case of flows.

Consider a C^1-invariant compact submanifold N. We may hope, because of hyperbolicity, that the splitting passes on to a splitting for TN; that is, $TN = E^s \cap TN \bigoplus E^u \cap TN \bigoplus E^0 \cap TN$. It is easy to see that this is equivalent to the subsystem (N, ϕ) itself being of Anosov type. So our question can be formulated as follows: Does a subsystem of an Anosov system have to be Anosov? From the infinitesimal point of view, we could generalize the question by assuming that the splitting of TM is defined only over N, i.e. N is a hyperbolic set of (M, ϕ). Then the question (asked by Hirsch in [Hir]) becomes: Is the restriction of a dynamical system to a hyperbolic set which is a C^1 submanifold an Anosov system?

Mañé [Mañ1] characterized such systems and called them quasi-Anosov. The last question becomes then: Does quasi-Anosov imply Anosov? The answer given by Franks and Robinson [FR] was negative. But the question remains open for subsystems of Anosov systems (not just hyperbolic sets). By the example of [FR], we know that this question is global in nature and not just infinitesimal or local. The aim of this paper is to provide a positive answer to this problem in the case of classical Anosov systems. For this we recall that an Anosov system is said to be splitting [Fra1] if its local product structure is in fact global in the universal cover (see §4). For example, this is the case for Anosov diffeomorphisms on tori. Indeed, by a well-known theorem of Manning, they are all topologically conjugate to linear diffeomorphisms, for which the splitting property is obvious. As usual, we say that two dynamical systems are topologically equivalent if there is a homeomorphism sending the orbits of one to that of the other. The torus case in the following result was proved by Mañé [Mañ2].

THEOREM A. *Let (M, ϕ) be an Anosov system which is splitting or topologically equivalent to the geodesic flow of a compact negatively curved manifold. Let N be a closed invariant C^1-submanifold (of nontrivial dimension, that is, 0 for diffeomorphisms and 1 for flows). Then (N, ϕ) is a transitive Anosov system.*

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Remarks.

1. Note that all known Anosov diffeomorphisms are splitting (more precisely, topologically conjugate to infra-nil-automorphisms).

2. Although the question is not topological (invariant C^1-submanifolds are not mapped by a topological equivalence to invariant C^1-submanifolds), the answer given here is! We mean by this that the above result covers a category of Anosov systems, closed by topological equivalence.

Geodesic flows. Let V be a compact Riemannian manifold of negative curvature and $\tilde{V}$ its universal cover. Denote by (T^1V, ϕ) and $(T^1\tilde{V}, \tilde{\phi})$ the geodesic flows on the unitary tangent bundle of V and $\tilde{V}$ respectively. Let S^∞ be the sphere at infinity of $\tilde{V}$. For a closed subset $A \subset S^\infty$, we denote by $[A]$ the subset of $T^1\tilde{V}$ of all vectors for which (both) endpoints at infinity are in A (the projection of $[A]$ in $\tilde{V}$ is the union of all geodesics joining points of A). We say that a closed invariant subset N of (T^1V, ϕ) is of *quasi-Fuchsian* type if it is the projection in T^1V of a subset of $T^1\tilde{V}$ of the form $[L_\Gamma]$. Here L_Γ is the limit set of a subgroup Γ of $\pi_1(V)$. If we assumed that Γ is exactly the stabilizer in $\pi_1(V)$ of L_Γ, N would be homeomorphic to $[L_\Gamma]/\Gamma$. Thus Γ would be a special case of quasi convex cocompact groups, i.e. those for which the last quotient is compact (see §5). Note that in this definition we do not assume as in the classical case that L_Γ is a topological sphere. This last situation happens exactly when N is a topological manifold.

THEOREM B. *A closed invariant C^1 submanifold of the geodesic flow of a Riemannian compact manifold of negative curvature is of quasi-Fuchsian type.*

Geometric rigidity. The last theorem characterizes neither invariant topological submanifolds (as it is false in this case, see for instance [Zeg2]) nor invariant C^1 submanifolds (since there are examples where N is just a topological submanifold of quasi-Fuchsian type but not C^1). In the classical case (i.e. 3-hyperbolic space), obviously $N = [L_\Gamma]$ is C^1 if and only if L_Γ is. In general the sphere at infinity has no natural differentiable structure, and we can not distinguish the topological and C^1 cases by looking only at infinity. But in the classical case we know that, when it is C^1, the limit set L_Γ is in fact a round circle. That is, Γ is Fuchsian. The analogous geometric rigidity question which would characterize the C^1 case could be: Is a closed invariant C^1-submanifold N (which is of quasi-Fuchsian type by the previous result) of the form T^1S, where S is a totally geodesic submanifold of V? An affirmative answer to this question was given in [Zeg1] (see also [Zeg4]) when V is locally symmetric. Although we have no counter-example, we believe this is wrong for general negatively curved manifolds. Such geometric rigidity would be much stronger (at least in some interesting cases) than other types of rigidities, like geodesic flow rigidity and boundary rigidity for Riemannian metrics. For example, take (V_0, g_0) to be a compact hyperbolic manifold (i.e. of constant curvature -1). In the Grassmann bundle $Gr_2(V_0)$, of 2-tangent planes to V_0, there is a tautological geodesic foliation. The leaf of a tangent plane $P \subset T_xV_0$, is the set of tangent planes of the immersed geodesic surface $S = \exp_x P$ ($\exp_x$ denote the exponential map at x). Suppose that the set of compact leaves of this foliation is dense in $Gr_2(V_0)$. Such V_0 exists (in all dimensions) by arithmetic constructions (thanks to the Harish-Chandra-Borel Theorem [Bor]). Let now (V_1, g_1) be another Riemannian manifold with geodesic flow C^1-equivalent to that of (V_0, g_0). Suppose that the geometric rigidity holds for the C^1 invariant submanifolds of the geodesic flow of (V_1, g_1). From the above density, we get by elementary arguments that for each $x \in V$ and $P \subset T_xV_1$, $S = \exp_x P$ is a totally geodesic (not just geodesic at x) surface. This implies (by standard Riemannian geometry) that (V_1, g_1) is of constant curvature. A dynamical argument shows that (V_1, g_1) is hyperbolic, and then isometric to (V_0, g_0) (by the Mostow rigidity, for example). The boundary rigidity problem can be treated by the same method. Let (B, g) be a Riemannian metric in a ball B of a hyperbolic space, which induces the same boundary distance function on $\partial B \times \partial B$. Consider a hyperbolic manifold like V_0 above, which contains an embedded ball isometric to B. Assume the ball B is convex with respect to g. By [Mic] we can glue (B, g) to $(V_0 - B, g_0)$ and obtain a C^2 Riemannian manifold (V_1, g_1). Moreover, there is a canonical C^1 isomorphism between geodesic flows of (V_0, g_0) and (V_1, g_1). As above, the hypothesis of geometric rigidity for (V_1, g_1) implies that (B, g) is a hyperbolic ball. This would answer the boundary rigidity conjecture in this case.

Remark. There is a natural analogous geometric rigidity property for subsystems of infra-nil Anosov systems. In the case of linear Anosov diffeomorphisms on a tori, C^1 invariant submanifolds are geometric tori [Fra2] [Mañ2]. In the general infra-nil case, C^1 invariant submanifolds are determined by Lie subgroups [Zeg3].

Topological rigidity. Weaker than the geometric rigidity is the requirement for the subsystem (N, ϕ) (N is a C^1 submanifold or a topological manifold of quasi-Fuchsian type) to be topologically equivalent to the geodesic flow of some negatively curved manifold. This would not be so far from geometric rigidity, if furthermore the manifold were isometrically immersed in V. The following result deals with the case when the dimension of N is 3. Unfortunately, we were not able to get an immersed submanifold (a surface in this case), but only a branched immersed one (see for instance [GOR] for notions on branched immersed surfaces, and §5 for the definition of their geodesic flows).

THEOREM C. *Let V be a compact manifold of negative curvature, and N an invariant topological 3-manifold in the geodesic flow of V. Then (N, ϕ) is of quasi-Fuchsian type if and only if it is topologically equivalent to the geodesic flow of* *a branched immersed negatively curved surface S in V. Moreover, the topological entropy of the geodesic flow of any such S of negative curvature is less than that of (N, ϕ), with equality if and only if S is geodesic. In general, if V is locally CAT$(-a^2)$, i.e. its curvature is bounded by $-a^2$, then we can find S with the same property.*

We believe that the following strong statement holds. There is a nonnegative $\epsilon = e(N, \phi)$ such that the topological entropy of the geodesic flow of any such S (which has negative curvature) is less than $Ent_{top}(N, \phi) + \epsilon$, and $\epsilon = 0$, if and only if we can find an S being geodesic. This ϵ may have some geometrical meaning (related to the width of the convex hull) for classical quasi-Fuchsian groups. We also believe that the appearance of branched immersed surfaces in the statement above is just because of technical difficulties. One may expect and imagine that everything is regular!

We deduce from the entropy statement a new proof of a special case of the geometric rigidity of invariant C^1 submanifolds of the geodesic flow of hyperbolic manifolds (see [Zeg1] or [Zeg4]).

COROLLARY. *A C^1 closed invariant 3-submanifold in the geodesic flow of a hyperbolic compact manifold (i.e. curvature $= -1$), is the unitary tangent bundle of a geodesic surface in this manifold.*

Proof. By the theorem, we get a branched surface S which is CAT(-1). The topological entropy of its geodesic flow is then at least 1 (this works in the CAT(-1) case exactly as for Riemannian manifolds of curvature less than -1 [Bou]). On the other hand, the only (exact) positive Lyapunov exponent (for any measure) of the geodesic flow of a hyperbolic manifold is $+1$. It then follows, by the Ruelle formula [Rue], that the topological entropy of (N, ϕ) is at most 1. Hence we have equality and S is geodesic. □

Invariant subsets of quasi-Fuchsian type. Now we give a dynamical characterization of invariant subsets of quasi-Fuchsian type, with locally connected limit sets. That is, the limit set L_Γ in the above definition of quasi-Fuchsian type is locally connected (the result can perhaps be extended when L_Γ is only assumed to be connected). It is formulated with the notion of local product structure (see §2), which plays a crucial role in the proofs of Theorems A and B.

THEOREM D. *A locally connected closed invariant subset of the geodesic flow of a negatively curved compact manifold with a local product structure is of quasi-Fuchsian type.*

This is a consequence of the following result. In order to formulate it we shall use the word "manifold" for all kinds of stable and unstable sets (although they are not topological manifolds).

THEOREM E. *Let N be an invariant subset of the geodesic flow of a negatively curved compact manifold with a local product structure. Suppose that stable or unstable manifolds of N are locally connected. Then (N, ϕ) is of quasi-Fuchsian type.*

Applications to geodesic and Riemannian foliations. A tangent plane field $\mathcal{P}$ in a Riemannian manifold V is said to be geodesic if any geodesic γ tangent at some point to $\mathcal{P}$ is everywhere tangent to it (if $\gamma'(t_0) \in P_{\gamma(t_0)}$, then $\gamma'(t) \in P_{\gamma(t)}$ for any t). When $\mathcal{P}$ is integrable this means that its leaves are (totally) geodesic. A tangent plane field is said to be Riemannian if its orthogonal is geodesic.

THEOREM F. *A compact negatively curved manifold has no C^1 geodesic or Riemannian plane field (of nontrivial dimension).*

Proof (from Theorem B). It is sufficient to consider the geodesic case. Suppose that $\mathcal{P}$ is a C^1 geodesic plane field. Let $N = T^1\mathcal{P}$ be the set of all unit vectors tangent to $\mathcal{P}$. By definition, this is a C^1 invariant submanifold of the geodesic flow of V. Let $\tilde{N}$ be the inverse image of N in $T^1\tilde{V}$. This is nothing but $T^1\tilde{\mathcal{P}}$, the set of unit vectors tangent to the associated plane field $\tilde{\mathcal{P}}$. In particular $\tilde{N}$ is connected and $\pi_1(V)$-invariant. By Theorem B, $\tilde{N}$ must be of the form $[L_\Gamma]$ for some subgroup $\Gamma \subset \pi_1(V)$. The $\pi_1(V)$-invariance of $\tilde{N}$ implies that L_Γ is $\pi_1(V)$ invariant. By minimality of the $\pi_1(V)$ action we have : $L_\Gamma = S^\infty$. Hence $\tilde{N} = [S^\infty]$. That is, $\mathcal{P}$ is the trivial tangent space of V (i.e. $\operatorname{codim} \mathcal{P} = 0$). □

Results similar to Theorem F (for geodesic foliations and some cases of Riemannian foliations) were announced by P. Walczac in [Wal1] and [Wal2]. But as it was indicated in [Wal3], the proofs were not correct. Our proof above applies even for the so-called (C^1-) quasi geodesic and quasi Riemannian foliations, in the references given above. For example, a C^1 quasi geodesic foliation $\mathcal{P}$ is one for which the second fundamental tensor is small in the C^1 norm. The geodesic flow of the foliation, defined in the set $N = T^1\mathcal{P} \subset T^1V$, can then be extended to a flow in T^1V which is C^1 near the geodesic flow of V. It is then topologically equivalent to it. We may then apply the same argument as above.

More recently G. Walschap [Wal] announced a nonexistence result for Riemannian foliations in the particular case of locally symmetric manifolds (compact and with negative curvature). Of course this immediately follows from the geometric rigidity in this case, or just from the analogous of Theorem B in the locally symmetric case, which was already proved in [Zeg1]. The proof of Walschap consists of proving that the orthogonal of a Riemannian foliation (in this case) is integrable and thus gives rise to a geodesic foliation. Next, to conclude, he uses the result of [Wal1] (that we know now is incomplete) about nonexistence of geodesic foliations. He also proposes an alternative "elementary" proof of this last result for locally symmetric spaces, when furthermore the orthogonal of the geodesic foliation is integrable (that is, when it is derived from a Riemannian foliation). Unfortunately, this proof, also, was not correct (or at least incomplete). All of these facts let us believe that a (semi-) local or (semi-) infinitesimal proof of Theorem F is perhaps not available.

Remark. The theorems A, B, D, E and F extend to the case when V is not necessarily of negative curvature but only has its geodesic flow of Anosov type (observe that in particular the universal cover of such a manifold is hyperbolic in the sense of [Gro] and has a sphere as ideal boundary, which allows us to speak about subsets of quasi-Fuchsian type).

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2. Quasi Anosov systems. To simplify notations we shall restrict ourself everywhere to flows. This does not imply any loss of generality, as we can associate to a diffeomorphism its suspension flow and, in view of Theorem A, splitting diffeomorphisms correspond to splitting flows (see the definition below). The discussion in this section is classical. One possible definition of quasi Anosov systems is as follows: (N,ϕ) is quasi Anosov if it can be embedded in a system (M,ϕ), for which N is an hyperbolic set. Here the most important property is the following.

PROPOSITION 2.1. ([HP], [Mañ1], [Sel]) *A quasi Anosov system is Axiom A (that is, the no wandering set $\Omega(N,\phi)$ is hyperbolic and periodic orbits are dense in it).*

From hyperbolicity we get strong stable, stable, local strong stable and local stable manifolds at points of Ω: $W^{ss}(x) = \{y \in N/d(\phi^t x, \phi^t y) \to 0$ when $t \to +\infty\}$; $W^s(x) = \cup\{W^{ss}(\phi^t x)/t \in \mathbf{R}\}$; $W^{ss}_\eta(x) = \{y \in W^{ss}(x)/d(\phi^t x, \phi^t y) < \eta$ for $t > 0\}$ and $W^s_\eta(x) = \{y \in W^s(x)/d(\phi^t x, \phi^t y) < \eta$ for $t > 0\}$ (here d is some Riemannian distance in N). They are injectively immersed C^1 submanifolds. Reversing time, one defines the unstable analogues. We shall fix η, being the half of an expansivity constant for (M,ϕ). That is, two distinct orbits cannot stay (even after reparametrization) at a distance less than 2η.

Local product structure. For any $x \in N$, $\phi^t x \to \Omega$, when $t \to \infty$, i.e. all limits of $\phi^t x$ are in Ω. Axiom A implies a more precise convergence: There are elements y and z of Ω (not unique) such that $x \in W^s(y)$ and $x \in W^u(z)$. In other words, $W^s(x) = W^s(y)$ and $W^u(x) = W^u(z)$ (in particular $W^s(x)$ and $W^u(x)$ are C^1 immersed submanifolds of N for any x). This follows from the shadowing property of Ω, which again follows from the local product structure of Ω. That is: There exist ϵ and η positive such that if $y, z \in \Omega$, and $d(y,z) < \epsilon$, then $W^s_\eta(y) \cap W^{su}_\eta(z)$

contains exactly one point $[y,z]$ which belongs to Ω (the uniqueness follows from the choice of η as an expansivity constant). More precisely the bracket map determines a homeomorphism:

$$(W^s_\eta(x) \cap \Omega) \times (W^{su}_\eta(x) \cap \Omega) \to U_x$$

$$(y,z) \to [y,z] = W^s_\eta(y) \cap W^{su}_\eta(z),$$

where U_x is a neighborhood of x in Ω, containing the ball $B(x,\epsilon) \cap \Omega$.

This looks like a topological Anosov picture: Instead of the stable and unstable supplementary foliations we have the two laminations of Ω: $W^s(x) \cap \Omega$ and $W^{su}(x) \cap \Omega$. Any such ϵ will be called a local product or an Anosov constant of Ω.

Attractors. In general, Ω is thought of (if it is not all N) as a "fractal" set. But some parts of Ω are less fractal than others. To justify this, recall that Ω has a spectral decomposition, $\Omega = \Lambda_0 \cup \ldots \Lambda_n$, into closed disjoint invariant sets such that each has a dense orbit. Denote $W^s(\Lambda_i) = \{x \in N/\phi^t x \to \Lambda_i \text{ when } t \to +\infty\}$. Then N is a disjoint union of the $W^s(\Lambda_i)$. As above (in the case of Ω), because each Λ_i has a local product structure, we have: $W^s(\Lambda_i) = \cup\{W^s(x)/x \in \Lambda_i\}$. We define similarly $W^s_\eta(\Lambda_i) = \cup\{W^s_\eta(x)/x \in \Lambda_i\}$. Continuity properties of stable laminations and the Baire Theorem imply that there is some Λ_0 such that $W^s(\Lambda_0)$ has nonempty interior. Such a Λ_0 is called an *attractor*. We can prove that it verifies the two stronger equivalent characteristic properties:

(i) $W^s_\eta(\Lambda_0)$ is an open neighborhood of Λ_0.

(ii) Λ_0 is W^{su}-invariant: If $x \in \Lambda_0$, $W^{su}(x) \subset \Lambda_0$. This also holds for $W^u(x)$ as Λ_0 is ϕ-invariant (justifying why we can say that Λ_0 is less fractal, at least in a topological sense, than general hyperbolic sets).

CLAIM 2.2. *(N,ϕ) is a transitive (i.e. with a dense orbit) Anosov system, if and only if for some Λ_i, we have $W^s(\Lambda_i) = N$.*

Proof. The condition means that the spectral decomposition reduces to a single element $\Lambda_i = \Omega$. By definition of the spectral decomposition, this is obviously the case when (N,ϕ) is transitive. Conversely, if the spectral decomposition reduces to a single element, then Λ_i is at the same time an attractor and a repeller (defined analogously by reversing time). Thus Λ_i is W^u and W^s invariant. By hyperbolicity, Λ_i is open in N. It is closed by definition; hence $\Lambda_i = N$, because $N = W^s(\Lambda_i)$ is connected (recall that we consider only flows and that (Λ_i,ϕ) is transitive and hence Λ_i is connected). Hence (N,ϕ) is a transitive Anosov system. □

3. Invariant submanifolds of Anosov systems. Now let (M, ϕ) be an Anosov flow, $(\tilde{M}, \tilde{\phi})$ its universal cover and $\pi : \tilde{M} \to M$ the natural projection. Let $Q^s = \tilde{M}/W^s$ be the space of stable leaves of $(\tilde{M}, \tilde{\phi})$, that we assume to be Hausdorff, and $\pi^s : \tilde{M} \to Q^s$ the projection. For any x, the restriction of π^s to $W^{su}(x)$ is a local homeomorphism onto an open subset of Q^s.

We denote for a subset $X \subset \tilde{M}$ and $x \in X$, $W^s(x, X) = W^s(x) \cap X$ (with $W^s(x)$ is the stable leaf of x with respect to the flow $(\tilde{M}, \tilde{\phi})$). Similar notations hold for other stable and unstable manifolds.

ϵ-components. Let N be a C^1-invariant submanifold in M and $\tilde{N}$ be a connected component of $\pi^{-1}(N)$. Let Λ be an attractor of (N, ϕ) and $\tilde{\Lambda} = \pi^{-1}(\Lambda) \cap \tilde{N}$. It is not (a priori) locally connected. In order to choose components in $\tilde{\Lambda}$, we introduce the following notion. For a metric space (X, d) and a positive real ϵ, we say that a subset Y is ϵ-open if Y contains its ϵ-neighborhood: $d(y, Y) < \epsilon \Longrightarrow y \in Y$. In this case the complement of Y is also ϵ-open. We say that Y is ϵ-connected if it does not contain a proper ϵ-open subset. This is equivalent to saying that Y is connected by ϵ-chains: If $x, y \in Y$, then there are points of Y, $x_0 = x, x_1, , x_n = y$, with $d(x_i, x_{i+1}) < \epsilon$, for $0 \leq i < n$. Any X is decomposed into ϵ-connected components which are in ϵ-open. Take now $\tilde{\Lambda}_0$ to be an ϵ-connected component of $\tilde{\Lambda}$. It is invariant by $\tilde{\phi}$ and W^{su} because their leaves are connected and contained in $\tilde{\Lambda}$ and is an attractor of $(\tilde{N}, \tilde{\phi})$. Here, the interesting ϵ is an Anosov constant of Λ. Thus $\tilde{\Lambda}$ and even $\tilde{\Lambda}_0$ have local product structures of size (at least) ϵ.

PROPOSITION 3.1. *Suppose that for every x in $\tilde{M}$ the projection π^s: $W^u(x) \to Q^s$ is injective. Then $\pi^s(\tilde{\Lambda}_0)$ is a connected injectively immersed topological submanifold in Q^s.*

Proof. For $x \in \tilde{\Lambda}_0$, choose as in §2, a neighborhood U_x of size ϵ with a product structure. By the product structure property, we have: $\pi^s(U_x) = \pi^s(W^{su}_\eta(x) \cap \tilde{\Lambda}_0)$. Thus $\pi^s(U_x) = \pi^s(W^{su}_\eta(x, \tilde{N}))$ because $\tilde{\Lambda}_0$ is an attractor in $\tilde{N}$. The injectivity condition in the proposition implies that $V_x = \pi^s(W^{su}_\eta(x, \tilde{N}))$ is homeomorphic to $W^{su}_\eta(x, \tilde{N})$ which is homeomorphic to an open set of $\mathbf{R}^d$, for $d = \dim W^{su}(x, \tilde{N})$. Choose a covering of $\tilde{\Lambda}_0$ by the open sets U_x for x running through a countable subset S. Consider P', the disjoint union of the U_x, $x \in S$, and P its quotient by π^s. This P is the same as the quotient of the disjoint union of the U_x, by the relation $y \sim z$ iff $\pi^s(y) = \pi^s(z)$. It is also the same as the space obtained by gluing the V_x, $(x \in S)$ over their intersections.

In order to prove that P is a topological manifold we have to show that there is no "branching": for any x and x' in S, $V_x \cap V_{x'}$ is open in both V_x and $V_{x'}$ (this guarantees that the V_x are open in P). This follows from the next lemma:

LEMMA 3.2. *Assume that there are $y \in U_x$ and $y' \in U_{x'}$ with $\pi^s(y) = \pi^s(y')$. Then there are neighborhoods $A_y \subset U_x$ and $A_{y'} \subset U_{x'}$ of y and y' in $\tilde{\Lambda}_0$ such that $\pi^s(A_y) = \pi^s(A_{y'})$.*

Proof. The points y and y' are in the same W^s-manifold. Thus there are t and t' such that $\tilde{\phi}^t(y)$ and $\tilde{\phi}^{t'}(y')$ are in the same $U_{x''}$ for some $x'' \in \tilde{\Lambda}_0$. Hence $\tilde{\phi}^t(U_x) \cap U_{x''}$ and $\tilde{\phi}^t(U_{x'}) \cap U_{x''}$ are open in $U_{x''}$. Because of the product structure of $U_{x''}$, the projections of the latter open subsets are open in $V_{x''}$. Their intersection, which is nonempty since $\pi^s(y) = \pi^s(y')$, is then open in $V_{x''}$. Its inverse image $U_{yy'}$ is then open in $U_{x''}$. For the lemma, we take $A_y = \tilde{\phi}^{-t}(U_{yy'} \cap U_x)$ and $A_{y'} = \tilde{\phi}^{-t'}(U_{yy'} \cap U_{x'})$. □

Now for P a topological manifold, to see that it is connected, we observe that if $d(x, x') < \epsilon$, then $V_x \cap V_{x'}$ is not empty. Hence $\pi^s(x)$ and $\pi^s(x')$ are in the same connected component of P. We conclude by the ϵ-connectedness of $\tilde{\Lambda}_0$.

It is clear that π^s induces a topological immersion (because this is the case in each V_x) of P into Q^s, which is injective (because we have quotiented by π^s). Thus $\pi^s(\tilde{\Lambda}_0)$ is, as claimed in the proposition, a topological injectively immersed submanifold. □

Remark. In the lemma, the uniformity of the local product structure was crucial. Indeed, the example in [Zeg2] of an invariant topological manifold of the geodesic flow of a hyperbolic 3-manifold (not of quasi-Fuchsian type) was "pseudo-Anosov." The local product structure is defined in the open subset, being the complement of the finite set of singular periodic orbits. The quotient space (like P above) is a tree rather than a 1-manifold.

4. Proofs of Theorems A and B. For dynamical systems as in Theorem A the last proposition applies, as, for any x, the projection $\pi^s \colon W^{su}(x) \to Q^s$ is injective (and hence a homeomorphism onto its image). Moreover, for x in $\tilde{\Lambda}_0, \pi^s(W^{su}(x, \tilde{N}))$ is a *closed* topological submanifold of $\pi^s(W^{su}(x))$. Indeed, $W^{su}(x, \tilde{N}) = W^{su}(x) \cap \tilde{N}$ is a closed C^1 submanifold of $W^{su}(x)$.

4.1. Splitting systems. We recall that (M, ϕ) is splitting if the local product in $\tilde{M}$ is in fact global. This is equivalent to stating that for any x the projection $\pi^s \colon W^{su}(x) \to Q^s$ and the analogous projection in the space of unstable leaves is bijective. Let $x \in \tilde{\Lambda}_0$, then $Q^s = \pi^s(W^{su}(x))$. By Proposition 3.1, $\pi^s(\tilde{\Lambda}_0)$ is a connected submanifold of $\pi^s(W^{su}(x))$ containing $\pi^s(W^{su}(x, \tilde{N}))$. As $\pi^s(W^{su}(x, \tilde{N}))$. is closed in $\pi^s(W^{su}(x))$, we get: $\pi^s(\tilde{\Lambda}_0) = \pi^s(W^{su}(x, \tilde{N}))$ ($\pi^s(W^{su}(x, \tilde{N}))$ is a closed submanifold of the same dimension in $\pi^s(\tilde{\Lambda}_0)$). Hence $\pi^s(\tilde{\Lambda}_0)$ is closed in Q^s. In particular, $W^s(\tilde{\Lambda}_0, \tilde{N}) = (\pi^s)^{-1}(\pi^s(\tilde{\Lambda}_0)) \cap \tilde{N}$ is closed in $\tilde{N}$. But this is open in $\tilde{N}$, because $\tilde{\Lambda}_0$ was an attractor. Since $\tilde{N}$ is connected, we get: $W^s(\tilde{\Lambda}_0, \tilde{N}) = \tilde{N}$. Hence, in N: $W^s(\Lambda) = N$. Thus, by 2.2, (N, ϕ) is a transitive Anosov system. This completes the proof for this case.

4.2. Geodesic flows. Now (M, ϕ) is topologically equivalent to the geodesic flow of a negatively curved manifold V. Here Q^s is identified with the sphere at infinity S^∞. The unstable quotient Q^u is also identified with the same S^∞. We denote by π^s and π^u: $\tilde{M} \to S^\infty$ the corresponding projections. In the case of the geodesic flow itself, π^s and π^u associate to a geodesic its positive and negative ends at infinity.

The proof of the theorem will be achieved as above by proving that $\pi^s(\tilde{\Lambda}_0)$ is closed in Q^s. There are two cases: $d = \dim \pi^s(W^{su}(x,\tilde{N})) = \dim W^{su}(x,\tilde{N}) = 0$, or $d > 0$. We claim that if $d = 0, \tilde{\Lambda}_0$ is reduced to a single orbit of $\tilde{\phi}$ which projects to a periodic orbit of Λ. Indeed, if not, we could find two points x and y in $\tilde{\Lambda}_0$ with different $\tilde{\phi}$-orbits such that $d(x,y) < \epsilon$ (because $\tilde{\Lambda}_0$ is ϵ-connected). The bracket $[x,y]$ belongs to $\tilde{\Lambda}_0$. But $d = 0$ means that strong unstable manifolds are singletons, thus $[x,y] = y$. That is, x and y are in the same stable manifold. As periodic points are dense (Axiom A), we can choose x and y corresponding to periodic orbits. But two distinct periodic orbits cannot lie in the same stable manifold. Hence $\tilde{\Lambda}_0$ is a single orbit and $\pi^s(\tilde{\Lambda}_0)$ is a point and in particular closed. We consider now the case $d > 0$. For any x in $\tilde{M}$, we have $\pi^s(W^{su}(x)) = S^\infty - \{\pi^u(x)\}$. Thus, if $x \in \tilde{\Lambda}_0, \pi^s(W^{su}(x,\tilde{N}))$ is a closed submanifold of $S^\infty - \{\pi^u(x)\}$, homeomorphic to an Euclidean space of dimension d. Thus (because $d > 0$), $\pi^u(x)$ is an accumulation point of $\pi^s(W^{su}(x,\tilde{N}))$. Hence, for the injectively immersed submanifold $\pi^s(\tilde{\Lambda}_0)$ containing $\pi^s(W^{su}(x,\tilde{N}))$ and with the same dimension, we have two possibilities:

(i) $\pi^s(\tilde{\Lambda}_0) = \pi^s(W^{su}(x,\tilde{N}))$, or

(ii) $\pi^s(\tilde{\Lambda}_0) = \pi^s(W^{su}(x,\tilde{N})) \cup \{\pi^u(x)\}$

In the last case $\pi^s(\tilde{\Lambda}_0)$ will be a topological sphere (it is precisely the Alexandroff compactification of a Euclidian space of dimension d) and in particular closed in S^∞. To finish the proof of the theorem, we have to show that case (i) cannot occur. Assume that we have case (i) for some point x; then the same is true for any other point y in $\tilde{\Lambda}_0$. In fact, the two cases are distinguished by whether or not $\pi^s(\tilde{\Lambda}_0)$ (which does not depend on x) is a topological sphere. But $\pi^u(x)$ is the unique accumulation point of $\pi^s(\tilde{\Lambda}_0)$. Hence for any point y of $\tilde{\Lambda}_0$, we have $\pi^u(y) = \pi^u(x)$. That is, all points of $\tilde{\Lambda}_0$ are in the same unstable manifold. As for the case $d = 0$, we have proved that $\tilde{\Lambda}_0$ is a single orbit which contradicts: $d > 0$. Hence we are in case (ii) and $\pi^s(\tilde{\Lambda}_0)$ is a topological sphere. This completes the proof of Theorem A. □

4.3. Proof of Theorem B. Denote $L = \pi^s(\tilde{\Lambda}_0) = \pi^s(\tilde{N})$ (the last equality follows from $\tilde{\Lambda}_0 = \tilde{N}$). From the last part of the above proof we have, for any $x \in \tilde{N} : L = \pi^s(W^{su}(x,\tilde{N})) \cup \{\pi^u(x)\}$. In particular $\pi^u(x) \in L$. Hence $\pi^u(\tilde{N}) \subset L = \pi^s(\tilde{N})$. We have the inverse inclusion for the same reasons. Thus $\pi^s(\tilde{N}) = \pi^u(\tilde{N}) = L$. That is, any geodesic in $\tilde{N}$ has both endpoints in L, i.e. $\tilde{N} \subset [L]$ (with the notation of §1). To prove the inverse inclusion we use again the last equality for L. It implies that for any $a \in L$, there is $x \in \tilde{N}$ such that $a = \pi^u(x)$. But from the last proof $L - \{a\} = \pi^s(W^{su}(x,\tilde{N}))$. Thus, for $b \in L - \{a\}$, there is y

in $W^{su}(x, \tilde{N})$. with $b = \pi^s(y)$. This gives by definition of $[L], [L] \subset \tilde{N}$ and then $[L] = \tilde{N}$. Let now Γ be the group of elements of $\pi_1(V)$ that respect $\tilde{N}$. It respects $\tilde{N}$, and thus also L. Since it is closed, L contains the limit set L_Γ.

It is clear that if a geodesic in $\tilde{N}$ projects to a dense orbit in (N, ϕ), then the Γ-orbit of this geodesic is dense in $\tilde{N}$ and consequently its π^s or π^u projections in L have dense Γ-orbits. This implies $L \subset L_\Gamma$. Hence $\tilde{N} = [L_\Gamma]$, and we have Theorem B. □

5. Proof of Theorem C. Let N be a closed invariant subset of the geodesic flow of a negatively curved manifold V. By the topological invariance of stable and unstable manifolds, if (N, ϕ) is topologically equivalent to an Anosov flow, then it has a local product structure. This is in particular the case when it is topologically equivalent to the geodesic flow of a negatively curved surface. We can then apply Theorem D (§6) and deduce that (N, ϕ) is of quasi-Fuchsian type. To prove the converse, assume that (N, ϕ) is of quasi-Fuchsian type: N is homeomorphic to some quotient $[L_\Gamma]/\Gamma$ for some subgroup $\Gamma \subset \pi_1(V)$. If, in addition, N is a topological 3-manifold, $[L_\Gamma]$ must be a compact 1-manifold and hence a topological circle. Indeed, in general $[L_\Gamma]$ is canonically a fibration over $L_\Gamma \times L_\Gamma - \{diagonal\}$ with fiber $\mathbf{R}$. Thus $[L_\Gamma]$ is a topological manifold if and only if L_Γ is, and $\dim [L_\Gamma] = 2 \dim L_\Gamma - 1$.

5.1. Generalities on quasi convex cocompact groups. Let $\tilde{V}$ be the universal cover of V and Γ a discrete group of isometries of $\tilde{V}$. We say that it is quasi convex cocompact if $Q(L_\Gamma)/\Gamma$ is compact where $Q(L_\Gamma)$ is the union of geodesics with endpoints in the limit set L_Γ. This is equivalent to the compactness of $[L_\Gamma]/\Gamma \subset T^1\tilde{V}/\Gamma$ because $Q(L_\Gamma)$ is nothing but the projection in $\tilde{V}$ of $[L_\Gamma]$.

PROPOSITION 5.1. ([Bou], [Coo]) Γ *is quasi convex cocompact if and only if any orbit of* Γ *is quasi convex. That is, if* X *is a such orbit, there is a constant* C *such that for any points* x *and* y *in* X*, the segment* $[x, y]$ *joining them is at a distance less than* C *from* X*. A quasi convex cocompact group* Γ *is hyperbolic. The induced metric (from* $\tilde{V}$*) to any orbit of* Γ *is equivalent to a word metric on* Γ*. The limit set* L_Γ *is canonically identified with the hyperbolic boundary* $\partial\Gamma$*.*

The proof of this proposition uses essentially the notion of quasi-convexity. The following proposition deals with the case where L_Γ is a topological circle.

PROPOSITION 5.2. *Let* $\tilde{V}$ *be the universal covering of a compact manifold* V *of negative curvature, and* Γ *a discrete torsion-free subgroup of isometries of* $\tilde{V}$*. Suppose that* $[L_\Gamma]/\Gamma$ *is compact and that* $[L_\Gamma]$ *is a topological circle. Then* Γ *is the fundamental group of a compact surface* W *of genus greater than* 2*. Suppose that* Γ *is a subgroup of* $\pi_1(V)$*. Let* $f : W \to V$ *be a* C^1 *generic map realizing the injection between fundamental groups. Let* $S = f(W)$ *and* $\tilde{S}$ *a lifting of* S *in* $\tilde{V}$*. Then the intrinsic and extrinsic distances on* $\tilde{S}$ *are (globally) equivalent.*

Proof. We first prove that Γ is the fundamental group of a closed surface W. This is a general property for hyperbolic torsion-free groups whose boundary is a topological circle [GM]. Its proof uses weak versions of the celebrated Theorem "convergence groups are Fuchsian groups" of Gabai and Casson-Jungreis. But in our case of torsion-free discrete isometry group of the universal cover of a compact negatively curved manifold (which implies that all elements are hyperbolic), the proof reduces to some elementary cases of this Theorem (see for instance [Tuk]).

Now let $f : W \to V$ be a C^1 map realizing the injection between fundamental groups. If f has isolated singularities (i.e. where f is not an immersion), then, by considering paths on $S = f(W)$, we get an induced length structure. For generic f, i.e. with isolated Morse singularities (here we assume $\dim V > 2$, for if not all things are trivial), the resulting distance is locally equivalent to the extrinsic distance induced from V. The same is true for the lifting $\tilde{S}$. Hence, by cocompactness (of Γ on $\tilde{S}$), to prove the (global) equivalence of intrinsic and extrinsic distances on $\tilde{S}$, it is sufficient to prove that they induce equivalent distances on the Γ-orbit of some point in $\tilde{S}$. But in general, by co-compactness, the intrinsic distance induces a distance equivalent to any word metric on Γ. The same is true for the extrinsic distance by Proposition 5.1. □

5.2. Canonical semi equivalence. Suppose now that f is a C^2 immersion. It then defines a C^2 pulled back metric. Denote by ϕ_S and $\tilde{\phi}_S$ the geodesic flows on T^1S and $T^1\tilde{S}$, respectively. Let $\tilde{\mathcal{M}}$ be the set of unit vectors tangent to $\tilde{S}$, which determine lines (globally minimizing geodesics) in $\tilde{S}$. By the equivalence of intrinsic and extrinsic distances on $\tilde{S}$, these lines are K-quasi geodesics of $\tilde{V}$, for some K. They are then shadowed by (i.e. at bounded distance from) true geodesics in $\tilde{V}$. We easily see that these geodesics have endpoints in L_Γ. By considering orthogonal projection from the lines to their asymptotic geodesics, we get a semi-equivalence between the geodesic flow on $(\tilde{\mathcal{M}}, \tilde{\phi}_S)$, and $(\tilde{N}, \tilde{\phi})$. By 5.1, the semi-equivalence is surjective. After diffusion [Gro], this semi-equivalence becomes injective along orbits. If $\tilde{S}$ has no distinct asymptotic geodesics, the semi-equivalence is injective. If, moreover, $\tilde{S}$ has negative curvature, then all geodesics are lines, and we get an equivalence between $(T^1\tilde{S}, \tilde{\phi}_S)$ and $(\tilde{N}, \tilde{\phi})$. By naturality, this gives an equivalence between (T^1S, ϕ_S) and (N, ϕ).

5.3. Minimal surfaces. We may now apply the result of [SY] or [SU] on existence of incompressible minimal surfaces. It asserts, since by definition $\Gamma = \pi_1(W)$ is injected in $\pi_1(W)$, that we can choose f to be harmonic (for some hyperbolic structure on W), minimizing area among all maps with the same action on $\pi_1(W)$. This map is C^∞ (we assume that all data are C^∞) and it is moreover a branched minimal immersion [GOR]. For x regular (i.e. an immersion point), the Gauss equation states:

$$K_S(x) = K_V(T_xS) + \det(II_x),$$

where $K_S(x)$ is the sectional curvature of S at x, $K_V(T_xS)$ is the sectional curvature of the 2-plane T_xS in V, and II_x is the second fundamental tensor of S at x. This is a vector valued symmetric tensor. The minimality condition: $tr(II_x) = 0$ implies $\det(II_x) \leq 0$. It follows that, if V is locally $CAT(-a^2)$, then S itself is locally $CAT(-a^2)$ at its regular points. On the other hand, branch points are isolated conical singularities. Furthermore, their angles are integer multiples of 2π [GOR]. and hence S verifies locally $CAT(-a^2)$ (see for instance [Gro] or [Bou]).

5.4. The geodesic flow of a branched immersed surface. In order to apply the semi-equivalence above, we have to make precise the meaning of the geodesic flow of a branched immersed surface. In fact, this notion may be defined in a quite general context as in [Gro], but the $CAT(-a^2)$ case is simpler and presents no pathology. Indeed, by the $CAT(-a^2)$ property, $\tilde{S}$ is (globally) convex in the sense that two points are joined by a unique geodesic (in the sense of the induced length structure). The phase space of the geodesic flow of $\tilde{S}$ is the space $Isom(\mathbf{R}, \tilde{S})$ of (geodesic) isometric immersion of $\mathbf{R}$ into $\tilde{S}$. The geodesic flow $\tilde{\phi}_S$ is just the tautological action of $\mathbf{R}$: $\tilde{\phi}_S^t(c)(s) = c(s+t)$, for $c : \mathbf{R} \to \tilde{S}$, an element of $Isom(\mathbf{R}, \tilde{S})$. The same construction yields the geodesic flow of S. This enjoys all the classical geometric and dynamical properties, except the differentiable structure.

Remark. If $\dim V = 3$, f is actually an immersion [GOR]. If $\dim V \geq 4$, a generic map $g : W \to V$ is an immersion (because $\dim W = 2$). We believe that a suitable perturbation of f gives an immersion whose image is at least locally $CAT(-a^2 + \epsilon)$, where ϵ is arbitrary small.

5.5. Entropy estimates. To simplify notations, we shall write the proof for $\tilde{S}$ immersed. The branched case only needs some additional notations. Let $\pi : T^1\tilde{V} \to \tilde{V}$ be the projection, and $\tilde{p} : T^1\tilde{S} \to \tilde{N}$ the map which, restricted to a geodesic of $\tilde{S}$, is the orthogonal projection on its asymptotic geodesic (that is, the map which may be diffused to give the previous equivalence between geodesic flows). Let $\tilde{f} : T^1\tilde{S} \to \mathbf{R}$ be the derivative of $\tilde{p}$ along the geodesic flow $\tilde{\phi}_S$. For v in $T\tilde{S}$, we denote by $\tilde{d}(v)$ the distance from $\pi(v)$ to the geodesic of $\tilde{V}$ determined by $\tilde{p}(v)$. All these maps are Γ-invariant, and define analogous maps: p, f, d on T^1S. It is known that the orthogonal projection on a geodesic in $\tilde{V}$ has a contracting coefficient decreasing exponentially with the distance from this geodesic. More precisely, there is $c > 0$ (depending on the curvature of V) such that: $f(v) \leq \exp -cd(v)$, for any v in T^1S. For a closed geodesic g of S, denote by g^* its corresponding geodesic of V, i.e. its image by p. We then have:

$$length(g^*) \leq \int_g \exp -c\, d \leq length(g)(*).$$

Let: $G_T = \{g/length(g) \leq T\}$ and $G_T^* = \{g^*/length(g^*) \leq T\}$. The previous inequality implies: $card(G_T) \leq card(G_T^*)$. In particular, we get for the topological entropies:

$$h_{top}(N, \phi) = \lim_{T\to\infty} \log(card(G_T^*))/T \leq \lim_{T\to\infty} \log(card(G_T))/T = h_{top}(T^1S, \phi_S).$$

(see for instance [Bou] for the last equality for the topological entropy in the $CAT(-a^2)$ case). Suppose now that equality holds and denote by h the common entropy. Consider: $X_\epsilon = \{g/\int_g \exp -c\, d \leq (1-\epsilon)length(g)\}$ and denote by X'_ϵ its complementary subset in the set of closed geodesics. By $(*)$, if $g \in X_\epsilon \cap G_T, g^* \in G^*_{(1-\epsilon)T.}$ In particular

$$card(X_\epsilon \cap G_T) \leq card(G^*_{(1-\epsilon)T}).$$

Hence:

$$\lim_{T\to\infty} \log card(X_\epsilon \cap G_T)/T \leq (1-\epsilon)h(**).$$

Denote by δ_g the Dirac measure determined by the closed geodesic g. Then the Margulis probability measure μ, which maximizes the entropy, is the weak limit of the probability measures μ_T defined by: $\mu_T = \sum \delta_g / \sum length(g)$, where the sum is taken over G_T.

In fact, the Margulis measure also equals the limit of the measures μ'_T, defined as μ_T, with restriction to elements in $X'_\epsilon \cap G_T$. Indeed, by $(**)$, the contribution in $\mu_T = \sum \delta_g/\sum length(g)$, from the elements of $X_\epsilon \cap G_T$ is bounded by a quantity equivalent to: $T(\exp((1-\epsilon)hT))/\sum length(g) \sim T(\exp((1-\epsilon)hT))/\exp(hT)$, which tends to 0 when T goes to ∞.

Now by definition, if $g \in X'_\epsilon$, then: $\int_g \exp(-cd) \geq (1-\epsilon)length(g)$. Thus $\int \exp(-cd)\, d\mu'_T \geq (1-\epsilon)$, and consequently: $\int \exp(-cd)\, d\mu \geq (1-\epsilon)$. It follows, since ϵ is arbitrary, that: $\int(\exp -cd)\, d\mu \geq 1$. Since $cd > 0$ and μ is a probability, we get $\int(\exp -cd)\, d\mu = 1$. That is, μ-almost everywhere, $d = 0$. This is in fact true everywhere in T^1S because d is continuous and μ has full support. Hence any $v \in T^1S$ is at distance 0 from its asymptotic geodesic in V. This means that this last geodesic of V is in S. Thus, S is geodesic in V. This completes the proof of the theorem. □

6. Proof of Theorem E. We first make the following remarks:

(i) Subsets with local product structures are invariant by topological equivalence. The last results could then be reformulated in the context of systems topologically equivalent to geodesic flows.

(ii) We have an analogous result for splitting systems; i.e. with the same hypotheses, we can choose components in the universal cover (in fact any connected component if our set is locally connected as in Theorem F) with global product structure.

Sketch of the proof. The proof here is less synthetic than that of §3, where not only unstable manifolds were locally connected but also topological manifolds. We denote (as in §3 for the attractor Λ) by $\tilde{N}_0$ some ϵ component of $\tilde{N}$, where ϵ is an Anosov constant. To determine $\pi^s(\tilde{N}_0)$, we consider for two points x and y in $\tilde{N}_0$ the subset:

$$\mathcal{E}_{yx}(\tilde{N}_0) = \{z \in W^{su}(x, \tilde{N}_0)/W^s(z, \tilde{N}_0) \cap W^{su}(y, \tilde{N}_0) \neq \phi\}.$$

As $\tilde{N}_0$ is closed in $\tilde{M}, \mathcal{E}_{yx}(\tilde{N}_0)$. is closed in the analogue set $\mathcal{E}_{yx}(\tilde{M})$. The crucial point, equivalent to Lemma 3.2, which follows from the uniformity of the product structure, is that $\mathcal{E}_{yx}(\tilde{N}_0)$ is open in $W^{su}(x, \tilde{N}_0)$. It is then open in $\mathcal{E}_{yx}(\tilde{M}) \cap W^{su}(x, \tilde{N}_0)$. We observe that local connectedness of unstable manifolds in N implies local connectedness and connectedness of unstable manifolds in $\tilde{N}$.

For splitting systems we have $\mathcal{E}_{yx}(\tilde{M}) = W^{su}(x, \tilde{M})$. Hence by connectedness $\mathcal{E}_{yx}(\tilde{N}_0) = W^{su}(x, \tilde{N}_0)$. This proves the statement of Remark (ii) above. □

Let us now consider in more detail the geodesic flow case. We shall make use of notations introduced in §4. Observe first that as in the proof of §4, for x in $\tilde{N}_0, \pi^u(x)$ is an accumulation point of $\pi^s(W^{su}(x, \tilde{N}_0))$. Indeed, if not, $W^{su}(x, N)$ would be a compact subset of $W^{su}(x, M)$, and hence for t big enough, $W^{su}(\phi^{-t}x, N)$, would be so small, which contradicts the local product property (unless we are in the trivial case of a single closed orbit).

Denote $L^u(x) = \pi^s(W^{su}(x, \tilde{N}_0)) = \pi^s(W^u(x, \tilde{N}_0))$ and $\overline{L}^u(x)$ its closure, which is just $\pi^s(W^u(x, \tilde{N}_0)) \cup \{\pi^u(x)\} = L^u(x) \cup \{\pi^u(x)\}$, since $\pi^u(x)$ is the unique accumulation point of $L^u(x)$. Observe that $L^u(x) = L^u(y)$ if and only if x and y are in the same weak unstable leaf.

Notice that in $\tilde{M}$, we-have: $\mathcal{E}_{yx}(\tilde{M}) = W^{su}(x, \tilde{M}) - \{(\pi^s)^{-1}(\pi^u(y))\}$. This translates at infinity as:

FACT 6.1. *For x and y in $\tilde{N}_0, L^u(x) \cap L^u(y)$ is an open-closed subset of $L^u(x) - \{\pi^u(y)\}$ and $L^u(y) - \{\pi^u(x)\}$. It then follows (from the connectedness and local connectedness of $L^u(x)$ and $L^u(y)$) that if $L^u(x) \neq L^u(y)$ (that is x and y are not in the same weak unstable leaf), then we have exactly one of the following three possibilities:*

(i) *$L^u(x) \cap L^u(y)$ is empty.*

(ii) *$\pi^u(y) \in L^u(x)$, and $\pi^u(x) \in L^u(y)$, and thus both $\pi^u(x)$ and $\pi^u(y)$ are accumulation points of $L^u(x) \cap L^u(y)$.*

(iii) *$\pi^u(y) \notin L^u(x), \pi^u(x) \in L^u(y)$ and $\overline{L}^u(x) \subset L^u(y)$.* □

Definition. We say that x is *maximal* if $\overline{L}^u(x)$ is maximal, that is, not strictly contained in another set $\overline{L}^u(y)$. We say that x is *almost-maximal* if $\pi^u(y) \in L^u(x)$, whenever $L^u(x) \cap L^u(y)$ is not empty and $y \neq x$.

One can show in a standard way (for instance, by contradiction) that the mapping $x \to \overline{L}^u(x)$, is continuous in the sense of Hausdorff topology on the space of closed subsets of the sphere at infinity S^∞.

It then follows that any compact set $K \subset \tilde{N}_0$ has a maximal point x, that is, $\overline{L}^u(x)$ is not strictly contained in another $\overline{L}^u(y)$, for $y \in K$. In fact, we have:

FACT 6.2. *For any x_0, there is y_0 maximal, such that $L^u(x_0) \subset L^u(y_0)$.*

Proof. We shall in fact show that there is a compact K, such that for any y with $\overline{L}^u(y)$ strictly containing $\overline{L}^u(x_0)$, there is $x \in K$, such that $L^u(x) = L^u(y)$. Indeed, the hypothesis implies that the geodesics $]\pi^u(y), \pi^u(x_0)[$ and $]\pi^u(y), \pi^s(x_0)[$ are contained in $W^u(y, \tilde{N}_0)$. Thus the hyperbolicity (of $\tilde{V}$ or equivalently $T^1\tilde{V}$) applied to the ideal triangle joining $\pi^u(y), \pi^u(x_0)$ and $\pi^s(x_0)$ implies that some point of $]\pi^u(y), \pi^u(x_0)[\cup]\pi^u(y), \pi^s(x_0)[$ is at a distance less than δ, the constant of hyperbolicity (of $T^1\tilde{V}$) from x_0 (see [Gro]). That is $W^u(y, \tilde{N}_0)$ passes trough a compact neighborhood of x_0. □

Let $\tilde{AM}$ be the set of almost-maximal points, $\tilde{AM}^*$ its complementary in $\tilde{N}_0$, and AM, AM^* their respective projections in N.

FACT 6.3. *$\tilde{AM}^*$ is open. If AM^* is not empty then it contains periodic orbits.*

Proof. Let $x \in \tilde{AM}^*$, then there is some y, with $L^u(x) \cap L^u(y) \neq \phi$ and $\pi^u(y) \notin L^u(x)$. By continuity of $L^u, \pi^u(y) \notin L^u(x')$, for x' near x. Thus, to show that $\tilde{AM}^*$ is open it suffices to observe that the set $\{x'/L^u(x') \cap L^u(y) \neq \phi\}$ is open: If $\pi^s(x'') \in L^u(y)$, then $\pi^s(U) \subset L^u(y)$, for some neighborhood U of x''. The fact that AM^* contains periodic orbits (if it is not empty) is true for any open subset of N saturated by W^u. Let O be such a subset, then it contains an open subset of the nonwandering set $\Omega(N)$. Indeed, since N has a local product structure, any unstable leaf $W^u(x, N)$ coincides with an unstable leaf $W^u(x', N)$ of some $x' \in \Omega(N)$. Hence O intersects $\Omega(N)$ in a nontrivial open subset since it is open and W^u-invariant. Therefore O contains a periodic orbit since the set of such orbits is dense in $\Omega(N)$. □

FACT 6.4. *AM^* is empty, that is, all points of $\tilde{N}_0$ are almost-maximal. In particular for any x and y with $L^u(x) \cap L^u(y) \neq \phi$ and $\pi^u(x) \neq \pi^u(y)$, the two opposite geodesics $]\pi^u(y), \pi^u(x)[$ and $]\pi^u(x), \pi^u(y)[$ are contained in $\tilde{N}_0$.*

Proof. Let $x_0 \in \tilde{AM}^*$ be a point projecting to a periodic orbit, and y_0 as above, that is, $L^u(x_0) \subset L^u(y_0)$, and y_0 maximal, in particular $y_0 \in \tilde{AM}$. Thus the geodesic $]\pi^u(y_0), \pi^s(x_0)[$ is contained in $W^u(y_0, \tilde{N}_0)$ since $\pi^s(x_0) \in L^u(x_0) \subset L^u(y_0)$. This geodesic is positively asymptotic to that determined by x_0. Let $p : \tilde{N}_0 \to N$ be the projection and ϕ^t be the geodesic flow. Then, there are t (big enough) such that $\phi^t(p(y_0))$ belongs to any fixed neighborhood of $p(x_0)$, and in particular to AM^*. Contradiction, since $p(y_0)$ and so also $\phi^t(p(y_0))$ belong to AM. □

End of the proof. We similarly define $L^s(x) = \pi^u((W^s(x, \tilde{N}_0))$. We always have $\overline{L}^s(x) = L^s(x) \cup \{\pi^s(x)\}$, but we do not know further about L^s, because we assume no connectedness conditions on stable leaves. So we start by showing that $\tilde{N}_0$ is in fact invariant by the flip $\sigma : T^1\tilde{V} \to T^1\tilde{V}$, that associates to a vector its opposite. Indeed, let $x \in \tilde{N}_0$ and $y \in W^s(x, \tilde{N}_0)$ such that $\pi^u(y)$ is near $\pi^s(x)$. From Fact 6.4, the geodesic $]\pi^u(y), \pi^u(x)[$ is contained in $\tilde{N}_0$, since $\pi^s(x) \in L^u(x) \cap L^u(y) \neq \phi$. This converges to $]\pi^s(x), \pi^u(x)[$, i.e. the opposite to the geodesic determined by x, when $\pi^u(y)$ tends to $\pi^s(x)$.

We now show that $\overline{L}^s(x) = \overline{L}^u(x)$. It suffices for this to prove that $L^s(x) \subset L^u(x)$, and thus to apply σ to get the inverse inclusion. Let $y \in W^s(x, \tilde{N}_0)$, then $L^u(x) \cap L^u(y) \neq \phi$ since it contains $\pi^s(x)$. It then follows since x is almost-maximal that $\pi^u(y) \in L^u(x)$. Hence $L^s(x) = \pi^u((W^s(x, \tilde{N}_0))$ is contained in $\overline{L}^u(x)$.

From the equality $\overline{L}^u = \overline{L}^s$ it follows that the mapping $x \to \overline{L}^u(x)$ (or $\overline{L}^s(x)$ is constant along both stable and unstable leaves. Therefore it equals a constant L, that is, for any $x \in \tilde{N}_0 : \overline{L}^u(x) = \overline{L}^s(x) = L$. It is now straightforward to prove that $\tilde{N}_0$ is quasi-Fuchsian, with L as limit set. □

Examples. It is not obvious in general how to construct closed locally connected invariant subsets with local product structures. But without the local connectedness condition, invariant subsets which are not of quasi-Fuchsian type can be constructed as follows. Let V be a compact negatively curved Riemannian manifold. Let a and b be two elements of $\pi_1(V)$ with different axis (i.e. a and b have no equal nontrivial powers). Then, one can prove that for m and n big enough, the group Γ generated by $a' = a^m$ an $b' = b^n$ is a Schottky group [Thu]. It is in particular free, quasi convex cocompact, and L_Γ is a cantor set. Thus $[L_\Gamma]/\Gamma$ is an invariant subset of quasi-Fuchsian type in (T^1V, ϕ). Any element of Γ is a word in a' and b'. Denote by Γ^+ those words with all exponents of a' and b' positive. We view the axis of an element γ in Γ, as a geodesic $A_\gamma \subset T\tilde{N}$. Let $\tilde{N}_0$ be the Closure of $\cup\{A_\gamma, \gamma \in \Gamma^+\}$. We can prove that the projection N of $\tilde{N}_0$ in $[L_\Gamma]/\Gamma \subset T^1V$ is a closed and connected invariant subset. Moreover, (N, ϕ) has a local product structure, but is not of quasi-Fuchsian type. But N is not locally connected since $[L_\Gamma]/\Gamma$ itself is not locally connected (it is locally modelled on a product of a Cantor set by an interval). Thus Theorem D is not valid without the local connectedness assumption.

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