Kaehler manifolds with a big automorphism group

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(joint work with Serge Cantat)

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Rather, Henri Cartan could be concerned!

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From a letter of A. Weil to Henri Cartan:
... il me dit de son ton tranquille: "J'apprends l'analysis situs, je crois que je pourrai en tirer quelque chose"

He (E. Cartan) told me with his quiet tone: " I am currently learning the analysis situs (which means Topology), with hope to take benifit of it"
(published in an Astérisque volume dedicated to Cartan, 1985)

The Theorem

Warning: almost all statements are, up to a finite cover for spaces, and finite index for groups!

Theorem

Let M be a compact Kaehler manifold of dimension n.

Let Γ be a lattice in a simple Lie group G of real rank $n-1$.

Let Γ acts on M holomorphically. Then, either

1) The action extends to an action of the full Lie group G.

2) or M is birational to a complex torus.

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More precisely, M is a Kummer variety: it is obtained form an abelian orbifold A/F by blow ups and resolution of singularities.

Introduction: Meeting of two worlds

Introduction 1: Automorphism groups of Kähler manifolds

The discrete factor of the automorphism group

Let M be a complex manifold.

$Aut(M)$ the group of holomorphic diffeomorphisms of M

– If M is compact, then $Aut(M)$ is a complex Lie group (of finite dimension).

– The Lie algebra of the identity component $Aut^0(M)$ is the space of holomorphic fields on M.

(Bochner-Montgomery...?)

A complex structure is not a rigid geometric structure!

The Lie group property (for compact spaces) follows from "ellipticity"

(Ref: Kobayashi's book, transformation groups)

the space of holomorphic sections of a holomorphic bundle over a compact complex manifold, has finite dimension (Cauchy formula):

L^∞ boundeness $\Longrightarrow$ C^1 boundeness

Automorphism groups in the Kähler case

If M is Kaehler, the dynamics of $Aut^0(M)$ (the connected component of 1) is poor (in particular non-chaotic)

Explanations:

- Elements of $Aut^0(M)$ have vanishing topological entropy.

– In the projective case, i.e. $M \subset \mathbb{C}P^N$,

$Aut^0(M) = \{g \in PGL_{N+1}(\mathbb{C}),\ gM = M\}$,

i.e. automorphisms of M are restrictions of global linear automorphisms of $\mathbb{C}P^N$

(actually we need to consider a bigger N)

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$\rightarrow$ it is more important to consider the discrete factor $\Gamma_M = Aut^{\#}(M) = Aut(M)/Aut^0(M)$.

• Which discrete groups are equal to $\Gamma_M = Aut^{\#}(M)$ for some M?

• For fixed dimension n, find M for which $Aut^{\#}(M)$ is as big as possible?

What is and Why Kähler?

Definition: (M, J) is Kähler, if there exists a Hermitian metric g, such that $\omega(u, v) = g(u, Jv)$ is a closed 2-form.

• Kähler: compatibility of complex geometry and Riemannian geometry:

– "holomorphic $\sim$ harmonic"

– Any complex submanifold is minimal (in the sense of Riemannian geometry)

• Kähler: it gives a natural, almost optimal condition for holomorphic embedding in $\mathbb{C}P^n$

Non-Kähler examples

$M = G/\Gamma$, homogeneous with Γ discrete, G complex

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$M = G/\Gamma$, homogeneous with Γ discrete, G complex

M admits a Kähler metric $\iff$ G is abelian!

Kähler dynamics

Two facts:

• The action of $\mathrm{Aut}^0(M)$ looks like an algebraic action of an algebraic group on an algebraic manifold

• The action of Γ_M on the cohomology is (virtually) faithful.

Γ acts on $H^*(M, \mathbb{C})$,

$W = H^{1,1}(M, \mathbb{R}) \subset H^2(M, \mathbb{R})$

$\rho : \Gamma \to \mathrm{GL}\,(W)$

– $Aut(M)$, in fact $Aut^{\#}(M) = Aut(M)/Aut^0(M)$, acts on W.

Fundamental Kaehler Fact: The action of $Aut^{\#}(M)$ is virtually faithful: its kernel is finite $\iff$ if an automorphism acts trivially on W, then a power of it belongs a flow.

(authors: Liebermann, Fujiki...)

Lieberman-Fujiki

(M, ω)

$\mathrm{Aut}_{[\omega]}(M) = \{f \in \mathrm{Aut}(M)$ such that $f^*\omega$ is cohomologeous to ω $\}$

ω Kaehler form

Fact: $Aut_{[\omega]}(M)$ has a finite number of connected components

Idea of proof:

– $\mathrm{Graph}(f^n) \subset M \times M$ have a bounded volume:

$Graph(f) = \{(x, f(x))\ x \in M\}$

$\omega^n = \omega \wedge \ldots \wedge \omega$ (n-times)

$$\int_{Graph(f)} \omega^n = \int_{Graph(Identity)} \omega^n$$

Kähler character: The Riemannian volume of any complex submanifold Y of dimension d equals $\int_Y \omega^d$

In particular, complex submanifolds are minimal submanifolds in the sense of Riemannian geometry

Chow or Hilbert scheme: $\mathcal{C}_v$ the space of complex analytic sets of a bounded volume v = it is a (singular) complex space:

Example: for $\mathbb{C}P^n$: bounded volume $\iff$ bounded degree
$\mathcal{C}_v$ has a finite number of connected components. (one basic property of algebraic sets)

Comparison with the affine torus

$M = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$

$Aff(M) = \mathrm{SL}_n(\mathbb{Z}) \ltimes \mathbb{T}^n$

$Aff^0(M) = \mathbb{T}^n$ acts with Zen dynamics

Elements of $Aff^{\#}(M) = \mathrm{SL}_n(\mathbb{Z})$ act with a violent (generally chaotic) dynamics,

• Affine transformations minimize topological entropy in their homotopy classes and are optimal mechanical model in this class

• all this thoughts extend to Kähler structures, without being a rigid geometric structure!!!

Introduction 2: Actions of arithmetic groups, Zimmer program

Margulis super-rigidity

G a semi-simple (real) group (e.g. $G = \mathrm{SL}_n(\mathbb{R})$...)

$\Gamma \subset G$ a lattice: G/Γ has finite volume, e.g. co-compact.

Example $\mathrm{SL}_n(\mathbb{Z})$ is non co-compact lattice of $\mathrm{SL}_n(\mathbb{R})$.

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The world of (simple Lie) groups:

$\mathcal{F} = \{O(n,1); SU(n,1); \}$

$\mathcal{R} = \{$the others, e.g., $Sp(n,1), SL_n(\mathbb{R}), SO(p,q), p,q > 1...\}$

A Γ a lattice of G, and $G \in \mathcal{R}$, is **super-rigid**: any $h : \Gamma \to \mathrm{GL}_N(\mathbb{R})$ extends to a homomorphism $G \to \mathrm{GL}_N(\mathbb{R})$, unless it is bounded...

The authors: Margulis if $\mathrm{rk}_{\mathbb{R}} G \geq 2$ (e.g. $\mathrm{SL}_n(\mathbb{R})$, $n \geq 3$...)
Gromov-Shoen: for the rk-one: $Sp(n, 1)$ and the isometry group of the hyperbolic Cayley plane.

Zimmer program

– Super-rigidity solves linear representation theory of Γ

– Zimmer program, a tentative to understand "non-linear representations", i.e. $\Gamma \to \mathrm{Diff}(M)$, where M is compact, i.e. differentiable actions of Γ.

Question

Let Γ be a lattice in a simple Lie group G of real rank ≥ 2.

– Find the minimal dimension d_Γ of compact manifolds on which Γ acts, but not via a finite group.

- Describe all actions at this dimension.

Remark

Zimmer proves a " super-rigidity of cocycles".
– In general, one deals (in the question above) with volume preserving actions.

Example: $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ (and congruence groups)
– The minimal linear representation is the standard one in $\mathrm{SL}_n(\mathbb{R})$, or its dual.
– Γ acts on the (real) torus $M = \mathbb{R}^n/\mathbb{Z}^n$.

Rigidity question (variant): prove that all smooth actions of Γ on the torus are smoothly conjugate to the standard one. (Authors: Zimmer, Margulis, Katok, Spatzier, Hurder, Lewis, Kanai...)

Strategy

Fix a kind of geometric structure, and restrict himself to actions preserving such a structure.

Our theorem: solves the question in the holomorphic Kaehler case.

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Conjecture

(???) If Γ a lattice in a higher semi-simple Lie group acts on a compact Kaehler manifold, and a Zariski generic point has a Zariski dense orbit, then M is birationnal to a torus?

Remark: another connection (Kähler and arithmetics): mapping class group

$Teic(M)$ space of complex structures up to (smooth) isotopy
$Mod(M) = \mathrm{Diff}(M)/Diff^0(M)$ acts on $Teic(M)$

Sullivan: $Mod(M)$ is an arithmetic group...
$Aut^{\#}(M, c) \cong$ stabilzer of $c \in Teic(M)$ in $Mod(M)$.

Special points: those with a big stabilizer.

More details about the statement

Complex tori: some arithmetics

Space of tori

Torus $X = X_\Lambda = \mathbb{C}^n/\Lambda$

Λ a lattice in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ (there is no way to define Λ a complex lattice)

Space of Lattices: $\mathcal{L} = \mathrm{SL}_{2n}(\mathbb{R})/\mathrm{SL}_{2n}(\mathbb{Z})$

$G = \mathrm{SL}_n(\mathbb{C})$ acts on $\mathcal{L}$.

($\mathrm{SL}_{2n}(\mathbb{R})$ acts transitively on $\mathcal{L}$ but not $\mathrm{SL}_n(\mathbb{C})$)

$Aut^{\#}(X_\Lambda) =$ stabilizer of Λ in $G = \mathrm{SL}_n(\mathbb{C}) =$

$$\Gamma_\Lambda = \mathrm{Aut}(X_\Lambda)/\mathrm{Aut}^0(X_\Lambda) = \{g \in \mathrm{SL}_n(\mathbb{C}),\ g\Lambda = \Lambda\}$$

Remark (dual point of view): the Teichmuller space is $\mathrm{SL}_{2n}(\mathbb{R})/\mathrm{SL}_{n}(\mathbb{C})$, endowed with the action of the modular group $\mathrm{SL}_{2n}(\mathbb{Z})$.

Generically: $\Gamma_\Lambda = \{1\}$

Our case: classify Λ such that: Γ_Λ is isomorphic to a lattice in a semi-simple Lie group of rang $n-1$

$\iff$ its Zariski closure $G \subset \mathrm{SL}_n(\mathbb{C})$ has real rank $= n-1$

$\iff$ $G = \mathrm{SL}_n(\mathbb{C})$, or G conjugate to the standard copy $SL_n(\mathbb{R}) \subset \mathrm{SL}_n(\mathbb{C})$.

Proposition

Let G be the Zariski closure of Γ_Λ

1) If $G = \mathrm{SL}_n(\mathbb{C})$, then $\Lambda = R^n$, $R = \mathbb{Z} + \sqrt{-d}\mathbb{Z}$, and $\Gamma = \mathrm{SL}_n(\mathbb{Z} + \sqrt{-d}\mathbb{Z})$.

2) If $G = \mathrm{SL}_n(\mathbb{R})$, then, either

2.1) $\Lambda = \mathbb{Z}^n + \delta\mathbb{Z}^n = (\mathbb{Z} + \delta\mathbb{Z})^n$, and $\Gamma \cong \mathrm{SL}_n(\mathbb{Z})$, or

2.2) $n = 2d$, $\Lambda = R^n$, where R is lattice in $\mathbb{R}^4 = \mathbb{C}^2$, $R = H_{a,b}(\mathbb{Z})$ the ring of integer points of a quaternion algebra over $\mathbb{Q}$.

$$\Gamma = \mathrm{SL}_n(H_{a,b}(\mathbb{Z})) \subset \mathrm{SL}_n(H_{a,b}(\mathbb{R})) = \mathrm{SL}_n(Mat_2(\mathbb{R})) = \mathrm{SL}_{2n}(\mathbb{R}).$$

In the first two case: $X = Y^n$, Y an an elliptic curve,

In the last case: $X = Z^n$, $Z = \mathbb{C}^2/H_{a,b}(\mathbb{Z})$

Z is an abelian surface.

More details

$\mathbf{H}_{a,b}(\mathbb{Q}))$ quaternion algebra over $\mathbb{Q}$ defined by its basis $(1, \mathrm{i}, \mathrm{j}, \mathrm{k})$, with

$$\mathrm{i}^2 = a,\ \mathrm{j}^2 = b,\ \mathrm{ij} = \mathrm{k} = -\mathrm{ji},\ \ (a, b > 0)$$

It embeds into $Mat_2(\mathbb{Q}(\sqrt{a}))$ by mapping i and j to the matrices

$$\begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}.$$

$\mathbf{H}_{a,b}(\mathbb{R}) = \mathbf{H}_{a,b} \otimes_{\mathbb{Q}} \mathbb{R}$

The matrix associated to $q = x + y\mathrm{i} + z\mathrm{j} + t\mathrm{k}$ has determinent $\mathrm{Nrd}(q) = x^2 - ay^2 - bz^2 + abt^2$.

$H_{a,b}(\mathbb{Z})$ is the set of norm 1 and points of integer coordinates

Case $a = b = -1$, we get $\mathrm{SL}_2(\mathbb{Z})$ a lattice (in the n on-linear sense) of $\mathrm{SL}_2(\mathbb{R})$

General case, $H_{a,b}(\mathbb{Z})$ is big in $\mathrm{SL}_2(\mathbb{R})$, it is a lattice, usually co-compact
(first cases of Harish-Chandra-Borel Theorem)t

Complex multiplication

Quoting Hilbert:

Complex multiplication is not only the most beautiful theory in mathematics, but in all sciences!

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Surprise: not our human multiplication,

Book by Serge Lang: complex multiplication,...

$X = \mathbb{C}^n/\Lambda$

$End(X)$ complex endomorphsim ring of X

$End(X) \supset \mathbb{Z}$

- If $\dim X = 1$, then

$End(X) = \mathbb{Z}$, or

$\mathbb{Z} + \sqrt{-d}\mathbb{Z}$, in which case: Λ is a sub-ring of $\mathbb{C}$

In the last case: X is of CM type,

Higher dimension ...

Abelian orbifolds with a Lattice action

$Y = X/F$

F abelian finite generated by a rotation $\overrightarrow{z} \rightarrow \eta \overrightarrow{z}$

η a root of unity,

$\eta^k = 1$,

$k = 1, 2, 3, 4, 6$

This is Calabi-Yau if dim $Y = k$ (in particular M is simply connected)

Actions of Lie groups

Actions of Lie groups

Let M be a connected compact complex manifold of dimension $n \geq 3$. Let H be an almost simple complex Lie group with $\mathrm{rk}_{\mathbb{C}}(H) = n - 1$.
If there exists an injective morphism $H \to \mathrm{Aut}(M)^0$, then M is one of the following:

(1) a projective bundle $\mathbb{P}(E)$ for some rank 2 vector bundle E over $\mathbb{P}^{n-1}(\mathbb{C})$, and then H is isogenous to $\mathrm{PGL}_n(\mathbb{C})$;

(2) a principal torus bundle over $\mathbb{P}^{n-1}(\mathbb{C})$, and H is isogenous to $\mathrm{PGL}_n(\mathbb{C})$;

(3) a product of $\mathbb{P}^{n-1}(\mathbb{C})$ with a curve B of genus $g(B) \geq 2$, and then H is isogenous to $\mathrm{PGL}_n(\mathbb{C})$;

(4) the projective space $\mathbb{P}^n(\mathbb{C})$, and H is isogenous to $\mathrm{PGL}_n(\mathbb{C})$ or to $PSO_5(\mathbb{C})$ when $n = 3$;

(5) a smooth quadric of dimension 3 or 4 and H is isogenous to $\mathrm{SO}_5(\mathbb{C})$ or to $\mathrm{SO}_6(\mathbb{C})$ respectively.

Cohomological Actions: a major ingredient

The cohomological automorphism group

A diffeomorphism f of M acts on its cohomology

$H^*(M, \mathbb{R}) = \Sigma H^i(M, \mathbb{R})$

f_* preserves the cup products $H^i \times H^j \to H^{i+j}$

and PoincarŐ duality: $H^i \cong (H^{n-i})^*$

if M Kähler, $H^{p,q}$ Hodge decomposition

$\text{CoAut}(M)$ = the Linear subgroup of $\text{GL}\,(H^(M, \mathbb{R})$ preserving

Hodge decomposition, the cup product and Poincaré duality,

This is an algebraic group,

Why it is non-trivial?

For instance, $W = H^{1,1}(M, R) \neq 0$

General question: understand $\text{CoAut}(M)$?

Using Super-rigidity

$\mathrm{Aut}(M) \to \mathrm{CoAut}(M)$

If $\Gamma \subset \mathrm{Aut}(M)$, then

$\rho : \Gamma \to \mathrm{CoAut}(M)$

If $G \subset \Gamma$ is a lattice in G, e.g. $G \cong \mathrm{SL}_n(\mathbb{R})$, $n \geq 3$,

then ρ extends to G

Example $\mathrm{SL}_n(\mathbb{Z})$ acts on $\mathbb{T}^n$, and $\mathrm{SL}_n(\mathbb{R})$ acts on $H^*(\mathbb{T}^n, \mathbb{R})$,

This appears as a Zariski closure!

Question: How $\mathrm{CoAut}(M)$, where $\dim M = n$ contains $\mathrm{SL}_m(\mathbb{R})$, $m \geq n$?

Is this impossible for $m > n$

Surface case,

In dim $= 2$, the cup product is a quadratic form: $b : W \times W \to \mathbb{R}$.

Hodge index theorem (Noether theorem): b has (anti-) Lorentz signature $+ - \ldots -$ (or $+$).

Fact If M is a surface (Kähler or not)

A semi-simple Lie group (with no compact factor) can be embedded in $O(1, N)$ iff it is (locally) isomorphic to some $SO(1, m)$, $m \leq N$

In particular,

CoAut(M) is either $\cong SO(1, m)$,

or contained in a parabolic group P of $O(1, N)$,

$P = \mathrm{SO}\,(n-1) \ltimes \mathbb{R}^n$

If Γ a lattice in G of higher rank, acts,
then G embeds in $O(1, N)$,
impossible!

Higher dimension

$c : W \times \ldots W \to W \to \mathbb{R}$

- Is there a kind of Nother theorem for c?

– Can the "signature" be bounded by means of the dimension?

Case: dimension $= 3$,

– "Trilinear forms are a challenge for mathematics" !!!

(The quotient space under the $\mathrm{GL}_3(\mathbb{R})$-action is infinite...)

Hodge index theorem, Hodge-Riemann bilinear relations

$q_\omega : \alpha \in H^{1,1} \to \int \alpha \wedge \alpha \wedge \omega^{n-1}$

If ω is a Kähler, then q_ω is negative definite on the primitive space

$[\omega]^\perp = \{\alpha \in W, \alpha \wedge \omega \wedge \omega = 0\}$

Dimension 3

Fact

(Lorentz-like property)

$b : W \times W \to W^*$ *satisfies, if* $E \subset W$ *is isotropic for* c*, then* $\dim E \leq 1$*.*

This allows one to classify ρ assuming $\mathrm{rk}_{\mathbb{R}}(G) \geq 2$.

(for instance, G can not contain $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$...)

– Proof of the Fact: If $\dim E \neq 0$, then, $E \cap [\omega]^\perp \neq 0$, and $q(a, b) = \omega \wedge a \wedge b$ negative definite.

Question: classify the orthogonal group of a vectorial bilinear (or equivalently a trilinear form) satisfying the Lorentz-like property?

Representation of $SL_2(\mathbb{R})$

Let $H = S \ltimes R$ be the orthogonal group of a Lorentz-like trilinear form,
S semi-simple, thus contains locally $\mathrm{SL}_2(\mathbb{R})$

Bounds of the $\mathrm{SL}_2(\mathbb{R})$ in S

R_k representation in

P_k= Polynômes homogènes de degré $k = \{p = \Sigma x^i y^{k-i}\}$

A diagonal matrix (λ, λ^{-1})

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

A_k action on P_k

Weights (eigenvalues) $\lambda^k, \lambda^{k-2}, \lambda^{2-k}, \lambda^k,$

Eigen-vectors: $e_k, \ldots e_{-k}$

- The wedges $e_r \wedge e_s$ are eigenvalues (or 0)

$A_k(e_r \wedge e_s) = \lambda^{r+s}(e_r \wedge e_s) = A_k^*(e_r \wedge e_s) = \lambda^l(e_r \wedge e_s)$

So:

– either $e_r \wedge e_s = 0$

– or $r + s$ is a weight of the dual $R_k^* \cong R_k$ and thus $-k \leq r + s \leq k$

- Necessarily $e_k \wedge e_k = 0$ (since $2k > k$)

By the Lorentz-like property, we can not have

$e_k \wedge e_{k-2} = 0$, and $e_{k-2} \wedge e_{k-2} = 0$

Hence $2(k-2) \leq k$, i.e. $k \leq 4$.

Conclusion

– In dimension 3, by representation theory, we get information on the cohomology

– Essentially, $H^{1,1}(M, \mathbb{R}) = H^{1,1}(\mathbb{T}^3, \mathbb{R}) \oplus T$, where the representation on T is trivial,

– Other structures on the cohomology are needed, e.g. The Kähler cone....

– All this is a crucial step in the proof...

In dimension > 3, other approach...

Non-algebraic structures on the cohomology: The Kaehler cone

W is a ordered linear space:

$\mathcal{K} \subset W$ the space of $\alpha \in W$ having a reprensentative $\omega \in [\alpha]$, which is Kaehler, i.e. $g(u, v) = \omega(u, Jv)$ is positive definite. (so $h = g + \omega$ is a hermitian metric)
$\mathcal{K}$ is a convex non-degenerate cone with a non-empty interior.

The nef cone is the closure $\overline{\mathcal{K}}$

Preserved sub-cones

Proposition

Let Γ be a lattice in a semi-simple group G, and $\rho : G \to \mathrm{GL}(W)$. Assume $\rho(\Gamma)$ preserves a non-degenerate cone $\mathcal{K}$. Then $\rho(G)$ preserves a non-degenerate cone $\mathcal{K}'$.

Remarks:

1) The cone is unique if ρ is irreducible.

2) This is not true if Γ is merely a Zariski dense subgroup.

Algebraic geometry

What is the torus

M a Kähler manifold,

M is covered by a torus $\iff$

$c_1(M) = c_2(M) = 0$

$c_1(M) \in W = H^{1,1}$,
$c_2(M) \in H^{2,2} = W^*$
c_1, c_2 are invariant by $\mathrm{Aut}(M)$
$c_1 \in T$

How to kill the fixed space T of the cohomological representation $\Gamma \to \mathrm{GL}\,(W)$?

This is possible if c_1 corresponds to:

1) an effective (i.e positive) divisor,

2) this divisor is rigid $\mathbb{C}P^2$...