Around Zimmer program: rigidity of actions of higher rank lattices

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1. Introduction
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   - Zimmer program
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   - Local rigidity on boundaries
   - With a rigid geometric structure
   - Holomorphic case
   - On the torus
4. SL $_n(\mathbb{Z})$

Warning : almost all statements are, up to a finite cover for spaces, and finite index for groups !

From representations to actions

(see two survey articles by D. Fisher :

– Local rigidity of group actions : past, present, futur

– Groups acting on Manifolds : around the Zimmer program

– ....)

Margulis super-rigidity

G a semi-simple (real) group (e.g. $G = \mathrm{SL}_n(\mathbb{R})$...)

$\Gamma \subset G$ a lattice : G/Γ has finite volume, e.g. co-compact.

Example $\mathrm{SL}_n(\mathbb{Z})$ is non co-compact lattice of $\mathrm{SL}_n(\mathbb{R})$.

Congruence groups : a subgroup of $\mathrm{SL}_n(\mathbb{Z})$ that contains the kernel of the projection : $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/p\mathbb{Z}) \to 1$

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The world of (simple Lie) groups :

$\mathcal{F} = \{O(n,1); SU(n,1); \}$

$\mathcal{R} = \{$the others, e.g., $Sp(n,1), SL_n(\mathbb{R}), SO(p,q), p,q > 1...\}$

A a lattice Γ of G, and $G \in \mathcal{R}$, is **super-rigid** : any $h : \Gamma \to \mathrm{GL}_N(\mathbb{R})$ extends to a homomorphism $G \to \mathrm{GL}_N(\mathbb{R})$, unless it is bounded...

The authors : Margulis if $\mathrm{rk}_{\mathbb{R}} G \geq 2$ (e.g. $\mathrm{SL}_n(\mathbb{R})$, $n \geq 3$...)
Gromov-Shoen : for the rk-one : $Sp(n, 1)$ and the isometry group of the hyperbolic Cayley plane.

Remarks : other partitions

$\mathcal{H}$ = hyperbolic groups = rank 1

$\mathcal{H}^*$ non-hyperbolic ...

$\mathcal{H}$ is partioned into non-Khazdan $NT\mathcal{H}$ and Khazdan groups $T\mathcal{H}$

$NT\mathcal{H}$ = Non-Kaehler groups $NKNT\mathcal{H}$ and Kaehler $KNT\mathcal{H}$

$NKNT\mathcal{H} = \{SO(1,n), n > 2\}$

$KNT\mathcal{H} = \{\text{SL}_2(\mathbb{R})$, the others $\}$, i.e. $SU(1,n)$, $n > 1$

Remark : Similar statements in the semi-simple case...

Zimmer program

– Super-rigidity solves linear representation theory of Γ

– Zimmer program, a tentative to understand "non-linear representations", i.e. $\Gamma \to \mathrm{Diff}(M)$, where M is compact, i.e. differentiable actions of Γ.

Non-linear superrigidity : $\mathrm{Diff}(M)$ is a non-linear group, i.e. not a subgoup of $\mathrm{GL}_N(\mathbb{R})$, yet it is simple...
(It does not mean extension of actions to the ambient Lie group)

A typical question

Question (Minimal dimension)

Let Γ be a lattice in a simple Lie group G of real rank ≥ 2.
– Find the minimal dimension d_Γ of compact manifolds on which Γ acts, but not via a finite group.
– Same question assuming the action measure preserving ?
– Describe all actions at this dimension.

Example : $\Gamma = \mathrm{SL}_n(\mathbb{Z})$

The minimal dimension of representations in that of minimal dimension of $\mathrm{SL}_n(\mathbb{R})$ and equals n.

Basic examples

1. Affine (automorphic) action on the torus

$\Gamma = \mathrm{SL}_n(\mathbb{Z})$ acts on the torus $\mathbb{R}^n/\mathbb{Z}^n$

If $A \in \mathrm{GL}_n(\mathbb{Z})$, then it acts as an automorphism of $\mathbb{R}^n$ preserving $\mathbb{Z}^n$, hence acts as automorphism of $\mathbb{T}^n$

2. Projective action on the sphere :

$\mathrm{SL}_{n+1}(\mathbb{R})$ linearily acts on $\mathbb{R}^{n+1} - 0$, and hence on $\mathbb{R}P^n = (\mathbb{R}^{n+1} - 0)/\mathbb{R}^*$, as well as on the sphere $\mathbb{R}^{n+1} - 0/\mathbb{R}^+$,

If $A \in \mathrm{SL}_{n+1}$, it acts on the sphere by $A.x = \frac{Ax}{\|A(x)\|}$

3. Affine actions on a homogeneous space : $M = H/\Lambda$, Λ a co-compact lattice,

H acts by left translation of M : $h.x\Lambda = (hx)\Lambda$

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The so generated group is $Aff(H/\Lambda)$.
It contains H

If fact, this is the symmetry group of the canonical connection on H/Λ, which goes down from H since Λ is discrete.

• Affine action of $\Gamma \iff$ a homomorphism $\Gamma \to Aff(H/\Lambda)$

e.g. a homomorphism defined on G in $H \subset Aff(H/\Lambda)$

4. Isometric actions : $\rho : \Gamma \in K$ a compact Lie group actions on M.

Example : K a finite group, they exist since Γ is residually finite

e.g. SL $_n(\mathbb{Z}) \rightarrow$ SL $_n(\mathbb{Z}/p\mathbb{Z}) \rightarrow 1$

5. Quasi-affine actions : the isometric actions are dynamically trivial, but can be combined with affine actions $\rightarrow$ **quasi affine**...

Unimodular vs non-unimodular

• As usual, there are differences between measure preserving and non-preserving cases,

In many rigidity contexts, one works with a volume preserving hypothesis

• Example : SL $_n(\mathbb{R})$ acts on $\mathbb{R}^n$, and so on $M = \mathbb{R}^n/(x \to \alpha x)$

$\alpha > 1$ (e.g. $\alpha = 2$),

$M \cong S^1 \times S^{n-1}$, It preserves no-measure

e.g. $n = 2$, M a torus,

One-parameter groups generate Reeb flows, with poor dynamics,

Suspension

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G acts on M : $g.(g', x) = (gg', x)$

$M \to G/L$ is a bundle with fiber N

If N compact, and G/L compact, then M is compact,

A non-unimodular example

Take L parabolic,

- let it acts on N via a homomorphism $L \to \mathbb{R}$,

- $\mathbb{R}$ acts as a one parameter group of diffeomorphisms ϕ^t

– The dynamics of G is equivalent to that of ϕ^t....

Zimmer dream !

- Actions of Γ (a lattice in a higher rank simple group) are classifiable...

— They are all of algebraic nature

— They are all quasi-affine : up to a compact noise, they are affine !

—– In particular, the unique volume preserving action of a lattice Γ in SL $_n(\mathbb{R})$ on a compact manifold of dimension n, is that of finite index subgroups **on the n-torus ?**

——- In particular the unique volume preserving action of a lattice in SL $_n(\mathbb{R})$ on $\mathbb{T}^n$ is the usual one.

Gromov-Zimmer vague conjecture

It seems that Gromov invented his theory because he meet difficulties in undestanding Zimmer !

Conjecture (Old conjecture)

Whenever a lattice Γ acts on a compact smooth manifolds, it preserves some rigid geometric structure !

Conjecture (Gromov Vague conjecture)

Compact manifolds having a rigid geometric structure with a **non-compact** symmetry group, are special, and classifiable !

Rigid geometric structures

(rigid geometric structures $\neq$ (global) rigidity of actions !)

Rigidity vs flexibility of geometric structures ?

Rigid (Solid) : Riemannian metric — Pseudo-Riemannian metric — Connection — Conformal structure on dimension > 2....

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• Rigid $\cong$ the pseudo-group of local isometries is a LIE GROUP (of finite dimension).

More explanations on rigid geometric structures

Rigid : $\cong$ give rise naturally to Generalized geodesics

– there is a kind of naturally associated connection,

– there is a naturally associated family of curve, generalized like geodesics,

Rigid at order k $\rightarrow$ a differential equation of order $k+1$

Example : a connection (order 1)

$\ddot{x}^i = \Sigma \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k$

($x = (x^1, \dots, x^n)$ a coordinate system)

More formally : $P(M)$ the GL $_n(\mathbb{R})$-principal bundle of frames of M

$P^k(M) = P(P^{k-1}(M)$

Parallelism (or framing) on a manifold N : $x \to r(x)$ a base of T_xN

Rigidity at order $k \cong$ there is a naturally associated parallelism of $P^k(M)$

Example : a connection on $M \to$ tautological parallelism of $P(M)$

Rigidity

$\mathrm{Iso}_x^{\mathrm{Loc}} = \{$f local isometry defined around x such that $f(x) = x\}$

$\Phi_x^k : f \in \mathrm{Iso}_x^{\mathrm{Loc}} \to \mathrm{jet}_x^k(f) \in \ldots$ (some complicated algebraic space)

"Local rigidity at order k" : Φ_x^k is injective

Example : A Riemannian isometry f : $f(x) = x$, and $D_x f = 1$, then $f = id$.

Infinitesimal rigidity

True definition, stronger than the local rigidity :
$\Psi_x^k : \mathrm{Iso}_x^{k+1} \longrightarrow \mathrm{Iso}_x^k$ is injective.

• Definition : f is an isometry up to order j, if $f^*g - g$ vanishes up to order j at x (The Taylor development vanishes up to order j, in some, and hence any, chart)

Supperrigidity of cocycles

What Zimmer did proved ?

Supperrigidity of cocycles

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Super-rigidity of cocycles

This implies the “conjecture” but at a measurable level !

Zimmer's hope : “measurable $\Longrightarrow$ smooth”

Remark

Margulis super-rigidity theorem for Γ a lattice in G is equivalent to the regularity fact :
any measurable Γ-equivariant map between boundaries $G/P \to H/L$ is smooth !

Cocycles

The derivative cocycle

Γ acts on M.

Choose a frame field on M (this exists in the measurable category)

$x \to F(x) = (e_1(x), \ldots, e_n(x))$

$\iff$ parallelism $\iff$ trivialization $TM = M \times \mathbb{R}^n$

$c(\gamma, x) =$ the matrix of $D_x\gamma : T_x \to T_{\gamma(x)}M$ with respect to the bases $F(x)$ and $F(\gamma(x))$,

Chain rule $c(\gamma\gamma', x) = c(\gamma, \gamma'(x))c(\gamma', x)$

This is the definition of a cocycle : over the Γ-action with value in GL $_n(\mathbb{R})$.

Constant cocycle : $c(\gamma, x) = c(\gamma)$, thus $c : \Gamma \to \mathrm{GL}_n(\mathbb{R})$ homomorphism

Theorem (Zimmer's cocycle superrigidity)

There is a choice of frame field such that the cocycle becomes a product of two commuting cocycles : one constant and the other has values in a compact group.

A measurable connection

If the frame field is smooth, and the compact noise is trivial, then define a connection ∇ by decreeing that :
the e_i are parallel

$\gamma^*(e_i)$ is a constant combination of the e_j $\Longrightarrow$
Γ preserves the connection

The connection is flat, but may have torsion...

RDS

Let $c : \Gamma \times M \to L$ a cocycle and L acts on X.

Skew-product : (= RDS= Random dynamical system)

Γ acts on $M \times X$: by

$(\gamma(x, u) = (\gamma x, c(\gamma, x)u)$

The formula gives rise to an action $\iff$ c is a cocycle,

The action on $\mathcal{M} \times X$ covers the action on M

– Example $\Gamma = \mathbb{Z}$, $L = X = S^1$ acting by rotation on itself

$(x, u) \to (f(x), u + \theta(x))$

Terminology : Random rotation on the circle

• Definition of **Quasi-affine** : skew product over an affine action, with L compact.

Low regularity (is not enough) !

• Furstenberg example : a random rotation that is measurably constant but not continuously constant !

• Another situation : geodesic flows of compact negatively curved Riemmannian manifolds preserve C^0- pseudo-Riemannian metrics (even $C^{1+Zygmud}$?) :
This is Kanai's construction :

Somme precise questions and conjectures

Conjecture (Minimal dimension ?)

If Γ acts on M, then $\dim M \geq \mathrm{rk} G$.
In the equality case, the action extends to G

The conjecture is true for actions of G at least in the transitive case ?

Conjecture (Minmal dimension in the measure preserving case)

If Γ acts by preserving a measure (say if necessary it charges open sets), then $\dim M \geq \mathrm{rk} G + 1$

Conjecture (Geometrization ?)

If Γ acts topologically transitively, then it preserves somme rigid geometric structure defined on an open invariant set.

Conjecture (Paradigm)

A smooth Γ-action on the torus of dimension n (where $n-1 = \mathrm{rk} G$) is smoothly conjugate to the usual action of finite index subgroups of SL $_n(\mathbb{Z})$.

Reminiscence : Thurston geometrization conjecture

A smooth volume preserving action of Γ on M can be geometrized :

There is a closed invariant set F and $M - F = U_1 \cup \ldots U_k$

Any U_i possess a Γ-invariant geometry...

Secondary task : find the possible geometries ?

Approches

•• Local rigidity

exactly as in the theory of representation, one starts with local questions ?

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•• Strong rigidity assuming existence of an extra structure,

e.g.

– The action is Anosov

– The action already preserves a rigid geometric structure (e.g. an affine connection, a pseudo-Riemannian metric)

– The action is holomorphic

Contents

Objective : Γ finite index subgroup of SL $_n(\mathbb{Z})$

- understand its actions on $\mathbb{T}^n$?

- its actions of the sphere S^{n-1}

- Answer the geometric structure conjecture in the following case :

Proposition

Let Γ be a lattice in SL $_n(\mathbb{R})$, *$n \geq 3$ acting on M^n.*
Assume :
- The action is analytic
- The action fixes some point x_0
Then, there is an open Γ-invariant set containing x_0 on which Γ preserves a flat connection,
Further details later on...

Apply this to prove homotopic rigidity.

More on Examples

Blow-up

$\hat{\mathbb{R}^n}_0 \subset \mathbb{R}^n \times \mathbb{R}P^{n-1}$
$= \{(x,d)/\ x \in d\}$

$\pi : \hat{\mathbb{R}^n}_0 \longrightarrow \mathbb{R}^n$
π is a a diffeomorphism over $\mathbb{R}^n - 0$
$\pi^{-1}(0) = \mathbb{R}P^{n-1}$: the singular divisor

$\hat{\mathbb{R}^n}_0$ is the total space of the tautological line bundle over $\mathbb{R}P^{n-1}$
(the unique non-trivial $\mathbb{R}$-line bundle)

$n = 2$, this is a Moebuis strip

The construction can be localised
$(M, x_0) \to \hat{M}_{x_0}$

$n = 2$, topologically : remove a ball around x_0 and glue a Moebuis band with boundary (its boundary is one circle)

The diffeomorphism group of (M, x_0) acts on $\hat{M}_{x_0}$
Its action on the exceptional divisor is the projectivization of the derivative action
$f \in \mathrm{Diff}(M, x_0) \to D_{x_0} f$ acting on $\mathbb{R}P^{n-1} = T_{x_0} M - 0/\mathbb{R}^*$

Similar blow up construction for a finite set F of M

Γ finite index subgroup of SL $_n(\mathbb{Z})$

Apply this to get a Γ action on the blow-up of any periodic orbit.

Periodic orbits of SL $_n(\mathbb{Z})$ are exactly rational points of $\mathbb{R}^n/\mathbb{Z}^n$.

Local coordinates

Affine coordinates for $\mathbb{R}P^{n-1}$

$d = [y_1 : \ldots : y_n]$

$U_1 = \{d, y_1 \neq 0\}$, neighborhood of $d_1 = [1 : 0, \ldots, 0]$

$c_1 : d \in U_1 \to (\frac{y_2}{y_1}, \ldots, \frac{y_n}{y_1}) \in \mathbb{R}^n$

Chart arount $(x, d_1) \in \hat{\mathbb{R}^n}_0$

$(x, y) = (x_1, \ldots, x_n; y_1, \ldots, y_n) : x \in [y] \iff x_i y_j = x_j y_i$

Chart $C_1 : (x,[y]) \to (x_1, \frac{y_2}{y_1}, \dots, \frac{y_n}{y_1}) = (z_1, z_2, \dots z_n)$

Thus : $x_i = x_1 z_i$, for $i \geq 2$

$dx_i = x_1 dz_i + z_i dx_1$

$\omega = dx_1 \wedge \dots \wedge dx_n = z_1^{n-1} dz_1 \wedge dz_2 \wedge \dots \wedge dz_n$

$\pi^*(dx_1 \wedge \dots \wedge dx_n)$ is an n-form vanishing exactly along the exceptional divisor,

It defines the Lebesgue measure

Desingularize it by changing the differentiable structure

Similar situation : on $\mathbb{R}$, $\omega = x^p dx$,

consider $f : \mathbb{R} \to \mathbb{R}$ a smooth homeomorphism but not a diffeomorphism, such that

$df = \omega$

New coordinates $x \to f(x) = y$

ω becomes dy

Example : $\omega = x^2 dx$, $f : x \to (1/3)x^3$

This is possible in the orientable case, i.e. n odd.

The construction in this framework it due to Katok-Lewis

But well know in the complex case...

Filling in the torus

If instead of $\mathbb{R}P^{n-1}$, one considers S^{n-1}, one gets a manifold with boundary S^{n-1}

The action of Γ on the boundary S^{n-1} is the projective one.

Glue an closed n-ball B with a $\mathrm{SL}_n(\mathbb{R})$ action :
$(A, x) \in \mathrm{SL}_n(\mathbb{R}) \times B \to (\| x \| \frac{A(x)}{\|A(x)\|})$
(this is the projective action on any sphere $\{\| x \|= r\}$)

The so obtained action is a C^0 on the torus (no change of the space !)

Question : is it possible to construct a smooth action like this ?

Holomorphic situation : Kummer varieties

Orbifolds : consider $\overline{\mathbb{T}^n} = \mathbb{T}^n/(x \to -x)$

Γ acts on $\overline{\mathbb{T}^n}$

$n = 2$, $\overline{\mathbb{T}^2}$ is a (regular) topological surface,

It has an affine Euclidean structure with 4 conical singularities

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Topologically : $\overline{\mathbb{T}^2} = S^2$

Desingularize $\overline{\mathbb{T}^2}$

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Desingularize $\overline{\mathbb{T}^2}$

One comes back to $\mathbb{T}^2$!

If M is a complex torus

The desingularization of $\overline{M}$ is a Kummer surface :

It has an action of $\mathrm{SL}_2(\mathbb{Z})$

It is simply connected

It has a holomorphic volume form : a holomorphic non-singular 2-form,

It is a K3 (complex) surface

Calabi-Yau examples

Generalize the construction by performing a quotient $x \to \eta x$

η a root of unity

To keep an action of a lattice, the order of η must be : 1, 2, 3, 4 or 6.

In dimension 2, 3, 4, 6, there exists η giving rise to a Calabi-Yau example :

i.e. it has a holomorphic volume form and is simply connected !

Non-existence of rigid geometric structures

Beneveniste-Fisher :

FRom Gromov's Theory of rigid transformation group : the symmetry group of a rigid geometric structure on **simply connected compact analytic** has finite number of connected components...

The action of Γ extends to G...
This is easily seen to be impossible since one knows that Γ acts non-trivially on the cohomology (H^2)

A (biased) quick survey on results

Local rigidity in the volume preserving case : Names and synthesis

Hurder, Katok, Lewis, Spatzier, Fisher, Margulis, Quian, Zimmer....

Last result :

Theorem (Fisher-Margulis)

Quasi-affine actions of higher rank lattices are smoothly locally-rigid.

Tools

In general, before the last theorem, working in a (weakly) hyperbolic framework, e.g.

ρ_0 the standard linear action of SL $_n(\mathbb{Z})$ on $\mathbb{T}^n$

ρ C^1-near ρ_0,

Two major steps :

•• structural stability : i.e. C^0 conjugacy (between the original and perturbed actions)

– There are many γ for which $\rho(\gamma)$ is Anosov

- By structural stability : $\Phi_\gamma \circ \rho(\gamma) = \rho_0(\gamma) \circ \Phi_\gamma$

How to get from individual conjugacy to a collective one : use higher rank

•• Regularity : smoothness of the conjugacy Φ

Recalls from hyperbolicity, Anosov...

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ a diffeomorphism fixing 0,

Def : f hyperbolic $\iff$ if $D_0 f$ has no eigenvalue of modulus 1.

In this case f is topologically conjugate to $D_0 f$

More generally, f is structurally stable : any g C^1 near f is topologically conjugate to f,

Same theory for vector fields

Now, $f : M \to M$ a diffeomorphism, with no fixed point,

A linearization (and quantification of f :

$f_* : \xi(M) \to \xi(M)$, the associated action of f on the space of vector fields

Consider here bounded vector fields with the sup-norm

Similarly f_* hyperbolic if $Spectre(f_*) \cap S^1 = \emptyset$

• By definition f is **Anosov** in this case

• It turns out that Anosov is essentially equivalent to being structurally stable...

Regularity

•• Idea to that Φ is smooth :

A similar situation :

Fact

Let $C = \{f, g\}$ be a commutative subgroup of two (non-linear) contractions , topologically conjugate via topologically conjugate by Φ ; independent to a subgroup $B = \{H_a, H_b\}$.
Assume B is dense in $\mathbb{R}^+$ the group of (all) homotheties, i.e. $\ln a$ and $\ln b$ are rationally independent.
Then Φ is smooth.

Let $H_a : x \to ax$ a contracting homothety

Observations :

• Any two homotheties are topologically conjugate, but smoothly conjugate only if they are equal !

• H_a has a big centralizer in $Homeo(\mathbb{R})$, but a small one in $\mathrm{Diff}(\mathbb{R})$:

$g(ax) = ag(x)$, g smooth at 0 $\Longrightarrow$ g is a homothety

By Sternberg, we assume f smoothly conjugate to H_c, and thus $g = H_d$

The problem becomes : a conjugacy between $\{H_a, H_b\}$ and $\{H_d, H_c\}$ is a homothety ?

Remarks

- Exotic actions can be made non locally rigid

Boundaries

Boundary of $G \to$ a compact homogeneous space G/P, where P is parabolic, i.e. it contains a Borel subgroup...

Example : S^{n-1} is a boundary for $SO(1,n)$,
This is the Gromov boundary of any co-compact lattice in $SO(1,n)$

Also the projectivied frame bundles over S^{n-1}

$\mathbb{R}P^n$ and S^n are boundary of SL $_{n+1}(\mathbb{R})$

Local-rigidity action : Let Γ be a lattice in G acting smoothly by ρ on G/P, and ρ is near the left-translation action ρ_0. Is ρ smoothly conjugate to ρ ?

A weaker version taking into account the $SL_2(\mathbb{R})$-case : does ρ extends to an action of G ?

Results

— Γ co-compact lattice in G

• higher rank case :

- Kanai : a new non-hyperbolic technique for the projective action

+ conditions

- Katok-Spatzier : structural stability + regularity (all boundaries)

• Rank 1 case :

Early : Ghys, $G = \mathrm{SL}_2(\mathbb{R})$

Kanai : $SO(n, 1)$-case

.... (Yue)

— Γ non co-compact : we shall consider the case $\mathrm{SL}_n(\mathbb{Z})$

Structural stability

- In the rank one case : structural stability :

$\rho : \Gamma \to \mathrm{Diff}(S^{n-1})$, $\rho_0 : \Gamma \to SO(n,1)$

Consider the suspension $M_\rho = H^n \times S^{n-1}/(x,u)\tilde{(}\gamma x, \rho(\gamma)(u))$

– $\mathcal{F}_\rho$ the horizontal foliation,

– $M_{\rho_0} = T^1(H^n/\rho_0(\Gamma))$, the phase space of the geodesic flow,

– $M_\rho \cong M_{\rho_0}$

– The intersection of $\mathcal{F}_\rho$ with the unstable foliation of the geodesic flow $\rightarrow$ a direction field $\rightarrow$ Anosov flow, near the geodesic flow, The topological conjugacy between ρ and ρ_0 follows from structural stability.

Homotopic rigidity

The smooth conjugacy is more delicate in the rank 1 case...

- Ghys : in dimension 2,

ρ is not assumed near ρ but just in its homotopy component : it has maximal Euler number

More

– There are local rigidity results of actions of abelian groups,

– Gloabl rigidity results of Anosov actions of higher rank lattices and abelian groups,
Matsumoto : global rigidity of (split) Anosov actions of $\mathbb{Z}^{n-1}$ on n-manifolds,
– Anosov actions of Γ a lattice in SL $_n(\mathbb{R})$ on n-manifolds ?
(Feres-Labourie, Katok-Rodriguez-Hertz, Damjanovic-Katok,)

Gromov's approach

Pseudo-Riemannian metric

Affine connections

A sample result

Theorem (Zimmer-Feres-Goetze-Zeghib)

If Γ a finite index sub-group acts on a compact n-manifold preserving an unimodular affine connection, then it is the usual torus.

One proves M is flat, but how to get a standard torus $(= \mathbb{R}^n/\Lambda)$?

Affine group of affine flat manifolds ?

M affine unimodular : it has a $(\mathbb{R}^n, SAff(\mathbb{R}^n))$-structure $\iff$ torsion free flat connection with unimodular parallel transport, $Aff(M)$ transformation preserving the connection,

Question

- When does $Aff(M)$ contain a higher rank lattice ?
- When does $Aff(M)$ contain an Anosov element ?

(Benoist-Labourie work treated the case where an additive symplectic structure is preserved)...

Example : surfaces with a finite set removed may have an affine group with is a lattice in $\mathrm{SL}_2(\mathbb{R})$ (exactly as $\mathbb{T}^2$),

The Zimmer dimension conjecture in the Kähler case

Theorem (Cantat-Zeghib)

Let Γ be a lattice in a simple Lie group G of real rank $n-1$ (e.g. $\Gamma \subset \mathrm{SL}_n(\mathbb{R})$ or $\mathrm{SL}_n(\mathbb{C})$). Let M be a compact Kaehler manifold of dimension $\leq n$, and Γ acts on it holomorphically, but not via a finite group.

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• *If* $\dim M = n-1$, *then, the action extends to the usual action of* $\mathrm{SL}_n(\mathbb{C})$ *on* $\mathbb{C}P^n$.
• *If* $\dim M = n$, *then either*
1) The action extends to an action of the full Lie group G (and we have a complete clasification)
***2) or M is birational to a complex torus.*

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Then, necessarily, $\dim M \geq \mathrm{rk} G = n-1$, i.e. $\dim M = n-1$ or n.

- *If $\dim M = n-1$, then, the action extends to the usual action of $\mathrm{SL}_n(\mathbb{C})$ on $\mathbb{C}P^n$.*
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1) The action extends to an action of the full Lie group G (and we have a complete clasification)

***2) or M is birational to a complex torus.*

More precisely, M is a Kummer variety : it is obtained form an abelian orbifold A/F by blow ups and resolution of singularities.

The discrete factor of the automorphism group of a complex manifold

Let M be a complex manifold.

$Aut(M)$ the group of holomorphic diffeomorphism

– If M is compact, then $Aut(M)$ is a complex Lie group (of finite dimension).

– The Lie algebra of $Aut^0(M)$ is the space of holomorphic fields on M.

A complex structure is not a rigid geometric structure!
The Lie group property (for compact spaces) follows from "ellipticity" : the space of holomorphic sections of a holomorphic bundle over a compact complex manifold, has finite dimension (Cauchy formula) : ∞ boundeness $\Longrightarrow$ C^1 boundeness

- If M is Kaehler, the dynamics of $Aut^0(M)$ is poor

Explanations :

- Elements of $Aut^0(M)$ have vanishing topological entropy.

– In the projective case, $M \subset \mathbb{C}P^N$,

$Aut^0(M) = \{g \in PGL_{N+1}(\mathbb{C}),\ gM = M\}$

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$\rightarrow$ it is more important to consider the discrete factor

$Aut^{\#}(M) = Aut(M)/Aut^0(M)$.

• Which discrete groups are equal to $Aut^{\#}(M)$ for some M ? When is this SL $_n(\mathbb{Z})$.... ?

• For fixed dimension n, find M for which $Aut^{\#}(M)$ is as big as possible ?

Remark : following Sullivan : The mapping class group of a Kähler manifold of dimension with nilpotent homotopy (e.g. simply connected of with an abelian π_1) is a arithmetic group...

A Comparison with affine-Riemannian structures

Let (M, g) be a compact Riemannian manifold

Yano Theorem $Affin^0(M) = Isom^0(M)$: a flow preserving the Levi-Civita connection preserves the metric !

The dynamics is encoded in $Aff^{\#}(M) = Affin(M)/Affin^0(M)$, e.g. $Aff^{\#}(\mathbb{T}^n) = \mathrm{SL}_n(\mathbb{Z})$

• Affine transformations minimize topological entropy in their homotopy classes and are optimal mechanical model in this class

• all this thoughts extend to Kähler structures, without being a rigid geometric structure ! ! !

What is and Why Kähler ?

• Kähler : compatibility of complex geometry and Riemannian geometry :

– "holomorphic $\sim$ harmonic"

– Any complex submanifold is minimal (in the sense of Riemannian geometry)

• Kähler : it gives a natural, almost optimal condition for holomorphic embedding in $\mathbb{C}P^n$

Some tools

Γ acts on $H^*(M, \mathbb{C})$, by preserving the cup product $H^* \times H^* \to H^*$

$W = H^{1,1}(M, \mathbb{R}) \subset H^2(M, \mathbb{R})$

$\rho : \Gamma \to \mathrm{GL}(W)$

– $Aut(M)$, in fact $Aut^{\#}(M) = Aut(M)/Aut^0(M)$, acts on W.

Fundamental Kaehler Fact : The action of $Aut^{\#}(M)$ is virtually faithful : its kernel is finite $\iff$ if an automorphism acts trivially on W, then a power of it belongs a flow.

(authors : Liebermann, Fujiki...)

Lieberman-Fujiki

(M, ω)

$\mathrm{Aut}_{[\omega]}(M) = \{f \in \mathrm{Aut}(M)$ such that $f^*\omega$ is cohomologeous to ω $\}$

ω Kaehler form

Fact : $Aut_{[\omega]}(M)$ has a finite number of connected components

Idea of proof :

– Graph$(f^n) \subset M \times M$ have a bounded volume :

$Graph(f) = \{(x, f(x))\ x \in M\}$

$\omega^n = \omega \wedge \ldots \wedge \omega$ (n-times)

$$\int_{Graph(f)} \omega^n = \int_{Graph(Identity)} \omega^n$$

Kähler character : The Riemannian volume of any complex submanifold Y of dimension d equals $\int_Y \omega^d$

In particular, complex submanifolds are minimal submanifolds in the sense of Riemannian geometry

Chow or Hilbert scheme : $\mathcal{C}_v$ the space of complex analytic sets of a bounded volume v = it is a (singular) complex space :

Example : for $\mathbb{C}P^n$: bounded volume $\iff$ bounded degree

$\mathcal{C}_v$ has a finite number of connected components. (one basic property of algebraic sets)

Cohomological actions

Γ acts on M and hence $H^2(M, \mathbb{R})$

The action is non-finite, otherwise, up to a finite index,

$\rho(\Gamma) \subset Aut^0(M)$= identity component

Apply super-rigidity to get an action of the Lie group G.

Henceforth, we assume the action on the cohomology infinite

By Margulis super-rigidity, the ambient Lie group G acts on $H^*(M, \mathbb{R})$ preserving all algebraic structures :

– The Hodge decomposition,

– The cup product

– The Poincaré duality

In particluar,

$\rho : G \to \mathrm{GL}(W)$

ρ preserves a n-linear form $W \times \ldots W \to \mathbb{R}$

Surface case,

In dim $= 2$, the cup product is a quadratic form : $b : W \times W \to \mathbb{R}$.

Hodge index theorem (Noether theorem) : b has (anti-) Lorentz signature $+ - \ldots -$ (or $+$).

Thus : $\rho : G \to O(1, N)$.

Fact A semi-simple Lie group (with no compact factor) can be embedded in $O(1, N)$ iff it has the form $O(1, m)$.

Higher dimension

$c : W \times \dots W \to W \to \mathbb{R}$

- Is there a kind of Nother theorem for c ?

– Can the "signature" be bounded by means of the dimension ?

Case : dimension $= 3$,

– "Trilinear forms are a challenge for mathematics" ! ! !

(The quotient space under the GL $_3(\mathbb{R})$-action is infinite...)

Hodge index theorem, Hodge-Riemann bilinear relations

$q_\omega : \alpha \in H^{1,1} \to \int \alpha \wedge \alpha \wedge \omega^{n-1}$

If ω is a Kähler, then q_ω is negative definite on the primitive space $[\omega]^\perp = \{\alpha \in W, \alpha \wedge \omega \wedge \omega = 0\}$

Dimension 3

Fact

(Lorentz-like property)
$b : W \times W \to W^*$ satisfies, if $E \subset W$ is isotropic for c, then $\dim E \leq 1$.

This allows one to classify ρ assuming $\mathrm{rk}_{\mathbb{R}}(G) \geq 2$.
(for instance, G can not contain $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$...)

– Proof of the Fact : If $\dim E \neq 0$, then, $E \cap [\omega]^{\perp} \neq 0$, and $q(a, b) = \omega \wedge a \wedge b$ negative definite.

Question : classify the orthogonal group of a vectorial bilinear (or equivalently a trilinear form) satisfying the Lorentz-like property ?

Representation of $SL_2(\mathbb{R})$

R_k representation in

P_k= Polynômes homogènes de degré $k = \{p = \Sigma x^i y^{k-i}\}$

A diagonal matrix (λ, λ^{-1})

$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$

A_k action on P_k

Weights (eigenvalues) $\lambda^k, \lambda^{k-2}, \lambda^{2-k}, \lambda^k,$

Eigen-vectors : $e_k, \ldots e_{-k}$

- The wedges $e_r \wedge e_s$ are eigenvalues (or 0)

$A_k(e_r \wedge e_s) = \lambda^{r+s}(e_r \wedge e_s) = A_k^*(e_r \wedge e_s) = \lambda^l(e_r \wedge e_s)$

So :

– either $e_r \wedge e_s = 0$

– or $r+s$ is a weight of the dual $R_k^* \cong R_k$ and thus $-k \leq r+s \leq k$

- Necessarily $e_k \wedge e_k = 0$ (since $2k > k$)

By the Lorentz-like property, we can not have

$e_k \wedge e_{k-2} = 0$, and $e_{k-2} \wedge e_{k-2} = 0$

Hence $2(k-2) \leq k$, i.e. $k \leq 4$.

Conclusion

– In dimension 3, by representation theory, we get information on the cohomology

– Other structures on the cohomology are needed, e.g. The Kähler cone....

– All this is a crucial step in the proof...

In dimension > 3, other approach...

The torus

Γ a finite index subgroup of SL $_n(\mathbb{Z})$

Affine action on $\mathbb{T}^n$:

$Aff^+(\mathbb{T}^n) = \mathrm{SL}_n(\mathbb{Z}) \ltimes \mathbb{T}^n$

Remarks :

1) In $\rho : \Gamma \to Aff(\mathbb{T}^n)$, there is always a periodic point : up to a finite index, the image is in SL $_n(\mathbb{Z})$

2) There is an exterior automorphism $\sigma : A \to (A^t)^{-1} \to$ the dual representation,

we consider ρ and $\rho \circ \sigma$ as conjugate !

– Hence : up to this, Γ has a unique affine action ρ_0

Conjecture

Any infinite action : $\rho : \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^n)$ is conjugate to its affine action ρ_0

Action on the cohomology,

Γ acts on $H^1(\mathbb{T}^n, \mathbb{R}) = \mathbb{R}^n$ by preserving $H^1(\mathbb{T}^n, \mathbb{Z}) = \mathbb{Z}^n$.

$h : \Gamma \to \mathrm{SL}_n(\mathbb{Z})$

- The delicate case : h has a finite image, say trivial

In this case, Γ acts on the universal cover $\mathbb{R}^n$ commuting with the translation $\mathbb{Z}^n$-action !

Question : Prove this can not hold

Assuming homotopic standard data

If h is infinite, by superrigidity, it equals the usual embedding in SL $_n(\mathbb{Z})$: say it induces standard homotopy data

Conjecture

Any **analytic** action Γ on the torus inducing the **standard homotopy data** is analytically conjugate to the standard one.

Weaker

Conjecture

Any **analytic** action Γ on the torus inducing the **standard homotopy data** and preserving some measure is analytically conjugate to the standard one.

Results

(1). Zeghib (1999) : Yes if the action has a (global) fixed point

(2). Katok – Rodriguez-Hertz (2009) : Assume existence of a large invariant measure μ (say μ charges open sets)
They prove existence of a continuous semi-conjugacy, bijective and analytic on an open invariant set
(3). Applying (1) $\rightarrow$ the semi-conjugacy is an analytic conjugacy

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Summarising : the answer to the last conjecture is yes if the measure is large

To get some familiarity with $\mathrm{SL}_n(\mathbb{Z})$ and its family

$$A = \begin{pmatrix} 0 & 0 & -c_0 \\ 1 & 0 & -c_1 \\ 0 & 1 & -c_2 \end{pmatrix}$$

The characteristic polynomial of A is $p(X) = c_0 + c_1 X + c_2 X^2$.

• Similar construction in any dimension :

Corollary :

– Any integer polynomial corresponds to a matrix of Mat $_n(\mathbb{Z})$

– Any algebraic integer of degree $n-1$ is the eigenvalue of some element of SL $_n(\mathbb{Z})$.

Heisenberg discrete group

In dimension 3, the (integer) discrete Heisenberg $\mathrm{Heis}_{\mathbb{Z}}$ consists of :

$$A = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

$a, b, c \in \mathbb{Z}$

• In any dimension $\mathrm{SL}_n(\mathbb{Z})$ contains $U^+(\mathbb{Z})$ the group of integer triangular unipotent matrices.

Semi-direct products

$\mathrm{SL}_3(\mathbb{Z})$ conatins $\mathrm{SL}_2 \ltimes \mathbb{Z}^2$:

$$A = \begin{pmatrix} 1 & x & y \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

$a, b, c, d, x, y \in \mathbb{Z}$

- In any dimension $\mathrm{SL}_n(\mathbb{Z})$ contains $\mathrm{SL}_{n-1}(\mathbb{Z}) \ltimes \mathbb{Z}^{n-1}$.