COSMOLOGICAL TIME VERSUS CMC TIME I: FLAT SPACETIMES

LARS ANDERSSON*, THIERRY BARBOT†, FRANÇOIS BÉGUIN‡, AND ABDELGHANI ZEGHIB§

Abstract. This paper gives a new proof that maximal, globally hyperbolic, flat spacetimes of dimension $n \geq 3$ with compact Cauchy hypersurfaces are globally foliated by Cauchy hypersurfaces of constant mean curvature, and that such spacetimes admit a globally defined constant mean curvature time function precisely when they are causally incomplete. The proof, which is based on using the level sets of the cosmological time function as barriers, is conceptually simple and will provide the basis for future work on constant mean curvature time functions in general constant curvature spacetimes, as well for an analysis of the asymptotics of constant mean foliations.

1. Introduction

The study of the global properties of spacetimes solving the Einstein equations plays a central role in differential geometry and general relativity. However, with the exception of results which rely on small data assumptions (nonlinear stability results) or the assumption of symmetries, many fundamental questions about the global structure of Einstein spacetimes remain open, including cosmic censorship, structure of singularities, and existence of global foliations by Cauchy hypersurfaces with controlled geometry. The Einstein equation is hyperbolic only in a weak sense, and therefore in order to approach its Cauchy problem from a PDE point of view, it is neccessary either to impose gauge conditions, or extract a hyperbolic system by modifying the equation. The constant mean curvature (CMC) condition is an important gauge condition in the study of the Cauchy problem of the Einstein equation, and hence in general relativity. The CMC time gauge is known to lead to a well-posed Cauchy problem in conjunction with the zero shift condition [22] as well as with the spatial harmonic gauge condition [9]. In the Hamiltonian formulation of the Einstein equation, the volume of a CMC hypersurface can be viewed as the canonical dual to the CMC time, see [25]. In the case of 2+1 dimensional spacetimes, this point of view leads

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to a formulation of the Einstein equation in CMC gauge as a time-dependent Hamiltonian system on the cotangent bundle of Teichmuller space [32].

There are numerous results concerning the existence of global CMC foliations and CMC time functions under various symmetry conditions, for spacetimes with and without matter. See [2, 33] for recent surveys. It should be noted that examples of Ricci flat spacetimes which do not contain any CMC Cauchy hypersurface were recently constructed [23]. However, it is not yet known if these examples are stable.

Spacetimes with constant sectional curvature constitute an important subclass of spacetimes, where one may expect to understand the fundamental questions, including the cosmic censorship problem completely. However, even within this subclass, there are still open questions relating to the existence and properties of constant mean curvature foliations, and the asymptotic structure at cosmological singularities is not fully understood.

The systematic study of spacetimes of constant sectional curvature was initiated by Mess [31], following work by among others Margulis [30] and Fried [26]. The classification of maximal globally hyperbolic flat spacetimes with complete Cauchy hypersurfaces has recently been completed by Barbot [11], following work of Bonsante [18, 19] and others.

The purpose of this paper is to give a proof that maximal, globally hyperbolic, flat spacetimes of dimension $n \geq 3$ with compact Cauchy hypersurfaces are globally foliated by CMC Cauchy hypersurfaces, and that such spacetimes admit a global CMC time function precisely when they are causally incomplete, see Theorem 1.2 below. The proof is based on using the level sets of the cosmological time function as barriers. This result is not new, see remark 1.3, but the method of proof presented here is conceptually simple and will provide the basis for an analysis CMC time functions in general constant curvature spacetimes, as well as of the asymptotics of CMC foliations in future work.

Recall that a Lorentz manifold, or spacetime, (M, g) is *globally hyperbolic* if it contains a Cauchy hypersurface S, i.e. a weakly spacelike hypersurface such that each inextendible Causal curve in M intersects S. The hypersurface S may without loss of generality be assumed to be smooth and strictly spacelike [16, 17]. A globally hyperbolic spacetime is *maximal* if it cannot be extended in the class of globally hyperbolic spacetimes. For brevity we use the acronym MGHF for maximal, globally hyperbolic, flat spacetimes. Let $S \subset M$ be a spacelike hypersurface in a spacetime of dimension n and let ν be its future directed unitary normal. Then for X, Y tangent to S, the second fundamental form is given by $\mathrm{II}(X, Y) = \langle \nu, \nabla_X Y \rangle$. The mean curvature of S is defined by $H = \mathrm{tr}\mathrm{II}/(n-1)$. The hypersurface S is said to have *constant mean curvature* (CMC) if $H\big|_S$ is constant. If M satisfies the *timelike convergence condition* (i.e. if $\mathrm{Ric}(V, V) \geq 0$ for timelike vectors V) and has compact Cauchy hypersurfaces, then for each $p \in M$ and for each $\tau \neq 0$, there is at most one CMC surface containing x with mean curvature τ. A compact spacelike hypersurface in a globally hyperbolic spacetime is a Cauchy hypersurface [20], so the leaves of a CMC foliation are always Cauchy hypersurfaces if they are compact. A time function $t : M \to I$ is a *CMC time function* if the level sets of t are CMC Cauchy hypersurfaces with $H(t^{-1}(\tau)) = \tau$ for all $\tau \in I$. In contrast to the situation for CMC hypersurfaces and foliations, a globally defined CMC time function with compact level sets is unique, even if the timelike convergence condition fails to hold. The proof is a straightforward application of the maximum principle, see [13, §2] for details.

It is a basic fact that if an MGHF spacetime (M, g) with compact Cauchy hypersurfaces is causally complete, then it is a quotient of the Minkowski space $\mathbb{R}^{1,n-1}$. In this case M is foliated by flat, totally geodesic Cauchy hypersurfaces. Therefore we may focus on the case when M is causally incomplete. Without loss of generality, assume that M is past causally incomplete. Then M is future complete, and is the quotient of a convex strict subset of $\mathbb{R}^{1,n-1}$ by a group of isometries acting freely and properly discontinuously. This subset is in fact a future regular domain $E^+(\Lambda)$, cf. definition 2.2. The past boundary, or Cauchy horizon, of $E^+(\Lambda)$ represents in some sense the universal cover of the past cosmological singularity of M.

The cosmological time function $\tau(p)$ is defined as the maximal Lorentzian length of past directed causal curves starting at p. The cosmological time function is a $C^{1,1}_{\mathrm{Loc}}$ function, but not C^2 in general, and therefore the mean curvature of its level sets must be interpreted in the weak sense, in terms of supporting hypersurfaces. An analysis of the the weak mean curvature of the level sets of the cosmological time function of flat spacetimes, and an application of the strong maximum principle of [7], enables us to show that the level sets of τ can be used as barriers for CMC hypersurfaces. An important role in this analysis is played by the notion of regular domain in $\mathbb{R}^{1,n-1}$, introduced by Bonsante [19].

A future regular domain $E^+(\Lambda)$ is the intersection of the future of a family Λ of lightlike hyperplanes. It can be shown that the universal cover of a past causally incomplete MGHF spacetime is isometric to a future regular domain. If Λ has at least two elements, then $E^+(\Lambda)$ has regular cosmological time function, in the sense that τ is bounded from below and the limit of τ along past inextendible causal geodesics is zero. In particular this is true for the universal cover of an incomplete MGHF spacetime M, as well as for M itself. See §2 for details. The level sets of τ have interesting geometric properties. Benedetti and Guadagnini [14] showed that in a 2+1 dimensional MGHF spacetime with compact Cauchy hypersurface of genus > 1, the geometry induced on the level sets of τ precisely corresponds to a Thurston earthquake deformation defined in terms of the holonomy data of M.

1.1. **Statement of results.** We now state the main results in this paper. The first result characterizes the generalized mean curvature of the level sets of the cosmological time function in a regular domain.

Theorem 1.1. *Consider a (future complete flat) regular domain $E^+(\Lambda)$ in $\mathbb{R}^{1,n-1}$, and the associated cosmological time $\tau : E^+(\Lambda) \to (0, +\infty)$. Then, for every $a \in (0, +\infty)$, the level hypersurface $S_a = \tau^{-1}(a)$ has generalized mean curvature bounded from below by $-\frac{1}{a}$, and from above by $-\frac{1}{(n-1)a}$.*

Our convention for second fundamental form and mean curvature are such that the future hyperboloids in Minkowski space have negative mean curvature with respect to the future directed normal, see section 4. Clearly, Theorem 1.1 holds for quotients of regular domains, and such spaces therefore have crushing singularity, since the level sets of the cosmological time function provide a sequence of Cauchy hypersurfaces with uniformly diverging mean curvature.

For the case of spacetimes with compact Cauchy hypersurface, a standard barrier argument yields existence of a CMC foliation.

Theorem 1.2. *Let (M, g) be a MGHF spacetime with compact Cauchy hypersurfaces.*

(1) *If (M, g) is both past and future geodesically complete then it does not admit any globally defined CMC time function, but it admits a unique CMC foliation.*
(2) *If (M, g) is future geodesically complete, then it admits a globally defined CMC time function $\tau_{cmc} : M \to I$ where $I = (-\infty, 0)$. Furthermore, the CMC and cosmological times are comparable:*
$$\tau \leq -\frac{1}{\tau_{cmc}} \leq (n-1)\tau.$$
(3) *A similar statement, but with a time range $I = (0, +\infty)$, is true in the past geodesically complete case.*

In all cases, these foliations are analytic.

Remark 1.3. This result is not new. It was proved in [10] in the 2+1 dimensionsal case, assuming the existence of one CMC Cauchy hypersurface. In [1], a proof was given for the case of spacetimes with hyperbolic spatial topology. Finally, it has been observed in [11], that the general case follows from the classification of MGHF spacetimes with compact Cauchy hypersurfaces.

The proof provided here is conceptually much simpler that the arguments given in the above mentioned papers. More importantly, this proof can be adapted to the general constant curvature case. The proof of the main part of Theorem 1.2, the case when M is causally incomplete, makes use of the level sets of the cosmological time function of the universal cover of M, which is a regular domain, as barriers in the construction of CMC hypersurfaces. In principle, this idea generalizes immediately also to the case of constant nonzero curvature. However, the geometry and global causality in the non-flat case are sufficiently complicated that the technical details require a separate paper [4]. There, we will in particular investigate the structure of non-flat regular domains.

Further, the level sets of the cosmological asymptotic behavior of the level sets of the cosmological time function is intimately related to the geometry of the singularity itself, i.e. the boundary of the universal cover of the spacetime. This will enable us in a forthcoming paper to analyze the asymptotic behavor of the CMC foliation at the cosmological singularity of constant curvature spacetimes, see [5]. In particular, in the case of flat spacetimes, we are able to prove in [5] the conjecture of Benedetti and Guadagnini [14] that the limit of the geometry of the level sets of the CMC time function in the Gromov sense is the same as the limit of the geometry of the level sets of the cosmological time function. In the 2+1 dimensional case, this limit can be identified as a point on the Thurston boundary of Teichmuller space. While one expects the limiting geometry of the cosmic time levels to be the same as the CMC time levels in general, there is not yet a clear identification of the limiting geometry except in the 2+1 dimensional flat case.

Remark 1.4. There is no compactness condition on Cauchy hypersurfaces in Theorem 1.1. However, a direct proof of existence of CMC hypersurfaces given barriers requires compactness. In a noncompact situation, it is necessary to consider a sequence of Plateau problems, following ideas developed in [34]. It is natural to ask whether any flat regular domain has a CMC foliation. In particular, given two level hypersurfaces of the cosmological time function with mean curvatures bounded above and below by c, is there a CMC hypersurface with mean curvature c between them? Similarly, given an isometry group of a regular domain, does there exist CMC hypersurfaces, or CMC foliations, invariant under the isometry group action?

Overview of the paper. The proof of Theorem 1.1 is given in section 4, which is the central part of this article. In the preceding sections, we review introduce some notions and preliminary results which will be needed there. In section 2, some basic facts about regular domains are recalled. The results here are mainly due to Bonsante [19]. The definition and properties of the cosmological time are given in section 2.1. The classification of MGHF spacetimes with compact Cauchy hypersurface is given in section 2.2. Section 3 discusses the past horizon, and the retraction to the singularity of a future complete regular domain. In §5, we will explain how to get from hypersurfaces with prescribed mean curvature to a CMC foliation. This technique is well known to experts in the field, but since the details are somewhat scattered in the literature, we include them for the convenience of the reader. Along the way, we also check that this works with our notion of generalized mean curvature. In particular, in the literature the strong energy condition is often assumed, but we consider also the case of positive curvature (corresponding to spacetimes of deSitter type), for future use in [4]. Finally, in section 6 we give the proof of Theorem 1.2.

2. Flat regular domains

Regular domains in Minkowski spacetime $\mathbb{R}^{1,n-1}$ were first defined by F. Bonsante in [18, 19] (generalizing a construction of G. Mess in the 2+1-dimensional case, see [31]). Here we will use an equivalent definition introduced in [11], since it appears to be slightly more adapted to our purpose. For more details, we refer to section 4.1 of [11].

The importance of flat regular domains for our purpose comes from the fact that they have regular cosmological time function, see Proposition 2.8, and that each MGHF spacetime with compact, or more generally complete, Cauchy hypersurface is a quotient of a flat regular domain, see Theorem 2.10. Thus, the analysis of the singularity of MGHF spacetimes can be carried out by studying the past boundary of flat regular domains. This will be carried out in section 3.

Definition 2.1. The Penrose boundary $\mathcal{J}_{n-1}$ of the Minkowski spacetime $\mathbb{R}^{1,n-1}$ is the space of null affine hyperplanes of $\mathbb{R}^{1,n-1}$

Let N be an auxiliary euclidean metric on $\mathbb{R}^{1,n-1}$. Let $\mathcal{S}^{n-2}$ be the set of future oriented null elements of $\mathbb{R}^{1,n-1}$ with N-norm 1. Then the map which associates to a pair (u,a) the null hyperplane $H(u,a) = \{x | \langle x, u\rangle = a\}$ is a bijection between $\mathcal{S}^{n-2} \times \mathbb{R}$ and $\mathcal{J}_{n-1}$. It defines a topology on $\mathcal{J}_{n-1}$, which coincides with the topology of $\mathcal{J}_{n-1}$ as a homogeneous space under the action of the Poincaré group; $\mathcal{J}_{n-1}$ is then homeomorphic to $\mathbb{S}^{n-2} \times \mathbb{R}$.

For every element p of $\mathcal{J}_{n-1}$, we denote by $E^+(p)$ the future of p in $\mathbb{R}^{1,n-1}$, and by $E^-(p)$ the past of of p. If p is the null hyperplane $H(u,a)$, then $E^+(p) = \{x | \langle x, u\rangle < a\}$ and $E^-(p) = \{x | \langle x, u\rangle > a\}$. They are half-spaces, respectively future-complete and past-complete. For every closed subset Λ of $\mathcal{J}_{n-1}$, we define

$$E^{\pm}(\Lambda) = \bigcap_{p\in\Lambda} E^{\pm}(p).$$

Definition 2.2. A closed subset Λ of $\mathcal{J}_{n-1}$ is said to be *future regular* (resp. *past regular*) if it contains at least two elements and if $E^+(\Lambda)$ (resp. $E^-(\Lambda)$) is non-empty.

A *future complete flat regular domain* is a domain of the form $E^+(\Lambda)$ were Λ is a future regular closed subset of $\mathcal{J}_{n-1}$. Similarly, a *past complete flat regular domain* is a domain of the form $E^-(\Lambda)$ were Λ is a past regular closed subset of $\mathcal{J}_{n-1}$. A *flat regular domain* is a future complete regular domain or a past complete regular domain.

See §4.2 of [11] where it is proved in particular that this definition of flat regular domains coincides with Bonsante's definition.

Remark 2.3. A past regular closed set Λ is not necessarily future regular. Actually, a closed subset of $\mathcal{J}_{n-1}$ is past regular and future regular if and only if it is compact (and contains at least two points). See Corollary 4.11 in [11].

Remark 2.4. In the rest of the paper, we will mainly be dealing with a past incomplete, future complete spacetimes, and many statements have an obvious time reversed analog. In the following we will not make any explicit statements concerning the time reversed situation, and leave it to the reader to rephrase the relevant definitions and results.

2.1. **Cosmological time.** In any spacetime (M, g), we can define the *cosmological time* (see [6]):

Definition 2.5. The cosmological time of a spacetime (M, g) is the function $\tau : M \to [0, +\infty]$ defined by

$$\tau(x) = \mathrm{Sup}\{L(\gamma) \mid \gamma \in \mathcal{R}^-(x)\},$$

where $\mathcal{R}^-(x)$ is the set of all past-oriented causal curves starting at x, and $L(\gamma)$ the lorentzian length of the causal curve γ.

In general, this function has a very bad behavior: for example, if (M, g) is Minkowski spacetime, then $\tau(x) = +\infty$ for every x.

Definition 2.6. A spacetime (M, g) is said to have *regular cosmological time* if

(1) M has *finite existence time,* i.e. $\tau(x) < +\infty$ for every x in M,
(2) for every past-oriented inextendible causal curve $\gamma : [0, +\infty) \to M$, $\lim_{t\to\infty} \tau(\gamma(t)) = 0$.

The following result gives a charaterization of spacetimes with regular cosmological time.

Theorem 2.7 ([6, Theorem 1.2])**.** *If (M, g) has regular cosmological time, then:*

(1) *M is globally hyperbolic,*
(2) *The cosmological time τ is a time function, i.e. τ is continuous and is strictly increasing along future-oriented causal curves,*
(3) *for each x in M there is a future-oriented timelike ray $\gamma : [0, \tau(x)] \to M$ realizing the distance from the "initial singularity", that is, γ is a unit speed geodesic which is maximal on each segment and satisfies:*

$$\gamma(\tau(x))) = x \quad \tau(\gamma(t)) = t$$

(4) *τ is locally Lipschitz, and admits first and second derivative almost everywhere.*

One of the cornerstones of Bonsante's work on flat regular domains is the following proposition:

Proposition 2.8. *Future complete flat regular domains have regular cosmological time.*

Proof. See [19, Proposition 4.3 and Corollary 4.4]. ☐

2.2. **Maximal globally hyperbolic flat spacetimes with compact Cauchy hypersurfaces.**

Proposition 2.9. *Let $E^+(\Lambda) \subset \mathbb{R}^{1,n-1}$ be a future complete flat regular domain. Let Γ be a discrete torsion free group of isometries of Minkoswki spacetime $\mathbb{R}^{1,n-1}$ preserving $E^+(\Lambda)$. Then, the action of Γ on $E^+(\Lambda)$ is free and properly discontinuous, and the quotient space $M^+_\Lambda(\Gamma) = \Gamma\backslash E^+(\Lambda)$ is a globally hyperbolic spacetime with regular cosmological time.*

Sketch of proof. The proof that the action is free and properly discontinuous can be found in [11, Proposition 4.16]. The cosmological time τ is obviously Γ-invariant. Hence, it induces a map $\hat{\tau}$ on the quotient $M^+_\Lambda(\Gamma)$. Since inextendible causal curves in $M^+_\Lambda(\Gamma)$ are projections of causal curves in $E^+(\overline{\Lambda})$, the cosmological time on the quotient $M^+_\Lambda(\Gamma)$ is the map $\hat{\tau}$. It follows easily that $M^+_\Lambda(\Gamma)$ has regular cosmological time. □

Conversely:

Theorem 2.10. *Every MGHF with compact Cauchy hypersurfaces is the quotient of a flat regular domain or of the entire Minkowski space by a torsion-free discrete subgroup of isometries. More precisely, let (M, g) be a n-dimensional MGHF spacetime with compact Cauchy hypersurfaces.*

1. *If (M, g) is not past (resp. future) geodesically complete, then (M, g) is the quotient of a future (resp. past) complete regular domain in $\mathbb{R}^{1,n-1}$ by a torsion-free discrete subgroup of Isom($\mathbb{R}^{1,n-1}$).*
2. *If (M, g) is geodesically complete then it is the quotient of $\mathbb{R}^{n-1,1}$ by a subgroup of Isom($\mathbb{R}^{1,n-1}$) containing a finite index free abelian subgroup generated by $n - 1$ spacelike translations.*

Proof. It follows from the classification of MGHF spacetimes with compact Cauchy hypersurfaces given in [11]. The result in [11] is more precise: it characterizes up to finite index the possible torsion-free discrete subgroups. □

Remark 2.11. The natural setting for a result like Theorem 2.10 is not really spacetimes with compact Cauchy hypersurfaces, but rather MGHF spacetimes with complete Cauchy hypersurfaces. Indeed, every flat regular domain admits a complete Cauchy hypersurface (see [11, Proposition 4.14]). Conversely, according to [11, Theorem 1.1], every MGHF spacetime with complete Cauchy hypersurface can be tamely embedded in the quotient of a flat regular domain by a discrete group of isometries of Minkowski except if it is geodesically complete or if it is *an unipotent spacetime.* Geodesically complete MGHF spacetimes with complete Cauchy hypersurfaces are quotients of the entire Minkoswki space $\mathbb{R}^{1,n-1}$ by a commutative discrete group of spacelike translations. Flat unipotent spacetimes are defined and described in §3.3 of [11] (see also [26]); every flat unipotent spacetime is the quotient of a domain $\Omega \subset \mathbb{R}^{1,n-1}$ by a unipotent discrete subgroup of Isom($\mathbb{R}^{1,n-1}$), where Ω is of one of the three following forms: $\Omega = E^+(p)$,

$\Omega = E^-(p)$ or $\Omega = E^+(p) \cap E^-(p')$ where p and p' are two parallel null hyperplanes.

3. Past horizon and initial singularity of a future complete flat regular domain

In this section, we consider a future complete flat regular domain $E^+(\Lambda)$. We will describe the past horizon, the initial singularity, and the so-called "retraction to the initial singularity" of $E^+(\Lambda)$.

3.1. **Horizons.** According to Proposition 2.8 and Theorem 2.7, $E^+(\Lambda)$ is globally hyperbolic. Since $E^+(\Lambda)$ is a future complete convex open domain in Minkowski space, its boundary $\mathcal{H}^-(\Lambda)$ is a past horizon (and thus enjoys all the known properties of horizons).

Since $\mathcal{H}^-(\Lambda)$ is the boundary of a convex domain, it admits support hyperplanes at each of its points. And since $E^+(\Lambda)$ is future complete, the future in $\mathbb{R}^{1,n-1}$ of any point p in $\mathcal{H}^-(\Lambda)$ is contained in $E^+(\Lambda)$. But, timelike hyperplanes containing p all intersect the future of p, it then follows that support hyperplanes to $\mathcal{H}^-(\Lambda)$ are non-timelike.

Lemma 3.1. *Let p a point of the past horizon $\mathcal{H}^-(\Lambda)$ of a future complete flat regular domain $E^+(\Lambda)$. Let $C(p) \subset T_pX$ be the set of future oriented tangent vectors orthogonal to support hyperplanes to $\mathcal{H}^-(\Lambda)$ at p. Then $C(p)$ is the convex hull of its null elements. Moreover, the null elements of $C(p)$ are precisely the normals to elements of Λ tangent to $\mathcal{H}^-(\Lambda)$ at p.*

Proof. See [19, corollary 4.12] (see also [31, Proposition 11]). □

3.2. **Retraction to the initial singularity.** According to point (3) in Theorem 2.7, for every point x in a flat regular domain there is a unique maximal timelike geodesic ray with future endpoint x realizing the "distance to the initial singularity": we call such a geodesic ray a *realizing* geodesic for x.

Proposition 3.2. *Let x be an element of a future complete flat regular domain $E^+(\Lambda)$. Then, there is an unique realizing geodesic for x.*

Proof. See [19, Proposition 4.3]. □

Definition 3.3. A unit speed future oriented timelike geodesic $\gamma : [0,T] \to E^+(\Lambda)$ is tight if for every t in $[0,T]$ the restriction $\gamma : [0,t] \to E^+(\Lambda)$ is a realizing geodesic for $\gamma(t)$.

Proposition 3.4. *Let $\gamma : [0,T] \to E^+(\Lambda)$ be an unit speed future oriented timelike geodesic with initial point in the past horizon. Then the following assertions are equivalent:*

(1) *γ is tight,*
(2) *the derivative of γ at 0 is orthogonal to a support hyperplane at $\gamma(0)$.*

Proof. See [19, Proposition 4.3]. □

Definition 3.5. The initial singularity of a future complete flat regular domain $E^+(\Lambda)$ is the set of points in the past horizon admitting at least two support hyperplanes; it will be denoted by $\Sigma^-(\Lambda)$.

Proposition 3.6. *The map which associates to any point x of a regular domain $E^+(\Lambda)$, the initial singularity of the unique realizing geodesic for x is a continuous map taking value in $\Sigma^-(\Lambda)$. This map is denoted r, and called "retraction to the initial singularity".*

Proof. See [19, Proposition 4.3 and 4.12]. □

3.3. Description of the retraction map.

Proposition 3.7. *For every p in the past singularity Σ^-, the preimage $r^{-1}(p)$ in $E^+(\Lambda)$ is the union of complete timelike geodesic rays with initial point at p.*

Proof. The Proposition is an immediate corollary of Proposition 3.2 and 3.4. □

Corollary 3.8. *Let p be an element of the past horizon of $E^+(\Lambda)$ such that the convex hull $C(p)$ of the null generators has non-empty interior in the space of timelike tangent vectors at p. Then, $r^{-1}(p)$ is open in $E^+(\Lambda)$.* □

4. Cosmological levels as barriers, Proof of Theorem 1.1

If S is a spacelike hypersurface in a spacetime (M, g), then the second fundamental form (also known as the extrinsic curvature) of S at a point x is defined as $\mathrm{II}(X, Y) = \langle \nu, \nabla_X Y\rangle = -\langle \nabla_X \nu, Y\rangle$ where X, Y are tangent vectors to S at x and ν is the future oriented timelike normal of S (with lorentzian norm -1). The mean curvature is defined in terms of the trace of II with respect to the induced metric as $H_S = \mathrm{tr}\mathrm{II}/(n-1)$. This definition requires S to be at least C^2. Nevertheless, in certain cases, one can give a meaning to the assertion "a topological hypersurface has mean curvature bounded from below (or above) by some constant c". A definition of this notion for rough spacelike hypersurfaces was given in [7, Definition 3.3], making use of the notion of supporting hypersurfaces with one-sided Hessian bound. The following definition, which does not include the one-sided Hessian bound, is sufficient for our purposes in this paper. We will say that S is a C^0-spacelike hypersurface in M if for each $x \in S$, there is a neighborhood U of x so that $S \cap U$ is edgeless and acausal in U, see [7, Definition 3.1].

Definition 4.1. Let S be a C^0-spacelike hypersurface in a spacetime (M, g). Given a real number c, we will say that S *has generalized mean curvature bounded from above by c at x*, denoted $H_S(x) \leq c$, if there is a geodesically convex open neighborhood V of x in M and a smooth spacelike hypersurface $\mathbb{S}_x^-$ in V such that :

– $x \in \mathbb{S}_x^-$ and $\mathbb{S}_x^-$ is contained in the past of $S \cap V$ (in V),

– the mean curvature of $\mathbb{S}_x^-$ at x is bounded from above by c.

Similarly, we will say that S *has generalized mean curvature is bounded from below by* c *at* x, denoted $H_S(x) \geq c$, if, there is a geodesically convex open neighborhood V of x in M and a smooth spacelike hypersurface $\mathbb{S}_x^+$ in V such that :

- $x \in \mathbb{S}_x^+$ and $\mathbb{S}_x^+$ is contained in the past of $S \cap V$ (with respect to V),
- the mean curvature of $\mathbb{S}_x^+$ at x is bounded from below by c.

We will write $H_S \geq c$ and $H_S \leq c$ to denote that S has generalized mean curvature bounded from below respectively above by c for all $x \in S$.

Remark 4.2. Let S be a smooth spacelike hypersurface in a spacetime (M, g), and c be a real number. If $H_S \leq c$ or $H_S \geq c$ in the sense of the definition above, then the maximum principle, see Proposition 5.4 below, implies that the same bounds hold in terms of the usual sense.

Remark 4.3. Let S be a C^0-spacelike hypersurface, and let x be a point of S. Assume that there exists two numbers c^-, c^+ such that S has generalized mean curvature bounded from below by c^- and from above by c^+ at x. Then S has a tangent plane at x. Indeed, the point x belong to two smooth hypersurfaces $\mathbb{S}_x^-$ and $\mathbb{S}_x^+$ which are (locally) respectively in the past and in the future S. In particular, $\mathbb{S}_x^-$ is locally in the past of $\mathbb{S}_x^+$. This implies that the tangent hyperplane of $\mathbb{S}_x^-$ at x coincides with the tangent hyperplane of $\mathbb{S}_x^+$. And since S is between $\mathbb{S}_x^-$ and $\mathbb{S}_x^+$, this hyperplane is also tangent to S.

Let us recall the statement of Theorem 1.1:

Theorem 4.4. *Consider a future complete flat regular domain $E^+(\Lambda)$ and the associated cosmological time $\tau : E^+(\Lambda) \to (0, +\infty)$. Then, for every $a \in (0, +\infty)$, the hypersurface $S_a = \tau^{-1}(a)$ has generalized mean curvature satisfying $-\frac{1}{a} \leq H_{S_a} \leq -\frac{1}{(n-1)a}$.*

Remark 4.5. What is important in the proof of Theorem 1.2 is just the fact that the hypersurface S_a has generalized mean curvature satisfying $\alpha(a) \leq H_{S_a} \leq \beta(a)$, where $\alpha(a), \beta(a) \to -\infty$ when $a \to 0$, and $\alpha(a), \beta(a) \to 0$ when $a \to +\infty$.

Proof. Let x be a point on the level set S_a. We denote by $\gamma : [0, a] \to E^+(\Lambda)$ the unique realizing geodesic for x, with initial point $p = r(x)$. Let v be the future oriented unit speed tangent vector of γ at p. We denote as before by $C(p)$ the set of vectors in T_pX orthogonal to support hyperplanes of the past horizon at p.

Construction of $\mathbb{S}_x^+$. Define $\mathbb{S}_x^+$ as the hyperboloid $\{z | d(p, z) = a\}$. Since $E^+(\Lambda)$ is geodesically convex, for any z in $\mathbb{S}_x^+$ the timelike geodesic (p, z) is contained in $E^+(\Lambda)$. Hence, its length a is less than $\tau(z)$. The unique realizing geodesic for z must therefore intersect S_a. Hence, $\mathbb{S}_x^+$ is contained in the future of S_a. The tangent hyperplane to $\mathbb{S}_x^+$ at x is the hyperplane orthogonal to c at x. Hence, $\mathbb{S}_x^+$ is tangent to S_a at x. Finally, the mean curvature of $\mathbb{S}_x^+$ is obviously $-\frac{1}{a}$ everywhere. As a consequence, S_a has generalized mean curvature satisfying $H_{S_a} \geq -\frac{1}{a}$.

Construction of $\mathbb{S}_x^-$. According to Lemma 3.1, the tangent vector v of the realizing geodesic γ introduced above, belongs to the convex hull $C(p)$. Let B be a finite subset of the null elements of $C(p)$ such that v lies in the convex hull of B. We choose moreover B minimal, i.e. such that for any proper subset $B' \subset B$, v does not belong to the convex hull of B'. An equivalent statement is that v belongs to the relative interior $\mathrm{Conv}(B)$.

The null hyperplanes $p + w^\perp$ for w in B form a finite subset Λ_B of Λ. Observe that since the convex hull of B contains the timelike vector v, B contains at least two elements. Hence, $E^+(\Lambda_B)$ is a future complete flat regular domain.

Obviously, $E^+(\Lambda_B)$ contains $E^+(\Lambda)$. Hence $\mathcal{H}^-(\Lambda_B)$ is contained in the causal past of $E^+(\Lambda)$. Moreover, $E^+(\Lambda_B)$ contains the timelike geodesic γ, and also x, and its past horizon $\mathcal{H}^-(\Lambda_B)$ contains p. According to Lemma 3.1, support hyperplanes to $\mathcal{H}^-(\Lambda_B)$ at p are hyperplanes orthogonal to vectors in the convex hull of B. In particular, the hyperplane orthogonal to the timelike vector v is a spacelike support hyperplane. It follows that γ is a realizing geodesic for x in $E^+(\Lambda_B)$. Hence, $\tau_B(x) = a$, where τ_B is the cosmological time for $E^+(\Lambda_B)$.

Let S'_B be the level set $\{\tau_B = a\}$ in $E^+(\Lambda_B)$, and define $\mathbb{S}_x^-$ as a small open neighborhood of x in $S'_B \cap E^+(\Lambda)$. Let V be a geodesically convex neighborhood of x containing $\mathbb{S}_x^-$ (for example, the Cauchy development of $\mathbb{S}_x^-$ in $E^+(\Lambda)$). For any z in S'_B let c be the unique realizing geodesic for z in $E^+(\Lambda)$. Since $\mathcal{H}^-(\Lambda_B)$ is in the causal past of $\mathcal{H}^-(\Lambda)$ there is a past extension of c with past endpoint in $\mathcal{H}^-(\Lambda_B)$. Hence, $\tau(z) \leq a$. It follows that $\mathbb{S}_x^-$ lies in the causal past of S_a in V.

To complete the proof, we must prove that $\mathbb{S}_x^-$ near x is smooth, admits at x the same tangent hyperplane $(x-p) + v^\perp$, and that it has constant mean curvature $-\frac{d}{(n-1)a}$ for some integer $1 \leq d \leq n-1$.

Consider $\mathbb{R}^{1,n-1}$ as a vector space, with origin $p = 0$. Let F be the vector space spanned by $\mathrm{Conv}(B)$. Then F is a timelike subspace, with dimension $2 \leq k \leq n$, and we have a splitting $\mathbb{R}^{1,n-1} = F \oplus F^\perp$. The subspace $F^\perp$ is spacelike. Every element of Λ_B is a null hyperplane containing $F^\perp$. It follows easily that $E^+(\Lambda_B)$ is the sum $E'(\Lambda_B) \oplus F^\perp$, where $E'(\Lambda_B) = F \cap E^+(\Lambda_B)$. For every element H of Λ_B, $H \cap F$ is a null hyperplane in $F \approx \mathbb{R}^{1,k-1}$.

Let $\Lambda'_B = \{H \cap F \mid H \in \Lambda_B\}$. Then Λ'_B is a finite subset of the Penrose boundary of F. Clearly $E'(\Lambda_B)$ is precisely the flat regular domain $E(\Lambda'_B) \subset F$. Now we observe that restricting to F, v is in the interior of $\mathrm{Conv}(\Lambda'_B)$. Hence, for some small neighborhood V' of x in $E(\Lambda'_B)$, which can be selected geodesically convex, the image by the retraction r of each point y in V' is p. Shrinking V if necessary, we can assume that V is contained in $V' \oplus F^\perp$. According to Corollary 3.8, $\mathbb{S}_x^-$ has the form $\mathbb{H} \oplus F^\perp$, where $\mathbb{H}$ is the hyperboloid consisting of points in F in the future of p and at lorentzian distance a from p. Hence, $\mathbb{S}_x^-$ is smooth and admits at x the same tangent hyperplane than S_a (the orthogonal $x+v^\perp$). Moreover, since $x+F^\perp$ is totally geodesic, and since the principal directions of $\mathbb{H}$ are all equal to $-\frac{1}{a}$, the mean curvature of $\mathbb{S}_x^-$ is equal to $-\frac{1}{a}.\frac{d}{n-1}$ where $d=k-1$. This shows that S_a has generalized mean curvature satisfying $H_{S_a} \leq -\frac{1}{(n-1)a}$. □

Remark 4.6. The proof of theorem 4.4 shows that the second fundamental forms of $\mathbb{S}_x^-, \mathbb{S}_x^+$ have eigenvalues $-1/a, 0$ (in the case of $\mathbb{S}_x^-$) and $-1/a$ (in the case of $\mathbb{S}_x^+$. Therefore the level sets of τ have mean curvature satisfying $-1/a \leq H_{S_a} \leq -1/((n-1)a)$ with one-sided Hessian bound as in [7, Definition 3.3], and hence the strong maximum principle for spacelike hypersurfaces given in [7, Theorem 3.6] applies in our situation. However, we shall not need the full strength of this result here. See proposition 5.4 below for the version of the maximum principle which we shall make use of.

The eigenvalue bounds stated in remark 4.6 allow us to give a more precise characterization of the regularity of the cosmological time function. The Hessian bounds for the height function implied by the bounds on the second fundamental form of the supporting hypersurfaces, together with an application of the case $p=\infty$ of [21, Proposition 1.1] proves

Corollary 4.7. $\tau \in C^{1,1}_{\mathrm{Loc}}$

We leave it to the reader to formulate the obvious analogs of theorem 4.4 and corollary 4.7 for *past complete* flat regular domains $E^-(\Lambda)$ which hold in terms of the *reverse* cosmological time $\widehat{\tau} : E^-(\Lambda) \to (0, +\infty)$.

5. From barriers to CMC time functions

In this section, we consider a n-dimensional, $n \geq 3$, maximal globally hyperbolic spacetime (M, g) with compact Cauchy hypersurfaces and constant curvature equal to k. We emphasize that many of the proofs that we give are not valid without the assumption that M has compact Cauchy surfaces. Recall that (M, g) has curvature k if the Riemann tensor satisfies

$$\langle \mathrm{Riem}(X,Y)Y, X\rangle = k(\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2)$$

for any vector fields X, Y. Then the Ricci tensor satisfies $\mathrm{Ric} = (n-1)kg$. We will define a notion of *sequence of asymptotic barriers*, and prove (using quite classical arguments) that (M, g) admits a CMC time function provided that it admits a sequence of asymptotic barriers.

Definition 5.1. Let c be a real number. A *pair of c-barriers* is a pair of C^0-spacelike Cauchy hypersurfaces (Σ^-, Σ^+) in M such that

– Σ^+ is in the future of Σ^-,
– $H_{\Sigma^-} \leq c \leq H_{\Sigma^+}$ in the sense of definition 4.1.

Definition 5.2. Let α be a real number. A *sequence of asymptotic past α-barriers* is a sequence of C^0-spacelike Cauchy hypersurfaces $(\Sigma_m^-)_{m\in\mathbb{N}}$ in M such that

- Σ_m^- tends to the past end of M when $m \to +\infty$ (*i.e.* given any compact subset K of M, there exists m_0 such that K is in the future of Σ_m^- for every $m \geq m_0$),
- $a_m^- \leq H_{\Sigma_m^-} \leq a_m^+$, where a_m^- and a_m^+ are real numbers such that $\alpha < a_m^- \leq a_m^+$, and such that $a_m^+ \to \alpha$ when $m \to +\infty$.

Similarly, a *sequence of asymptotic future β-barriers* is a sequence of C^0-spacelike Cauchy hypersurfaces $(\Sigma_m^+)_{m\in\mathbb{N}}$ in M such that

- Σ_m^+ tends to the future end of M when $m \to +\infty$,
- $b_m^- \leq H_{\Sigma_m^+} \leq b_m^+$, where b_m^- and b_m^+ are real numbers such that $b_m^- \leq b_m^+ < b$, and such that $b_m^- \to \beta$ when $m \to +\infty$.

Theorem 5.3. *Let (M, g) be an n-dimensional, $n \geq 3$, maximal globally hyperbolic spacetime, with compact Cauchy hypersurfaces and constant curvature k, and such that (M, g) admits a sequence of asymptotic past α-barriers and a sequence of asymptotic future β-barriers. If $k \geq 0$, assume moreover that $(\alpha, \beta) \cap [-\sqrt{k}, \sqrt{k}] = \emptyset$. Then, (M, g) admits a CMC time function $\tau_{cmc} : M \to (\alpha, \beta)$.*

Theorem 5.3 follows easily from known facts in case the barriers are smooth, and introducing C^0 barriers is not difficult given the results above. Nevertheless, since we are not aware of a reference for this precise statment, we include a proof below. The following are the two main technical steps in the proof. In the case of smooth barriers and hypersurfaces, they were proved in this formulation by Gerhardt [28].

- a proposition which states that any CMC hypersurface of mean curvature c' lies in the future of any CMC hypersurface of mean curvature c whenever $c' > c$ (Proposition 5.6);
- a theorem which ensures the existence of a Cauchy hypersurface of constant mean curvature c, assuming the existence of a pair of c-barriers (Theorem 5.9).

Let us start with a slight generalization of the classical *maximum principle*.

Proposition 5.4. *Let Σ and Σ' be two C^0-spacelike hypersurfaces. Assume that these hypersurface have one point x in common, and assume that Σ is in the past of Σ'. Assume that Σ has generalized mean curvature bounded from above by c at x, and Σ' has generalized mean curvature bounded from below by c' at x. Then $c \geq c'$.*

Remark 5.5. Proposition 5.4, which may be viewed as a comparison principle, follows from the strong maximum principle for C^0 hypersurfaces satisfying a one-sided Hessian bound, see [7, Theorem 3.6]. The notion of generalized mean curvature we are using here does not included this requirement and we therefore include the simple proof of the proposition.

Proof. Since Σ has generalized mean curvature bounded from above by c at x, there exists a smooth spacelike hypersurface S_x such that $x \in S_x$, S_x is in the past of Σ and the mean curvature of S_x at x is at most c. Similarly, there exists a smooth spacelike hypersurface S'_x such that $x \in S'_x$, S'_x is in the future of Σ' and the mean curvature of S'_x at x is at least c'. Since Σ is in the past of Σ', this implies that S_x is in the past of S'_x. And since the point x belongs to both S_x and S'_x, we deduce that S_x and S'_x share the same tangent hyperplane at x. Now the classical maximum principle can be applied to show that $c \geq c'$. □

The following result was proved by Gerhardt for the case of spacetimes with a lower bound on the Ricci curvature on timelike vectors, see [28, Lemma 2.1].

Proposition 5.6. *Let (M, g) be an n-dimensional, $n \geq 3$, maximal globally hyperbolic spacetime, with compact Cauchy hypersurfaces and constant curvature k. Let Σ and Σ' be two smooth Cauchy hypersurfaces in M. Assume that $H_\Sigma \leq c$ and $H_{\Sigma'} \geq c'$, with $c \leq c'$. If k is non-negative, assume moreover that $c < -\sqrt{k}$ or that $c' > \sqrt{k}$. Then Σ' is in the future of Σ.*

We will give a proof of Proposition 5.6 below, as we shall make use of some of the details in the proof of theorem 5.3.

Let Σ_0 be a smooth Cauchy hypersurface with future unit normal ν_0. Recall that the orbit of the *Gauss flow* of smooth Cauchy hypersurface Σ_0 in the direction ν_0 consists of the Cauchy hypersurfaces $\Sigma_t = F_t(\Sigma_0)$ where $F : I \times \Sigma_0 \to M$ is defined as $F_t(x) = exp_x(t\nu_0)$ for $x \in \Sigma_0$, for $t \in I$. Here I is the maximal time interval where F_t is regular. The core of the proof of Proposition 5.6 is the following standard comparison lemma, see for example [8, corollary 2.4].

Lemma 5.7. *We consider the orbit $(\Sigma_t)_{t\in I}$ of a smooth Cauchy hypersurface Σ_0 under the Gauss flow. We consider a geodesic γ which is orthogonal to the Σ_t's, and we denote by $p(t)$ the point of intersection of the geodesic γ with the hypersurface Σ_t. The mean curvature $H(t)$ of Σ_t at $p(t)$ satisfies the differential inequality*

$$\frac{dH(t)}{dt} \geq (n-1)(H(t)^2 - k).$$

Proof of Proposition 5.6. Assume that Σ' is not in the future of Σ. Then, we can consider a future-directed timelike geodesic segment γ going from a point of Σ' to a point of Σ having maximal length among all such geodesic segments. It is well-known that γ is orthogonal to both Σ' and Σ, and that there is no focal point to Σ' or Σ along γ (see e.g. [29, Proposition 4.5.9]). We will denote by $p' \in \Sigma'$ and $p \in \Sigma$ the ends of γ, and by δ be the length of γ.

If k is non-negative, we have to distinguish two differents cases, according to whether $c' > \sqrt{k}$ or $c < -\sqrt{k}$. Let us consider the first case. Since there is no focal point to Σ' along γ, the image Σ'_t of Σ' by the time t of the Gauss flow is well-defined for $t \in [0, \delta]$ in a neighbourhood of γ. Denote by $p'(t)$ the point of intersection of the hypersurface Σ'_t with the geodesic segment γ, and by $H'(t)$ the mean curvature of Σ'_t at $p'(t)$. By Lemma 5.7, $t \mapsto H'(t)$ satisfies the differential inequality $\frac{dH'}{dt} \geq (n-1)(H'^2 - k)$. This implies that H' increases along γ (note that $H'(t)^2$ is strictly greater than k for every t, since $H'(0) = c' > \sqrt{k}$ by assumption and since $H'(t)$ increases). In particular, we have $H'(\delta) > H'(0) = c'$. But now, recall that, by definition of Σ'_δ, every point of Σ'_δ in a neighbourhood of $\gamma(\delta) = p$ is at distance exactly δ of Σ'. Also recall that γ is the longest geodesic segment joining a point of Σ' to a point of Σ. This implies that Σ is in the past of Σ'_δ. Hence, by Proposition 5.4, the mean curvature of Σ at p is bounded from below by the mean curvature of Σ'_δ, which itself is strictly greater than the mean curvature of Σ'. This contradicts the assumption $c \leq c'$.

The proof is the same in the case where $c < -\sqrt{k}$ (except that one considers the backward orbit of Σ for the Gauss flow, instead of the forward orbit of Σ'). □

Remark 5.8. Proposition 5.6 implies that, for every $c \in \mathbb{R} \setminus [-\sqrt{k}, \sqrt{k}]$, there exists at most one Cauchy hypersurface in M with constant mean curvature equal to c. In particular, for any open interval (α, β), which if $k \geq 0$ satisfies the condition $(\alpha, \beta) \cap [-\sqrt{k}, \sqrt{k}] = \emptyset$, there exists at most one function $t_{cmc} : M \to (\alpha, \beta)$ such that $t_{cmc}^{-1}(c)$ is a smooth Cauchy hypersurface with constant mean curvature equal to c for every $c \in (\alpha, \beta)$. Note that we are not assuming here that t_{cmc} is a time function (recall that, if t_{cmc} is a time function, then it is automatically unique, without any assumption on (α, β)).

Further, it is easy to see using a maximum principle argument, that in the standard deSitter space with topology $S^{n-1} \times \mathbb{R}$ and curvature $k > 0$, there is no Cauchy hypersurface with mean curvature $c \in \mathbb{R} \setminus [-\sqrt{k}, \sqrt{k}]$. Therefore Proposition 5.6 is vacuous in this case.

Theorem 5.9. *Let (M, g) be an n-dimensional, $n \geq 3$, maximal globally hyperbolic spacetime, with compact Cauchy hypersurfaces. Let c be any real number, and assume that there exists a pair of c-barriers (Σ^-, Σ^+) in M. Then, there exists a smooth Cauchy hypersurface Σ with constant mean curvature equal to c. Moreover, Σ is in the future of Σ^- and in the past of Σ^+.*

Proof. The result is proved e.g. in [27] in the case where the barriers Σ^- and Σ^+ are smooth. The only way the barriers Σ^- and Σ^+ are used in Gerhardt's proof is via the maximum principle (to show that a family of Cauchy hypersurfaces whose mean curvature approaches c cannot "escape to infinity"). Since the maximum principle is still valid for C^0 hypersurfaces (Proposition 5.4), Gerhardt's proof also applies in the case where the barriers are not smooth. □

Proof of Theorem 5.3. We consider a sequence $(\Sigma_m^-)_{m\in\mathbb{N}}$ of asymptotic past α-barriers, and a sequence $(\Sigma_m^+)_{m\in\mathbb{N}}$ of asymptotic future β-barriers.

Construction of the function τ_{cmc}. Fix $c \in (\alpha, \beta)$. For m large enough, the pair of Cauchy hypersurfaces (Σ_m^-, Σ_m^+) is a pair of c-barriers. Thus, by Theorem 5.9, for any $c \in (\alpha, \beta)$, there exists a Cauchy hypersurface S_c with constant mean curvature equal to c. Proposition 5.6 implies that the S_c's are pairwise disjoint, and that S_c is in the past of $S_{c'}$ if $c < c'$ (let us call this "property $(\star)$").

Now, let us prove that the set $\bigcup_{c\in(\alpha,\beta)} S_c$ is connected. Assume the contrary. Because of property $(\star)$, there are only two possible cases :

(i) there exists $c_0 \in (\alpha, \beta)$ such that $\bigcup_{c>c_0} I^+(S_c) \subsetneq I^+(S_{c_0})$,
(ii) or there exists $c_0 \in (\alpha, \beta)$ such that $\bigcup_{c<c_0} I^-(S_c) \subsetneq I^-(S_{c_0})$.

Let us consider, for example, case (i). Using the Gauss flow, we can push the hypersurface S_{c_0} towards the future, in order to obtain a Cauchy hypersurface S'_{c_0} which is in the future of Σ_{c_0}, but as close to S_{c_0} as we want. In particular, we can assume that S'_{c_0} is not in the future of S_c for any $c > c_0$. Moreover, according to Lemma 5.7, the mean curvature of S'_{c_0} is bounded from below by some number $c'_0 > c_0$. But this contradicts Proposition 5.6. Case (ii) can be treated similarly. As a consequence, the set $\bigcup_{c\in(\alpha,\beta)} S_c$ is connected. Note that this implies that the hypersurface S_c depends continuously on c

Now, let us prove that the union $\bigcup_{c\in(\alpha,\beta)} S_c$ is equal to the whole M. Assume that there exists a point $x \in M \setminus \bigcup_{c\in(\alpha,\beta)} S_c$. Since the hypersurface S_c depends continuously on c, there are only two possible cases :

(i) either x is in the future of S_c for every $c \in (\alpha, \beta)$,
(ii) or x is in the past of S_c for every $c \in (\alpha, \beta)$.

Now, recall that we have a sequence $(S_m^+)_{m\in\mathbb{N}}$ of asymptotic future β-barriers. By definition, this means that S_m^+ has generalized mean curvature bounded from below by some b_m^- and smaller than some b_m^+ where $b_m^- \leq b_m^+ < \beta$ and $b_m^- \to_{m\to\infty} \beta$. Fix some integer p. One can find $q > p$ such that $b_q^- > b_p^+$. Then (S_p^+, S_q^-) is a pair of b_p^+-barriers. By Theorem 5.9, one can find a Cauchy hypersurface with constant mean curvature equal to b_p^+ between S_p^+ and S_q^-, and by uniqueness (see remark 5.8), this hypersurface is the hypersurface S_c for $c = b_p^+$. In particular, for $c \geq b_p^+$, the hypersurface S_c is in the future of the barrier S_p^+. Now, recall that, by definition of a sequence of asymptotic future barriers, S_p^+ tends to the future end of M when $p \to \infty$. This shows that case (i) cannot happen. Of course, one can exclude case (ii) using similar arguments. Therefore we have proved that $\bigcup_{c\in(\alpha,\beta)} S_c = M$.

Now, we can define define the function $\tau_{cmc} : m \to (\alpha, \beta)$ as follows : for every $x \in M$, we set $\tau_{cmc}(x) = c$ where c is the unique number such that $x \in S_c$.

Properties of the function τ_{cmc}. The fact the hypersurface S_c depends continuously on c implies that the function τ_{cmc} is continuous. The fact that the hypersurface $S_{c'}$ is in the strict future of the hypersurface S_c when $c' > c$ implies that the function τ_{cmc} is strictly increasing along any future directed timelike curve. Hence, τ_{cmc} is a time function. □

Remark 5.10. The function τ_{cmc} is also a time function in the following stronger sense : for every future directed timelike curve $\gamma : I \to \mathbb{R}$, one has

$$\frac{d}{dt}\tau_{cmc}(\gamma(t)) > 0.$$

Indeed, fix such a curve γ and some $t_0 \in I$, let $x_0 = \gamma(t_0)$ and $c_0 = \tau_{cmc}(x_0)$. For t small enough, denote by $S^t_{c_0}$ the image of the hypersurface S_{c_0} by the time t of the Gauss flow. Since the derivative γ is future-oriented timelike vector, there exists a constant $\lambda_1 > 0$ such that, for $h > 0$ small enough, the point $\gamma(t_0 + h)$ is in the future of the image of the hypersurface $S^{\lambda_1.h}_{c_0}$. Now Lemma 5.7 implies that there exists a constant $\lambda_2 > 0$ such that the mean curvature of the hypersurface $S^{\lambda_1.h}_{c_0}$ is bounded from below by $c_0 + \lambda_1.\lambda_2.h$ (for h small enough). Then Proposition 5.6 implies that $S_{c_0+\lambda_1.\lambda_2.h}$ is in the past of $S^{\lambda_1.h}_{c_0}$. In particular, for h small enough, the point $\gamma(t_0 + h)$ is the future of the hypersurface $S_{c_0+\lambda_1.\lambda_2.h}$. In other words, we have $\tau_{cmc}(t_0+h) > c_0 + \lambda_1.\lambda_2.h$. This implies $\frac{d}{dt}\tau_{cmc}(\gamma(t)) > \lambda_1.\lambda_2 > 0$.

Remark 5.11. Using the same arguments as above, one can prove the following result:

Let (M, g) be an n-dimensional, $n \geq 3$, maximal globally hyperbolic spacetime, with compact Cauchy hypersurfaces and constant curvature k. Assume that (M, g) admits a sequence of asymptotic past α-barriers. If $k \geq 0$, assume moreover that $\alpha \notin [-\sqrt{k}, \sqrt{k}]$. Then, (M, g) admits a partially defined CMC time function $\tau_{cmc} : U \to (\alpha, \beta)$ where U is a neighbourhood of the past end of M (i.e. the past of a Cauchy hypersurface in M) and β is a real number greater than α.

Proposition 5.12. *Let (M, g) be an n-dimensional, $n \geq 3$, maximal globally hyperbolic spacetime, with compact Cauchy hypersurfaces and constant curvature k. Suppose that there exists a function $t_{cmc} : M \to (\alpha, \beta)$ such that $t^{-1}_{cmc}(c)$ is a Cauchy hypersurface with constant mean curvature equal to c for every $c \in (\alpha, \beta)$. Assume moreover that one of the following hypotheses is satisfied:*

- *t_{cmc} is a time function,*
- *the curvature k is negative,*
- *the curvature k is non-negative and $(\alpha, \beta) \cap [-\sqrt{k}, \sqrt{k}] = \emptyset$.*

Then t_{cmc} is real analytic.

Sketch of proof. Under the stated conditions, there is exactly one CMC Cauchy hypersurface for each $c \in (\alpha, \beta)$. CMC hypersurfaces in a real analytic spacetime are real analytic, since they are solutions of a quasi-linear elliptic PDE. Given a CMC Cauchy hypersurface S_0 with mean curvature $c_0 \in (\alpha, \beta)$, let u be the Lorentz distance to S_0. For c close to c_0, a Cauchy hypersurface S_c with mean curvature c is a graph over S_0, defined by the level function $w = u\big|_{S_c}$. The function w solves the mean curvature equation $H[w] = c$, which is a quasilinear elliptic system with real analytic dependence on the coefficients. It follows that S_c depends in a real-analytic manner on c, and that the function t_{cmc} is a real analytic function on M. □

6. Proof of Theorem 1.2

Let (M, g) be a n-dimensional MGHF spacetime with compact Cauchy hypersurface. We first consider the case where (M, g) is not past geodesically complete. Then Theorem 2.10 states that (M, g) is the quotient of a future complete regular domain $E^+(\Lambda) \subset \mathbb{R}^{1,n-1}$ by a torsion-free discrete subgroup Γ of $\mathrm{Isom}(\mathbb{R}^{1,n-1})$. Let $\tau : E^+(\Lambda) \to (0, +\infty)$ be the cosmological time of $E^+(\Lambda)$. It follows from Theorem 2.7 and its proof, see [6, Proposition 2.2], that for every $a \in (0, +\infty)$, the level set $S_a = \tau^{-1}(a)$ is a closed strictly achronal edgeless hypersurface in $E^+(\Lambda)$. Moreover, τ is obviously invariant under every element of $\mathrm{Isom}(\mathbb{R}^{1,n-1})$ preserving $E^+(\Lambda)$. Hence, for every $a \in (0, +\infty)$, the projection Σ_a of S_a in $M \equiv \Gamma \setminus E^+(\Lambda)$ is a closed strictly achronal edgeless hypersurface in M. Since M is globally hyperbolic with compact Cauchy hypersurfaces, this implies that Σ_a is a compact strictly achronal hypersurface in M, and thus is a topological Cauchy hypersurface in M. Theorem 4.4 implies that, for every $a \in (0, +\infty)$, Σ_a has generalized mean curvature bounded from below by $-1/a$, and bounded from above by $-1/((n-1)a)$. Let $(a_m)_{m\in\mathbb{N}}$ be a decreasing sequence of positive real numbers such that $a_m \to 0$ when $m \to +\infty$, and $(b_m)_{m\in\mathbb{N}}$ be a increasing sequence of positive real numbers such that $b_m \to +\infty$ when $m \to +\infty$. Observe that $(\Sigma_{a_m})_{m\in\mathbb{N}}$ is a sequence of past asymptotic α-barrier in M for $\alpha = -\infty$ (indeed $-\infty < -1/a_m < -1/((n-1)a_m)$ for every m, and since $-1/((n-1)a_m) \to -\infty$ when $m \to \infty$), and $(\Sigma_{b_m})_{m\in\mathbb{N}}$ is a sequence of future asymptotic β-barrier in M for $\beta = 0$ (indeed $-1/b_m < -1/((n-1)b_m) < 0$) for every m, and since $-1/b_m \to 0$ when $m \to \infty$). Hence Theorem 5.3 implies that M admits a globally defined CMC time function $\tau_{cmc} : M \to (-\infty, 0)$.

Next, we prove that τ and τ_{cmc} are comparable. It follows from theorem 4.4 that for every $a > 0$, the pair of hypersurfaces $\left(\Sigma_{a/(n-1)}, \Sigma_a\right)$ is a pair of $-1/a$-barriers. Hence, theorem 5.9 and remark 5.8 imply that the hypersurface $\tau_{cmc}^{-1}(-1/a)$ is in the future of $\Sigma_{a/(n-1)} = \tau^{-1}(a/(n-1))$ and in the past of $\Sigma_a = \tau^{-1}(a)$. Equivalently, one has

$$\tau \leq -\frac{1}{\tau_{cmc}} \leq (n-1)\tau.$$

The case where (M, g) is future geodesically incomplete is similar (except that (M, g) is the quotient of a *past complete* flat regular domain $E^-(\Lambda)$, and that one has to consider the *reverse* cosmological time of $E^-(\Lambda)$).

Finally, let us consider the case where (M, g) is geodesically complete. Then Theorem 2.10 states that up to a finite covering (M, g) is a quotient of $\mathbb{R}^{1,n-1}$ by a commutative subgroup Γ of $\mathrm{Isom}(\mathbb{R}^{1,n-1})$ generated by $n-1$ spacelike linearly independant translations $t_{\overrightarrow{u_1}}, \dots, t_{\overrightarrow{u_n}}$. Let $\overrightarrow{v}$ be any (say future-directed) timelike vector. Then, for every $t \in \mathbb{R}$, the affine plane $P_t := t.\overrightarrow{v} + \mathbb{R}.\overrightarrow{u_1} + \cdots + \mathbb{R}.\overrightarrow{u_n}$ is Γ-invariant. Hence it induces a totally geodesic spacelike hypersurface $\Sigma_t := \Gamma \backslash P_t$ in $M \simeq \Gamma \backslash \mathbb{R}^{1,n-1}$. The family of hypersurfaces $(\Sigma_t)_{t \in \mathbb{R}}$ is a foliation of M whose leaves are by totally geodesic (in particular, CMC) spacelike hypersurfaces.

In order to complete the proof of Theorem 1.2, we only need to prove that in the case where (M, g) is geodesically complete, every CMC Cauchy hypersurface Σ in M is a leaf of the totally geodesic foliation $(\Sigma_t)_{t \in \mathbb{R}}$ constructed above. Indeed, let $t^- = \inf\{t \text{ such that } \Sigma \cap \Sigma_t \neq \emptyset\}$ and $t^+ = \sup\{t \text{ such that } \Sigma \cap \Sigma_t \neq \emptyset\}$. Then, Σ is tangent to Σ_{t^-} at some point and is in the future of Σ_{t^-}. Hence, the maximum principle (Proposition 5.4) implies that the mean curvature of Σ is smaller or equal than those of Σ_{t^-}, i.e. is non-positive. Similarly, Σ is tangent to Σ_{t^+} at some point and is in the past of Σ_{t^+}, so by the maximum principle, the mean curvature of Σ is non-negative. So, we know that the mean curvature of Σ is equal to 0. And now, we use the equality case of the maximum principle (see, e.g., [7, Theorem 3.6]): *if S and S' are two CMC Cauchy hypersurfaces with the same mean curvature, which are tangent at some point, and such that S' is in the future of S, then $S = S'$.* This shows that $\Sigma = \Sigma_{t^-} = \Sigma_{t^+}$; in particular, Σ is a leaf of the totally geodesic foliation $(\Sigma_t)_{t \in \mathbb{R}}$.

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E-mail address: larsa@math.miami.edu

Albert Einstein Institute, Am Mühlenberg 1, D-14476 Potsdam, Germany and Department of Mathematics, University of Miami, Coral Gables, FL 33124, USA

E-mail address: Thierry.BARBOT@umpa.ens-lyon.fr

CNRS, UMPA, École Normale Supérieure de Lyon

E-mail address: Francois.Beguin@math.u-psud.fr

Laboratoire de Mathématiques, Université Paris Sud.

E-mail address: Abdelghani.ZEGHIB@umpa.ens-lyon.fr

CNRS, UMPA, École Normale Supérieure de Lyon