Homogeneous spaces, dynamics, cosmology: Geometric flows and rational dynamics

by

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ABSTRACT. The Ricci flow is a parabolic evolution equation in the space of Riemannian metrics of a smooth manifold. To some extent, Einstein equations give rise to a similar hyperbolic evolution. The present text is an introductory exposition to Bianchi-Ricci and Bianchi-Einstein flows, that is, the restricted finitely dimensional dynamical systems, obtained by considering homogeneous metrics.

1 Introduction

These notes are variations around the homogeneous space

$$X_n = Sym_n^+ = GL(n, \mathbb{R})/O(n)$$

that is, the space of $n \times n$ positive definite symmetric matrices, and sometimes, its subspace of those matrices with determinant 1,

$$Y_n = SSym_n^+ = SL(n, \mathbb{R})/SO(n)$$

Besides their striking beauty, these spaces modelize many structures and support fascinating geometry and dynamics. It is surely very interesting to investigate interplays between these aspects. We will not do it systematically here, but rather briefly note some of them[1].

[1]We would like to emphasis on that this is a preliminary short non-finished work. This explains in particular why many proofs are left as exercises. Also, this text may be considered as expository, although the method here does not follow any existing approach.

Our modest remark here is that "Bianchi" Ricci flows and Bianchi cosmologies are better seen as natural dynamical systems (i.e. differential equations) on X_n, respectively of first and second orders. To be more precise, Bianchi spaces are special homogeneous Riemannian spaces, those given by left invariant metrics on Lie groups. For a given Lie group G, the space of such metrics is identified with X_n, $n = \dim G$. The Ricci flow acting of the space of Riemannian metrics on a manifold, becomes here a (gradient-like) flow on X_n. Similarly, the Cauchy problem for the (vaccum) Einstein equations becomes here a dynamical system on TX_n. The group G acts preserving all these dynamical systems.

References

There are now abundant references on the Ricci flow techniques, since their use by Perelman in his program on the geometrization conjecture on 3-manifolds. We may quote as an example [10] as an interesting recent reference. The "toy" homogeneous case (with which we are dealing here) was in particular investigated in [13, 14].

The history of cosmology from a relativistic point of view, i.e. applying Einstein equations, is very old, and was in fact usually considered within a homogeneous framework, even an isotropic one, as in the standard big-bang model [12, 24, 25]. Even, concrete interplay between cosmology and the mathematical theory of dynamical systems, were involved in the literature, but this seems to be not well known by "mathematicians" (which gives motivation for our text here). As recent references, we may quote [2, 21, 20].

For the general material on homogeneous spaces, we quote [6, 9, 17, 18].

Finally, in these proceedings, recommended references would include [15, 22].

2 A multifaceted space

The general linear group $GL(n, \mathbb{R})$ acts transitively on the space of positive scalar products on $\mathbb{R}^n$, the stabilizer of the canonical scalar product being the orthogonal group $O(n)$. This space is therefore identified to the homogeneous space $GL(n, \mathbb{R})/O(n)$. It will be sometimes more convenient to deal with a reduced variant:

$$Y_n = SL(n, \mathbb{R})/SO(n)$$

which then represents conformal scalar products, or equivalently, scalar products with unit volume (i.e. their unit ball has volume 1 with respect to the canonical volume).

For a more intrinsic treatment, we start with a real vector space E, and consider $Sym(E)$ the space of its quadratic forms. Inside it, we have the open subspace of all non-degenerate ones $Sym^*(E)$, and $Sym^+(E)$ (or $X(E)$) the open cone of positive definite ones. We also get spaces of conformal structures by taking quotient by $\mathbb{R}$ acting by homothety; in particular $SSym^+(E)$ (or $Y(E)$) will denote the space of conformal positive definite structures. In the case of $E = \mathbb{R}^n$, we use the notations:

$$Sym_n,\ Sym_n^*,\ Sym_n^+(= X_n), SSym_n^+(=\ Y_n)$$

They are identified to subspaces of symmetric matrices $\{A = A^* \in M_{nn}\}$. The last three spaces correspond respectively to: $\det A \neq 0$, A has positive eigenvalues, and positive eigenvalues with $\det A = 1$.

Exercise 1. Show that the $GL(n, \mathbb{R})$-action on Sym_n is given by $g.A = (g^*)^{-1}Ag$ (where g^* is the transpose of g).

A metric on the space of metrics

A Euclidean structure q on a vector space E induces similar ones on associated spaces, in particular on the dual E^* and on $E^* \otimes E^*$. If (e_i) is a q-orthonormal basis, then its dual basis (e_i^*) is also orthonormal, and also is the basis $(e_i^* \otimes e_j^*)$.

Now, since Sym_n^+ is open in Sym_n, its tangent space at any point q is naturally identified to Sym_n, which is thus endowed with the scalar product $\langle, \rangle_q$. Therefore, Sym_n^+ becomes (tautologically) a Riemannian space.

Exercise 2.

- Show that

$$\langle p, p \rangle_q = tr(q^{-1}pq^{-1}p) \ (= tr(pq^{-1}pq^{-1})) \tag{1}$$

(where $p \in T_q(Sym_n^+)$is identified with a matrix $\in Sym_n$).

- Show that Sym_n^+ is isometric to the product $\mathbb{R}_+^* \times Y_n$, more precisely, one has an isometry: $A \in Sym_n^+ \mapsto (\log \det A, \frac{A}{\det A}) \in (\mathbb{R}, n\text{can}) \times Y_n$ (where can stands for the canonical metric of $\mathbb{R}$).

- Show that $q \mapsto q^{-1}$ is an isometry, that is, Y_n as well as X_n are Riemannian symmetric spaces.

Exercise 3. For $n = 1$, Y_n is $\mathbb{R}_+^*$ endowed with $\frac{dx^2}{x^2}$.

- Y_2 is "a hyperbolic plane", i.e. a homogeneous simply connected surface with (constant) negative curvature. Compute this constant. Does this correspond to a classical model of the hyperbolic plane?

- Show that the $SL(n, \mathbb{R})$-action on Y_n factors through a faithful action of $PSL(n, \mathbb{R})$.

Symmetric matrices vs quadratic forms

We guess it is worthwhile to seize the opportunity and clarify the relationship between quadratic forms and their representations as matrices. Let E be a vector space, and as above $Sym(E)$ and $Sym^+(E)$ its spaces of quadratic forms, and those which are positive definite, respectively. A basis (e_i) of E yields a matricial representation isomorphism

$$P \in Sym(E) \mapsto p = (P(e_i), P(e_j))_{ij} \in Sym_n (= Sym(\mathbb{R}^n))$$

Justified by a latter use (see § 8), the restriction to $Sym^+(E)$, will be denoted by: $Q \in Sym^+(E) \mapsto q \in Sym_n^+$.

In fact, these representations depend only on the scalar product on E for which the given basis is orthonormal.

Now, given $(Q, P) \in Sym^+(E) \times Sym(E)$, that is a pair of a scalar product together with a quadratic form on E, one associates a Q-autoadjoint endomorphism $f : E \to E$, representing P by means of Q, that is $P(x, y) = Q(x, f(y)) = Q(f(x), y)$, where P and Q are understood here as symmetric bilinear forms.

Fact 2.1 *Given (Q, P) and their associates (q, p), the endormorphism f has a matrix representation $A = q^{-1}p$.*

Conversely, given Q and a Q-autoadjoint endomorphism f, its corresponding quadratic form P has a matrix representation $p = qA$.

2. 1 Flats

Let $\mathcal{B} = (e_i)$ be a basis of E. The **flat** $F_{\mathcal{B}} \subset Sym^+(E)$ is the space of scalar products on E for which $\mathcal{B}$ is orthogonal. It is parametrized by n positive reals x_i, its elements have the form: $\Sigma x_i e_i^* \otimes e_i^*$.

Exercise 4. Show that the metric induced on $F_{\mathcal{B}}$ is given by $\Sigma_i \frac{dx_i^2}{x_i^2}$, and that

$$(t_1, \ldots, t_n) \in (\mathbb{R}^n, \text{can}) \mapsto \Sigma \exp(t_i) e_i^* \otimes e_i^* \in F_{\mathcal{B}}$$

is an isometric immersion.

Prove that:

- $F_{\mathcal{B}}$ is totally geodesic in $Sym^+(E)$.(Hint: make use of the isometries of Sym_n^+, $p \mapsto \sigma_i^* p \sigma_i$, where σ_i is the reflection fixing all the e_j, $j \neq i$, and $\sigma_i(e_i) = -e_i$).
- $GL(E)$ acts transitively on the set of flats of the form $F_{\mathcal{B}}$ (Hint: relate this to the simultaneous diagonalization of quadratic forms).
- Any geodesic of $Sym^+(E)$ is contained in some $F_{\mathcal{B}}$.

3 An individual left invariant metric

Standard references for this section and the following one are [6, 9, 17, 18].

Equation of Killing fileds

Let $(M, \langle , \rangle)$ be a pseudo-Riemannian manifold. A Killing field X is a vector field that generates a (local) flow of isometries. Any vector field X has a covariant derivative which is an endomorphism of TM defined by: $D_x X : u \in T_x M \mapsto \nabla_u X(x) \in T_x M$, where ∇ is the Levi-Civita connection of the metric.

Fact 3.1 *X is a Killing field, iff, $D_x X$ is skew-symmetric with respect to $\langle , \rangle_x$, for any $x \in M$.*

Proof. Let us first recall that this is the case for the Euclidean space: in fact, this is equivalent to that the Lie algebra of the orthogonal group is the space of skew-symmetric matrices.

In the general case, assume x generic, that is, $X(x) \neq 0$, and consider N a small transverse submanifold (to X at x). Let Y and Z be two vector fields defined on N, and extend them on open neighborhood of x, by applying the flow of X, that is by definition: $[X, Y] = [X, Z] = 0$.

If X is Killing, then $\langle Y, Z\rangle$ is constant along X: $X.\langle Y, Z\rangle = 0$. Thus, by definition of the Levi-Civita connection, $\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = 0$. Apply the commutations $\nabla_X Y = \nabla_Y X$, $\nabla_X Z = \nabla_Z X$, to get: $0 = \langle \nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle$, that is $D_x X$ is skew-symmetric.

It is also easy to use those arguments backwards, that is, if $D_x X$ skew-symmetric for any x, then X is a Killing field. □

Three Killing fields

Let now X, Y and Z be **three Killing fields**. Apply skew-symmetry for all their covariant derivatives, and get (at the end of substitutions) the following formula:

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle + \langle [Y, Z], X\rangle - \langle [Z, X], Y\rangle \tag{2}$$

Remark 3.1. Observe the beauty (= symmetry) and easiness of the formula!

Remark 3.2. Recall Koszul's formula for the Levi-Civita connection:

$$\begin{aligned} 2\langle \nabla_Y Z, X\rangle = \quad & Y\langle Z, X\rangle + Z\langle X, Y\rangle - X\langle Y, Z\rangle - \\ & \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle + \langle X, [Y, Z]\rangle \end{aligned} \tag{3}$$

It implies the previous formula of three Killing fields. Conversely, this last formula yields Koszul's one for any combination of Killing fields with coefficients (non-necessarily constant) functions on M. In particular (2) yields (3) if M is homogeneous. In fact, as (2) holds as X, Y and Z are pointwise Killing at order 1, (2) yields (3) also in the general non-homogeneous case.

3. 1 Left invariant metrics

We are interested now on Riemannian metrics on a Lie group G which are left invariant, i.e any **left** translation $x \mapsto gx$ is isometric. This is in particular the case of any flow $\varphi^t(x) = g^t x$, where $\{g^t\}$ is a one-parameter subgroup of G. Its infinitesimal generator $X(x) = \frac{\partial \varphi^t}{\partial t}(x)|_{t=0}$ is a Killing field. This is a **right** invariant vector field: $X(g) = X(1)g$.

Therefore, *a left invariant metric is exactly a metric admitting the right invariant fields as Killing fields.*

Another characterization is that *a left invariant metric is one for which left invariant fields have a constant length.* (but they are not necessarily Killing).

Such a metric is equivalent to giving a scalar product on the tangent space of one point in G, say $T_1 G$, i.e. the Lie algebra of G. We keep the same notation $\langle , \rangle$ for both the metric on G and the scalar product on its Lie algebra $\mathcal{G}$.

Here, to be precise, we define the Lie algebra $\mathcal{G}$ *as the space of right* invariant vector fields on G. (The other choice, i.e. that of left invariant vector fields, would induce modification of signs in some formulae).

From the formula of three Killing fields, one sees that the connection on G is expressed by means of the scalar product on $\mathcal{G}$ and the Lie bracket. In other words, one can forget the group G and see all things on the Lie algebra. For instance the Riemann curvature tensor is a 4-tensor on $\mathcal{G}$. The Ricci curvature is just a symmetric endomorphism of $\mathcal{G}$.

3. 2 The connection

Any Lie group has a canonical torsion free connection defined for right invariant vector fields by:

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

Any other left invariant connection can be written $\nabla_X Y = \frac{1}{2}[X, Y] + C(X, Y)$, where $C : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a bilinear map, and the connection is torsion free, iff, $C(X, Y) = C(Y, X)$. In the case of the Levi-Civita connection of a metric, one deduces from the formula of three Killing fields

$$\begin{aligned} 2\langle C(X, Y), Z\rangle &= \langle [X, Z], Y\rangle + \langle X, [Y, Z]\rangle \\ &= \langle ad^*_X Y, Z\rangle + \langle ad^*_Y X, Z\rangle \end{aligned} \tag{4}$$

(Here as usually $ad_u v = [u, v]$, and ad^* is its adjoint with respect to $\langle,\rangle$).
It then follows:

$$\nabla_X Y = \frac{1}{2}([X, Y] + ad^*_X Y + ad^*_Y X \tag{5}$$

Sectional curvature

The following formulae for various curvatures follow from Eq. (4), see for instance [6] for detailed proofs.

$$\begin{aligned} \langle R(X, Y)Y, X\rangle &= -\frac{3}{4}\langle [X, Y], [X, Y]\rangle - \frac{1}{2}\langle [X, [X, Y], Y\rangle \\ &\quad -\frac{1}{2}\langle [Y, [Y, X], X]\rangle + \langle C(X, Y), C(X, Y)\rangle \\ &\quad -\langle C(X, X), C(Y, Y)\rangle + \langle Y, [[X, Y], X]\rangle \end{aligned}$$

Ricci curvature

The tensor C disappears in the expression of the Ricci and scalar curvatures!

$$\begin{aligned} Ric(X, X) = &- \frac{1}{2}B(X, X) - \langle [Z, X], X\rangle \\ &- \frac{1}{2}\Sigma_i |[X, e_i]|^2 + \frac{1}{4}\Sigma_{ij}\langle [e_i, e_j], X\rangle^2 \end{aligned} \tag{6}$$

where:

- B is the Killing form of $\mathcal{G}$, that is the bilinear form $(X, Y) \mapsto tr(ad_X ad_Y)$,
- (e_i) is any orthonormal basis of $(\mathcal{G}, \langle,\rangle)$,
- and finally, Z is the vector of $\mathcal{G}$ defined by $\langle Z, Y\rangle = trace(ad_Y)$, that is, it measures the unimodularity defect of $\mathcal{G}$ by means of $\langle,\rangle$.

Scalar curvature

$$r = -\frac{1}{4}\Sigma_{ij}|[e_i, e_j]|^2 - \frac{1}{2}\Sigma_i B(e_i, e_i) - |Z|^2 \tag{7}$$

3. 3 Warning: left vs right

A metric is bi-invariant i.e. invariant under both left and right translations of G if and only if its associated scalar product is invariant under the Ad representation. In this case, the connection is the canonical one given by $\nabla_X Y = \frac{1}{2}[X, Y]$, for X and Y right-invariant vector fields. As said above, this torsion free connection exists on any Lie group, but does not in general derive from a Riemannian or a pseudo-Riemannian metric (i.e. it is not a Levi Civita connection). Its geodesics trough the neutral element are one-parameter groups, and its curvature is given by $R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$, and has a Ricci curvature $Ric(X, Y) = \frac{1}{4}B(X, Y)$, where B is the Killing form (recall that a connection, not necessarily pseudo-Riemannian, has a Ricci curvature, $Ric(X, Y) = tr(Z \mapsto R(X, Z)Y)$.

Other quantities

The following fact, left as an exercise, gives characterization of bi-invariant metrics:

Fact 3.2 *A left invariant metric is right invariant (and hence bi-invariant) iff it satisfies one of the following conditions:*

(1) For any right invariant (Killing) field X, $\langle X, X\rangle$ is constant on G.

(2) The orbits of any such X are geodesic

(3) The orbit of 1 ($\in G$) under any such X is a one parameter group.

All this suggests the possibility to define other quantities essentially equivalent to the Ricci curvature. Say, in the bi-invariant case, the Ricci curvature is essentially the Killing form, and hence, in the general left invariant case, the remaining part Re (of the Ricci curvature) is an obstruction to the constancy of $\langle X, X\rangle$, or (equivalently) an obstruction for one parameter groups to be geodesic. A naive construction goes as follows. For X a right invariant vector field, let $l^X : x \in G \mapsto \langle X(x), X(x)\rangle$, its length function, and dl_1^X its differential at 1. Define $Re(X, X) = tr(dl_1^X \otimes dl_1^X)$...

4 Curvature mappings on X_n

4. 1 All scalar products together: The space $Sym^+(\mathcal{G})$ et al

We are now considering all left invariant Riemannian metrics on G. As said above the space of such metrics can be identified with $Sym^+(\mathcal{G})$, the space of positive definite scalar products on $\mathcal{G}$. Let $Sym(\mathcal{G})$ be the space of all quadratic forms on $\mathcal{G}$. Then, the above formula for the Ricci curvature determines a map:

$$Ric : Sym^+(\mathcal{G}) \to Sym(\mathcal{G})$$

4. 2 $Aut(\mathcal{G})$-action on $Sym^+(\mathcal{G})$

Not only interior automorphisms of $\mathcal{G}$, but also "exterior" ones, i.e. general automorphisms act on $Sym(\mathcal{G})$. Their group $Aut(\mathcal{G})$ is sometimes identified with $Aut(G)$, assuming implicitly that G is simply connected. Of course, this action is compatible with all mappings that will be discussed below.

4. 3 All Lie algebras together

We can now go a step further and deal with all Lie algebras of a given dimension n. As vector spaces they are identified with $\mathbb{R}^n$. The space of quadratic forms and the positive definite one are Sym_n and Sym_n^+. In short, for any Lie algebra $\mathcal{G}$ (endowed with a basis allowing one to identify it with $\mathbb{R}^n$), we have a Ricci and a scalar curvature mapping, involving the bracket structure of $\mathcal{G}$:

$$Ric_{\mathcal{G}} : Sym_n^+ \to Sym_n$$

$$r_{\mathcal{G}} : Sym_n^+ \to \mathbb{R}$$

4. 4 Formulae

The Lie algebra $\mathcal{G}$ has a basis (e_i). An element of Sym_n is denoted by $p = (x_{ij})_{1\leq,i,j,\leq n}$. The structure constants C_{ij}^k, are defined by $[e_i, e_j] = C_{ij}^k e_k$.

(Here and everywhere in this paper, if a letter is repeated as a lower and an upper index, like k is the last equation, we use the Einstein summation convention, for example, $a_i^j b_j$ stands for $\Sigma_j a_i^j b_j$, etc...).

From Formula (6), we have:

$$Ric_{\mathcal{G}} : q = (x_{ab}) \in Sym_n^+ \mapsto p = (X_{ab}) \in Sym_n$$

Where:

$$X_{ab} = -\frac{1}{2}B_{ab} - \frac{1}{2}C_{ai}^k C_{bj}^l x_{kl} x^{ij} + \frac{1}{4}C_{ik}^p C_{jl}^q x_{pa} x_{qb} x^{ij} x^{kl} + \frac{1}{2}E_{ab} \tag{8}$$

and, $B_{ab} = C_{ai}^j C_{bj}^i$ is the matrix representing the Killing form, (x^{ij}) is the inverse matrix of (x_{ij}), and

$$E_{ab} = C_{ij}^j x^{is}(C_{sa}^l x_{lb} + C_{sb}^l x_{la})$$

This last term depends on q, but vanishes identically if $\mathcal{G}$ is unimodular. So, assuming $\mathcal{G}$ unimodular will simplify and shorten the formula.

4. 5 Parameter

To simplify, we will restrict ourselves to unimodular algebras, and so (E_{ab}) disappears. Any Lie algebra is defined by a system (C_{ij}^k) which furthermore satisfies the Jacobi identity. We can then consider a mapping,

$$Ric : (\overrightarrow{C}, q) = ((C_{ij}^k), (x_{ab})) \in \mathbb{R}^{n^3} \times Sym_n^+ \mapsto p = (X_{ab}) \in Sym_n$$
$$X_{ab} = -\frac{1}{2} C_{ai}^j C_{bj}^i - \frac{1}{2} C_{ai}^k C_{bj}^l x_{kl} x^{ij} + \frac{1}{4} C_{ik}^p C_{jl}^q x_{pa} x_{qb} x^{ij} x^{kl} \tag{9}$$

This map is equivariant with respect to the $GL(n, \mathbb{R}) \times GL(n, \mathbb{R})$-action on the source and the $GL(n, \mathbb{R})$-action on the target.

5 Three-dimensional case: Bianchi geometries (of class A)

A **Bianchi space** (or geometry) is a homogeneous Riemannian 3-manifold together with a transitive free action of a Lie group. This is therefore equivalent to giving a left invariant Riemannian metric on a 3-dimensional Lie group. These groups have been classified by Bianchi (see for instance [9, 20]). They split into classes A and B, according to they are unimodular or not. To simplify we will consider here only class A.

Milnor (or Bracket cyclic) bases of a 3-dimensional Lie algebra

Let $\mathcal{G}_{(a,b,c)}$ be the Lie algebra generated by $\{u, v, w\}$ with relations:

$$[u, v] = aw, [v, w] = bu, [w, u] = cv$$

It is easy to show that this is actually a Lie algebra, i.e. that the Jacobi identity is satisfied.

Conversely, a basis $\mathcal{B} = \{u, v, w\}$ of a Lie algebra $\mathcal{G}$ is called a **Milnor basis** ([9]) if it satisfies the previous relations: $[u, v] = aw, [v, w] = bu, [w, u] = cv$. In particular, $\mathcal{G}$ is then isomorphic to $\mathcal{G}_{(a,b,c)}$.

5. 1 Invariance of Milnor flats under Ric

Let us recall here that in dimension 3, giving the Ricci curvature is equivalent to giving the full Riemann curvature tensor (in higher dimension, Ricci is too weaker than Riemann).

Recall that the **flat** $F_\mathcal{B}$ determined by $\mathcal{B}$ is the space of scalar products on $\mathcal{G}$ for which $\mathcal{B}$ is orthogonal.

We will say that $F_\mathcal{B}$ is a **Milnor flat** if $\mathcal{B}$ is a Milnor basis.

The flat $F_\mathcal{B}$ is parametrized by 3 positive reals x, y and z, where:

$$x = \langle u, u \rangle, y = \langle v, v \rangle, \text{ and } z = \langle w, w \rangle.$$

The corresponding metric will be denoted by $(x, y, z) \in (\mathbb{R}^+)^3$

Define the **extended flat** $\overline{F_\mathcal{B}}$ to be the set of all quadratic forms which are diagonal in the basis $\mathcal{B}$, i.e. they have the same form as elements of $F_\mathcal{B}$, but x, y and z are allowed to be any real numbers.

One uses the orthonormal basis $\{\frac{u}{\sqrt{x}}, \frac{v}{\sqrt{y}}, \frac{w}{\sqrt{z}}\}$ for the metric (x, y, z), and computes from Formula (6) (or (9)),

$$\begin{cases} Ric(u,u) = & \frac{1}{2}(b^2\frac{x^2}{yz} - a^2\frac{z}{y} - c^2\frac{y}{z}) + ac \\ Ric(v,v) = & \frac{1}{2}(c^2\frac{y^2}{xz} - a^2\frac{z}{x} - b^2\frac{x}{z}) + ab \\ Ric(w,w) = & \frac{1}{2}(a^2\frac{z^2}{xy} - b^2\frac{x}{y} - c^2\frac{y}{x}) + bc \end{cases} \tag{10}$$

In fact, Ric is diagonal in the basis $\mathcal{B}$, i.e. $Ric(u,v) = \ldots = 0$. We can on the other hand perform some simplification, for instance, for $Ric(u,u)$ we have:

$$\begin{aligned} \frac{1}{2}(b^2\frac{x^2}{yz} - a^2\frac{z}{y} - c^2\frac{y}{z}) + ac &= \frac{1}{2yz}(b^2x^2 - c^2y^2 - a^2z^2 + 2acyz) \\ &= \frac{1}{2yz}(b^2x^2 - (cy - az)^2) \end{aligned}$$

Proposition 5.1 *Consider a 3-Lie algebra $\mathcal{G}$, and a Milnor flat $F_{\mathcal{B}} \subset Sym_3^+$. Then, the Ricci map sends:*

$$(x,y,z) \in F_{\mathcal{B}} \mapsto (X,Y,Z) \in \overline{F_{\mathcal{B}}}$$

and is given by:

$$\begin{cases} X = & \frac{1}{2yz}(b^2x^2 - (cy - az)^2) \\ Y = & \frac{1}{2xz}(c^2y^2 - (az - bx)^2) \\ Z = & \frac{1}{2xy}(a^2z^2 - (bx - cy)^2) \end{cases} \tag{11}$$

Remark 5.1. Observe the complete symmetry of these equations: there is for instance "a duality" $x \mapsto b$: everywhere the coefficient of x (resp. x^2) is b (resp. b^2). Recall that the coordinate x correponds to the vector u, and they are both related to the coefficient b, by the fact that the unique bracket proportional to u is $[v,w] = bu$. The same observation applies to the other variables, following a same correspondence: $(x,y,z) \mapsto (b,c,a)$.

5. 2 Scalar curvature

Similarly, from Formula (7), we infer:

$$r = \frac{1}{2xyz}(-b^2x^2 - c^2y^2 - a^2z^2 + 2acyz + 2abxz + 2bcxy). \tag{12}$$

6 Structure of unimodular Lie algebras in dimension 3

Proposition 6.1 *Any unimodular 3-Lie algebra has a Milnor basis. More precisely the map* $(a, b, c) \in \mathbb{R}^3 \mapsto \mathcal{G}_{(a,b,c)} \in \mathcal{L}$ *gives a parametrization of the space of unimodular 3-Lie algebras* $\mathcal{L}$*. The diagonal action of* $\mathbb{R}^{*3}$ *on* $\mathbb{R}^3$*, and that of the permutation group* S_3 *by permutation of coordinates, preserve the isomorphism classes of algebras.*

In fact, this is equivalent to the more precise:

Corollary 6.2 *There is six isomorphism classes of unimodular 3-algebras, represented as follows, where below* $+$ *(resp.* $-$*) means any positive (resp. negative) number, e.g.* $+1$ *(resp.* -1*).*

(1) $(0, 0, 0)$*: the abelian algebra* $\mathbb{R}^3$.

(2) $(0, 0, +)$*: the Lie algebra of* $G = Heis$*, the Heisenberg group.*

(3) $(0, -, +)$*: the Lie algebra of* $G = Euc$*, the group of rigid motions of the Euclidean plane (i.e. the isometry group of the affine Euclidean plane). It is a semi-direct product* $\mathbb{R} \ltimes \mathbb{R}^2$*, where* $\mathbb{R}$ *acts on* $\mathbb{R}^2$*, by*

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

(4) $(0, +, +)$*: the algebra of the group* $G = SOL$*, the group of rigid motions of the Minkowski plane (i.e. the Minkowski space of dimension* $1 + 1$*),* $G = \mathbb{R} \ltimes \mathbb{R}^2$*, where* $\mathbb{R}$ *acts on* $\mathbb{R}^2$ *by*

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

(5) $(-, +, +)$*:* $sl(2, \mathbb{R})$.

(6) $(+, +, +)$*:* $so(3)$.

They are labeled respectively, Bianchi: I*,* II*,* VII_0*,* VI_0*,* $VIII$ *and* IX*.*

Proof. (see [17]). In dimension 3, there are exactly two semi-simple Lie algebras, $so(3)$ and $sl(2, \mathbb{R})$, and there are no semi-simple Lie algebra of dimension ≤ 2. Therefore, if a 3-algebra $\mathcal{G}$ is not semisimple, then it contains no semisimple algebra, that is, $\mathcal{G}$ is solvable.

Assume $\mathcal{G}$ contains an (abelian) ideal isomorphic to $\mathbb{R}^2$. Take any supplementary one dimensional subspace (hence a subalgebra) $\mathbb{R}$. Then, $\mathcal{G}$ writes as a semi-direct product of $\mathbb{R}$ acting on $\mathbb{R}^2$. Since $\mathcal{G}$ is assumed to be unimodular, this action is via a one parameter group of $SL(2, \mathbb{R})$. These one parameter groups split into: elliptic, parabolic and hyperbolic types. We obtain respectively: Euc, $Heis$ and SOL.

It suffices therefore to show the existence of such an ideal $\mathbb{R}^2$. For this, let us remark that there exists always an abelian ideal of dimension 1, say $\mathcal{A} \cong \mathbb{R}$. Indeed, if the commutator ideal has dimension 2, then since it is solvable, its commutator has dimension 1 or 0...

Now, since $\mathcal{G}$ acts on $\mathcal{A}$, the Kernel $\mathcal{L}$ has at least dimension 2 and contains $\mathcal{A}$. Let us consider here the case $\dim \mathcal{L} = 2$, since the dimension 3 case is easier. By definition (of the Kernel) this is an ideal, and since it has dimension 2 and has a non-trivial center (it contains $\mathcal{A}$), then it is abelian. $\square$

7 Summary; further comments

In this section, we present a general setup where the previous constructions can be defined. In other words, we "summarize" how one associates to a Lie group various rational dynamical systems.

7. 1 A rational map

(§ 4. 4). Let $\mathcal{G}$ be a Lie algebra of dimension n, with a basis (e_i), such that $[e_i, e_j] = C_{ij}^k e_k$, and assume to simplify that it is unimodular. Then, we have a rational map:

$$\begin{array}{rcl} Ric_{\mathcal{G}} & : & (x_{ab}) \in Sym_n^+ \mapsto (X_{ab}) \in Sym_n \\ X_{ab} & = & -\frac{1}{2}B_{ab} - \frac{1}{2}C_{ai}^k C_{bj}^l x_{kl} x^{ij} + \frac{1}{4}C_{ik}^p C_{jl}^q x_{pa} x_{qb} x^{ij} x^{kl} \end{array} \tag{13}$$

as well as a rational (scalar curvature) function:

$$r_{\mathcal{G}} : (x_{ab}) \in Sym_n^+ \mapsto X_{ab} x^{ab} \tag{14}$$

(recall that (x^{ij}) is the inverse matrix of (x_{ij})).

7. 2 $Aut(\mathcal{G})$-equivariance

Both $Ric_{\mathcal{G}}$ and $r_{\mathcal{G}}$ are respectively equivariant and invariant under the $Aut(\mathcal{G})$-action on Sym_n (identified with $Sym(\mathcal{G})$, the space of quadratic forms on $\mathcal{G}$).

7. 3 Extensions

Actually, everything extends to left invariant pseudo-Riemannian metrics on G, or equivalently to $Sym^*(\mathcal{G})$ the space of non-degenerate quadratic forms on $\mathcal{G}$. The same formulae allow one to calculate the Ricci and scalar curvature of such metrics.

- Now, the formulae may have sense even for some degenerate quadratic forms!
- We can also consider complex quadratic forms on the complexification of $\mathcal{G}$. They form a space $Sym(\mathcal{G}) \otimes \mathbb{C}$ identified with $Sym_n(\mathbb{C})$, the space of symmetric complex matrices. In other words, to a Lie group G of dimension n is associated an $Aut(G)$-equivariant rational transformation Ric_G on $Sym_n(\mathbb{C})$. We also have an $Aut(G)$-invariant meromorphic function r_G.
- Ric_G can be written as a rational vectorial map $(\frac{P_i}{Q_i})_i : \mathbb{C}^{n(n+1)/2} \to \mathbb{C}^{n(n+1)/2}$ where P_i and Q_i are homogeneous polynomials of a same degree $2n$ (on $n(n+1)/2$ variables).
- From this, one gets a rational transformation of the projective space $\mathbb{C}P^{n(n+1)/2-1} \cong PSym_n(\mathbb{C})$. We will denote it by $\mathbf{Ric}_G$, to emphasize that it is the extension to the complex projective space. For instance, if $n = 3$, then we have a rational map on $\mathbb{C}P^5$.

- Ric_G is equivariant under scalar multiplication. In fact, as this follows from its defining formula, Ric_G is polynomial when restricted to matrices with $\det = 1$. Therefore, a representative of $\mathbf{Ric}_G$ (on the projective space) is the polynomial map $(P_i)_i$ (of degree $2n$).

- $\mathbf{Ric}_G$ is invariant under the (algebraic) action of (the complexification of) $Aut(\mathcal{G})$ on $\mathbb{C}P^{n(n+1)/2-1}(\cong PSym(\mathcal{G}) \otimes \mathbb{C})$. It then determines a map on the quotient space. It depends however on the meaning to give to such a quotient space (by $Aut(\mathcal{G})$). As an algebraic action, it has a poor dynamics, and a nice quotient space can be thus constructed for it. There is in particular a notion of "algebraic quotient". In dimension $n = 3$, the algebraic quotient space has dimension $5-3$, more exactly, it is a (singular) compact complex surface S_G, say. We have then associated to a 3-dimensional Lie group a rational map on a compact complex surface S_G. This map seems to have a "poor dynamics", for instance, it has in general a vanishing entropy. We guess nevertheless that (other) "dynamical invariants" of it can characterize the group G (i.e. two different groups have different invariants). It is also worthwhile to see what happens in higher dimension case.

7. 4 Forget invariance

In the formula defining $\mathbf{Ric}_G$, we can consider any system of parameters (C_{ij}^k), not necessarily satisfying the Jacobi identity of Lie algebras. We obtain a big family of rational transformations generalizing those associated to Lie groups. In this case, various dynamical types may appear. We think it is worthwhile to investigate the structure of this parameter space, and to understand inside it, the (algebraic) set of Lie algebras, the algebraic actions on it...

7. 5 Cross sections, Flats

Let us call a cross section $\mathcal{S}$ for Ric_G or $\mathbf{Ric}_G$ a submanifold in Sym_n^+ (resp. $PSym_n(\mathbb{C})$) which is invariant under Ric_G (resp. $\mathbf{Ric}_G$) and such that $\mathcal{S}$ meets any G-orbit in a non-empty discrete set. The last condition implies in particular that $\mathcal{S}$ is transversal (at least in a topological sense) to the G-orbits. In fact there are weaker variants of this definition which can be useful, in particular in a geometric algebraic context.

• It is in the case where G is unimodular and has dimension 3 that cross sections occur easily. In this case, one can in fact find them, as affine (resp. projective) subspaces for Ric_G (resp. $\mathbf{Ric}_G$) of dimension 3 (resp. 2). Indeed, as explained in § 6, the Lie algebra of such a group has a Milnor basis $\mathcal{B} = \{u, v, w\}$ (§ 5), and the flat $F_\mathcal{B}$ (§ 2. 1), or more formally its "extension" $\overline{F_\mathcal{B}}$, i.e. the space of quadratic forms diagonalizable in $\mathcal{B}$, is invariant under Ric_G. In order to see that one gets in this way a cross section, it remains to show the abundance of Milnor bases as in the following statement,

Exercise 5. Prove that any quadratic form can be digonalized in some Milnor basis of $\mathcal{G}$. *(Hint: this can be done by checking case by case. For instance, for the group SOL, its Lie algebra is generated by X, Y, Z, with relations $[X, Y] = Y$, $[X, Z] = -Z$ and $[Y, Z] = 0$. Consider $u = X + T$, where T belongs the the plane $\mathcal{P}$ generated by Y and Z. Then, the* *restriction of ad_u on $\mathcal{P}$, satisfies $ad_u^2 = -1$. Choose u to be orthogonal to $\mathcal{P}$ (with respect to the given metric). Consider a non-vanishing vector $v \in \mathcal{P}$, and let $w = ad_u(v)$. Then $\{u, v, w\}$ is a Milnor basis, because of the fact $ad_u^2 = 1$. We claim that v can be chosen such that w is orthogonal to v. This is a calculation in the basis $\{Y, Z\}$).*

• Let us point out the following polynomial presentation of Ric_G. As was said above, Ric_G is invariant under scalar multiplication, that is, it suffices to consider its restriction on unimodular matrices $SSym_n$, in which case, it becomes polynomial. In particular, from § 5. 1, $\mathbf{Ric}_G$ has the following form as a cubic homogeneous polynomial map:

$$\mathbf{Ric}_G : \mathbb{C}^3 \to \mathbb{C}^3 \tag{15}$$

$$(x, y, z) \mapsto \frac{1}{2}(x(b^2x^2 - (cy - az)^2), y(c^2y^2 - (az - bx)^2), z(a^2z^2 - (bx - cy)^2)).$$

7. 6 Formula in each case

Case of $SO(3)$:

$a = b = c = 1$.

$$Ric(x, y, z) = \frac{1}{2}(x(x^2 - (y - z)^2), y(y^2 - (z - x)^2), z(z^2 - (x - y)^2)).$$

Case of $SL(2, \mathbb{R})$:

$a = b = 1$, and $c = -1$.

$$Ric(x, y, z) = \frac{1}{2}(x(x^2 - (y + z)^2), y(y^2 - (z - x)^2), z(z^2 - (x + y)^2)).$$

Case of the Heisenberg group $Heis$:

$a = b = 0, c = 1$

$$Ric(x, y, z) = \frac{1}{2}y^2(-x, y, -z).$$

Case of Euc:

$a = 0, b = -1, c = 1$

$$Ric(x, y, z) = \frac{1}{2}(x(x^2 - y^2), y(y^2 - x^2), -z(x + y)^2)$$

Case of SOL:

$a = 0, b = c = 1$

$$Ric(x, y, z) = \frac{1}{2}(x(x^2 - y^2), y(y^2 - x^2), -z(x - y)^2)$$

7. 7 Bianchi-Ricci flow

Recall that the Ricci flow associated to a compact manifold M (of finite volume) is an evolution equation on its space $Met(M)$ of Riemannian metrics:

$$\frac{\partial g_t}{\partial t} = -2Ric(g_t) + 2\frac{< r(g_t) >}{n} g_t$$

where $n = \dim M$ and $< r(g) >= \int r(g) dv_g / Vol(M, g)$ is the average scalar curvature of g [9].

The vector field $\mathcal{R}ic_G$

Now, if G is an n-dimensional Lie group, then this gives a classical differential equation on the space of its left invariant Riemannian metrics, where one takes a punctual value of the scalar curvature instead of its average (since this scalar curvature is constant). Equivalently, this is a vector field on Sym_n^+. In fact, all this is derived from our previous rational map Ric_G. Since Sym_n^+ is an open set in the vector space Sym_n, the vectorial map Ric_G : $Sym_n^+ \to Sym_n$ can be alternatively seen as a **vector field** on Sym_n^+, say, $\mathcal{R}ic_G$.

Observe that $\mathcal{R}ic_G$ is invariant under the $Aut(G)$-action on Sym_n^+ (which is equivalent to the fact that Ric_G is equivariant under the (linear) action of $Aut(G)$).

Let us denote a generic point of Sym_n^+ by q, and consider the radial vector field $\mathcal{V}(q) = q$. The previous differential equation, which we will call the Bianchi-Ricci flow associated to G is the vector field

$$-2\mathcal{R}ic_G + 2\frac{r_G}{n}\mathcal{V}$$

Commutation

Consider the bracket $[\mathcal{R}ic_G, \mathcal{V}] = D_{\mathcal{V}} Ric_G - D_{Ric_G}\mathcal{V}$, where D_u denotes the usual derivation in the u-direction. This equals $0 - Ric_G$, since, $D_{\mathcal{V}} Ric_G = 0$, i.e. the map Ric_G is invariant under multiplication; and $D\mathcal{V} = Identity$, everywhere. Therefore, $[\mathcal{R}ic_G, \mathcal{V}] = -\mathcal{R}ic_G$. Because of this commutation rule (that is, the two vector fields generate a local action of the affine group), the essential dynamics of the Bianchi-Ricci flow comes from the $\mathcal{R}ic_G$-part.

Bianchi-Hilbert-Ricci flow

The remark applies to any combination of $\mathcal{R}ic_G$ and $\mathcal{V}$: understanding one combination allows one to understand the others. A famous one is $\mathcal{E}in = \mathcal{R}ic_G - \frac{r}{2}\mathcal{V}$, which can be called in this context the “Bianchi-Einstein flow”, since the tensor $Ric(g) - \frac{r}{2}g$ of a Riemannian manifold (M, g) is called Einstein tensor (this is, essentially, the unique combination of $Ric(g)$ and g which is divergence free). However, in order to prevent confusion with “Einstein equations” and some related flows which will be considered below, $\mathcal{E}in$ could be better called Bianchi-Hibert flow. Indeed, the function

$$\mathcal{H} : p \in Sym_n^+ \mapsto r(p)\sqrt{\det(p)} \in \mathbb{R}$$

is the substitute of the classical Hilbert action in the case of left invariant metrics. Indeed:

Exercise 6. Show that $\mathcal{E}in$ is a gradient vector field. More exactly, $\mathcal{E}in = \nabla\mathcal{H}$, where the gradient ∇ is taken with respect to the metric of Sym_n^+.

Remark 7.1. The computation can be handled in a more explicit way on a Milnor flat $F_{\mathcal{B}}$ (§ 5. 1), where the Hilbert action has the form:

$$\mathcal{H}(x,y,z) = \frac{1}{2\sqrt{xyz}}(-b^2x^2 - c^2y^2 - a^2z^2 + 2acyz + 2abxz + 2bcxy)$$

and the metric is

$$\frac{dx^2}{x^2} + \frac{dy^2}{y^2} + \frac{dz^2}{z^2}$$

Restriction on $SSym_n^+$

The interest of the normalization in the definition of the Ricci flow is to let it preserving the volume of the Riemannian metric, that is, the total volume remains constant under evolution. In the case of left invariant metrics, this is equivalent to the fact that the vector field $\mathcal{R}ic_G - \frac{r}{n}\mathcal{V}$ is tangent to $SSym_n$. This in turn is equivalent to the fact, that for any $q \in SSym_n$, $Ric_G(q) - \frac{r(q)}{n}q$ is trace free, which follows from the very definition of the scalar curvature r. This allows one to justify the following simplification: write equations assuming $q \in SSym_n$, i.e. $\det(q) = 1$, which gives polynomial equation. However, in order to keep this polynomial natural, do not take reduction of variables from the equation $\det(q) = 1$. To be more concrete, consider a flat $F_{\mathcal{B}}$, then instead of the rational forms of $Ric_G(x,y,z)$ and $r_G(x,y,z)$, we assume $xyz = 1$ which leads to polynomial forms: §§ 5. 1 and Formula 12 (but we do not go further and eliminate one variable, say $z = \frac{1}{xy}$). We can then write the Bianchi-Ricci flow as follows

$$-2\mathcal{R}ic_G + 2\frac{r}{3}\mathcal{V}(x,y,z) = \begin{cases} x(\frac{2}{3}bx(-2bx + cy + az) + \frac{2}{3}c^2y^2 + \frac{2}{3}a^2z^2 - \frac{4}{3}acyz) \\ y(\frac{2}{3}cy(-2cy + bx + az) + \frac{2}{3}b^2x^2 + \frac{2}{3}a^2z^2 - \frac{4}{3}bcxz) \\ z(\frac{2}{3}az(-2az + bx + cy) + \frac{2}{3}b^2x^2 + \frac{2}{3}c^2y^2 - \frac{4}{3}bcxy) \end{cases} \tag{16}$$

Differential equations on a projective space

In the same way, we associate to a Lie group G an $Aut(G)$-invariant one dimensional complex algebraic foliation on the projective space $PSym_n(\mathbb{C})$. Here, among combinations of the vector fields $\mathcal{R}ic_G$ and $\mathcal{V}$, only $\mathcal{R}ic_G$ is relevant, since the radial vector field $\mathcal{V}$ becomes trivial on the projective space. In the case of a unimodular 3-group, we have the following homogeneous cubic differential system on $\mathbb{C}^3$:

$$
\left\{
\begin{array}{rl}
\frac{dx}{dt} = & x(b^2x^2 - (cy - az)^2) \\
\\
\frac{dy}{dt} = & y((c^2y^2 - (az - bx)^2) \\
\\
\frac{dz}{dt} = & z(a^2z^2 - (bx - cy)^2)
\end{array}
\right.
\tag{17}
$$

Dynamics, compactifications

It is the dynamics of the Bianchi-Ricci flow $-2\mathcal{R}ic_G + \frac{2r}{3}\mathcal{V}$ which was investigated in the literature [9, 13]. As we argued above, this is essentially the same as that of the Einstein-Hilbert field $\nabla\mathcal{H}$. But, as a gradient flow, its dynamics is completely trivial on Sym_n^+... The point is to study the behavior of orbits when they go to an infinity boundary $\partial_\infty Sym_n^+$. There is however several ways to attach such a boundary to (the non-positively curved Riemannian symmetric space) Sym_n^+. One naturally wants to interpret ideal points as collapsed Riemannian metrics. With respect to this, the Hadamard compactification seems to be the most pertinent (see for instance [16]). On the other hand, the advantage of algebraic compactifications (e.g. the projective space) is to extend the dynamics...

8 Hamiltonian dynamics on Sym_n^+

After consideration of some maps and vector fields, we are now going to study second order differential equations on Sym_n^+, the prototype of which is the geodesic flow of Sym_n^+, and then the "Einstein flow" associated to a Lie group.

8. 1 Geodesic flow

Write the metric on Sym_n^+ as: $L(q,p) = \langle p,p\rangle_q = tr(q^{-1}pq^{-1}p)$. Since Sym_n^+ is open in Sym_n, its tangent bundle trivializes $TSym_n^+ = Sym_n^+ \times Sym_n$. We will use the usual notations $\frac{\partial L}{\partial q}$, $\frac{\partial L}{\partial p}$ for the horizontal and vertical differentials d_pL and d_qL.

We have:

$$\frac{\partial L}{\partial q}(\delta q) = tr(-q^{-1}(\delta q)q^{-1}pq^{-1}p - q^{-1}pq^{-1}(\delta q)q^{-1}p) = -2tr((\delta q)q^{-1}pq^{-1}pq^{-1})$$

where δq is a horizontal tangent vector, i.e. an element of Sym_n.

$$\frac{\partial L}{\partial p}(\delta p) = 2tr((\delta p)q^{-1}pq^{-1}).$$

Now, write: $q = q(t),\ p(t) = \dot{q} = \frac{\partial q}{\partial t}$, and compute

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial p}(\delta p) = 2tr((\delta p)q^{-1}[-2\dot{q}q^{-1}\dot{q} + \ddot{q}]q^{-1}).$$

The Euler-Lagrange equation is obtained by taking $\delta q = \delta p = A$, and writing for any A,

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial p}A - \frac{\partial L}{\partial q}A = 0.$$

This reads:

$$tr(2Aq^{-1}(-\dot{q}q^{-1}\dot{q} + \ddot{q})q^{-1}) = 0,\ \forall A \in Sym_n.$$

and therefore,

Fact 8.1 *The equation of geodesics of Sym_n^+ is the second order matricial equation on Sym_n^+:*

$$\ddot{q} = \dot{q}q^{-1}\dot{q}$$

or equivalently (in the phase space):

$$\begin{cases} \dot{q} = & p \\ \dot{p} = & pq^{-1}p. \end{cases} \tag{18}$$

8. 2 Other pseudo-Riemannian and Finsler metrics on Sym_n^+

There is a canonical $GL(n, \mathbb{R})$-invariant form ω on Sym_n^+:

$$\omega_q(p) = tr(q^{-1}p).$$

We can then associate to any reals α and β a Lagrangian:

$$L_{\alpha,\beta}(q,p) = \alpha(\omega_q(p))^2 + \beta\langle p, p\rangle_q = \alpha(tr(q^{-1}p))^2 + \beta tr(q^{-1}pq^{-1}p).$$

For generic α and β, this is a homogeneous pseudo-Riemannian metric, but it can degenerate for some values.

Similarly, there are homogeneous Finsler metrics:

$$F_{\alpha,\beta}(q,p) = \alpha\omega_q(p) + \beta\sqrt{\langle p, p\rangle_q} = \alpha(tr(q^{-1}p)) + \beta\sqrt{tr(q^{-1}pq^{-1}p}.$$

Exercise 7. Write the Euler-Lagrange equation for $L_{\alpha,\beta}$ and $F_{\alpha,\beta}$.

Solve the geodesic equation for Sym_2.

9 Einstein Equations in a Gauss gauge

Cylinders

Let M be a differentiable n-manifold endowed with a family of Riemannian metrics g_t, t is a ”time” parameter lying in an interval I. Consider the Lorentz manifold $\bar{M} = I \times M$ endowed with the metric

$$\langle , \rangle = \bar{g} = -dt^2 + g_t, \text{ i.e. } \bar{g}_{(t,x)} = -dt^2 + (g_t)_x$$

Such a structure is sometimes called a cylinder. Our purpose is to relate geometric (e.g. curvature) quantities on $\bar{M}$ and M. For a fixed point (t, x), R, Ric, and r will denote the Riemann, Ricci and scalar curvatures of (M, g_t) at x and ∇ its Levi-Civita connection. The corresponding quantities for $\bar{M}$ are noted by $\bar{R}$, $\bar{Ric}$ and $\bar{r}$ and $\bar{\nabla}$.

9. 1 Second fundamental form

The (scalar) second fundamental form of $\{t\} \times M$ is denoted k_t (or sometimes simply k). Actually, the second fundamental form is defined as a vectorial form: $II(X, Y)$ equals the orthogonal projection of $\bar{\nabla}_X Y$ on $\mathbb{R}e_0$, where $e_0 = \frac{\partial}{\partial t}$.

The scalar second fundamental form is defined by

$$k(X, Y) = \langle II(X, Y), e_0\rangle = \langle \bar{\nabla}_X Y, e_0\rangle.$$

The Weingarten map $a = a_{e_0}$ is defined by:

$$a(X) = -\bar{\nabla}_X e_0.$$

We have:

$$k(X, Y) = \langle \bar{\nabla}_X Y, e_0\rangle = X\langle Y, e_0\rangle - \langle \bar{\nabla}_X e_0, Y\rangle = 0 + \langle a(X), Y\rangle.$$

In other words, a is the symmetric endomorphism associated to k by means of the metric g (we will use sometimes the notation a_t as well as g_t and k_t, in order to emphasize the dependence on t).(The definition of k and a coincides with that in the Riemannian case. The unique difference is that here, $II = -ke_0$, since e_0 is unit timelike, i.e. $\langle e_0, e_0\rangle = -1$).

9. 2 Geometry of the product

Consider $e_1, \dots, e_n$ a frame of vector fields on M, that we also consider as horizontal vector fields on $\bar{M}$. By definition, they commute with $e_0(= \frac{\partial}{\partial t})$.

Fact 9.1 *We have:*

$$\begin{aligned}
\bar{\nabla}_{e_0} e_0 &= 0 \; (\textit{the trajectories of } e_0 \textit{ are geodesic}), && (19)\\
k_t &= (-1/2)\frac{\partial}{\partial t} g_t, && (20)\\
\langle \bar{R}(e_0, e_i)e_i, e_0\rangle &= \frac{\partial}{\partial t}\langle a_t(e_i), e_i\rangle + \langle a_t^2(e_i), e_i\rangle, && (21)\\
\bar{Ric}(e_0, e_0) &= \frac{\partial}{\partial t} tr(a_t) + tr(a_t^2). && (22)
\end{aligned}$$

Proof. • We have

$$0 = \partial/\partial t\langle e_0, e_i\rangle = \langle \bar{\nabla}_{e_0} e_0, e_i\rangle + \langle e_0, \bar{\nabla}_{e_0} e_i\rangle.$$

But

$$\langle e_0, \bar{\nabla}_{e_0} e_i\rangle = (1/2)e_i.\langle e_0, e_0\rangle = 0,$$

since e_0 and e_i commute. Therefore $\langle \bar{\nabla}_{e_0} e_0, e_i\rangle = 0$, $\forall i$.

• We have

$$\frac{\partial}{\partial t} g_t(e_i, e_j) = e_0\langle e_j, e_j\rangle = \langle \bar{\nabla}_{e_0} e_i, e_j\rangle + \langle \bar{\nabla}_{e_0} e_j, e_i\rangle$$

Since e_0 commutes with e_i and e_j, this also equals:

$$-\langle (a(e_i), e_j\rangle - \langle a(e_j), e_i\rangle = -2k_t(e_i, e_j)$$

• *Computation of* $\bar{R}(e_0, e_i)e_i, e_0\rangle$ï¿½: Because of the commutation relations, and because e_0 is geodesic, we have, by definition of the curvature:

$$\bar{R}(e_0, e_i)e_0 = -\bar{\nabla}_{e_0} a(e_i),$$

and thus:

$$\langle \bar{R}(e_0, e_i)e_0, e_i\rangle = -\langle \bar{\nabla}_{e_0} a(e_i), e_i\rangle = -e_0\langle a(e_i), e_i\rangle - \langle a(e_i), a(e_i)\rangle,$$

and since a is symmetric, this also equals:

$$-\frac{\partial}{\partial t}\langle a(e_i), e_i\rangle - \langle a^2(e_i), e_i\rangle$$

And hence,

$$\langle \bar{R}(e_0, e_i)e_i, e_0\rangle = \frac{\partial}{\partial t}\langle a_t(e_i), e_i\rangle + \langle a_t^2(e_i), e_i\rangle$$

• We can assume that at a fixed point (t, x), the basis $(e_i)_{i\geq 1}$ is orthonormal, and taking the sum (over $i > 0$) we get:

$$\bar{Ric}(e_0, e_0) = \frac{\partial}{\partial t} tr(a_t) + tr(a_t^2).$$

□

Remark 9.1. In fact, the meaning of "Gauss gauge" is nothing but that e_0 is unit and has geodesic orbits.

9. 3 Gauss equation

It describes the relationship between the sectional curvatures for R and $\bar{R}$:

$$\begin{aligned}\langle \bar{R}(e_i, e_j)e_j, e_i\rangle = \langle R(e_i, e_j)e_j, e_i\rangle &+ k(e_i, e_i)k(e_j, e_j)\\ &- k(e_i, e_j)k(e_i, e_j)\end{aligned} \tag{23}$$

(observe this difference of sign of the k-term, in comparison with the Riemannian case).

9. 4 Einstein evolution equation for k_t

Again, assume (e_i) orthonormal, fix i, and take the sum over $j > 0$. We first have:

$$\Sigma_j k(e_i, e_i)k(e_j, e_j) = k(e_i, e_i)tr(a) = tr(a)\langle(a(e_i), e_i)\rangle$$

and

$$\Sigma_j k(e_i, e_j)k(e_i, e_j) = \langle a^2(e_i), e_i)\rangle$$

(Indeed in matricial notations, $a_{ij} = a_{ji} = k(e_i, e_j)$, and thus $(a^2)_{ii} = \Sigma_j a_{ij}a_{ji}$).

Therefore, if we consider the quadratic form l, defined by:

$$l(e_i, e_i) = \Sigma_j(k(e_i, e_i)k(e_j, e_j) - k(e_i, e_j)k(e_i, e_j))$$

then its associated endomorphism is:

$$tr(a)a - a^2$$

• $\bar{Ric}(e_i, e_i)$ equals the trace of $u \to R(u, e_i)e_i$. Remember, (e_i) is a Lorentz orthonormal basis, i.e. $\langle e_i, e_j\rangle = 0$, for $i \neq j$, $\langle e_0, e_0\rangle = -1$ and $\langle e_j, e_j\rangle = +1$, for $j > 0$. It then follows that

$$\bar{R}(e_i, e_i) = \sum_{j>0}\langle\bar{R}(e_i, e_j)e_j, e_i\rangle - \langle\bar{R}(e_0, e_i)e_i, e_0\rangle$$

• Returning to the Gauss equation (23), and taking the sum over $j > 0$, we get:

$$\bar{Ric}(e_i, e_i) + \langle\bar{R}(e_0, e_i)e_i, e_0\rangle = Ric(e_i, e_i) + \langle(tr(a)a - a^2)(e_i), e_i\rangle$$

• Replacing $\langle\bar{R}(e_0, e_i)e_0, e_i\rangle$ by its previous value:

$$\bar{Ric}(e_i, e_i) + \frac{\partial}{\partial t}\langle a(e_i), e_i\rangle + \langle a^2(e_i), e_i\rangle = Ric(e_i, e_i) + \langle(tr(a)a - a^2)(e_i), e_i\rangle$$

Equivalently, for any $X, Y \in TM$:

$$\frac{\partial}{\partial t}k_t(X, Y) = -\bar{Ric}(X, Y) + Ric(X, Y) + \langle(tr(a_t)a_t - 2a_t^2)(X), Y\rangle.$$

Fact 9.2 *Define the square power $k_{t g_t} k_t$ to be the quadratic form associated by means of g_t with the matrix a_t^2 (where a_t is the matrix associated to k_t via g_t). Then:*

$$\frac{\partial}{\partial t}k_t = -\bar{Ric} + Ric + tr_{g_t}(k_t)k_t - 2k_{t g_t}k_t.$$

9. 5 Gauss constraints

Consider again the Gauss equation and take the sum over $i, j > 0$:

$$\bar{r} - 2\bar{Ric}(e_0, e_0) = r + (tr_{g_t}k_t)^2 - tr(k_{t g_t}k_t) = r + (tr_{g_t}k_t)^2 - |k_t|^2_{g_t}.$$

9. 6 Matrix equations

We are now going to write equations by means of symmetric matrices associated to the quadratic forms g_t and k_t (see § 2). For this, we fix x and a time t_0 and choose an orthonormal basis $(e_i(t_0))$ of T_xM. We denote by q_t (resp. p_t) the matrix associated with g_t (resp. $-2k_t$), and by $\bar{ric}_t$ and ric_t (or simply $\bar{ric}$ and ric) those associated with $\bar{Ric}$ and Ric (recall they are the Ricci curvatures of respectively $\bar{M}$, at (t, x), and (M, g_t), at x). With this, we have:

- Evolution equations:

$$\begin{cases} \dot{q} = & p \\ \dot{p} = & -\bar{ric} + ric + \frac{1}{4}tr(q^{-1}p)p - \frac{1}{2}q^{-1}pq^{-1}p \end{cases} \tag{24}$$

- Gauss constraints (actually said **Hamiltonian constraints**)

$$\begin{aligned} \bar{r} - 2\bar{ric}(e_0, e_0) &= r + (tr(q^{-1}p))^2 - tr(q^{-1}pq^{-1}p) \\ &= r + tr(q^{-1}p)^2 - \langle p, p\rangle_q \\ &= r - L_{-1,1}(q, p) \end{aligned}$$

  where $L_{-1,1}$ is the pseudo-Riemannian metric defined in § 8, for the value $(\alpha, \beta) = (-1, 1)$

10 Bianchi cosmology

We will now restrict ourselves to the vacuum case, i.e. $\bar{M}$ is Ricci-flat: $\bar{Ric} = 0$, and thus also $\bar{r} = 0$. We will also assume M is a Lie group G and the metrics on it (i.e. g_t) are left invariant. Therefore such a metric is identified with an element $q \in Sym_n^+$ ($n = \dim G$, the identification of $Sym^+(\mathcal{G})$ with Sym_n^+ comes from a choice of a basis). Now, ric becomes a map $ric : Sym_n^+ \mapsto Sym_n$, and $r : Sym_n \mapsto \mathbb{R}$. We get the (beautiful) ODE system with constraints:

$$\begin{cases} \dot{q} = & p \\ \dot{p} = & ric(q) + \frac{1}{4}tr(q^{-1}p)p - \frac{1}{2}q^{-1}pq^{-1}p \\ L_{-1,1}(q, p) = & r(q) \ \text{(Hamiltonian constraint)} \end{cases} \tag{25}$$

Remark 10.1. Observe that ric and r are basic functions, they depend only on q (and not on p).

10. 1 Isometric G-action on $\bar{M}$

Here $\bar{M} = I \times G$, with $\bar{g} = -dt^2 + g_t$. A left translation $x \in G \mapsto hx$ is isometric for all the metrics g_t, and therefore is isometric for $\bar{g}$ as well.

10. 2 The Bianchi-Einstein flow along and on a flat

Actually, there are other constraints to add to the ODE system above, in order to get what we will call the **Bianchi-Einstein flow**. These (momentum) constraints will be considered below. Before, let us consider a subsystem of it, the restriction (of everything) to a Milnor flat $F_{\mathcal{B}}$. The following proposition derives from Formulae (11) and (12).

Proposition 10.1 *Let $F_{\mathcal{B}}$ be a Milnor flat, and $TF_{\mathcal{B}}$ its tangent bundle, a point of which is denoted by (q, p), $q = (x, y, z)$, $p = (x', y', z')$. The* **Bianchi-Einstein** *flow on $F_{\mathcal{B}}$ is the following system of ODE on $TF_{\mathcal{B}}$, together with one algebraic constraint defined by a Lorentz metric on $TF_{\mathcal{B}}$ and a basic function on $F_{\mathcal{B}}$. (The phase space has thus dimension 5, and is a fiber bundle over $F_{\mathcal{B}}$):*

$$\begin{cases} \dot{x} = & x' \\ \dot{y} = & y' \\ \dot{z} = & z' \\ \dot{x'} = & \frac{1}{2yz}(b^2x^2 - (cy - az)^2) + \frac{1}{4}(\frac{x'}{x} + \frac{y'}{y} + \frac{z'}{z})x' - \frac{1}{2}\frac{x'^2}{x^2} \\ \dot{y'} = & \frac{1}{2xz}(c^2y^2 - (az - bx)^2) + \frac{1}{4}(\frac{x'}{x} + \frac{y'}{y} + \frac{z'}{z})y' - \frac{1}{2}\frac{y'^2}{y^2} \\ \dot{z'} = & \frac{1}{2xy}(a^2z^2 - (bx - cy)^2) + \frac{1}{4}(\frac{x'}{x} + \frac{y'}{y} + \frac{z'}{z})z' - \frac{1}{2}\frac{z'^2}{z^2} \end{cases} \tag{26}$$

The phase space is a hypersurface (maybe singular) N (in $TF_{\mathcal{B}}$) defined by the Hamiltonian equation:

$$l_{(x,y,z)}(x', y', z') = -\frac{r(x, y, z)}{2} \tag{27}$$

Where l is the Lorentz metric (on $F_{\mathcal{B}}$):

$$l_{(x,y,z)}(x', y', z') = \frac{x'y'}{xy} + \frac{x'z'}{xz} + \frac{y'z'}{yz}, \tag{28}$$

and r is given by Formula (12):

$$r(x, y, z) = \frac{1}{2xyz}(-b^2x^2 - c^2y^2 - a^2z^2 + 2acyz + 2abxz + 2bcxy)$$

Exercise 8. Show explicitly that the constraint is preserved by the dynamics, i.e. the vector field determined by the differential equations is tangent to the “submanifold” $N \subset TF_{\mathcal{B}}$ defined by the constraint.

10. 3 Codazzi (or Momentum) constraints

The Codazzi equation establishes a relation between the intrinsic and extrinsic curvatures of a submanifold M in a Riemanniann manifold $\bar{M}$, and is in fact valid in the general background of pseudo-Riemannian manifolds provided the induced metric on the submanifold is also pseudo-Riemannian, i.e. it is not degenerate. More precisely, it states that some "partial symmetrisation" of the covariant derivative of the second fundamental form (all this depends only upon data on M) equals the normal part of the Riemann curvature tensor (this depends on $\bar{M}$). The equation gives obstructions for a (vectorial) 2-tensor to be the second fundamental form of a submanifold.

In the case where M is a **CMC** spacelike hypersurface (i.e. with a constant mean curvature) in a Ricci flat Lorentz manifold $\bar{M}$, one can deduce from Codazzi equation, by taking a trace, that the second fundamental form k is a divergence free 2-tensor. This applies in particular to our case (our hypersurfaces are G-orbits and thus are CMC).

Let us recall some definitions. First, if k is a symmetric 2-tensor on M, then its covariant derivative $\nabla_X k$ with respect to a vector X, is a 2-tensor:

$$(\nabla_X k)(Y, Z) = Xk(Y, Z) - k(\nabla_X Y, Z) - k(Y, \nabla_X Z)$$

Now $\mathrm{div} k$ is a 1-form, the trace of ∇k (with respect to the metric of M), i.e. if (e_i) is an orthonormal basis:

$$\mathrm{div} k(X) = \Sigma_i \nabla_{e_i} k(e_i, X)$$

Divergence of left invariant quadratic forms on Lie groups

At first glance one can guess that left invariant objects are divergence free (with respect to left invariant Riemannian metrics). This is however false (apart from some trivial cases).

Let G be a 3-dimensional unimodular Lie group, endowed with a left invariant metric $\langle , \rangle = q \in Sym^+(\mathcal{G})$, with a Milnor q-orthonormal basis $\{u, v, w\}$: $[u, v] = aw$, $[v, w] = bu$ and $[w, u] = cv$ (see § 5). The proof of the following facts and corollaries is left as exercise. Let $p \in Sym(\mathcal{G})$ represent a left invariant quadratic form.

Fact 10.2 *Let X, Y and Z be right invariant vector fields, with $X(1) = e \in \mathcal{G}$. Let $g^t = \exp te$. Then the derivative $X.p(Y, Z)$ at $1 \in G$ is given by:*

$$X.p(X, Y) = \frac{\partial}{\partial t} p(Adg^t(Y), Adg^t(Z)) = p([X, Y], Z) + p(Y, [X, Z])$$

Fact 10.3 *For the basis $\{u, v, w\}$, we have:*
- $\nabla_u u = \nabla_v v = \nabla_w w = 0$,
- $2\nabla_u w = (-c + a - b)v$, $2\nabla_v w = (b - a + c)u$ ... *(Use Formula (2))*

Corollary 10.4 *Consider the left invariant quadratic form, $p_{12} = du \otimes dv + dv \otimes du$. Then:*
-

$$\begin{aligned} u.p_{12}(u, e) + v.p_{12}(v, e) + w.p_{12}(w, e) &= -c - a, \textit{for } e = w \\ &= 0, \textit{ for } e = u, \textit{ or } e = v \end{aligned}$$

• $p_{12}(u, \nabla_u w) + p_{12}(v, \nabla_v w) + p_{12}(w, \nabla_w w) = 0$

• *It then follows that* $\omega = \mathrm{div} p_{12}$ *is such that* $\omega(u) = \omega(v) = 0$*, and* $\omega(w) = -(c+a)$*, that is* $\omega = -(c+a)dw$.

Corollary 10.5 *Let us say a Milnor basis is* **generic** *if* $(a+c)(a+b)(b+c) \neq 0$*. Then, for a generic Milnor basis, any divergence free left invariant quadratic form (with respect to the metric for which this basis is orthonormal) is diagonalizable in this basis.*

In other words (keeping the previous notation), along a Milnor flat $F_{\mathcal{B}}$*, an element* $p \in T_q Sym_3^+$ *satisfies the momentum constraints, iff,* $p \in T_q(F_{\mathcal{B}})$ *(or in more linear words,* $p \in \overline{F_{\mathcal{B}}}$*)*

10. 4 Cross sections for the Bianchi-Einstein flow

On $TSym_3^+$, the group $Aut(G)$ acts, preserving the Bianchi-Einstein flow (determined by G). A cross section (§ 7. 5) will play the role of a flow on a quotient space (for the $Aut(G)$-action).

Proposition 10.6 *The Bianchi-Einstein flow on a generic Milnor flat is a cross section of the full Bianchi-Einstein flow (with constraints) on* Sym_3 *endowed with the* $Aut(G)$*-action. Generic flats exist except in the abelian and nilpotent cases, i.e. when G is* $\mathbb{R}^3$ *or the Heisenberg group* $Heis$.

Proof. Firstly, one easily sees that if there exists a Milnor basis for which a and $b \neq 0$, then after re-scaling, this basis becomes generic. This exists exactly when G is different from $\mathbb{R}^3$, and $Heis$. If we are not in these cases, then we can assume, after re-scaling if necessary, that all Milnor bases are generic. Let $\mathcal{B}$ be such a basis, and $(q,p) \in TSym_n^+$, then up to application of an element of $Aut(\mathcal{G})$, $q \in F_{\mathcal{B}}$. But since $\mathcal{B}$ is generic, if p satisfies the momentum constraints, then $p \in TF_{\mathcal{B}}$, which means that $TF_{\mathcal{B}}$ is a cross section. □

Case of $G = \mathbb{R}^3$

In the case of the Heisenberg group, there is exactly one momentum constraint which gives rise to invariant sets of the system. There is no such constraint in the case of $\mathbb{R}^3$, where we obtain the following system:

$$\begin{cases} \dot{q} = & p \\ \dot{p} = & \frac{1}{4}tr(q^{-1}p)p - \frac{1}{2}q^{-1}pq^{-1}p \\ L_{-1,1}(q,p) = & 0 \ \ \text{(the lightlike cone bundle of } L_{-1,1}\text{)} \end{cases} \tag{29}$$

The spacetime $\bar{M}$ has a metric

$$\bar{g} = -dt^2 + t^{2p_1}du^2 + t^{2p_2}dv^2 + t^{2p_3}dv^2.$$

This is called a **Kasner** spacetime (observe that in some cases, e.g. $p_1 = p_2 = p_3$, this is just the Minkowski space) [2, 7, 25].

Exercise 9. Prove the previous form of $\bar{g}$ and solve the same problem in the case of the Heisenberg group.

10. 5 Isometry group of $\bar{M}$

As said previously, the left action of G on itself induces, by definition of its metric, an isometric action on $\bar{M}$. In fact, if for some level (M, g_t), there are extra-isometries (i.e. other than left translations), then they extend to $\bar{M}$. More precisely, if the metric at some level, say $t = 0$, is identified with $q_0 \in Sym^+(\mathcal{G})$, and $\bar{M}$ corresponds to a point $(q_0, p_0) \in TSym_n^+ = Sym_n^+ \times Sym_n$, and $K \subset Aut(\mathcal{G})$ is the stabilizer of (q_0, p_0), then, on the one hand, K acts as an isometric isotropy group for (M, g_0) (g_0 corresponds to q_0). On the other hand, K preserves the Bianchi-Einstein trajectory of (q_0, p_0), and thus acts isometrically on $\bar{M}$ (as isotropy for any point identified with $1 \in G$).

10. 6 An example: Bianchi IX

This means $G = SO(3)$, or more precisely its universal cover the sphere S^3. In this case, there are Milnor bases with $a = b = c = 1$. Any other Milnor basis satisfies these equalities, up to re-scaling. Also, all such bases are equivalent up to conjugacy and re-scaling. Yet, this is the most challenging case of Bianchi cosmologies (see for instance [21]). As an example, **TAUB-NUT** spacetimes are exact solutions of the Bianchi-Einstein equations of class IX. They are characterized among Bianchi IX spacetimes as those having extra-symmetries, i.e. a non-trivial isotropy, which then must be $SO(2)$ (and thus their isometry group is $S^3 \times SO(2)$, up to a finite index). Nevertheless, their high complexity (at least among exact solutions) led people to describe them as “counter-examples to everything”! In a Milnor flat where $a = b = c = 1$, these spacetimes correspond (up to isometry) to $x = y$, and $x' = y'$. The left invariant metric on G (at any time) corresponds to a Berger sphere, i.e. (up to isometry) a metric on the sphere derived from the canonical one, by rescaling the length along the fibers of a Hopf fibration. In other words, the set of solutions of the Bianchi-Einstein flow, which are Berger spheres at any time, is closed and invariant, say, the TAUB-NUT set.

10. 7 Effect of a non-vanishing cosmological constant

Instead of requiring $\bar{M}$ to be Ricci-flat, let us merely assume it to be Einstein, i.e. $\bar{Ric} = \Lambda\bar{g}$. Its effect is essentially an additive constant (related to Λ) in all equations and constraints. This situation does not seem to be systematically investigated in the literature. In particular, one can wonder whether the introduction of Λ is “catastrophic” or in contrary produces only a moderate effect. A similar situation is that of the paradigmatic example in holomrophic dynamics, of the quadratic family $z \mapsto z^2 + c$. Here the variation of the parameter c generates a chaotic dynamics as well as a fractal geometry [4].

10. 8 Wick rotation

Here $\hat{M} = I \times M$ is endowed with the Riemannian metric $\hat{g} = +dt^2 + g_t$. Writing $\bar{Ric} = 0$, yields:

$$\begin{cases} \frac{\partial}{\partial t} g_t &= -2k_t \\ \frac{\partial}{\partial t} k_t &= -Ric + tr_{g_t}(k_t)k_t - 2k_{t_{g_t}} k_t \\ 0 &= r + |k_t|^2_{g_t} - (tr_{g_t} k_t)^2 \text{ (Constraint)} \end{cases} \tag{30}$$

For instance, this allows one to construct examples of Riemannian Ricci flat manifolds, of co-homogeneity 1, i.e. their isometry group has codimension 1 orbits.

Notice that the (true) Einstein equations (i.e. without symmetries) can not be solved in a Riemannian context (they cannot be transformed to a hyperbolic PDE system). Maybe, this Bianchi situation can give insights on the reasons behind this fact.

Finally, it does not seem there exists a "true Wick rotation", i.e. some correspondence between solutions of Bianchi-Einstein equations in the Lorentzian and Riemannian cases. (Compare with [5]).

10. 9 Orthonormal frames approach vs Metric approach

As a result of a search on fundamental references in this area, "dynamical systems and cosmology", one can get at least [7, 19] and [20] which are surely the most known and recent synthesis in this "emerging" domain. The authors adopted there an "orthonormal frames approach" in opposite to our "metric approach" here (see explanations therein). They obtained the following system of quadratic polynomial differential equations on $\mathbb{R}^5$.

$$\begin{array}{lcl} \Sigma'_+ &=& -(2-q)\Sigma_+ - \mathcal{S}_+ \\ \Sigma'_- &=& -(2-q)\Sigma_- - \mathcal{S}_- \\ N'_1 &=& (q - 4\Sigma_+)N_1 \\ N'_2 &=& (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2 \\ N'_3 &=& (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3 \end{array} \tag{31}$$

where:

$$\begin{array}{lcl} \mathcal{S}_+ &=& \frac{1}{6}\left[(N_2 - N_3)^2 - N_1(2N_1 - N_2 - N_3)\right] \\ \mathcal{S}_- &=& \frac{1}{2\sqrt{3}}(N_3 - N_2)(N_1 - N_2 - N_3) \\ q &=& \frac{1}{2}(3\gamma - 2)(1 - K) + \frac{3}{2}(2-\gamma)(\Sigma_+^2 + \Sigma_-^2) \\ K &=& \frac{1}{12}\left[N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_3N_1)\right] \end{array}$$

Here γ is a parameter: $2/3 < \gamma < 2$. (See for instance [22]).

Comparison

This system of differential equations must be "equivalent" to our equations (§ 10. 2) on the tangent bundle of a Milnor flat, which was a rational differential system on $\mathbb{R}^6$ with one constraint. A formal definition of equivalency of approaches is that the two systems are "bi-rationally equivalent". However, the transformation of our system to this polynomial system is by no means obvious. This last system was not a priori motivated by simplifying our more "naive" one, but rather by considering another point of view in considering Einstein equations. Instead of studying the evolution with time of the metrics on spacelike slices, one considers the evolution of brackets of *orthonormal* frames on these slices. The gauge freedom is more subtle in this case, but still this method is very clever, as shown by the simplified form of the equations here. In our Bianchi case, i.e. where spacelike slices are Lie groups with left invariant metrics, one can very roughly say that the bi-rational equivalence comes from the projection map $Mil(\mathcal{G}) \to Sym(\mathcal{G})$, where $Mil(\mathcal{G})$ is the space of Milnor bases of $\mathcal{G}$. The next step is to lift the Einstein equation (including a gauge choice) to $Mil(\mathcal{G})$ (more precisely an associated bundle) and to take the quotient by the G-action!

10. 10 Further remarks

This beauty of Sym_n appeals one to go beyond..., but as we said, our contribution here is essentially preliminary and expository. Let us mention some facts that were not considered here (with the hope to give details on some of them in the future).

Variants of Sym_n^+

First, one can generalize the discussion from Sym_n^+ to Sym_n^*, the space of all pseudo-Euclidean products, i.e. non-degenerate quadratic forms. Everything extends there, a pseudo-Riemannian metric (on Sym_n^*), Ricci maps, Bianchi-Einstein flows...

The components of Sym_n^* are spaces of quadratic forms of a given signature. As for Sym_n^+, each component is a pseudo-Riemannian symmetric space and plays a universal role in its class.

- *Complex case.* The same is true for complex spaces: $Sym_n^*(\mathbb{C})$, the space of complex non-degenerate quadratic forms on $\mathbb{C}^n$, is a holomorphic symmetric space...

- *Projectivization.* Taking the associated projective spaces will send all these spaces into compact ones, and hence compactify them, by attaching various boundaries, with more or less nice interpretations. A natural requirement is that ideal points correspond to collapsing of Riemannian metrics, say in the Gromov sense [11] (restricted here to homogeneous spaces). By algebraicity, all differential equations extend to the projective spaces.

- *Fiberwise constructions.* If $E \to B$ is a vector fiber bundle, then one can associate to it $Sym^+(E)$...

- *Configuration spaces.* Another interesting aspect of Sym_n^+ is its configuration space aspect. We mention here the case of " hydrodynamics", where a geometric formalism (a Riemannian metric, its geodesic flow...) was developed (see for instance [3]) following similar ideas as those presented here. There are also other non-linear and infinite dimensional situations, in particular a space $Sym^*(E)$ associated to a Hilbert space E could be exciting!

Geodesic flows of left invariant metrics

For a Lie group G, $Sym^+(\mathcal{G})$ plays a role of a parameter space of its left invariant metrics. The geodesic flow of any such metric on G is a second order quadratic ODE system on $\mathcal{G}$ [1]. It is intersting to study the dependence on parameter of the qualitative properties of these geodesic flows.

(Locally) Homogeneous, but "non-simply homogeneous" spaces.

Instead of left invariant metrics on Lie groups, one can consider general homogeneous spaces, say, those endowed with an isometric transitive, but not necessarily free action of a given group G. More important is the case of locally homogeneous spaces, i.e. when the metric varies in the space of all those locally modeled on a fixed space X endowed with a (non-fixed) G-invariant metric (but G is fixed). Here, G does not act (it acts only locally, as a pseudogroup). As an example, we have the Robertson-Walker-Friedman-Lemaitre space-times [12, 18, 24], which are warped products $\bar{M} = T \times_w N$, $\bar{g} = -dt^2 + w(t)g$, where N has a constant curvature.

Dimension $2 + 1$

So far, only the Gauss gauge has been considered. Maybe, this is because of its "deterministic character", i.e. it gives rise to autonomous differential equations, instead of non-autonomous ones, as in the generic case. There are however other situations where interesting gauges are available. As an example, in 't Hooft's theory of systems of particles in dimension $2 + 1$ [23], one has a flat polyhedral surface with singularities, evolving (locally) in a Minkowski space. The gauge here is fixed by the fact that time is locally equivalent to a "linear time" in the Minkowski space. In particular the time levels remain flat polyhedral. By consideration of suitable spaces of such surfaces, one may be convinced there is a configuration space approach similar to our situation here.

Non-empty spaces

Recall that $\bar{M} = I \times M$ is a perfect fluid, if $\bar{Ric} = (\mathfrak{p} + \rho)dt^2 + \mathfrak{p}\bar{g}$, where $\mathfrak{p}$ is the pressure and ρ is the density [12, 18, 24]. A Bianchi-Einstein flow can be defined in this case, when $\mathfrak{p}$ and ρ are functions on $TSym_n^+$, or more reasonably, when they are basic functions, i.e. they depend on the coordinate $q \in Sym_n^+$ alone. Robertson-Walker spacetimes ï¿½are examples of perfect fluids (strictly speaking, they would be covered by our approach, once we consider general locally homogeneous spaces, as discussed above).

Quantization of the Bianchi-Einstein flow

We strongly believe this is a natural case that can be treated by a quantum gravity theory (see for instance [8]), that is, a reasonable quantization of the Bianchi-Einstein flow should be possible...

A modified Einstein equation

World would be perhaps simpler if the Einstein equation on $TSym_n^+$ were given by the mechanical system determined by the Riemannian metric $\langle, \rangle$ on Sym_n^+ as a kinetic energy, and the Hilbert action $\mathcal{H}$ as a potential energy. Recall [1] that solutions of such a mechanical system are curves $q(t) \in Sym_n^+$, satisfying

$$\nabla_{q'(t)} q'(t) = -\nabla \mathcal{H}(q(t))$$

(∇ is the Riemannian-connection and $\nabla \mathcal{H}$ is the Riemannian gradient of $\mathcal{H}$). Other more "realistic" modified equations are obtained by replacing the Riemannian metric by a pseudo-Riemannian or a Finsler one of the form $L_{\alpha,\beta}$ or $F_{\alpha,\beta}$ (§ 8).

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