On the conformal group of Finsler manifolds

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Abstract

We generalize to the Finsler case, the Lelong-Ferrand-Obatta Theorem about the compactness of conformal groups of compact Riemannian manifolds, except, the standard sphere.

1 Introduction

The goal of this note is to present a proof of the following:

Theorem 1.1 *If the conformal group of a compact smooth Finsler manifold is not compact, then it is the canonical* Riemannian *sphere.*

To begin with let us recall that, roughly speaking, a smooth Finsler metric on a manifold M is a smooth field $x \to f_x$, where for each $x \in M$, f_x is a norm on the tangent space T_xM.

We shall discuss smoothness and variants in §3.

Our result says that, even in the wide world of compact Finsler manifolds, only the canonical Riemannian spheres are conformally "remarkable"!.

The analogous fact is no longer true in the non-compact case. A normed vector space E (of finite dimension) seen as a Finsler manifold is called a **Minkowskian** space (a generalization of Euclidean spaces). Homotheties act conformally on any Minkowskian space, which thus has an *essential* conformal group, that is, this group can not be reduced to an isometry group (for some other structure).

1.1 History, Ferrand's oeuvre

The Riemannian case is known in the literature as Lelong-Ferrand-Obatta Theorem answering a Lichnerowicz conjecture. We refer to [7] for a report on the history of the proof of this result, which is considered nowadays as a paradigmatic example of geometrico-dynamical rigidity. To up-date this reference, let us quote [11] where a purely dynamical proof is given. Actually, the former full proof was established by J. Ferrand. In fact, she also proved a non-compact version, claiming that a non-compact Riemannian manifold with a conformal group acting non-properly, is the Euclidean space [8].

1.2 Dynamics

Many previous partial cases of our result were proved by many authors (see for instance [1]). One of them is the *topological* rigidity of spheres among Finsler manifolds with *non-precompact* conformal group proved by J. Ferrand [9]. Recall that precompacteness means that the closure of the group is compact in the group of all homeomorphismes (endowed with the compact-open topology). We agree with Ferrand's opinion on the subtlety of this closeness question! For this, we find it interesting to split the result into two natural parts. The first one is:

Lemma 1.2 *If the conformal group of a compact smooth Finsler manifold is not precompact, then it is the canonical* Riemannian *sphere.*

1.3 Our regularity

The last part which allows us to complete the proof of Theorem 1.1 is:

Lemma 1.3 *A C^0 limit of C^∞ conformal transformations of a compact Finsler manifold, is a C^∞ conformal transformation.*

In particular if the conformal group is precompact (as a subgroup of the homeomorphisms), then it is compact.

Let us emphasize on the fact that everything here is C^∞. The Finsler metrics are smooth (as it will be precised in §3). Also, by the conformal group, we mean those conformal transformations which are C^∞.

1.4 General framework. Conformal Lichnérowicz problem on metric spaces

In reality, one can speak about C^1 conformal transformations on Finsler manifolds. Our result is true with a C^3 finite regularity. We assumed here all things C^∞, for the sake of simplicity, and also, because, anyway, this does not help to attack the true question, the topological one. In fact, more generaly, it is possible to define conformal homeomorphisms for metric spaces!

• In order to get a coherent theory, one naturally asks the following regularity question:

Is: *topologically conformal* $=$ *smoothly conformal ?*

(of course when the last notion is meaningful, that is, for metric spaces generated by C^0 Finsler metrics on smooth manifolds).

From the fact that the notion is topological, closeness of the topological conformal group with respect to the compact-open topology, becomes trivial. In particular, a positive answer to the regularity question in the Finsler case, would trivially imply Lemma 1.3.

The positive answer to the regularity question in the case of Euclidean spaces, and then the general case of smooth Riemannian manifolds, was a very deep result of geometric analysis (see for instance [13, 14]).

• The conformal group of a metric space (or the space itself) will be called **essential** if it can not be reduced to the isometry group (say, for an auxiliary metric). Equivalently, when the space is locally compact, the conformal group is essential if it is not equicontinuous.

A natural exciting question is:

Classify conformally essential metric spaces.

Some particular cases are:

– A compact conformally essential C^0 Riemannian (or Finsler) space is the usual sphere?

– A non-compact conformally essential C^0 Riemannian (resp. Finsler) space is the Euclidean (resp. a Minkowskian) space?

As was said in §1.1, J. Ferrand proved that the Euclidean space is the only smooth conformally essential non-compact Riemannian manifold.

The analogous Finslerian fact would be that, a conformally essential non-compact smooth Finsler space is Minkowskian?

1.5 Further developments. The "linear" group of a Finsler structure.

Let (M, f) and (M, f') be two Finsler structures on M. Say they are (pointwise) linearly equivalent, if for any $x \in M$, the norms f_x and f'_x are linearly equivalent on T_xM, i.e. $f'_x = f_x \circ l_x$, for some linear isomorphism l_x. For instance, two conformally related metrics are linearly equivalent.

Let $Lin(M, f)$ be the subgroup of diffeomorphisms of M preserving the equivalence class of (M, f). For instance, in the case of a Riemannian or more generally Berwald metric (see §2). $Lin(M, f)$ is the group of all diffeomorphisms. On the other hand, if $Lin(M, f)$ acts transitively, then the metric is necessarily Berwald. We think it is worthwhile to investigate this group. For example, can it be a non-trivial Lie group (of finite dimension)?

1.6 Content

As for Lemma 1.2, the given proof will be complete. In particular, no knowledge of Finsler geometry is supposed since we recall here rudimentary and major facts on it, especially in comparison with Riemannian geometry. The proof of Lemma 1.3 relies on "rigidity of conformal structures as geometric structures". The precise meaning of this (say, in the Riemannian case, to fix ideas) is that a conformal transformation is fully determined by its 2-jet at some point. "Everybody" knows this, or the essentially equivalent Liouville Theorem: a conformal transformation of an Euclidean space of dimension ≥ 3 is a composition of similarities and inversions (see for instance [17]). Our proof of Lemma 1.3 is complete up to details on this rigidity, for which we outline a synthetic approach. Anyway, we consider that Lemma 1.2 should be considered as the principal result of the article.

2 Dynamics: Proof of Lemma 1.2

2.1 Quasi-conformal dynamics

It is straightforward to define conformality, and more generally to measure conformality defect (or distortion) for C^1 diffeomorphisms on Finsler manifolds. A differentiable group action is called quasi-conformal, if all its elements have a (uniformly) bounded conformality distortion. In the compact case this notion does not depend on the Finsler metric.

J. Ferrand studied quasi-conformal actions and proved that they behave dynamically as Möbius transformations.

Lemma 2.1 *[10] (North-South dynamics for non-equicontinuous quasi-conformal mappings). Let (ϕ_n) be a quasi-conformal sequence of diffeomorphisms of a compact manifold M, which is not equicontinuous. Then, a subsequence of it, say (ϕ_n) itself (for the sake of notation simplicity) has a north-south dynamics, that is, there exist $a, b, a', b' \in M$, such that, on $M - \{a\}$, ϕ_n converges uniformly on compact sets to (the singleton) b', and similarly ϕ_n^{-1} converges uniformly on compact sets of $M - \{b\}$ to a'.*

Remarks 2.2

• The Lemma indicates a rough Möbius behaviour of quasi-conformal groups. In dimension 2, thanks to the measurable Riemann mapping theorem, we have more: a quasi-conformal group is conjugate to a conformal group (see for instance [18]). Our result here can be interpreted as a conjugacy with conformal action, of quasi-conformal groups, with the supplementary hypothesis that they are (exactly) conformal for some smooth Finsler metrics.

• This "abstract" lemma which implies (in a slightly stronger formulation) the conformal rigidity of spheres (among compact Riemannian manifolds) was actually brought out after the proof of this rigidity. Let us also point out here that many authors investigated degeneracy, uniform modulus of continuity, regularity, dynamics... of quasi-conformal maps, in both the classical case of the Riemann sphere (dimension = 2), the general case of Euclidean spaces, or more generally on Riemannian spaces (as references, one can may quote [14, 13, 15, 18]). A historical point is worthwhile on this subject!

2.2 Berwald spaces

A Finsler space (M, f) is called **Berwald**, if all the norms (T_xM, f_x), $x \in M$, are linearly equivalent. That is, there exists $x_0 \in M$ such that for any $x \in M$, there is a linear isomorphism $L_x : T_xM \to T_{x_0}M$, such that $f_{x_0} = f_x \circ L_x$ (see [3] for a nice modern account on Finsler geometry).

Lemma 2.3 *If a Finsler metric on a compact manifold has a non-precompact conformal group, then it is Berwald.*

Proof. Consider the notations of Lemma 2.1. Let $x \neq a$, and denote $x_n = \phi_n(x)$. Since x_n is near b', there is $K_n : T_{x_n}M \to T_{b'}M$, an almost isometry between f_{x_n} and $f_{b'}$. By conformality, we can write $D_x\phi_n = \lambda_n R_n : T_xM \to T_{x_n}M$, where R_n is an isometry. Consider $L_n = K_n R_n : T_xM \to T_{b'}M$. Obviously any limit of L_n gives rise to a linear isometry between f_x and $f_{b'}$. ♢

2.3 Riemann versus Finsler

A "folkloric" fact of "synthetic geometry of normed spaces" states that any normed space hides a Euclidean one! Unfortunately, this beautifull fact seems to not be taken over by modern differential geometry. To state this fact, consider $\mathrm{Conv}(E)$, the space of all compact convex and with nonempty interior subsets of a finite dimensional normed space E. Let $\mathrm{Ell}(E) \subset \mathrm{Conv}(E)$ be the subspace of (closed) ellipsoids (i.e. linear images of a Euclidean closed ball).

Fact 2.4 *There is a mapping: Euc : Conv(E) $\to$ Ell(E) (where, Euc stands for Euclid), defined by the condition that Euc(C) is the ellipsoid with minimal volume containing C. This mapping is equivariant under the action of the affine group Aff(E). It is also continuous in a natural way, say with respect to a Hausdorff metric.*

Proof. For $C \in \mathrm{Conv}(E)$, the subset of ellipsoids containing C and contained in some closed ball in E is compact. The existence of a volume minimizing ellipsoid follows then from the continuity of the volume function: $\mathrm{Conv}(E) \to \mathbb{R}$, (see for instance [4].) The point is then uniqueness of a volume minimizing ellipsoid. Observe also that this does not depend on the chosen volume form on E. For the sake of completeness, we present here a complete proof for uniqueness. We argue by contradiction. Assume that C is contained in two minimizing volume ellipsoids C_1 and C_2. Observe that C is then contained in the convex combination $tC_1 + (1-t)C_2$ (that is the unit ball of the convex combination of the associated Euclidean metrics). Let $f(t) = \mathrm{vol}(tC_1 + (1-t)C_2)$. To get a contradiction, one just shows that f is a convex function, and for this goal, we just compute it. Up to a linear transformation, one can assume that C_1 is the canonical unit ball $\Sigma x_i^2 \leq 1$. After diagonalizing the symmetric matrix representing C_2, one can write it in the form $\Sigma c_1^2 x_i^2$ in some orthogonal coordinate system.

In general, denote by $V(a_1, \dots, a_n)$ the volume of the ellipsoid $\Sigma a_i^2 x_i^2 \leq 1$. Any diagonal linear mapping $(x_1, \dots, x_n) \to (b_1 x_1, \dots, b_n x_n)$ preserves these equations of ellipsoids, and multiplies volume by $\prod b_i$. It then follows that, up to a positive constant, $V(a_1, \dots, a_n) = \prod_i \frac{1}{a_i}$, and then $f(t)$ has the form $(\prod_i (c_i^2 + t(1-c_i^2))^{-1}$. The convexity of f follows then easily. $\Diamond$ $\Diamond$

A minor remark is in order: the so obtained ellipsoid is not necessarily symmetric, since the initial convex is not supposed to be. However, when dealing with conformal transformations, one can, without worrying, symmetrize everything.

From this, one gets a functor from Finsler spaces to C^0 Riemannian spaces.

Lemma 2.5 *In the case of a smooth Berwald metric, its associated Riemanian metric is smooth.*

Proof. For the sake of simplicity, let us restrict ourselves to convex sets containing (a neighbourhood of) 0 and to the $GL(E)$-action. For such a convex C, consider its $GL(E)$-orbit,

$$\mathcal{O}(C) = \{A(C), A \in GL(E)\}$$

In an open set U of M where the tangent bundle is trivial, a smooth Berwald metric is given by a smooth mapping $s : U \to \mathcal{O}(C)$, for some convex C (the unit ball at some tangent space).

The associated Riemannian metric is just given by $Euc \circ s : U \to \mathrm{Ell}(E)$.

The question reduces to see that the restriction of Euc to an orbit $\mathcal{O}(C)$ is smooth. Such an orbit is a homogeneous space $GL(E)/K$, where K is the stabilizer of C, that is, the group of linear transformations preserving C. In the case of the unit ball of the Euclidean space of dimension n, the stabilizer is $O(n)$. The orbit is thus identified to "the universal" symmetric space $GL(n)/O(n)$.

In general, being compact, K can be seen after conjugacy inside $O(n)$. After this, the mappping Euc, becomes just the canonical projection

$$GL(n)/K \to GL(n)/O(n),$$

which is analytic. Therefore, the Riemannian metric is as regular as the Berwald one. $\diamondsuit$

2.4 End of the proof of Lemma 1.2

Recall the notation (M, f) for the Finsler (Berwald) metric under consideration, and let us denote by $(M, \mathrm{Euc}(f))$ its associated Riemannian structure. Note that if $F \in \mathrm{Conf}(M, f)$ then $Euc(F^* f) = F^*(Euc(f))$, this implies that $\mathrm{Conf}(M, f) \subset \mathrm{Conf}(M, Euc(f))$. Therefore, by Ferrand's Theorem for Riemannian manifolds, M is a usual sphere S^n, and $G = \mathrm{Conf}(M, f)$ is a subgroup of $O(1, n)$.

It is known that any precompact subgroup of $O(1, n)$ is conjugate to a subgroup of $O(n)$.

On the other hand a non-precompact subgroup contains an element B which is parabolic or hyperbolic (see for instance [19]). A quick definition of these notions goes as follows. In both cases the conformal transformation has (at least) a fixed point. It can thus be seen as a similarity of the Euclidean space $\mathbb{R}^n$. It has the form $x \to \lambda R(x) + a$, where $R \in O(n)$. The hyperbolic case corresponds to $|\lambda| \neq 1$, in particular there is another fixed point in $\mathbb{R}^n$. The parabolic case corresponds to the situation where there exists no fixed point in $\mathbb{R}^n$ (essentially a translation).

Now, we argue by a simple calculation to show that the Finsler metric must be Riemannian.

The proof for the parabolic and hyperbolic cases are similar. Therefore, we restrict ourselves to the case B hyperbolic.

We have a Berwald metric f on $\mathbb{R}^n$, invariant under a non-trivial linear similarity with a non-trivial distortion, $\lambda < 1$, say. It has therefore (a finite) fixed point, and is up to conjugacy conjugate to $x \to \lambda R(x)$, with $R \in O(n)$, a rotation.

The conformal invariance translates to the fact that the Finsler metric f at $\lambda^m R^m(x)$ is proportional to $f_x \circ (R^{-1})^m$, for any integer m.

Since, R is a rotation, one can choose a subsequence m_i such that $(R^{-1})^{m_i} \to 1$, and thus concludes (by continuity), that all norms f_x are proportional to f_0 (since $\lambda^m R^m(x) \to 0$). In other words, $(\mathbb{R}^n, f)$ is conformal to the Minkowskian space $(\mathbb{R}^n, f_0)$.

Now, the point is to see that a non-Euclidean norm on $\mathbb{R}^n$, say f_0, cannot extend conformally and continuously to ∞ in order to give a conformal Finsler structure on S^n, invariant under the hyperbolic transformation.

A chart around ∞ is given by the inversion $x \to \frac{x}{||x||^2}$, where $||x|| = \sqrt{< x, x >}$ denotes the Euclidean norm.

The derivative of this inversion at x, is given by:

$$u \to \frac{1}{||x||^2}(u - 2\langle u, \frac{x}{||x||}\rangle \frac{x}{||x||}) = \frac{1}{||x||^2} S_x(u),$$

where S_x is the orthogonal symmetry around the line $\mathbb{R}x$.

Therefore, the Minkowskian norm f_0 tranforms to a Finsler metric

$$F_x(u) = f_0(\frac{1}{||x||^2} S_x(u))$$

However, our hyperbolic transformation has a similar linear form at ∞ (with a distorsion λ^{-1}). Therefore, the same conclusion holds as above, that is, all the family of norms F_x are proportional to a same one, f_1, say. A standard argument allows one to see that f_1 must equal f_0, and its unit ball is invariant under all axial symmetries, and hence it is Euclidean. $\diamondsuit$

Remark 2.6 *(Ferrand's Theorem for C^0 Riemannian metrics?)* Observe that all developments about Berwald metrics could be avoided if Ferrand's Theorem was true for C^0 Riemannian metrics.

3 Regularity, Proof of Lemma 1.3

The essential idea behind the concept of a smooth Finsler metric on a manifold M, is that it is an assignment $x \to f_x$, a norm on T_xM, which varies in a *smooth* way with respect to x.

3.1 Usual (strong) definition

The standard definition is more precise, and different in a non-trivial meaner. For instance, from [3], a (smooth) Finsler metric on M is a mapping $f : TM \to [0, \infty[$ which is:

- smooth outside the null section,

- positively homogeneous $f(\lambda u_x) = \lambda f(u_x)$, for $\lambda \in [0, \infty[$, $x \in M$, $u_x \in T_xM$,

- The fiberwise Hessian of f is positive definite.

• *Some weakness.* Because of the absence of the symmetry condition $f(u_x) = f(-u_x)$ i.e. the fiberwise levels are not symmetric, f does not determine true norms on fibers, let us call them **non-symmetric norms** (they satisfy positive homogeneity and triangle inequality). As an example, Randers metrics are not symmetric. They are functions on TM of the form $g + \omega$, where g is a Riemannian metric on M and ω is a differential form of degree 1 on M (they are sometimes interpreted as Hamiltonians of electromagnetic fields).

• *Strength.* The definition means, in other words, that the unit ball of the non-symmetric norm is smooth and strongly convex. For instance, this (unfortunately) excludes beautiful norms like $\| \, . \, \|_1$ and $\| \, . \, \|_\infty$!

3.1.1 Differential calculus, geodesic flow...

We refer the interested reader to [3] to see which Riemannian notions extend or not to the Finsler case. For instance, it seems that Weyl tensors (the conformal and the projective ones) are difficult to imitate in the Finsler case. In contrast, new invariants appear in Finsler geometry, e.g. a tensor to measure if the structure is Berwald...

The condition of strong convexity is the right one to get a good Legendre transformation, and then a Hamiltonian formulation for the geodesic flow. Therefore, thanks to this precise definition, geodesics for Finsler metrics behave exactly as in the Riemannian case.

Example 3.1 *(Smoothness of isometries) A C^0 isometry is a homeomorphism preserving the distance (generated from the Finsler metric). It is Lipschitz, and hence almost everywhere differentiable. Because of tameness of geodesics, one can define an exponential map as in the Riemannian case. At any point where an isometry is differentiable, it becomes conjugate to its linear part, via the exponential map. A standard argument allows one to deduce that the isometry is then (everywhere) smooth.*

Remark 3.2 *(C^0 metrics)* The definition of C^0 Finsler metrics can not give rise to controversy: just a continuous $f : TM \to [0, \infty[$ whose restrictions to fibers are non-symmetric norms.

3.2 Weaker smoothness

We discuss now a weaker notion of smoothness of Finsler metrics, from which we would essentially like to allow norms on individual tangent spaces to be non-differentiable. Furthermore Lemma 1.2, and very presumably the full Theorem 1.1, apply with this weaker smoothness.

Let E be a vector space, and consider $\mathcal{H}(E)$ the vector space of positively homogeneous functions $h : E \to \mathbb{R}$ of degree 1, $h(\lambda x) = |\lambda| h(x)$. It can be endowed with a family of equivalent Banach structures: any bounded open subset B of E containing 0, determines a norm $\sup_B |h|$.

The space of non-symmetric norms $\mathcal{N}(E)$ (i.e. those positive h satisfying the triangle inequality) is a closed cone in $\mathcal{H}(E)$.

One can proceed by generalizing this construction to vector bundles, e.g. the tangent bundle of a manifold M, and defines smooth Finsler metrics in this way. Concretely, over an open set U, where TM is trivial, say tangent spaces are identified with E, a C^k Finsler metric is a C^k mapping $s : U \to \mathcal{H}(E)$, with values in $\mathcal{N}(E)$. In order to avoid confusion with the previous smoothness notion, let us call a Finsler metric **functionally smooth** if it is C^∞ as we have just defined it.

A naive example: visual metrics. Let Ω be an open convex set of $\mathbb{R}^n$. A "visual" metric on Ω can be defined by the rule: the unit ball in $T_x\Omega = \Omega - x$ $(= \{u - x, u \in \Omega\})$. The Finsler metric as a mapping with target the space of convex sets is $x \to \Omega - x$. It is therefore functionally smooth.

Now, observe that in our proof of Lemma 1.2, we used smoothness just in order to get a smooth Berwald metric and thus a smooth Riemannian one (and then apply the known result in this case). However, it is obvious that functional smoothness suffices with respect to this goal:

Theorem 3.3 *If the conformal group of a compact functionally smooth Finsler manifold is not precompact, then it is the canonical* Riemannian *sphere.*

3.3 Geometric structures

The notion of geometric structures seems to grow with time. A remarkable evolution was made by giving up "infinitesimal homogeneity" (as in G-structures) when M. Gromov introduced and successfully applied, a more general concept, just called "geometric structure" (see [6, 12, 5]. Unfortunately, this does not seem to include the case of Finsler metrics!!!

Nevertheless, a Finsler structure generate many geometric structures, in both classical and Gromov senses. Definition and study of them, is the true theme of Finsler differential geometry. Let us consider here two of such structures, which are defined on $TM - 0$. At any $u \in T_xM$, the Hessian of f at u determines a scalar product g_u on $\mathcal{V}_u$ where $\mathcal{V}$ is the vertical of the fibration $TM \to M$.

A partial Riemannian metric: The (g_u) determine a Riemannian metric along fibers. In other words, we get a Riemannian metric on the vector bundle $\mathcal{V}$ over $TM - 0$.

A degenerate Riemannian metric: One uses the identification $\mathcal{V}_u \approx T_x M$ to get a metric on $T_x M$ (depending of u). By means of the projection $\pi : TM \to M$, one gets a semi-definite scalar product on $T_u(TM)$: $\langle ., . \rangle_u = g_u(d_u\pi(.), d_u\pi(.))$. This gives rise to a degenerate Riemannian metric on TM, whose Kernel is exactly $\mathcal{V}$.

A nice fact in Gromov's theory of geometric structures is that the "union" of two geometric structures is a geometric structure... Therefore, these partial and degenerate Riemannian metrics can be seen together as a geometric structure on TM. Let us call it the **Riemannian-like** structure on TM associated to the Finsler metric f.

A remarkable fact is that the conformal group of (M, f) acts conformally with respect to this structure. This is the major advantage of this structure in comparison with the similar suggestive construction of a plain Riemannian metric on TM, by endowing the "horizontal" bundle with the degenerate metric. Indeed, it is possible to construct a horizontal bundle, as in the Riemannian case. However, the so obtained (full) metric is never conformally invariant.

Finally, from geometric structures point of view, our Riemannian-like structure enjoys all nice properties of Riemannian metrics. For instance, it is a rigid structure of algebraic type.

Remark 3.4 Being or not geometric structures, there is an obvious natural "superiority" of Finsler metrics relative to the Riemannian ones, see for instance [2] for their applications. This lack of structure is in fact a source of richness, involving all convex sets (instead of ellipsoids as in the Riemannian case).

3.4 Rigidity. Proof of Lemma 1.3

There are many approaches to Lemma 1.3 in the conformal Riemannian case. We now outline a proof which can be straightforwardly adapted to the general case of rigid geometric structures. Let (f_n) be smooth conformal transformation of a Riemannian manifold (M, g). The significance of rigidity (of conformal Riemannian structures) is the existence of a natural associated bundle, on which conformal transformations, become isometries (for some naturally associated Riemannian metric). The geodesics of this metric projects onto (parameterized) curves of M, called conformal geodesics (see for instance [16]). As an example, conformal geodesics of the Euclidean space are circles (and lines) parameterized homogarphically (that is, all parameterizations are obtained from a constant speed one, by means of projective changes). Now, if (f_n) converges in the C^0 topology to a homeomorphism f, then f also preserves these conformal geodesics. This allows one to check directional smoothness of f, and finally that it is fully smooth and conformal.

In the case of conformal Finsler structures, one considers conformal geodesics of their associated Riemannian-like metrics. $\Diamond \Diamond \Diamond$

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