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Global rigidity of holomorphic Riemannian metrics on compact complex 3-manifolds

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Abstract We study compact complex 3-manifolds M admitting a (locally homogeneous) holomorphic Riemannian metric g. We prove the following: (i) If the Killing Lie algebra of g has a non trivial semi-simple part, then it preserves some holomorphic Riemannian metric on M with constant sectional curvature; (ii) If the Killing Lie algebra of g is solvable, then, up to a finite unramified cover, M is a quotient $\Gamma\backslash G$, where Γ is a lattice in G and G is either the complex Heisenberg group, or the complex SOL group.

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1 Introduction

A *holomorphic Riemannian metric g* on a complex manifold M is a holomorphic field of non degenerate complex quadratic forms on the holomorphic tangent bundle TM. Formally, g is a holomorphic section of the bundle $S^2(T^*M)$ such that $g(m)$ is non degenerate for all $m \in M$. This has nothing to do with the more usual Hermitian metrics. It is in fact nothing but the complex version of Riemannian metrics. Observe that

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since complex quadratic forms have no signature, there is here no distinction between the Riemannian and pseudo-Riemannian cases. This observation was the origin of the nice use by Gauß of the complexification technic of (analytic) Riemannian metrics on surfaces, in order to find conformal coordinates for them.

As in the real case, a holomorphic Riemannian metric on M gives rise to a covariant differential calculus, i.e. a Levi-Civita (holomorphic) linear connection, and to geometric features: curvature tensors, geodesic (complex) curves [25,26].

Locally, a holomorphic Riemannian metric has the form $\Sigma g_{ij}(z) dz_i dz_j$, where $(g_{ij}(z))$ is a complex inversible symmetric matrix depending holomorphically on z. The standard example is that of the global flat holomorphic Riemannian metric $dz_1^2 + dz_2^2 + \cdots + dz_n^2$ on $\mathbb{C}^n$. This metric is translation-invariant and thus descends to any quotient of $\mathbb{C}^n$ by a lattice. Hence complex tori possess (flat) holomorphic Riemannian metrics. This is however a very special situation since, contrary to real case, *only few compact complex manifolds admit holomorphic Riemannian metrics.* In fact, Yau's proof of the Calabi conjecture shows that, up to finite unramified covers, complex tori are the only compact *Kaehler* manifolds admitting holomorphic Riemannian metrics [18].

However, very interesting examples, constructed by Ghys in [13], do exist on 3-dimensional complex non Kaehler manifolds and deserve classification.

Notice first that *parallelisable manifolds*, obtained as a (left) quotient of a complex Lie group G by a co-compact lattice Γ, bear holomorphic Riemannian metrics coming from left invariant holomorphic Riemannian metrics on G (which can be constructed by left translating any complex non-degenerate quadratic form defined on the Lie algebra $\mathcal{G}$.) Recall that parallelisable manifolds are Kaehler if and only G is abelian.

Ghys's examples of 3-dimensional compact complex manifolds endowed with holomorphic Riemannian metrics are obtained by deformation of the complex structure on parallelisable manifolds $\Gamma \backslash SL(2, \mathbb{C})$ [13]. They are *non standard, meaning they do not admit parallelisable manifolds as finite unramified covers*. Those non standard examples will be described in Sect. 2.

Our goal in this paper is to classify (non Kaehler) complex compact 3*-manifolds endowed with holomorphic Riemannian metrics.* Since tools coming from complex algebraic geometry are not available in this context, we essentially work in the (complex) differential geometry background.

The first step toward the classification is the main result of [9]:

Theorem 1.1 [9] *Any holomorphic Riemannian metric on a compact connected complex* 3*-manifold is locally homogeneous. More generally, if a compact connected complex* 3*-manifold M admits a holomorphic Riemannian metric, then any holomorphic geometric structure of affine type on M is locally homogeneous.*

Thanks to Theorem 1.1, M is locally modelled on a $(G, G/I)$-geometry in Thurston's sense [37], where I is a closed subgroup of the Lie group G such that the G-action on G/I preserves some holomorphic Riemannian metric (see Sect. 2 for more details and notice that the local *Killing Lie algebra* of the holomorphic Riemannian metric is the Lie algebra of G). In this context we get a *developing map* from the universal cover $\tilde{M}$ of M into G/I which is a local diffeomorphism and which is equivariant in respect to the action of the fundamental group on $\tilde{M}$ by deck transformations and on G/I via the *holonomy morphism* $\rho : \pi_1(M) \to G$ [37].

Recall that the $(G, G/I)$-geometry is called *complete* if the developing map is a diffeomorphism and, consequently, $\Gamma = \rho(\pi_1(M))$ acts properly on G/I such that M is a quotient $\Gamma\backslash G/I$.

The main result of this paper gives a completeness result in the case where G is solvable:

Theorem 1.2 *Let M be a compact connected complex 3-manifold which admits a (locally homogeneous) holomorphic Riemannian metric g. Then*:

(i) *If the Killing Lie algebra of g has a non trivial semi-simple part, then it preserves some holomorphic Riemannian metric on M with constant sectional curvature.*
(ii) *If the Killing Lie algebra of g is solvable, then, up to a finite unramified cover, M is a quotient either of the complex Heisenberg group, or of the complex SOL group by a lattice.*

Note that the group SOL is the complexification of the affine isometry group of the Minkowski plane $\mathbb{R}^{1,1}$ or equivalently the isometry group of $\mathbb{C}^2$ endowed with its flat holomorphic Riemannian metric (see Sect. 2).

Remark 1.3 If g is flat, its Killing Lie algebra corresponds to $O(3, \mathbb{C}) \ltimes \mathbb{C}^3$, which has non trivial semi-simple part. Thus, flat holomorphic Riemannian metrics on complex tori are part of point (i) in the main theorem.

The point (ii) of the previous theorem is not only about completeness, but also gives a rigidity result in Bieberbach's sense [38]: G contains a 3-dimensional closed subgroup H (either isomorphic to the complex Heisenberg group, or to the complex SOL group) which acts simply and transitively (and so identifies) with G/I and (up to a finite index) the image Γ of the holonomy morphism lies in H. It follows that, up to a finite cover, M identifies with $\Gamma\backslash H$.

Our present result does not end the story since *it remains to classify the compact complex 3-manifolds endowed with a holomorphic Riemannian metric of constant sectional curvature*.

Flat case. In this case M admits a $(O(3, \mathbb{C}) \ltimes \mathbb{C}^3, \mathbb{C}^3)$-geometry. The challenge remains:

1) *Markus conjecture*: Is M complete?
2) *Auslander conjecture*: Assuming M as above, is Γ solvable?

Note that these questions are settled in the setting of (real) flat Lorentz manifolds [5,11], but unsolved for general (real) pseudo-Riemannian metrics. The real part of the holomorphic Riemannian metric is a (real) pseudo-Riemannian metric of signature (3, 3) for which both previous conjectures are still open.

Non flat case. In this case $G = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ and $I = SL(2, \mathbb{C})$ is diagonally embedded in the product. The completeness of this geometry on compact complex manifolds is still an open problem, despite a local result of Ghys [13]. Recall that the real analogous of this problem, i.e. the completeness of compact manifolds endowed with Lorentz metrics of negative constant sectional curvature, was solved in [20], but the proof cannot generalize to other signatures. More details about the case of a non-zero constant sectional curvature are in Sect. 2.

Comparison with [10] The present article is naturally linked to our recent work on the classification of essential lorentz geometries in dimension 3. There are similarities in the algebraic classification of all possible local Killing algebras. However, we had to modify significantly our methods because in [10] we used global results about the classification of (real) Riemannian Killing fields [4] and about the classification of non-equicontinuous Lorentz Killing fields [39] which do not exist in the holomorphic setting.

Related works There are various works dealing with different holomorphic geometric structures, and sharing the same philosophy as ours here, that is, a "strong global rigidity" of such objects on compact complex manifolds. As an example, we can quote [7,17,18,21], and especially [32], about holomorphic conformal structures on *projective* 3-manifolds. As an extension of both their results and ours, we believe a global rigidity result is true for holomorphic conformal metrics in the framework of complex (not necessarily projective) 3-manifolds.

1.1 Plan of the proof

The first step of the proof consists on finding all 3-dimensional complex homogeneous spaces G/I such that the G-action on G/I preserves some holomorphic Riemannian metric (i.e. the adjoint representation of I preserves some non-degenerate complex quadratic form on the quotient $\mathcal{G}/\mathcal{I}$ of the corresponding Lie algebras). Despite a "quick" reduction to the case where G has dimension 4 and is solvable, our solution needs a geometric tool which is the existence of a codimension one geodesic foliation $\mathcal{F}$.

The second part is a standard problem: classify compact manifolds locally modelled on a given $(G, G/I)$-geometry. If G is solvable, we prove that M is complete and, up to a finite cover, it is a quotient of $Heis$ or SOL by a lattice.

2 Examples

Obstructions. Real Chern classes. A first obstruction to the existence of a holomorphic Riemannian metric on a compact complex manifold is its first Chern class. Indeed, a holomorphic Riemannian metric on M provides an isomorphism between TM and T^*M. In particular, the canonical bundle K is isomorphic to the anti-canonical bundle K^{-1} and K^2 is trivial. This means that the first Chern class of M vanishes and, up to a double unramified cover, M possesses a holomorphic volume form.

Since M admits a (Levi-Civita) holomorphic linear connection, the Chern-Weil method implies that the Chern class $c_i(M)$ can be represented by a holomorphic $2i$-form α. On the other hand, using the curvature of a hermitian metric on M, we can represent $c_i(M)$ by a real (i, i)-form β.

If M is Kaehler, two forms α and β of different type cannot be cohomologous, unless $\alpha = \beta = 0$. It follows that $c_i(M) = 0$.

Assume now that M is not necessarily Kaehler, of complex dimension n, and $2i = n$. Since β is real and cohomologous to α, it is also cohomologous to $\bar{\alpha}$. Hence, α and $\bar{\alpha}$ are cohomologous and $\int_M \alpha \wedge \bar{\alpha} = \int_M \alpha \wedge \alpha = 0$. Thus, we obtain $\alpha = 0$ and $c_i(M) = 0$.

Since the curvature of a holomorphic Riemann metric is a two-form taking values in the skew-symmetric endomorphisms of the holomorphic tangent space, Chern-Weil theory implies that $c_i(M) = 0$ as soon as i is not a multiple of 4. This is the complex version of the well known fact that (real) Pontryagin classes p_i of a (real) manifold vanishes as soon as i is not a multiple of 4. For instance if M is 3-dimensional, all (real) Chern classes vanish.

For an improvement of this method for complex manifolds bearing holomorphic conformal structures, one could see [21] (Theorem 3.20) and the recent (and more general) [29].

Surface case The classification of complex compact surfaces admitting holomorphic Riemannian metrics is an easy consequence of the main result in [21]: up to a finite cover, the only examples are (flat) translations-invariant holomorphic Riemannian metrics on complex tori. In particular, there is no compact surface having a holomorphic Riemannian metric of non zero constant sectional curvature.

Universal holomorphic Riemannian spaces of constant curvature One can multiply a holomorphic Riemannian metric by a complex constant λ which induces a multiplication by λ^{-2} of its sectional curvature. Therefore, only the vanishing or not (but not the sign) of the curvature is relevant.

The flat case The model $(\mathbb{C}^n, dz_1^2 + dz_2^2 + \cdots + dz_n^2)$ is (up to isometry) the unique n-dimensional complex simply-connected manifold endowed with a flat and geodesically complete holomorphic Riemannian metric. Its isometry group is $O(n, \mathbb{C}) \ltimes \mathbb{C}^n$. Any flat holomorphic Riemannian metric on a complex manifold of dimension n is locally isometric to this model, equivalently, it has a $(O(n, \mathbb{C}) \ltimes \mathbb{C}^n, \mathbb{C}^n)$-structure [37,38]. This geometry can be seen as a complexification of the Minkowski space $\mathbb{R}^{n-1,1}$.

- *Dimension 2.* For $n = 2$, the connected component of the identity in the isometry group is $SOL \simeq \mathbb{C} \ltimes \mathbb{C}^2$, where the action of $\mathbb{C}$ on $\mathbb{C}^2$ is given by the complex one-parameter group $I = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$.

The non-zero constant curvature case The model of the geometry of constant nonzero curvature, in dimension $n \geq 2$, is the "holomorphic sphere" $S_n = O(n + 1, \mathbb{C})/O(n, \mathbb{C})$. Indeed, up a to multiplicative constant, S_n admits a unique, $O(n + 1, \mathbb{C})$-invariant, holomorphic Riemannian metric g. It turns out that $O(n+1, \mathbb{C})$ is the full isometry group of g, that g has a constant sectional curvature and is geodesically complete. Therefore, any n-manifold endowed with a holomorphic Riemannian metric of non-vanishing constant sectional curvature is locally modelled on the geometry $(O(n + 1, \mathbb{C}), S_n)$ [37].

- *Dimension 2.* A model of S_2 is $P^1(\mathbb{C}) \times P^1(\mathbb{C}) \backslash Diag$ endowed with the holomorphic Riemannian metric $\frac{dz_1 dz_2}{(z_1 - z_2)^2}$, given in local affine coordinates. Here the isometry group is $SL(2, \mathbb{C})$ acting diagonally.

- *Dimension 3.* The unique case where $O(n, \mathbb{C})$ is not simple is when $n = 4$ and then, $O(4, \mathbb{C}) = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. The space S_3 is identified with the group $SL(2, \mathbb{C})$ endowed with a left invariant holomorphic Riemannian metric which equals the Killing form at the identity. But the invariance of the Killing form by the adjoint representation implies that this holomorphic Riemannian metric is also right invariant. Therefore, the right and left multiplicative action of $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ on $SL(2, \mathbb{C})$ is isometric. For more details about this geometry (geodesics...) one can see [13].

Parallelisable manifolds The following proposition describes holomorphic Riemannian metrics on parallelisable manifolds.

Proposition 2.1 *Let $M = \Gamma \backslash G$ a compact parallelisable manifold, with G a simply connected complex Lie group and Γ a uniform lattice in G. Then, any holomorphic Riemannian metric g on M comes from a non degenerate complex quadratic form on the Lie algebra $\mathcal{G}$ of G. In particular, the pull-back of g is left invariant on the universal covering G (and g is locally homogeneous on M).*

Moreover, any compact parallelisable 3-manifold admits a holomorphic Riemannian metric of constant sectional curvature. The metric is flat exactly when G is solvable.

Proof Consider $X_1, X_2, \ldots, X_n$, the fundamental vector fields corresponding to the simply transitive G-action on M. Let g be a holomorphic Riemannian metric on M and denote also by g the associated complex symmetric bilinear form. Then $g(X_i, X_j)$ is a holomorphic function on M and thus constant, for all $1 \leq i, j \leq n$. This implies that g comes from a left-invariant holomorphic Riemannian metric on G.

Assume now G is a simply connected complex unimodular Lie group of dimension 3. We have only four such Lie groups: $\mathbb{C}^3$, the complex Heisenberg group, the complex SOL group and $SL(2, \mathbb{C})$ [19].

It is an easy exercice to exhibit in the isometry group $O(3, \mathbb{C}) \ltimes \mathbb{C}^3$ of the flat holomorphic Riemannian space, copies of the Heisenberg group and of the SOL group which acts simply transitively. Thus the flat holomorphic Riemannian space also admits models which are given by right-invariant metrics on the Heisenberg group and on the SOL group. One can get explicit expression of these holomorphic Riemannian metrics by complexification of flat right-invariant Lorentz metrics on the real Heisenberg and SOL groups [33,34]. □

We describe now the isometry group (and hence the Killing Lie algebra) of a left invariant holomorphic Riemannian metric on a 3-dimensional simply connected unimodular complex Lie group G.

- $G = \mathbb{C}^3$. Then any translations-invariant metric on $\mathbb{C}^3$ is flat and the corresponding isometry group is $O(3, \mathbb{C}) \ltimes \mathbb{C}^3$.
- $G = SL(2, \mathbb{C})$. We have seen that G admits the left invariant holomorphic Riemannian metric of non zero constant sectional curvature coming from the Killing form. Proposition 4.1 will show that the isometry group of a left invariant holomorphic Riemannian metric on $SL(2, \mathbb{C})$ lies in the isometry group $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ of the metric of constant non zero curvature. The isometry group is either $SL(2, \mathbb{C})$, or $\mathbb{C} \times SL(2, \mathbb{C})$, or $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$.

- $G = Heis$. Then, either the metric is flat (and G is a copy of the complex Heisenberg group lying in the full isometry group $O(3, \mathbb{C}) \ltimes \mathbb{C}^3$), or the isometry group is $Heis$, or the extension $\mathbb{C} \ltimes Heis$ of the Heisenberg group described at the point (2) in Proposition 5.2 (see Proposition 7.1).
- $G = SOL$. Here, either the metric is flat, or the isometry group is SOL, or a group of dimension four which is an extension of the Heisenberg group described at the point (iii) in Proposition 5.7 (see Proposition 6.1 which shows that a copy of the complex SOL group acts freely and transitively).

Remark 2.2 In particular, the quotients $\Gamma \backslash Heis$ and $\Gamma \backslash SOL$ possess holomorphic Riemannian metrics with solvable Killing Lie algebra and, as well, flat holomorphic Riemannian metrics (for which the Killing Lie algebra admits a non trivial semi-simple part). Consequently, the two situations in the main Theorem 1.2 are not exclusive.

Ghys non standard examples As above, for any co-compact lattice Γ in $SL(2, \mathbb{C})$, the quotient $M = \Gamma \backslash SL(2, \mathbb{C})$ admits a holomorphic Riemannian metric of non-zero constant sectional curvature. It is convenient to consider M as a quotient of S_3 by Γ, seen as a subgroup of $O(4, \mathbb{C})$ by the trivial embedding $\gamma \in \Gamma \mapsto (\gamma, 1) \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$.

New interesting examples of manifolds admitting holomorphic Riemannian metrics of non-zero constant sectional curvature have been constructed in [13] by deformation of this embedding of Γ.

Those deformations are constructed choosing a morphism $u : \Gamma \to SL(2, \mathbb{C})$ and considering the embedding $\gamma \mapsto (\gamma, u(\gamma))$. Algebraically, the action is given by:

$$(\gamma, m) \in \Gamma \times SL(2, \mathbb{C}) \to \gamma m u(\gamma^{-1}) \in SL(2, \mathbb{C}).$$

It is proved in [13] that, for u close enough to the trivial morphism, Γ acts properly (and freely) on $S_3 (\cong SL(2, \mathbb{C}))$ such that the quotient M_u is a complex compact manifold (covered by $SL(2, \mathbb{C})$) admitting a holomorphic Riemannian metric of non-zero constant sectional curvature. *In general, these examples do not admit parallelisable manifolds as finite covers*.

Note that left-invariant holomorphic Riemannian metrics on $SL(2, \mathbb{C})$ which are not right-invariant, in general, will not descend on M_u.

Let us notice that despite this systematic study in [13], there are still many open questions regarding these examples (including the question of completeness). A real version of this study is in [15,23,36]. This story is also related to the study of Anosov flows with smooth distributions [14].

Non-zero constant curvature in higher dimension? One interesting problem in differential geometry is to decide whether a given homogeneous space G/I possesses or not a compact quotient. A more general related question is to decide whether there exist compact manifolds locally modelled on $(G, G/I)$ (see, for instance [1,2,22,24]).

The case $I = 1$, or more generally I compact, reduces to the classical question of existence of co-compact lattices in Lie groups. For homogeneous spaces of non-Riemannian type (i.e. I non-compact) the problem is much harder.

The case $S_n = O(n+1, \mathbb{C})/O(n, \mathbb{C})$ is a geometric situation where these questions can be tested. It turns out that compact quotients of S_n are known to exist only for

$n = 1, 3$ or 7. We discussed the case $n = 3$ above, and the existence of a compact quotient of S_7 was proved in [22]. Here, we dare ask with [22]:

Conjecture 2.3 *S_n has no compact quotients, for $n \neq 1, 3, 7$.*

A stronger version of our question was proved in [1] for S_n, if n has the form $4m+1$.

Keeping in mind our geometric approach, we generalize the question to manifolds locally modelled on S_n. More exactly:

Conjecture 2.4 *A compact complex manifold endowed with a holomorphic Riemannian metric of constant non-vanishing curvature is complete. In particular, such a manifold has dimension* 3 *or* 7.

3 Geometry of the Killing algebra

Recall that a holomorphic Riemannian metric g on M is said *locally homogeneous* if for all $m, n \in M$ there is a local biholomorphism from an open neighborhood of m to an open neighborhood of n which sends m to n and preserves g. Such a local biholomorphism preserving g is called a *local isometry*.

By Theorem 1.1, each holomorphic Riemannian metric on a *compact* complex 3-manifold is locally homogeneous. Equivalently the Lie algebra of local holomorphic Killing fields (i.e. local holomorphic vector fields whose local flows preserve g) is transitive on M. In particular, the Killing Lie algebra $\mathcal{G}$ of g is of dimension ≥ 3.

Moreover, for any holomorphic tensor field ϕ on M, the pseudo-group of local isometries of g preserving also ϕ acts transitively on M (i.e. if we put together g and ϕ, this yields a locally homogeneous geometric structure).

The set of local isometries I of g fixing a point $x_0 \in M$ generate a local group called the *isotropy group* of g. The corresponding Lie algebra $\mathcal{I}$ consists in the subalgebra of Killing fields vanishing at x_0. As an isometry fixing x_0 is uniquely determined by its differential at x_0 [38], the local group of isotropy at x_0 injects into the orthogonal group of $(T_{x_0}M, g_{x_0})$ and thus it is of dimension ≤ 3. It follows that $\mathcal{G}$ is of dimension ≤ 6.

Let G be the connected simply connected complex Lie group corresponding to $\mathcal{G}$ and I its subgroup corresponding to $\mathcal{I}$. By a Theorem of Mostow [31], I is closed in G (this will follow also from our classification of $\mathcal{G}$ and $\mathcal{I}$). Thus g is locally isometric to an algebraic model G/I endowed with a G-invariant holomorphic Riemannian metric. Since the (full) isometry group of G/I has at most finitely many connected components, up to a finite cover, M admits a $(G, G/I)$-geometry in Thurston's sense [37]: M admits an atlas with open sets in G/I and transition functions given by elements in G.

We will classify all possible models $(G, G/I)$. We settle first the easiest cases where $\mathcal{G}$ has dimension 3, 5 and 6.

3.1 dim $\mathcal{G} = 3$

Lemma 3.1 *Let M be a compact connected complex* 3*-manifold admitting a holomorphic Riemannian metric g. Assume one of the following assumptions holds*:

(i) *the Killing Lie algebra* $\mathcal{G}$ *of* g *has dimension* 3;
(ii) M *admits two linearly independent global holomorphic vector fields.*

Then, up to a finite unramified cover, M *is a quotient of a complex Lie group* G *by a lattice* Γ *(hence it admits some holomorphic Riemannian metric of constant sectional curvature) and the pull-back of* g *on the universal cover of* M *is a left invariant holomorphic Riemannian metric on* G.

Remark 3.2 If $G = \mathbb{C}^3$, then g is flat and its Killing Lie algebra is of dimension 6 (see Proposition 3.3).

Proof (i) As g is locally homogeneous and $\mathcal{G}$ is of dimension 3, the action of $\mathcal{G}$ on M is simply transitive. This gives a (G, G)-structure on M, where the complex Lie group G acts on itself by left translations. The compactness of M implies the completeness of the (G, G)-structure [37] and hence M is a quotient of G by a lattice Γ.

(ii) We apply Theorem 1.1 to the holomorphic geometric structure on M which is the combination of g with the two global vector fields . Consequently this geometric structure is locally homogeneous. Moreover, its Killing Lie algebra is easily seen to be of dimension 3. Indeed, the local isotropy group at $x_0 \in M$ is trivial because any element of it which fixes two linearly independent vectors in $T_{x_0}M$ is trivial. One has just to check directly the claim for the equivalent situation: $O(3, \mathbb{C})$ acting linearly on $\mathbb{C}^3$. Finally, we conclude as in the case (i). □

3.2 dim $\mathcal{G} = 6$

Here we have the following well-known

Proposition 3.3 *The dimension of* $\mathcal{G}$ *is* 6 *if and only if* g *is of constant sectional curvature.*

Remark 3.4 In this case $\mathcal{G}$ has a non trivial semi-simple part.

Proof The dimension of $\mathcal{G}$ is 6 if and only if the dimension of $\mathcal{I}$ is 3 and if and only if each element in the connected component of identity of the orthogonal group of $(T_{x_0}M, g_{x_0})$ extends to a local isometry. As the identity component of the orthogonal group of $(T_{x_0}M, g_{x_0})$ acts transitively on the set of non-degenerate planes in $T_{x_0}M$, all these planes have the same sectional curvature. By local homogeneity, this sectional curvature does not depend on the point x_0.

Conversely the two models of 3-dimensional spaces of constant sectional curvature have a Killing Lie algebra of dimension 6 which is the Lie algebra of $O(3, \mathbb{C}) \ltimes \mathbb{C}^3$, in the flat case, or the Lie algebra of $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$, in the non flat one. □

3.3 dim $\mathcal{G} = 5$

We will see this never happens.

Recall first that $SL(2, \mathbb{C})$ is locally isomorphic to $O(3, \mathbb{C})$. One way to see it is to consider the adjoint representation of $SL(2, \mathbb{C})$ into the 3-dimensional complex vector space $sl(2, \mathbb{C})$ and to note that this action preserves the Killing form. More precisely, we have $SO(3, \mathbb{C}) \simeq PSL(2, \mathbb{C})$, where $SO(3, \mathbb{C})$ is the connected component of the identity of the orthogonal group and $PSL(2, \mathbb{C})$ is the quotient of $SL(2, \mathbb{C})$ by the center $\{Id, -Id\}$.

Proposition 3.5 *The dimension of $\mathcal{G}$ is $\neq 5$.*

Proof Assume, by contradiction, that dim $\mathcal{G} = 5$ and, equivalently, the dimension of the isotropy $\mathcal{I}$ is 2. Consider the action of the local isotropy group at x_0 on $T_{x_0}M$ and identify this local isotropy to a 2-dimensional subgroup I of $SO(3, \mathbb{C}) \simeq PSL(2, \mathbb{C})$. The action of I on $T_{x_0}M$ preserves g_{x_0}, but also the curvature tensor and, in particular, the Ricci tensor $Ricci_{x_0}$ which is a complex quadratic form on $T_{x_0}M$.

Consider the action of $PSL(2, \mathbb{C})$ on the complex vector space of complex quadratic forms $S^2(T^*_{x_0}M)$. This action preserves g_{x_0} and gives an action of $PSL(2, \mathbb{C})$ on the quotient vector space $S^2(T^*_{x_0}M)/\mathbb{C}g_{x_0}$.

The isotropy group lies in the stabilizer of the class of $Ricci_{x_0}$ in the quotient $S^2(T^*_{x_0}M)/\mathbb{C}g_{x_0}$. But, for an algebraic action of $PSL(2, \mathbb{C})$ on an affine space, the stabilizer of an element can not be 1-dimensional. Indeed, by contradiction, up to an inner automorphism of $PSL(2, \mathbb{C})$, the stabilizer coincides with the subgroup $G' \subset PSL(2, \mathbb{C})$ of upper triangular matrices and thus the orbit $PSL(2, \mathbb{C})/G'$ is biholomorphic to the projective line $P^1(\mathbb{C})$, which is compact and so can not be holomorphically embedded in an affine space.

It follows that the stabilizer of the $Ricci_{x_0}$ class in $S^2(T^*_{x_0}M)/\mathbb{C}g_{x_0}$ is of dimension 3 and hence equal to $PSL(2, \mathbb{C})$. This implies that $Ricci_{x_0} = \lambda g_{x_0}$, with $\lambda \in \mathbb{C}$ and the function λ is constant on M by local homogeneity. But then, g has constant sectional curvature and so $\mathcal{G}$ is of dimension 6 which is contrary to our initial assumption. □

3.4 dim $\mathcal{G} = 4$

This is the most delicate case and all our analysis throughout the paper will be devoted to it.

Here $\mathcal{I}$ has dimension 1. The (local) isotropy group I is algebraic and has finitely many components. Up to a finite cover, we can assume it connected, i.e. a one parameter group. Therefore, I is conjugate in $PSL(2, \mathbb{C})$ to one of the following:

(1) A *unipotent* one-parameter subgroup $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ fixing in $T_{x_0}M$ a vector of norm 0;

(2) A *semi-simple* one-parameter subgroup $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ fixing in $T_{x_0}M$ a vector of norm 1.

Adapted basis In order to understand the action of I on $T_{x_0}M$ (as a subgroup of $O(3, \mathbb{C})$) we shall consider some adapted bases.

Let us first consider the case where the isotropy is semi-simple. Then the action of I on $T_{x_0}M$ fixes some vector e_1 of norm 1. The plane $e_1^{\perp}$ is non degenerate and, up to a multiplicative constant, the vectors $e_2, e_3 \in e_1^{\perp}$ are uniquely defined by the following conditions: e_2, e_3 generate the two isotropic directions in $e_1^{\perp}$ and $g(e_2, e_3) = 1$.

The time t of the flow generated by the isotropy $\mathcal{I}$ will be given in this adapted basis (e_1, e_2, e_3), by the formula $(e_1, e_2, e_3) \to (e_1, e^t e_2, e^{-t} e_3)$.

In the case of a unipotent isotropy, the action of I on $T_{x_0}M$ fixes an isotropic vector e_1 and so preserves the degenerate plane $e_1^{\perp}$ (of course $e_1 \in e_1^{\perp}$). In order to define an adapted basis, take two vectors $e_2, e_3 \in T_{x_0}M$ such that: $g(e_1, e_2) = 0$, $g(e_2, e_2) = 1$, $g(e_3, e_3) = 0$, $g(e_2, e_3) = 0$ and $g(e_3, e_1) = 1$.

Note that such an adapted basis is uniquely determined by the choice of an unitary vector $e_2 \in e_1^{\perp}$. Indeed, then e_3 is uniquely defined in $e_2^{\perp}$ by the relation $g(e_3, e_1) = 1$ (e_1 and e_3 generate the two isotropic directions in $e_2^{\perp}$).

The action of the isotropy I on $T_{x_0}M$ sends an adapted basis to an adapted basis. This action is given in the basis (e_1, e_2, e_3) by $\begin{pmatrix} 1 & t & -\frac{t^2}{2} \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{pmatrix}$.

Lemma 3.6 *(i) If $\mathcal{G}$ is of dimension* 4, *then, up to a finite cover, M admits a global holomorphic vector field X which is preserved by the action of $\mathcal{G}$. The norm of X is constant equal to* 0 *or constant equal to* 1, *according to that the isotropy is unipotent or semi-simple.*

(ii) The divergence of X (with respect of the volume form of g) is 0.

(iii) If the isotropy is semi-simple, then X is a Killing field.

Corollary 3.7 *If the isotropy is semi-simple, then $\mathcal{G}$ has a non trivial center.*

Remark 3.8 We will see further that the orthogonal $X^{\perp}$ of the Killing field X constructed at the point (iii) of the previous lemma is not always integrable. It follows that the dual one form associated to X is not closed. This is another difference with the Kaehler background where holomorphic forms are known to be closed (and one could study the Albanese map).

Proof (i) At x_0, X is defined by $X(x_0) = e_1$.

(ii) Denote by ϕ^t the complex flow generated by X. Recall that the divergence $div(X)$ of X, with respect to the volume form vol of g, is given by the formula $L_X vol = div(X) vol$, where L_X is the Lie derivative in the direction X. As $\mathcal{G}$ acts transitively on M preserving X (and also vol), the function $div(X)$ is holomorphic and so is a constant $\lambda \in \mathbb{C}$. This means that $(\phi^t)^* vol = e^{\lambda t} vol$, for all $t \in \mathbb{C}$. But the total real volume of M given by the integral on M of the real form $vol \wedge \overline{vol}$ has to be preserved by ϕ^t. Thus the modulus of $e^{\lambda t}$ equals 1 for all $t \in \mathbb{C}$. It then follows that $\lambda = 0$, that is $div(X) = 0$.

(iii) The action of $\mathcal{G}$ preserves X and so also $X^{\perp}$. We will show first that ϕ^t preserves $X^{\perp}$ as well. Take a point $x_0 \in M$ and consider its image $\phi^t(x_0)$. For each $t \in \mathbb{C}$ let us choose a local isometry g^t sending x_0 to $\phi^t(x_0)$.

The local diffeomorphism $(g^t)^{-1} \circ \phi^t$ fixes x_0 and the vector $X(x_0) \in T_{x_0}M$. Since X is $\mathcal{G}$-invariant, $(g^t)^{-1} \circ \phi^t$ commutes with all local isometries . In particular, the differential L_t of $(g^t)^{-1} \circ \phi^t$ at x_0 commutes with the action of the isotropy at x_0 and hence preserves the eigenspaces of the isotropy. Since the isotropy is supposed to be semi-simple, the differential L_t preserves the non-degenerate plane $X(x_0)^{\perp}$ and also its two isotropic directions.

As $div(X) = 0$, the differential L_t preserves the volume. It follows that the product of the two eigenvalues corresponding to the two isotropic directions of $X(x_0)^{\perp}$ equals 1. This implies that the differential of $(g^t)^{-1} \circ \phi^t$ at x_0 is an isometry. Consequently the flow of X acts by isometries and X is Killing. Hence $\mathbb{C}X$ is in the center of $\mathcal{G}$. □

Proposition 3.9 *If the isotropy is unipotent, then the holomorphic field of complex endomorphisms* $\nabla.X$ *of* TM*, in an adapted basis, is* $\begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, *with* α *a complex constant.*

Then X *is Killing if and only if* $\alpha = 0$.

Proof Let us fix $x_0 \in M$ and an adapted basis (e_1, e_2, e_3) of $T_{x_0}M$. In this basis the differential L_t of I at x_0 is given by the one-parameter group $\begin{pmatrix} 1 & t & -\frac{t^2}{2} \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{pmatrix}$.

First we show that any $\mathcal{G}$-invariant holomorphic field of complex endomorphisms Ψ of TM has, in our adapted basis, the following form: $\begin{pmatrix} \lambda & \beta & \alpha \\ 0 & \lambda & -\beta \\ 0 & 0 & \lambda \end{pmatrix}$, with α, β and $\gamma \in \mathbb{C}$.

Let B be the matrix of $\Psi(x_0)$ in the basis (e_1, e_2, e_3). Since Ψ is $\mathcal{I}$-invariant, B and L_t commute. Each eigenspace of B is preserved by L_t and conversely. As L_t does not preserve any non trivial splitting of $T_{x_0}M$, it follows that all eigenvalues of B are equal to some $\lambda \in \mathbb{C}$. A straightforward calculation shows that B has the previous form. As Ψ is $\mathcal{G}$-invariant, the parameters α, β and γ do not depend of x_0.

We apply this result to $\nabla.X$ (which is $\mathcal{G}$-invariant because X and ∇ are). As the trace of $\nabla.X$ is the divergence of X, lemma 3.6 implies $\lambda = 0$.

It will be (independently) shown in Proposition 5.4 that X is parallel on any direction tangent to $X^{\perp}$. It follows that $\nabla_{e_2}X = 0$ and $\beta = 0$.

The vector field X is Killing if and only if $\nabla.X$ is g-skew-symmetric [38]. But an endomorphism of rank ≤ 1 is skew-symmetric if and only if it is trivial. It follows that X is Killing if and only if $\alpha = 0$. □

Geodesic foliations The following lemma is just the complexification in the realm of holomorphic Riemannian metrics of a well-known fact remarked for the first time by M. Gromov [16] (see also the survey [6]) in the context of Lorentz geometry.

Lemma 3.10 *(i) If the isotropy is unipotent, then the plane field* $X^{\perp}$ *is integrable. Its tangent holomorphic foliation of codimension one* $\mathcal{F}$ *is geodesic,* g*-degenerate and* $\mathcal{G}$*-invariant.*

(ii) If the isotropy is semi-simple, then M *possesses two holomorphic foliations of codimension one* $\mathcal{F}_1$ *and* $\mathcal{F}_2$*, which are geodesic,* g*-degenerate and* $\mathcal{G}$*-invariant. The tangent space of each one of these two foliations is generated by* X *and by one of the two isotropic directions of* $X^{\perp}$.

Proof The idea of Gromov's proof is to consider the graph of a local isometry fixing $x_0 \in M$ as a (3-dimensional) submanifold in $M \times M$ passing through (x_0, x_0).

This submanifold is geodesic and isotropic for the holomorphic Riemannian metric $g \oplus (-g)$ on $M \times M$. If f_n is a sequence of elements in the local isotropy group at x_0 (identified with the orthogonal group of $(T_{x_0}M, g_{x_0})$) which tends to infinity in this orthogonal group, then the sequence of corresponding graphs tends to a 3-dimensional geodesic and isotropic submanifold F' which is no longer a graph. Nevertheless, the intersection of F' with the vertical space $\{x_0\} \times M$ is isotropic in M and thus has dimension ≤ 1. The projection F of F' on the horizontal space $M \times \{x_0\}$ is a geodesic surface passing through x_0.

In our situation I has dimension 1 and we can take a sequence of elements of the one-parameter group I in the orthogonal group going to infinity (one parameter groups are not compact, which contrasts with the real case). In exponential coordinates our local isometries are linear and in some adapted basis they have the form presented previously. We note that the limit of our sequence of (linear) graphs is the plane $X(x_0)^{\perp}$ if the isotropy is unipotent and the two planes generated by $X(x_0)$ and by each of the two isotropic directions of $X(x_0)^{\perp}$ if the isotropy is semi-simple.

These foliations are obviously $\mathcal{G}$-invariants, as everything is. □

We will also denote by X and $\mathcal{F}$ the corresponding vector field and foliation on the algebraic model G/I.

The stabilizer H of a leaf If the isotropy is unipotent, denote by $\mathcal{H}$ the subalgebra of $\mathcal{G}$ stabilizing the leaf $\mathcal{F}(x_0)$ of $\mathcal{F}$ passing through $x_0 \in M$ and by H the corresponding Lie subgroup of G. We keep the same notation for the stabilizer of $\mathcal{F}_1(x_0)$ if the isotropy is semi-simple.

Proposition 3.11 *The group H is of dimension 3 and acts transitively on $\mathcal{F}(x_0)$ (or $\mathcal{F}_1(x_0)$ accordingly). The isotropy I at x_0 lies in H.*

Corollary 3.12 *The leaf F is locally modelled on $(H, H/I)$.*

Proof We give the proof in the case of unipotent isotropy. Take $x_1 \in \mathcal{F}(x_0)$ and consider a local isometry ϕ sending x_0 on x_1. As ϕ preserves X and $X^{\perp}$ it has to send $exp_{x_0}(X^{\perp})$ onto $exp_{x_1}(X^{\perp})$. The leaf $\mathcal{F}(x_0)$ being geodesic, $exp_{x_0}(X^{\perp}) \subset \mathcal{F}(x_0)$ and $exp_{x_1}(X^{\perp}) \subset \mathcal{F}(x_0)$. That means that ϕ lies in the stabilizer of $\mathcal{F}(x_0)$. In particular, if ϕ fixes x_0 then ϕ lies in the stabilizer of $\mathcal{F}(x_0)$. This implies $\mathcal{I} \subset \mathcal{H}$.

As $\mathcal{G}$ acts transitively on $\mathcal{F}(x_0)$, the previous argument shows that $\mathcal{H}$ acts transitively on $\mathcal{F}(x_0)$ (with isotropy of dimension 1). It follows that $\mathcal{H}$ has dimension 3. □

4 Algebraic models for the local structure: the semi-simple case

In this section the Killing algebra $\mathcal{G}$ has dimension 4, and thus the isotropy $\mathcal{I}$ has dimension 1. We assume that $\mathcal{G}$ has a non-trivial semi-simple part.

Proposition 4.1 *Assume $\mathcal{G}$ has a non-trivial semi-simple part. Then, it is a direct product of Lie algebras $\mathbb{C} \oplus sl(2, \mathbb{C})$, and we have two possible models G/I:*

(1) *The holomorphic Riemannian metric is left invariant on the group $SL(2, \mathbb{C})$.*

The identity connected component of its isometry group is a direct product of $SL(2,\mathbb{C})$ *acting by left translations and some one parameter subgroup* $h^t \subset SL(2,\mathbb{C})$ *acting on by right translations. The isotropy group* I *is the image of the diagonal embedding* (h^t, h^t) *in* $\mathbb{C} \times SL(2,\mathbb{C})$.

(2) *The holomorphic Riemannian direct product* $\mathbb{C} \times S_2$*, where* S_2 *is the universal model of a surface with holomorphic Riemannian metric of non zero constant sectional curvature and* $\mathbb{C}$ *is endowed with its standard metric* dz^2.

The action of the isometry group $G = \mathbb{C} \times SL(2,\mathbb{C})$ *is split. The isotropy* I *is the one-parameter subgroup of* $SL(2,\mathbb{C})$ *given by* $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$.

Corollary 4.2 *In the case* (1) *the action of* $\mathcal{G}$ *on* M *preserves the holomorphic Riemannian metric of non zero constant sectional curvature coming from the Killing form on* $sl(2,\mathbb{C})$.

Remark 4.3 It will be shown in §7 that the situation (2) cannot occur on compact 3-manifolds.

Proof There is no semi-simple algebra of dimension 4, and $sl(2,\mathbb{C})$ is the unique semi-simple complex Lie algebra of dimension 3. Therefore, $\mathcal{G}$ is a direct product $\mathbb{C} \times sl(2,\mathbb{C})$ (see, for instance, [19]).

If the isotropy of some point intersects non-trivially the factor $SL(2,\mathbb{C})$, then this is the case for all points. In fact, since the isotrop I has dimension 1, it intersects $SL(2,\mathbb{C})$ iff it is contained inside it.

(1) Therefore, in the case of trivial intersection, the group $SL(2,\mathbb{C})$ acts freely transitively on M. The metric is thus identified to a left invariant one on $SL(2,\mathbb{C})$.

Consider the action of the isotropy I on $SL(2,\mathbb{C})$ (the base point being the neutral element Id in $SL(2,\mathbb{C})$). Our claim reduces to the fact that the I-action coincides with the adjoint action of some one parameter group h^t. For this, it suffices to show that the metric is preserved by the adjoint action of h^t on $sl(2,\mathbb{C})$. Indeed, if so, this integrates on the adjoint action of h^t on the group $SL(2,\mathbb{C})$ which is isometric. But, since the dimension of the isotropy is one, we get coincidence of I with the adjoint action of h^t.

The I-action on $sl(2,\mathbb{C})$ by the adjoint representation is done by Lie algebras isomorphisms.

On the other hand the previous action identifies with the I-action on $T_{Id}SL(2,\mathbb{C})$ and has to fixe some vector. It is easy to check that each one-parameter group of isomorphisms of the Lie algebra $sl(2,\mathbb{C})$ fixing a vector coincides with the adjoint representation of some one-parameter subgroup h^t of $SL(2,\mathbb{C})$.

(2) Assume now that $I \subset SL(2,\mathbb{C})$. The action of I on $\mathbb{C} \oplus sl(2,\mathbb{C})$ gives an I-invariant non trivial splitting of $T_{x_0}M$. It follows that I is semi-simple and the $SL(2,\mathbb{C})$-orbits are tangents to $X^{\perp}$ (in particular, they are g-non degenerate). Then, the $SL(2,\mathbb{C})$-orbits are complex homogeneous surfaces endowed with a $SL(2,\mathbb{C})$-invariant holomorphic Riemannian metric. They have in particular constant curvature, and obviously cannot be flat (because their Killing algebra contains $sl(2,\mathbb{C})$). Up to a multiplicative constant, they are isometric to S_2. □

5 Algebraic models for the local structure: the solvable case

We assume here that G is solvable (and of dimension 4).

Proposition 5.1 *(i) The derivative Lie algebra* $[\mathcal{H}, \mathcal{H}]$ *is 1-dimensional.*

(ii) The group H *is isomorphic either to the Heisenberg group or to the product* $\mathbb{C} \times AG$*, where* AG *is the universal covering of the affine group of the complex line.*

Recall that the affine group of the complex line is the group of transformations of $\mathbb{C}$, given by $z \to az + b$, with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. If Y is the infinitesimal generator of the homotheties and Z the infinitesimal generator of the translations, then $[Y, Z] = Z$.

Proof (i) It is a general fact that a derivative algebra of a solvable algebra is nilpotent. Remark first that $[\mathcal{H}, \mathcal{H}] \neq 0$. Indeed, if not $\mathcal{H}$ is abelian and the action of the isotropy $\mathcal{I} \subset \mathcal{H}$ would be trivial on $\mathcal{H}$ and hence on $T_{x_0}F$ which is identified to $\mathcal{H}/\mathcal{I}$. Since the restriction to the isotropy action to the tangent space of F is injective this implies that the isotropy action is trivial on $T_{x_0}G/I$ which is impossible.

As $\mathcal{H}$ is 3-dimensional, its derivative algebra $[\mathcal{H}, \mathcal{H}]$ is a nilpotent Lie algebra of dimension 1 or 2, hence $[\mathcal{H}, \mathcal{H}] \simeq \mathbb{C}$ or $[\mathcal{H}, \mathcal{H}] \simeq \mathbb{C}^2$.

Assume by contradiction that $[\mathcal{H}, \mathcal{H}] \simeq \mathbb{C}^2$.

We first prove that the isotropy $\mathcal{I}$ lies in $[\mathcal{H}, \mathcal{H}]$. If not, $[\mathcal{H}, \mathcal{H}] \simeq \mathbb{C}^2$ will act freely on F. Therefore F is locally identified with the group $\mathbb{C}^2$ endowed with a left invariant connection, a left invariant holomorphic degenerate Riemannian metric (compatible with the connection) and a left invariant holomorphic vector field (which is X).

We show now that the connection is flat. The local model for the left invariant degenerate metric on F is dh^2 in the coordinates (x, h) of $\mathbb{C}^2$. In this coordinates the left invariant vector field X coincides with $\frac{\partial}{\partial x}$, if the isotropy is unipotent and with $\frac{\partial}{\partial h}$, if the isotropy is semi-simple.

An easy calculation shows that any torsion-free and $\mathbb{C}^2$-invariant connection compatible with dh^2 is given by $\nabla_{\frac{\partial}{\partial h}} \frac{\partial}{\partial h} = a\frac{\partial}{\partial x}$, $\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = b\frac{\partial}{\partial x}$ and $\nabla_{\frac{\partial}{\partial h}} \frac{\partial}{\partial x} = \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial h} = c\frac{\partial}{\partial x}$, for some $a, b, c \in \mathbb{C}$. The invariance by the isotropy one-parameter group implies that at least two of the parameters a, b, c vanish. In this case the curvature of ∇ vanishes.

The isometry group of this model is $\mathbb{C} \ltimes \mathbb{C}^2$, where the action of the isotropy $I \simeq \mathbb{C}$ on $\mathbb{C}^2$ is given by the one parameter group of linear transformations $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, if I is unipotent, or by $\begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}$, if I is semi-simple. Our group is thus isomorphic to the Heisenberg group or to $AG \times \mathbb{C}$. In both cases the derivative group is 1-dimensional which contradicts our assumption, and hence $\mathcal{I} \subset [\mathcal{H}, \mathcal{H}]$.

It follows in particular that the orbits of $[\mathcal{H}, \mathcal{H}]$ on F are 1-dimensional. We prove now that the orbits of $[\mathcal{H}, \mathcal{H}]$ on F correspond to the isotropic direction in F and the isotropy I is unipotent.

Let Y be a generator of $\mathcal{I}$, $\{Y, X'\}$ be generators of $[\mathcal{H}, \mathcal{H}]$ and $\{Y, X', Z\}$ be a basis of $\mathcal{H}$. The tangent space of F at some point $x_0 \in F$ is identified with $\mathcal{H}/\mathcal{I}$ and the infinitesimal (isotropic) action of Y on this tangent space is given in the basis $\{X', Z\}$ by the matrix $ad(Y) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. This is because $[\mathcal{H}, \mathcal{H}] \simeq \mathbb{C}^2$ and $ad(Y)(\mathcal{H}) \subset [\mathcal{H}, \mathcal{H}]$. Moreover, $ad(Y) \neq 0$ since the restriction to the isotropy action to the tangent space of F is injective.

From this form of $ad(Y)$, we see that the isotropy is unipotent with fixed direction $\mathbb{C}X'$. This direction is exactly the tangent direction of the orbits of $[\mathcal{H}, \mathcal{H}]$ on F.

Denote by $\mathcal{L}$ the derivative algebra $[\mathcal{G}, \mathcal{G}]$ of $\mathcal{G}$. Then $\mathcal{L} \supset [\mathcal{H}, \mathcal{H}] \supset \mathcal{I}$. The dimension of $\mathcal{L}$ is 2 or 3 and the $\mathcal{L}$-orbits on G/I have dimension 1 or 2 accordingly.

Assume first that $\mathcal{L}$ is 3-dimensional and thus has 2-dimensional orbits on G/I. The foliation of G/I provided by the $\mathcal{L}$-action is 2-dimensional and invariant by the unipotent isotropy I. Since $X'^{\perp}$ is the only plane field on G/I preserved by the isotropy, it follows that the leafs of the $\mathcal{L}$-action coincide with those of the $\mathcal{H}$-action. So $\mathcal{L} = \mathcal{H}$, as Killing algebra of F. But this is impossible, since $\mathcal{L}$ is nilpotent (as a derivative algebra of a solvable algebra) and $\mathcal{H}$ is not (its derivative algebra is supposed to be 2-dimensional).

It remains to settle the case where $\mathcal{L}$ is 2-dimensional. We show in this case that the infinitesimal isometry $ad(Y)$ of $T_{x_0}G/I$ has rank 1, which is not possible for an infinitesimal isometry of a holomorphic Riemannian metric.

Since $\mathcal{L} = [\mathcal{G}, \mathcal{G}]$, the image of $\mathcal{G}$ by the isotropy action $ad(Y)$ at $x_0 \in G/I$ is contained in $\mathcal{L}$. Thus this image has at most dimension 2 and as $\mathcal{I} \subset \mathcal{L}$ and the tangent space at x_0 is identified with $\mathcal{G}/\mathcal{I}$, the image of $ad(Y)$ in $T_{x_0}G/I$ is of dimension at most 1.

This completes the proof of part (i) of the proposition.

(ii) Let Z be a generator of $[\mathcal{H}, \mathcal{H}]$ and consider its adjoint map $ad(Z) : \mathcal{H} \to \mathbb{C}Z$. If this map is trivial then, Z is central and $\mathcal{H}$ is nilpotent isomorphic to the Heisenberg group.

Consider now the case where $ad(Z)$ is not trivial. Let X' be a generator of the kernel of $ad(Z)$ and take $Y \in \mathcal{H}$ such that $\{Y, X', Z\}$ is a basis of $\mathcal{H}$. We can assume that $[Y, Z] = Z$. We also have $[X', Y] = aZ$, with $a \in \mathbb{C}$. After replacing X' by $X' + aZ$, we can assume that $a = 0$. It follows that $H = \mathbb{C} \times AG$, where the center of H is $exp(\mathbb{C}X')$ and AG is generated by $exp(\mathbb{C}Z)$ and $exp(\mathbb{C}Y)$. □

5.1 The case: $H = \mathbb{C} \times AG$

In this case, all possible algebraic models $(G, G/I)$ are described in the following:

Proposition 5.2 *The isotropy group I is semi-simple (it is generated by the infinitesimal generator of the homotheties in AG) and G is one of the following Lie groups*:

(1) $G = \mathbb{C} \times SOL$
(2) $G = \mathbb{C} \ltimes Heis$
(3) $G = \mathbb{C}^2 \ltimes \mathbb{C}^2$

In case (2) *the action of the first factor* $I \simeq \mathbb{C}$ *on* $Heis$, *is given by* $(X', Z, T) \to (X', e^t Z, e^{-t} T)$, *with respect of a basis* (X', Z, T), *such that* X' *is central and* $[T, Z] = X'$, *In case* (3) *the action of the first copy of* $\mathbb{C}^2$ *on the second one is given by the matrices* $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ *and* $\begin{pmatrix} 1 & 0 \\ 0 & e^{-t} \end{pmatrix}$.

Remark 5.3 As the center of $\mathcal{G} = \mathbb{C}^2 \ltimes \mathbb{C}^2$ is trivial, it follows from Lemma 3.6 that this Lie algebra cannot occur as a local Killing algebra for a holomorphic Riemannian metric on a compact complex 3-manifold.

Proof As before suppose that $\{X', Y, Z\}$ is a basis of $\mathcal{H}$ with X' central and Y, Z spanning the Lie algebra of AG such that $[Y, Z] = Z$. Denote by T a fourth generator of the Killing algebra $\mathcal{G}$.

We show that, up to an automorphism of $\mathcal{H}$ sending Y to $Y + aZ + bX'$, with $a, b \in \mathbb{C}$, *the isotropy algebra* $\mathcal{I}$ *is* $\mathbb{C}Y$.

Observe that $ad(\alpha X' + \beta Z)(\mathcal{H}) \subset \mathbb{C}X' \oplus \mathbb{C}Z$, for all $\alpha, \beta \in \mathbb{C}$. If the isotropy $\mathcal{I}$ is $\mathbb{C}(\alpha X' + \beta Z)$ then the action of $ad(\alpha X' + \beta Z)$ on $T_{x_0}F \simeq \mathcal{H}/\mathcal{I}$ is given by a matrix of rank 1. Consequently the isotropy is not semi-simple. We then proved that in the case where the isotropy is semi-simple, the isotropy $\mathcal{I}$ doesn't lie in $\mathbb{C}X' \oplus \mathbb{C}Z$ and, up to an automorphism of $\mathcal{H}$ sending Y to $Y + aZ + bX'$, we can assume that $\mathcal{I} = \mathbb{C}Y$.

Now, we show the same result in the case of unipotent isotropy. Observe first that $\mathcal{I} \neq \mathbb{C}X'$ since the central element X' acts trivially on $\mathcal{H}$ and hence also on $\mathcal{H}/\mathcal{I} \simeq T_{x_0}F$, which is impossible.

Assume, by contradiction, that $\mathcal{I} \subset \mathbb{C}X' \oplus \mathbb{C}Z$. Up to an automorphism of $\mathcal{H}$ sending Z to $Z + \alpha X'$, with $\alpha \in \mathbb{C}$, we can assume that $\mathcal{I} = \mathbb{C}Z$. Then, the abelian Lie algebra $\mathbb{C}X' \oplus \mathbb{C}Y$ intersects trivially $\mathcal{I}$ and will act freely on F. As in the proof of Proposition 5.1, this implies that F is flat and the Killing Lie algebra of F is $heis$. But, this is impossible, since the Heisenberg group is nilpotent and $H = \mathbb{C} \times AG$ is not.

It follows that, up to an automorphism of H, we have $\mathcal{I} = \mathbb{C}Y$. This is impossible in the unipotent isotropy case. Indeed, the abelian Lie algebra $\mathbb{C}X' \oplus \mathbb{C}Z$ acts freely on F and F is flat. If the isotropy was unipotent then, as before, H is isomorphic to the Heisenberg group which contradicts our hypothesis.

Therefore, the isotropy is semi-simple. As the isotropy $\mathbb{C}Y$ fixes X' and expands the direction $\mathbb{C}Z$ (because of the relation $[Y, Z] = Z$), we can choose as fourth generator T of $\mathcal{G}$ the second isotropic direction of the Lorentz plane $X'^{\perp}$. Then we will have $[Y, T] = -T + aY$, for some constant $a \in \mathbb{C}$ and we can replace T with $T - aY$ in order to get $[Y, T] = -T$.

In the following, we assume that $[Y, T] = -T$.

We will first show that $[T, Z] = aX' + bY$, with $a, b \in \mathbb{C}$ and $[T, X'] = cT$, for some $c \in \mathbb{C}$.

For the first relation we use the Jacobi relation $[Y, [T, Z]] = [[Y, T], Z] + [T, [Y, Z]] = [-T, Z] + [T, Z] = 0$ to get that $[T, Z]$ commutes with Y and consequently lies in $\mathbb{C}Y \oplus \mathbb{C}X'$.

To get the second one, observe that X' et Y commute, and thus T (which is an eigenvector of $ad(Y)$, is also an eigenvector of $ad(X')$). This gives $[T, X'] = cT$, for some $c \in \mathbb{C}$.

Consider now the derivative algebra $\mathcal{L} = [\mathcal{G}, \mathcal{G}]$ and recall it is nilpotent.

The relations $[Y, Z] = Z, [Y, T] = -T$ and $[T, Z] = aX' + bY$, show that $\mathcal{L}$ contains the Lie algebra generated by Z, T and $aX' + bY$. We have $[aX' + bY, Z] = bZ$ and this implies $b = 0$ (if not the Lie algebra generated by $aX' + bY$ and Z is isomorphic to the Lie algebra of AG, which is not nilpotent and so cannot be embedded into the nilpotent algebra $\mathcal{L}$). It follows that $b = 0$ and so $[T, Z] = aX'$.

We also have $[T, aX'] = acT$ and the same proof yields that $a = 0$ or $c = 0$.

Up to an automorphism of $\mathcal{G}$, if $a \neq 0$ we can assume $a = 1$, and if $c \neq 0$ we can assume $c = 1$.

Summarizing, we have the following three possibilities concerning the Lie algebra structure of $\mathcal{G}$:

(1) If $a = 0$ and $c = 0$, the Lie bracket relations are the following: $[Y, Z] = Z, [Y, T] = -T, [T, Z] = 0$ and $[T, X'] = 0$. Thus X' is central in $\mathcal{G}$. The Lie group generated by $\{Y, Z, T\}$ is isomorphic to SOL. It then follows that G is isomorphic to the direct product $\mathbb{C} \times SOL$, where X' generates the center. The isotropy $I = exp(\mathbb{C}Y)$ lies in SOL.

(2) If $a = 1$ and $c = 0$ the Lie bracket relations are $[Y, Z] = Z, [Y, T] = -T, [T, Z] = X'$ and $[T, X'] = 0$. The corresponding Lie group G is isomorphic to the semi-direct product $\mathbb{C} \ltimes Heis$, where the Lie algebra $heis$ of Heisenberg is generated by X', T and Z.

The first factor $\mathbb{C}$ is the isotropy $exp(\mathbb{C}Y)$, and its action on $heis$ is given by $(X', Z, T) \to (X', e^t Z, e^{-t} T)$, where X' is the generator of the center of $heis$. It follows that X' is central in $\mathcal{G}$. The factor $Heis$, intersects trivially the isotropy and hence acts freely and transitively on G/I.

(3) For $a = 0$ and $c = 1$, we have: $[Y, Z] = Z, [Y, T] = -T, [T, Z] = 0, [T, X'] = T$ and the Lie group G is a semi-direct product $G = \mathbb{C}^2 \ltimes \mathbb{C}^2$. The infinitesimal action of the first copy of $\mathbb{C}^2$ (generated by Y et X') on the second copy of $\mathbb{C}^2$ (generated by Z and T) is given by the matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$. □

5.2 The case: $H = Heis$

Under this assumption, we will describe first the geometry of the foliation $\mathcal{F}$ and then we will find all algebraic models $(G, G/I)$.

Proposition 5.4 *(i) The isotropy I is unipotent.*
(ii) The $\mathcal{F}$-leaves are flat and X is parallel along them.

Proof The action of the isotropy I on $\mathcal{H}/\mathcal{I}$ doesn't preserve any non trivially splitting. It follows that I is unipotent and I is different from the center of H (which acts trivialy). This implies that any copy of $\mathbb{C}^2$ transverse to the isotropy I in H acts freely on H/I (they exist since I is not central). This means that the H-leaves are flat (see the proof of Proposition 5.1) and that X is parallel along them. □

Proposition 5.5 *H is a normal subgroup of G.*

Corollary 5.6 *The H-foliation coincides with $\mathcal{F}$.*

Proof At the Lie algebra level we show that $\mathcal{H}$ is an ideal in $\mathcal{G}$. Take $A \in \mathcal{G}$ and let B be a local holomorphic vector field tangent to $X^{\perp}$ (recall $T\mathcal{F} = X^{\perp}$). We have to prove that $[A, B] = \nabla_A B - \nabla_B A$ lies in $X^{\perp}$. Note that $\nabla_A B \in X^{\perp}$: $g(B, X) = 0 \Longrightarrow g(\nabla_A B, X) = -g(\nabla_A X, B) = 0$ (because $\nabla_A X = \alpha X$ by Proposition 3.9). On the other hand the Killing field A preserves X and thus $\nabla_X A = \nabla_A X$. As $\nabla . A$ is skew-symmetric, it follows that $g(\nabla_B A, X) = -g(B, \nabla_X A) = -g(B, \nabla_A X) = 0$, because $\nabla_A X = \alpha X$. The second term $\nabla_B A$ lies in $X^{\perp}$, and thus $[A, B] \in X^{\perp}$. □

Algebraic structure of G. Therefore, G is an extension of the Heisenberg group H. In order to describe the algebraic structure of this extension denote by $\{X', Y, Z\}$ a basis of the Lie algebra $\mathcal{H}$ of H, such that Y is a generator of the isotropy $\mathcal{I}$, X' is a generator of the center and Z is such that: $[Y, Z] = X'$. We can assume that X' and Z generates the group of translations on the H-leaves.

Denote by T a fourth generator of $\mathcal{G}$. The action of the isotropy $\mathbb{C}Y$ on $\mathcal{G}/\mathbb{C}Y$ is such that $ad(Y)T = -Z$, which implies $[Y, T] = -Z + \beta Y$, for some $\beta \in \mathbb{C}$.

As the adjoint transformation of T acts on $\mathcal{H}$ preserving the center of $\mathcal{H}$ it follows that: $[T, X'] = cX'$, for some constant $c \in \mathbb{C}$.

We have the following

Proposition 5.7 *(i) There exists a H-invariant holomorphic function on G/I such that $X' = fX$ (f is only locally defined on M and constant on the leaves of $\mathcal{F}$).*

(ii) X is Killing (and f is constant) if and only if $c = 0$.

(iii) In the basis $\{X', Z, Y\}$ of $\mathcal{H}$ the action of T is given by $ad(T) = \begin{pmatrix} c & m & 0 \\ 0 & c+\beta & 1 \\ 0 & k & -\beta \end{pmatrix}$, *with $m, k \in \mathbb{C}$.*

(iv) If $c = 0$ and $k + \beta^2 = 0$, then g is flat.

Proof (i) As X' is in the center of $\mathcal{H}$ and $[T, X'] = cX'$, the direction $\mathbb{C}X'$ is $\mathcal{G}$-invariant. But in the case of unipotent isotropy the only direction in TM which is $\mathcal{G}$-invariant is $\mathbb{C}X$. Hence $X' = f \cdot X$, for some local holomorphic function f on M.

Moreover, the action of $\mathcal{H}$ is transitive on each leaf of $\mathcal{F}$ and preserves X' and X. It follows that f is constant on the leaves of $\mathcal{F}$.

(ii) As $\mathcal{G}$ preserves X, the vector field X is Killing if and only if it represents a non trivial element in the center of $\mathcal{G}$. It follows that X is Killing if and only if it is a multiple of X' and X' is in the center of $\mathcal{G}$. Equivalently, X' is a central element of $\mathcal{G}$ if and only if $c = 0$.

(iii) We apply the Jacobi relation to the vector fields Y, T and Z to verify that $ad(T)$ is a derivation if and only if $ad(T)Z$ is of the form $mX' + (c+\beta)Z + kY$, for $m, k \in \mathbb{C}$.

(iv) If $c = 0$ and $k + \beta^2 = 0$, then the vector fields X', $Z - \beta Y$ and T generate a Lie algebra isomorphic to the Heisenberg algebra $heis$, which acts freely on M. The center of this algebra is generated by X', which is colinear to X and hence isotropic. Then, g is locally modelled on a left invariant holomorphic Riemannian metric on the Heisenberg group which gives to the center of $heis$ the norm 0. These metrics are known to be flat [33]. □

6 Unipotent isotropy

In this section we deal to the case where the isotropy I is unipotent (and G is 4-dimensional and solvable). Then, Propositions 5.2 and 5.4 show that H is isomorphic to the Heisenberg group. The section is devoted to the proof of the following:

Proposition 6.1 *Up to a finite unramified cover, M is a quotient of SOL by some lattice* (*and* $c \neq 0$).

6.1 Completeness

Each leaf F of the H-foliation is a surface, on which the restriction of the vector field X is an (isotropic) Killing field for the $(H, H/I)$-structure (of the leaf). The vector field X generates the kernel $\mathcal{D}$ of the restriction of the metric g to the F. Furthermore, g determines a transverse holomorphic Riemannian structure on the foliation $\mathcal{D}$ (restricted to F), i.e. a $(\mathbb{C}, \mathbb{C})$-structure. For the basic facts concerning the study of foliations having a transverse $(G, G/I)$-structure one can see [30].

Lemma 6.2 (i) *The leaf F is $(H, H/I)$-complete, that is, the developing map $\tilde{F} \to H/I$, on the universal cover, is a diffeomorphism.*
(ii) *The $(G, G/I)$-structure of M is complete.*

Corollary 6.3 *The holonomy Γ acts properly on G/I.*

Proof (i) The $(H, H/I)$-structure on F is a combination of the Killing filed X and its transverse $(\mathbb{C}, \mathbb{C})$-structure. One directly sees, since X is complete (by compactness of M), that it suffices to prove completeness of the transverse $(\mathbb{C}, \mathbb{C})$-structure, i.e. completeness of the 1-dimensional holomorphic Riemannian metric induced on the quotient of F by X (or say, to prevent any pathology, the quotient of $\tilde{F}$ by $\tilde{X}$, where $\tilde{X}$ is the pull-back of X on $\tilde{F}$).

We will show that for any complex a, there is a *complete* vector field V_a on F with (constant) g-norm a. This would prove completeness, since such V_a come from translation vector fields on $\mathbb{C}$, and hence the V_a's commute, and they define a (complete) action of $\mathbb{R}^2$, and thus the leaf is homogeneous. This action commute with the developing map, which must be diffeomorphic.

In order to check existence of the complete vector fields V_a, we come back to our ambient compact manifold M and consider the space of vectors tangent to the H-foliation and having a norm a. For $a = 0$, this space is the vector bundle $\mathbb{C}X$ which is known to have the global section X. For $a \neq 0$, this space is a fiber bundle over M, with fiber two copies of $\mathbb{C}$ (endowed with a structure of an affine space). Up to a double cover, this bundle is trivial and provides a global vector field of norm a on M, and hence complete, by compactness of M.

(ii) Since $\mathcal{H}$ is an ideal of $\mathcal{G}$, the H-foliation has a transverse $(\mathbb{C}, \mathbb{C})$-structure, which is complete by compactness of M [30]. Combined with the completeness of the leaves, this proves completeness of the full $(G, G/I)$-structure. □

We can now prove:

Lemma 6.4 *(i)* Γ *is not abelian.*
(ii) *If* $c = 0$*, then* Γ *is not nilpotent.*

Proof Consider $\overline{\Gamma}$ the complex Zariski closure of Γ in G. As $\overline{\Gamma}$ has finitely many connected components, up to a finite cover of M, we may assume that the complex abelian Lie group $\overline{\Gamma}$ is connected.

Let us notice that $\overline{\Gamma}$ can not be contained in H. Indeed, if not, we get a well defined surjectif projection map $M \to \overline{\Gamma}\backslash G/I \to H\backslash G/I$. Since I is contained in H and H is normal, this last space is $\mathbb{C} = H\backslash G$. This contradicts the compactness of M.

(i) Assume by contradiction Γ is abelian. Then $\overline{\Gamma}$ is an abelian complex Lie group on which the action of Γ by adjoint representation is trivial.

Suppose first that the complex dimension of $\overline{\Gamma}$ is 1. As above, we get a projection from M to a double coset space $\overline{\Gamma}\backslash G/I$. Here $\overline{\Gamma}$ is a one-parameter complex group not included in H and this double coset space is diffeomorphic to H/I which is not compact. We get a contradiction.

Assume now that the complex dimension of $\overline{\Gamma}$ is >1. Any element of $\overline{\Gamma}$ is invariant by the holonomy Γ and it gives a globally defined holomorphic Killing field on M. With our assumption, M possesses at least two linearily independent holomorphic (Killing) vector fields and we can use Lemma 3.1. It follows that M is a quotient of a 3-dimensional complex Lie group C, by a lattice Γ. As Γ is supposed to be abelian, C is also abelian and isomorphic to $\mathbb{C}^3$. The holomorphic Riemannian metric g is left invariant on $\mathbb{C}^3$ and hence it is flat. This is absurde, since the Killing Lie algebra $\mathcal{G}$ of the flat model is of dimension 6 (and not of dimension 4).

(ii) Assume, by contradiction, Γ is nilpotent. Since Γ is not abelian, and supposed to be nilpotent, $\overline{\Gamma}$ is 3-dimensional (because the full group G is not nilpotent) and hence it is a complex Heisenberg Lie group, and its center is generated by X'. Take two linearily independent elements in the quotient of the Lie algebra of $\overline{\Gamma}$ outside its center. A straightforward computation (modulo $\mathbb{C}X'$) gives $[T + aY + bZ, T + a'Y + b'Z] = (a - a')(Z - \beta Y) + (b - b')(kY + \beta Z)$, for all $a, a', b, b' \in \mathbb{C}$ and shows that the Lie bracket of two such elements can be a multiple of X' only if the determinant $k + \beta^2$ of $\begin{pmatrix} 1 & -\beta \\ \beta & k \end{pmatrix}$ vanish. Then Proposition 5.7 implies that g is flat: absurde. □

Sub-holonomy group $\Delta = \Gamma \cap H$. Let $\overline{\Delta}$ be the real Zariski closure of Δ in H. Denote by δ the real Lie algebra of $\overline{\Delta}$, by $\delta_{\mathbb{C}}$ its complexified Lie algebra and by $\overline{\Delta}_{\mathbb{C}}$ the associated complex Lie group.

Recall that Γ acts on G by adjoint representation and has to preserve Δ and hence also $\overline{\Delta}$ and $\overline{\Delta}_{\mathbb{C}}$.

Proposition 6.5 *(i)* Δ *is not trivial and acts properly on* H/I.
(ii) $\overline{\Delta}$ *is of (real) dimension* ≤ 4.
(iii) $\overline{\Delta}_{\mathbb{C}}$ *is of (complex) dimension* ≤ 2.
(iv) Δ *is abelian.*

Proof (i) Assume, by contradiction, that Δ is trivial. Then the projection of Γ on $G/H \simeq \mathbb{C}$ is injective and Γ is abelian. This is in contradiction with Lemma 6.4.

Since the $(H, H/I)$-structure of a leaf F is complete, Δ is a discrete subgroup of H acting properly on H/I and the $\mathcal{F}$-leafs are diffeomorphic to $\Delta \backslash H/I$.

(ii) As H is nilpotent, Δ is also a nilpotent group and by Malcev Theorem Δ is a (co-compact) lattice in its real Zariski closure $\overline{\Delta}$ [35]. This means that $\overline{\Delta}$ acts properly on H/I as well. Thus $\overline{\Delta}$ has to intersect trivially the isotropy group $\mathbb{C}Y \simeq \mathbb{R}Y \oplus \mathbb{R}iY$. It follows that $\overline{\Delta}$ is a real Lie group of dimension ≤ 4.

(iii) A one-parameter complex group I' in H, not included in the subgroup of translations of F, has a fix point at $x_0' \in F$: it coincides with the isotropy at x_0'. As before, the isotropy at x_0' intersects trivially $\overline{\Delta}$. It follows that $\overline{\Delta}$ lies in the complex Lie group of translations, whose Lie algebra is $\mathbb{C}X' \oplus \mathbb{C}Z$. This implies $\delta_{\mathbb{C}} \subset \mathbb{C}X' \oplus \mathbb{C}Z$ and $\overline{\Delta}_{\mathbb{C}}$ is of dimension ≤ 2.

(iv) We have $\Delta \subset \overline{\Delta}_{\mathbb{C}}$, which is abelian by the previous point. □

Proposition 6.6 *The following facts are equivalent*:

- (i) *The $\mathcal{F}$-leaves are compact*;
- (ii) *$\overline{\Delta}$ is of (real) dimension* 4;
- (iii) *The projection of Γ on G/H has a discrete image.*

In this case M is biholomorphic to a holomorphic bundle over an elliptic curve with fiber type F isomorphic to a 2*-dimensional complex torus.*

Proof The $\mathcal{F}$-leaves are diffeomorphic to $\Delta \backslash H/I$. Since $\overline{\Delta}$ intersects trivially the isotropy, the action of $\overline{\Delta}$ on H/I is free and give a trivial foliation of $\Delta \backslash H/I$ with compact leaves (diffeomorphic to $\Delta \backslash \overline{\Delta}$). It follows that $\Delta \backslash H/I$ is compact if and only if the action of $\overline{\Delta}$ on H/I is transitive which means that the dimension of $\overline{\Delta}$ is 4.

The image of Γ by the projection $G \to G/H$ is the holonomy of the transverse $(\mathbb{C}, \mathbb{C})$-structure of the H-foliation $\mathcal{F}$. The image of Γ in $G/H \simeq \mathbb{C}$ is discrete if and only if the leaves of $\mathcal{F}$ are compact [30].

In this case, the general study of the developing map of the $(\mathbb{C}, \mathbb{C})$-transverse structure of $\mathcal{F}$ shows that M is a bundle over an elliptic curve with fiber F [30].

Since the leaves $F \simeq \Delta \backslash \overline{\Delta}$ are complex surfaces, $\overline{\Delta}$ is also a complex group: $\overline{\Delta} = \overline{\Delta}_{\mathbb{C}}$. It follows that $\overline{\Delta}_{\mathbb{C}} \simeq \mathbb{C}^2$ and F is diffeomorphic to $\Delta \backslash \mathbb{C}^2$, which is a complex torus. □

Proposition 6.7 *If the complex dimension of $\overline{\Delta}_{\mathbb{C}}$ is two, then $k = 0$. It follows that at least one of the parameters c and β are $\neq 0$ (see Proposition 5.7).*

Proof Here we have $\delta_{\mathbb{C}} = \mathbb{C}X' \oplus \mathbb{C}Z$.

Take $\gamma \in \Gamma$ not included in H and decompose it as $\gamma = exp(\alpha T)h$, with $h \in H$ and $\alpha \in \mathbb{C}^*$.

The holonomy group Γ lies in the normalizer $N_G(\overline{\Delta}_{\mathbb{C}})$ of $\overline{\Delta}_{\mathbb{C}}$ in G. The group H normalize $\overline{\Delta}_{\mathbb{C}}$ in G. We have then $exp(\alpha T) \in N_G(\overline{\Delta}_{\mathbb{C}})$. It follows that the action of $ad(T)$ on $\mathcal{G}$ preserves $\mathbb{C}X' \oplus \mathbb{C}Z$. Since (by Proposition 5.7) we have $[T, Z] = mX' + (c+\beta)Z + kY$, this implies $k = 0$. Moreover, if $c = \beta = 0$, then Proposition 5.7 implies g is flat: absurde. □

Proposition 6.8 *$\overline{\Delta}$ is of (real) dimension* 4.

Proof Assume, by contradiction, $\overline{\Delta}$ is of dimension <4. Up to a finite cover, $\overline{\Delta}$ is supposed to be connected.

The case: $\overline{\Delta}$ is 1-dimensional Then Δ is a discrete subgroup (isomorphic to $\mathbb{Z}$) of a real one parameter subgroup $\overline{\Delta}$ of H.

As $\mathbb{Z}$ does not admit non trivial automorphisms other than $z \to -z$, up to index 2, the action of Γ on Δ is trivial. This implies that the action of Γ on $\overline{\Delta}$ is trivial as well, and any infinitesimal generator Z' of $\overline{\Delta}$ is an element of the real Lie algebra $\mathcal{G}$ fixed by the holonomy. This element (seen as an element of the complex Lie algebra $\mathcal{G}$) gives a global holomorphic Killing field on M.

If the Killing field is a constant multiple of X, then $c = 0$ and X is given by a central element of $\mathcal{G}$. It follows then that Δ lies in the center of G and hence in the center of Γ. As $[\Gamma, \Gamma] \subset \Delta$, the holonomy Γ is a (two step) nilpotent group and Lemma 6.4 gives a contradiction.

Assume now the previous Killing field is not colinear with X. Note that Γ lies in the centralizer C of Z'. Since Z' is not a multiple of X', the centralizer C of Z' is at most 3-dimensional. It follows that, up to a finite cover, M admits a (C, C)-structure and M is a quotient of C by a lattice.

The Lie algebra of C is generated by Z', X' and some element $T' \in \mathcal{G}$ not contained in $\mathcal{H}$. We can assume that $T' = T(\text{modulo } \mathcal{H})$. In the Lie algebra of C, the element Z' is central, and $[T', X'] = cX'$. If $c \neq 0$, then $C \simeq \mathbb{C} \times AG$, which is impossible since this group is not unimodular and has no lattices.

It follows that $c = 0$ and $C \simeq \mathbb{C}^3$, which implies g is flat: absurde.

The case: $\overline{\Delta}$ is 2-dimensional The complex dimension of $\overline{\Delta}_{\mathbb{C}}$ is 1 or 2.

We assume first that $\overline{\Delta}_{\mathbb{C}}$ is 1-dimensional. In this case $\delta = \mathbb{R}X'' \oplus \mathbb{R}iX''$, for some $X'' \in \mathcal{G}$. The adjoint action of Γ on $\delta_{\mathbb{C}} = \mathbb{C}X''$ is $\mathbb{C}$-linear and preserves the lattice $exp^{-1}(\Delta)$. It follows that each element of Γ acts on $\delta_{\mathbb{C}}$ by homotheties given by roots of unity of order at most 6. Up to a finite covering of M, the holonomy Γ preserves X'' which gives a globally defined holomorphic Killing field on M. We conclude then as in the 1-dimensional case.

Assume now $\overline{\Delta}_{\mathbb{C}}$ is 2-dimensional: $\delta_{\mathbb{C}} = \delta \otimes \mathbb{C} = \mathbb{C}X' \oplus \mathbb{C}Z$.

We show that an element $\gamma \in \Gamma$, not contained H, acts trivially on $\mathbb{C}X'$ and on $\delta_{\mathbb{C}}/\mathbb{C}X'$, as soon as its projection on G/H is small enough. Such elements γ exist, since, by Proposition 6.6, the image of Γ in G/H is not discrete.

Consider $\gamma_n = r_n h_n$ a sequence of elements of Γ, with $h_n \in H$ and $r_n \notin H$ going to 0 in $G/H \simeq \mathbb{C}$, when n goes to infinity. We can assume $r_n = exp(\alpha_n T)$, with $\alpha_n \in \mathbb{C}^*$ going to 0 when n goes to infinity.

If $h_n = exp(a_n X')exp(b_n Y)exp(c_n Z)$, with $a_n, b_n, c_n \in \mathbb{C}$ then the adjoint action of h_n on $\overline{\Delta}_{\mathbb{C}}$ is exactly the action of $Ad(exp(b_n Y))$.

The action of $Ad(exp(b_n Y))$ on $\delta_{\mathbb{C}} = \mathbb{C}X' \oplus \mathbb{C}Z$ is given by the matrix $\begin{pmatrix} 1 & b_n \\ 0 & 1 \end{pmatrix}$.

By Proposition 5.7, $Ad(r_n) = Ad(exp(\alpha_n T))$ has the following matrix when acting on $\delta_{\mathbb{C}} = \mathbb{C}X' \oplus \mathbb{C}Z$: $\begin{pmatrix} e^{\alpha_n c} & * \\ 0 & e^{\alpha_n (c+\beta)} \end{pmatrix}$. The matrix of $Ad(\gamma_n) = Ad(r_n)Ad(h_n)$ has the same form.

Recall now that this action of $Ad(\gamma_n)$ preserves δ and the lattice $exp^{-1}(\Delta)$: it is conjugated to an element of $SL(2, \mathbb{Z})$. It follows that, for all $n \in \mathbb{N}$, the previous matrix of $Ad(\gamma_n)$ has a determinant which equals 1 and a trace which is an integer.

This implies that, for n large enough, the trace equals 2 and $e^{\alpha_n c} = e^{\alpha_n(c+\beta)} = 1$. It follows $c = 0$ and $\beta = 0$, which contradicts Proposition 6.7.

The case: $\overline{\Delta}$ is 3-dimensional As in the previous case, we have $\delta_{\mathbb{C}} = \mathbb{C}X' \oplus \mathbb{C}Z$. We can change the infinitesimal generator X' of the center of H and also Z into $Z + aX'$, with $a \in \mathbb{C}$, such that either $\delta = \mathbb{C}X' \oplus \mathbb{R}Z$, or $\delta = \mathbb{R}X' \oplus \mathbb{C}Z$. The previous transformation keeps unchange the Lie bracket relations.

Take as before a sequence $\gamma_n = exp(\alpha_n T)h_n$ of elements of Γ, such that $h_n \in H$ and $\alpha_n \in \mathbb{C}^*$ converges to 0. As before, the matrix of the $Ad(\gamma_n)$-action on $\delta_{\mathbb{C}} = \mathbb{C}X' \oplus \mathbb{C}Z$ is of the form $\begin{pmatrix} e^{\alpha_n c} & * \\ 0 & e^{\alpha_n(c+\beta)} \end{pmatrix}$.

Consider the restriction of $Ad(\gamma_n)$ to δ. For each $n \in \mathbb{N}$, the $Ad(\gamma_n)$-action on δ preserves some lattice, so it is conjugated to some element in $SL(3, \mathbb{Z})$. When n goes to infinity, the three eigenvalues of $Ad(\gamma_n)$ go to 1. By discreteness of $SL(3, \mathbb{Z})$, it follows that, for n large enough, all eigenvalues of $Ad(\gamma_n)$ equal 1. So, for n large enough, $e^{\alpha_n c} = e^{\alpha_n(c+\beta)} = 1$. It follows $c = 0$ and $\beta = 0$, which contradicts Proposition 6.7. □

We are now able to prove Proposition 6.1.

Proof By Proposition 6.6, M is a fiber bundle over an elliptic curve with fiber F biholomorphic to a 2-dimensional complex torus. We have seen that Δ is an abelian group isomorphic to $\mathbb{Z}^4$, $\overline{\Delta} \simeq \mathbb{R}^4$ and $\overline{\Delta}_{\mathbb{C}} = \mathbb{C}^2$. As before, we have $\delta_{\mathbb{C}} = \mathbb{C}X' \oplus \mathbb{C}Z$.

By Proposition 6.6, the projection of Γ on G/H is a discrete subgroup. This subgroup is isomorphic to the fundamental group of the basis of our fibration, so it is $\simeq \mathbb{Z}^2$. Take γ_1 and γ_2 two elements in Γ such that their projections in G/H span the previous $\mathbb{Z}^2$. Then any element of Γ decomposes as $\gamma_1^p \gamma_2^q d$, with $p, q \in \mathbb{Z}$ and $d \in \Delta$. Moreover, we can decompose γ_i as $exp(\alpha_i T)h_i$, where $i \in \{1, 2\}$, $h_i \in H$ and $\alpha_i \in \mathbb{C}$.

Assume by contradiction that $c = 0$. Then Proposition 5.7 implies that the action of $Ad(T)$ on the quotient $\mathcal{H}/\mathcal{I}$ is of (complex) determinant 1. Hence the determinant of the action of $Ad(\gamma_i)$ on $\delta_{\mathbb{C}}$ equals 1.

On the other hand the eigenvalues of $Ad(\gamma_i)$ are 1 and $e^{\alpha_i \beta}$ (see the proof of the case 2 in Proposition 6.8). It follows that $e^{\alpha_i \beta} = 1$, for $i \in \{1, 2\}$. This implies $\alpha_i \beta = 2i\pi k_i$, where $k_i \in \mathbb{Z}$. Since α_i are $\mathbb{Z}$-independent, we have $\beta = 0$. As before, this is in contradiction with Proposition 6.7.

It follows that $c \neq 0$.

We prove that there exists a basis of $\delta_{\mathbb{C}}$ in respect of which the actions of $Ad(\gamma_1)$ and $Ad(\gamma_2)$ are (both) diagonal. Recall that $\mathbb{C}X'$ is stable by the adjoint representation of G and, in particular, by $Ad(\gamma_1)$ and by $Ad(\gamma_2)$. Denote λ_i the corresponding eigenvalue of the restriction of $Ad(\gamma_i)$ to $\delta_{\mathbb{C}}$, $i \in \{1, 2\}$. We prove by contradiction that either the modulus of λ_1 or the modulus of λ_2 is $\neq 1$. Indeed, if not the modulus of the "quotient" f of X' over X (see Proposition 5.7) is preserved by the projection of Γ on G/H (which coincides with the holonomy of the transversal structure of the H-foliation). This means $|f|$ is globally defined on M. As M is compact and f is holomorphic, the maximum principle implies f is constant and, by Proposition 5.7, we have $c = 0$, which contradicts our assumption.

Assume now that the modulus of λ_1 is $\neq 1$. As $Ad(\gamma_1)$ acts on $\delta_{\mathbb{C}}$ preserving a lattice, this action is unimodular. It follows that the action of $Ad(\gamma_1)$ on $\delta_{\mathbb{C}}$ has distinct eigenvalues, and so it is diagonalizable over $\mathbb{C}$. Since γ_1 and γ_2 commutes (modulo Δ) and the action of Δ on $\delta_{\mathbb{C}}$ is trivial, then $Ad(\gamma_1)$ and $Ad(\gamma_2)$ commutes in restriction to $\delta_{\mathbb{C}}$. It follows that the two eigenvectors of $Ad(\gamma_1)$ are invariant by $Ad(\gamma_2)$ as well. Consequentely the two eigenvectors of $Ad(\gamma_1)$ are Γ-invariant. The holonomy group Γ lies in a subgroup of G for which the adjoint action on $\delta_{\mathbb{C}}$ preserves a non trivial splitting.

Take $T' \in \mathcal{G}$ such that $\gamma_1 = exp(T')$. We have proved that Γ lies in the 3-dimensional (solvable) complex Lie group C generated by $\mathbb{C}T'$ and $\delta_{\mathbb{C}}$. Thus, the manifold M possesses a (C, C)-structure and M is a quotient of C by a lattice (so C is unimodular). Since $c \neq 0$, the only compatible Lie group structure is SOL and so, up to a finite cover, M is a quotient of SOL by some lattice. □

7 Semi-simple isotropy

7.1 Solvable Killing algebra

We study separately the two possible models we got from Proposition 5.2. We prove the following:

Proposition 7.1 *Up to a finite unramified cover, M is a quotient of the Heisenberg group by a lattice (G is isomorphic to $\mathbb{C} \ltimes Heis$).*

Together with Proposition 6.1 this will prove part (ii) ($\mathcal{G}$ solvable) of the main Theorem 1.2.

The Case $G = \mathbb{C} \times SOL$.

Recall the Lie algebra of SOL is generated by $\{Z, T, Y\}$, with the Lie bracket relations $[Y, Z] = Z$, $[Y, T] = -T$ and $[T, Z] = 0$. The center of $\mathcal{G}$ is generated by X' and the 3-dimensional abelian Lie algebra generated by $\{X', Z, T\}$ acts freely on G/I. The holomorphic Riemannian metric g is locally identified with a translation-invariant holomorphic Riemannian metric on $\mathbb{C}^3$. Consequently g is flat, which is impossible.

The case $G = \mathbb{C} \ltimes Heis$

Recall that the Lie algebra of $Heis$ is generated by the central element X' and by Z, T such that $[Z, T] = X'$. We have seen that X' is fixed by the isotropy I and Z and T are the two isotropic directions expanded and contracted by I.

Here X' generates the global Killing field X of constant norm equal to 1 fixed by the isotropy. Denote ϕ^t, where $t \in \mathbb{C}$, the holomorphic flow of X. The flow ϕ^t preserves the orthogonal distribution $X^{\perp}$. This distribution has dimension 2 and it is non-degenerate in respect to g. Thus $X^{\perp}$ has exactly two isotropic line fields which are locally generated by Z and T. They are naturally preserved by ϕ^t. Since $[Z, T] \neq 0$, the distribution $X^{\perp}$ is not integrable.

We will say that X is *equicontinuous* if ϕ^t is. This means by definition that the closure K of ϕ^t in the group of homeomorphisms of M is a compact group. In this case K will be an abelian compact complex Lie group (a complex torus) acting on M and preserving g.

Assume first that X is equicontinuous. If K has complex dimension >1, the fundamental fields of the action of K on M give at least two linearily independent global holomorphic vector fields on M and Lemma 3.1 applies. So the centralizer C of K in G acts transitively on M, such that M is quotient of C by a lattice. The subgroup C of G is unimodular and has a center which is at least 1-dimensional. It follows that C is isomorphic to $Heis$.

Now consider the case where K is a 1-dimensional complex torus. The quotient of M by the action of K is a compact complex surface S which inherits a flat holomorphic Riemannian metric. Indeed, $G/exp(\mathbb{C}X) \simeq SOL$ and S is easily seen to be locally modelled on $(SOL, SOL/I')$, where $SOL \simeq \mathbb{C} \ltimes \mathbb{C}^2$ with the action of $\mathbb{C}$ on $\mathbb{C}^2$ given by the complex one-parameter group $I' = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$.

Up to a finite unramified cover, this surface is a 2-dimensional complex torus T^2 with a flat holomorphic Riemannian metric (see Theorem 4.3 in [8]). Consequently, up to a finite unramified cover, M is a principal bundle of elliptic curves over a complex torus and the projection of the holonomy Γ on $G/exp(\mathbb{C}X) \simeq SOL$ lies in the subgroup of translations $\mathbb{C}^2$. It follows that the holonomy Γ lies in a complex Lie group C of dimension 3 which is a central extension of $\mathbb{C}^2$ by $\mathbb{C}$ (isomorphic to $Heis$) and which acts freely and transitively on G/I. Up to a finite unramified cover, M is biholomorphic to a quotient of $Heis$ by a lattice.

It remains to settle the case where X is non-equicontinuous, for which we prove:

Proposition 7.2 *If the flow ϕ^t is non-equicontinuous, then it is holomorphic Anosov in Ghys sense* (*see* [12]).

Proof The closure of the one-parameter complex group ϕ^t in the group of homeomorphisms of M is a complex abelian Lie group which is supposed to be non compact. It follows that the closure of ϕ^t is isomorphic to $\mathbb{C}$ or to $\mathbb{C}/i\mathbb{Z}$ (normalizing the flow by multiplication with a complex number, we can assume that the stabilizer is $i\mathbb{Z}$). In the two cases all real one-parameter subgroups in ϕ^t are non-equicontinuous (except $\{\phi^t, t \in i\mathbb{R}\}$, in the last case). We will show that they act on M as the restrictions of a holomorphic Anosov flow in Ghys sense to a one parameter subgroup and so ϕ^t is holomorphic Anosov in Ghys sense.

By passing, if necessary, to a finite cover, we may assume that the two isotropic directions of $X^{\perp}$ are directed by two smooth vector fields T_1 and T_2. The ϕ^t-invariance of these isotropic directions shows that $D_x\phi^t(T_1(x)) = a(x,t)T_1(\phi^t(x))$ and $D_x\phi^t(T_2(x)) = b(x,t)T_2(\phi^t(x))$, for any $x \in M$ and $t \in \mathbb{C}$; a and b being some smooth complex valued functions on $M \times \mathbb{C}$. By the volume preserving property $a(x,t)b(x,t) = 1$.

Consider $\alpha \in \mathbb{C} \backslash i\mathbb{R}$ and $\{\phi^t, t \in \alpha\mathbb{R}\}$ a non-equicontinuous real one-parameter subgroup of $\{\phi^t, t \in \mathbb{C}\}$.

We now prove that for any $x \in M$, the orbit $\{D_x\phi^t(T_1(x)), t \in \alpha\mathbb{R}\}$ is not bounded in TM. Assume, by contradiction, that the modulus of the function a is upper bounded. If the modulus of $a(x,t)$ stays $\geq a' > 0$ for a sequence $t_n \in \alpha\mathbb{R}$ tending to $+\infty$ or $-\infty$, then $D_x\phi^{t_n}$ is equicontinuous and so, by Proposition 3.2 in [39], the flow itself is equicontinuous, which contradicts our hypothesis. It then follows that $a(x,t) \to 0$, when $t \to +\infty$ or $t \to -\infty$. Thus (by continuity of $|a|$) there are two sequences t_n

and $t_{n'}$ tending to $+\infty$, such that $|a(x, -t_n)| = |a(x, t'_n)|$. By the cocycle property of a, applied to $x_n = \phi^{-t_n}(x)$, we get: $|a(x_n, t'_n + t_n)| = |a(x, t'_n)a(x_n, t_n)|$. But $a(x_n, t_n)a(x, -t_n) = 1$, and hence $|a(x_n, t_n + t'_n)| = 1$. Hence $|b(x_n, t_n + t'_n)| = 1$, and consequently $D_{x_n}\phi^{t_n+t'_n}$ is equicontinuous. Since $t_n + t'_n$ tends to $+\infty$, Proposition 3.2 of [39] implies then that ϕ^t is equicontinuous which contradicts our assumption.

In the same way, the modulus of b is unbounded and hence the orbit of any non zero vector in $X^{\perp}$ under the action of $D\phi^t$ is not bounded. This means, by definition, that ϕ^t is quasi-Anosov and by an easy case of the main Theorem in [27] (see also [3] page 1345-1347) this implies that $\{\phi^t, t \in \alpha\mathbb{R}\}$ is the restriction of a holomorphic Anosov flow in Ghys sense [12].

Since $\{\phi^t, t \in i\mathbb{R}\}$ is equicontinuous, it preserves some Riemannian metric on M. In respect with this Riemannian metric we have $||\phi^t(v)|| = ||\phi^{Re(t)}(v)||$, for all tangent vectors v and $t \in \mathbb{C}$. Hence $\{\phi^t, t \in \mathbb{C}\}$ is holomorphic Anosov in Ghys sense. □

A simple case of the classification of holomorphic Anosov flows on compact complex 3-manifolds [12] shows that $\{\phi^t, t \in \mathbb{C}\}$ preserves some holomorphic Riemannian metric q of constant sectional curvature. As $X^{\perp}$ is not integrable, q is necessarily of non-zero constant sectional curvature [12]. By Theorem 1.1, the intersection $\mathcal{G}'$ of the Killing Lie algebra of g and the Killing Lie algebra of q acts transitively on M. This implies that the Heisenberg algebra is contained in the Killing Lie algebra $sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C})$ of q. This is absurde, and therefore, X is equicontinuous.

7.2 Semi-simple Killing algebra

Here $G = \mathbb{C} \times SL(2, \mathbb{C})$ and $I = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \subset SL(2, \mathbb{C})$.

We show the following

Proposition 7.3 *There are no compact manifolds locally modelled on* $(G, G/I)$.

This will complete the proof of the main Theorem 1.2.

Proof The factor $\mathbb{C}$ of G is generated by the flow of the Killing vector field X.

Assume first that X is equicontinuous and consider the complex Lie group K which is the closure of the flow of X in the group of homeomorphism of M. We have seen that if the complex dimension of K is >1 then, Lemma 3.1 implies that there exists a 3-dimensional complex subgroup C in G which acts freely and transitively on M and M identifies with a quotient of C by some lattice. This is impossible because the only 3-dimensional subgroups of G which act freely on M are isomorphic to $\mathbb{C} \times AG$ and they do not have lattices (they are not unimodular).

If K has dimension 1 the quotient of M by K is a complex compact surface locally modeled on $(SL(2, \mathbb{C}), SL(2, \mathbb{C})/I)$. This compact surface possesses a holomorphic Riemannian metric of non-zero constant sectional curvature. But, by Theorem 4.3 in [8], all holomorphic Riemannian metrics on compact complex surfaces are flat, which leads to a contradiction.

Consider now the case where X is non-equicontinuous. The proof of Proposition 7.2 implies that X is an Anosov flow with stable and instable directions given by the isotropic directions of $X^{\perp}$. Here the holomorphic plane field $X^{\perp}$ is integrable because it is tangent to the orbits of $sl(2,\mathbb{C})$-action. In this situation Ghys' classification [12] shows that, up to a finite cover, M is biholomorphic to a holomorphic suspension (given by the flow of X) of a complex hyperbolic linear automorphism of a complex torus T^2. In particular, the orbits of $sl(2,\mathbb{C})$ are 2-dimensional complex tori locally modelled on $(SL(2,\mathbb{C}), SL(2,\mathbb{C})/I)$. We get the same contradiction as before. □

References

1. Benoist, Y.: Actions propres sur les espaces homogènes réductifs. Ann. Math. **144**, 315–347 (1996)
2. Benoist, Y., Labourie, F.: Sur les espaces homogènes modèles de variétés compactes. Publ. Math. IHES **76**, 99–109 (1992)
3. Boubel, C., Mounoud, P., Tarquini, C.: Lorentzian foliations on 3-manifolds. Ergodic Theory Dyn. Syst. **26**, 1339–1362 (2006)
4. Carrière, Y.: Flots riemanniens, dans Structures transverses des feuilletages, Toulouse. Astérisque **116**, 31–52 (1984)
5. Carrière, Y.: Autour de la conjecture de L. Markus sur les variétés affines. Invent. Math. **95**, 615–628 (1989)
6. D'Ambra, G., Gromov, M.: Lectures on transformations groups: geometry and dynamics. In: Surveys in Differential Geometry (Cambridge), pp. 19–111. Leihigh University, Bethlehem (1990)
7. Dumitrescu, S.: Structures géométriques holomorphes sur les variétés complexes compactes. Ann. Sci. Ecole Norm. Sup. **34**(4), 557–571 (2001)
8. Dumitrescu, S.: Métriques riemanniennes holomorphes en petite dimension. Ann. Inst. Fourier, Grenoble **51**(6), 1663–1690 (2001)
9. Dumitrescu, S.: Homogénéité locale pour les métriques riemanniennes holomorphes en dimension 3. Ann. Instit. Fourier Grenoble **57**(3), 739–773 (2007)
10. Dumitrescu, S., Zeghib, A.: Géométries lorentziennes de dimension 3: classification et complétude, Arxiv math.DG/0703846
11. Fried, D., Goldman, W.: Three-dimensional affine christallographic groups. Adv. Math. **47**(1), 1–49 (1983)
12. Ghys, E.: Holomorphic Anosov systems. Invent. Math. **119**(3), 585–614 (1995)
13. Ghys, E.: Déformations des structures complexes sur les espaces homogènes de $SL(2,\mathbb{C})$. J. Reine Angew. Math. **468**, 113–138 (1995)
14. Ghys, E.: Flots d'Anosov dont les feuilletages stables sont différentiables. Ann. Sci. Ec. Norm. Sup. **20**, 251–270 (1987)
15. Goldman, W.: Nonstandard Lorentz space forms. J. Diff. Geom. **21**(2), 301–308 (1985)
16. Gromov, M.: Rigid transformation groups, Géométrie Différentielle. Bernard et Choquet-Bruhat, D., (ed.), Travaux en cours, Hermann, Paris, **33**, pp. 65–141 (1988)
17. Hwang, J.-M., Mok, N.: Uniruled projective manifolds with irreducible reductive G-structure. J. Reine Angew. Math. **490**, 55–64 (1997)
18. Inoue, M., Kobayashi, S., Ochiai, T.: Holomorphic affine connections on compact complex surfaces. J. Fac. Sci. Univ. Tokyo **27**(2), 247–264 (1980)
19. Kirilov, A.: Eléments de la théorie des représentations. MIR (1974)
20. Klingler, B.: Complétude des variétés Lorentziennes à courbure sectionnelle constante. Math. Ann. **306**, 353–370 (1996)
21. Kobayashi, S., Ochiai, T.: Holomorphic structures modeled after hyperquadrics. Tôhoku Math. J. **34**, 587–629 (1982)
22. Kobayashi, T., Yoshino, T.: Compact Clifford-Klein form of symmetric spaces -revisited. Pure Appl. Math. Q. **1**(3), 591–663 (2005)
23. Kulkarni, R., Raymond, F.: 3-dimensional Lorentz space-forms and Seifert fiber spaces. J. Diff. Geom. **21**(2), 231–268 (1985)

24. Labourie, F.: Quelques résultats récents sur les espaces localement homogènes compacts. In: Symposia Mathematica (en l'honneur d'Eugenio Calabi), pp. 267–283 (1996)
25. Lebrun, C.: Spaces of complex null geodesics in complex-Riemannian geometry. Trans. Amer. Math. Soc. **278**, 209–231 (1983)
26. Lebrun, C.: $\mathcal{H}$-spaces with a cosmological constant. Proc. Roy. Soc. Lond. Ser. A **380**(1778), 171–185 (1982)
27. Mané, R.: Quasi-Anosov diffeomorphisms and hyperbolic manifolds. Trans. Amer. Math. Soc. **229**, 351–370 (1977)
28. Markus, L.: Cosmological models in differential geometry, mimeographed notes. University of Minnesota, Minneapolis (1962)
29. McKay, B.: Characteristic forms of complex Cartan geometries. Arxiv math. DG/0704.2555
30. Molino, P.: Riemannian Foliations. Progress in Mathematics, Birkhauser, Boston (1988)
31. Mostow, G.: The extensibility of local Lie groups of transformations and groups on surfaces. Ann. Math. **52**(2), 606–636 (1950)
32. Priska, J., Radloff, I.: Projective threefolds with holomorphic conformal structure. Internat. J. Math. **16**(6), 595–607 (2005)
33. Rahmani, S.: Métriques de Lorentz sur les groupes de Lie unimodulaires de dimension 3. J. Geom. Phys. **9**, 295–302 (1992)
34. Rahmani, N., Rahmani, S.: Lorentzian geometry of the Heisenberg group. Geom. Dedicata **118**, 133–140 (2006)
35. Raghunathan, M.: Discrete subgroups of Lie groups. Springer, New York (1972)
36. Salein, F.: Variétés anti-de Sitter de dimension 3 exotiques. Ann. Inst. Fourier Grenoble **50**(1), 257–284 (2000)
37. Thurston, W.: The geometry and the topology of 3-manifolds. Princeton University Press, Princeton (1983)
38. Wolf, J.: Spaces of Constant Curvature. In: McGraw-Hill Series in Higher Math. (1967)
39. Zeghib, A.: Killing fields in compact Lorentz 3-manifolds. J. Differ. Geom. **43**, 859–894 (1996)