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ISOMETRY GROUPS AND GEODESIC FOLIATIONS OF LORENTZ MANIFOLDS. PART I: FOUNDATIONS OF LORENTZ DYNAMICS

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Abstract

This is the first part of a series on non-compact groups acting isometrically on compact Lorentz manifolds. This subject was recently investigated by many authors. In the present part we investigate the dynamics of affine, and especially Lorentz transformations. In particular we show how this is related to geodesic foliations. The existence of geodesic foliations was (very succinctly) mentioned for the first time by D'Ambra and Gromov, who suggested that this may help in the classification of compact Lorentz manifolds with non-compact isometry groups. In the Part II of the series, a partial classification of compact Lorentz manifolds with non-compact isometry group will be achieved with the aid of geometrical tools along with the dynamical ones presented here.

1 Lorentz Dynamics

We are interested here in the following question: when is the isometry group of a compact Lorentz manifold non-compact? Much progress has been made towards answering it, see for instance the works by Zimmer [Zi], Gromov ([Gr], [DGr]), D'Ambra [D], Kowalsky [Ko], Adams-Stuck ([ASt1,2]), and the author ([Z1,2]). At this stage, from the different investigations due to these authors, we know the list of non-compact *connected* Lie groups acting isometrically on compact Lorentz manifolds. For some groups in the list, we also understand completely the geometric structure of the underlying Lorentz manifold. However, we do not know enough things about this structure, in the case of the remaining groups, such as, for example, the case of abelian groups. In the case of *non-connected* groups, nothing significant is known about their algebraic structure or about the geometric structure of the Lorentz manifold on which they act.

Our approach here is to study the dynamics of Lorentz transformations, i.e. diffeomorphisms preserving Lorentz metrics. By this approach, we intend to understand, at the same time, the structure of Lorentz manifolds with large isometry groups, and the dynamics of the individual isometries themselves.

It is natural to think about Lorentz metrics as the nearest to Riemannian metrics, among all the geometric structures. This is why, among geometric dynamics, one might claim that the Lorentz dynamics is the simplest one after the Riemannian dynamics. It is worth saying that this latter dynamics is very poor. More precisely, let f be an isometry of a compact Riemannian manifold M. Then the closure of an orbit of f is diffeomorphic to a torus, on which f induces a linear translation. One can then say that there is only one pattern of Riemannian dynamics: a linear translation on a torus. On the other hand, one can (roughly) characterize Riemannian isometries, by topological properties, such as equicontinuity.

We know many patterns of Lorentz dynamics. For instance, on the unit tangent bundle of a hyperbolic compact surface, the circle, the horocycle and the geodesic flows coexist. All of them preserve the Killing Lorentz metric, although they present extremely different dynamics. Now the question is: find (fundamental) patterns with the help of which, one can build up any Lorentz isometry. Also, what are the topological properties characterizing diffeomorphisms that preserve Lorentz metrics?

Let us first observe that the Liapunov theory is not efficient for our purpose. Indeed, this theory is only sensitive to hyperbolicity, since it deals with the asymptotic exponential behavior of vectors under the derivative of a dynamical system. For instance, this theory does not distinguish between the circle and horocycle flows!

Here we will introduce another growth notion, which appears to be efficient in studying some dynamical systems with geometric properties (at least permitting us to detect a difference between the horocycle and circle flows !).

Essentially, our fundamental growth notion is the following: Consider a diffeomorphism f of a compact manifold M. We say that a vector $v \in TM$ is f-*approximately stable*, if v is a limit of a sequence (v_n) (in TM) such that the image sequence $(Df^n v_n)$ is bounded (i.e. lies in a compact subset of TM). We denote the set of such vectors by $AS(f) \subset TM$ (observe that $AS(x,f) = AS(f) \cap T_xM$ is not a priori a vector subspace of T_xM).

The weakness of this notion with regard to the Liapunov theory is clear, the set $AS(f)$ may behave very badly. Its efficiency appears in Lorentz dynamics. For a circle flow, the situation is trivial: $AS = TM$. For the horocycle flow, AS is the codimension 1 weakly stable or unstable bundle of the geodesic flow, that contains the given horocycle flow (so AS has dimension 2 and not 1). For the geodesic flow, AS is exactly its weak stable bundle.

The following theorem is one of our main results. It claims that, in Lorentz dynamics, the codimension 1 property of AS occurs for any isometry that is not contained in a compact group. It seems also, that this same property, gives a (rough) topological characterization of Lorentz isometries (however, we will not develop this point of view here).

Theorem 1.1. *Let M be a compact Lorentz manifold (i.e. M is endowed with a non-degenerate symmetric 2-tensor of signature $-+\cdots+$). Let f be an isometry of M generating a non-equicontinuous subgroup of isometries of M (that is $\{f^n/n \in \mathbf{Z}\}$ is not precompact in $\mathrm{Isom}(M)$). Then the following statements hold:*

Existence of approximately stable foliations. *The approximately stable set $AS(f)$ is a codimension one bundle tangent to a codimension 1 Lipschitz foliation $\mathcal{AS}(f)$, called the approximately stable foliation of f. The leaves of this foliation are geodesic and lightlike (that is the restriction of the Lorentz metric to them is degenerate). The isometry f preserves both its approximately stable foliation, and that of f^{-1}, which will be called the approximately unstable foliation of f.*

Dynamics of f. *In the subset of points where $AS(f)$ and $AS(f^{-1})$ are transverse, their (one dimensional) normal spaces $AS(f)^\perp$ and $AS(f^{-1})^\perp$ are respectively the negative and positive Liapunov spaces of f.*

The dynamics of Df on the projective tangent bundle $\mathbf{P}(TM)$ has the following "source-sink" description: Let U and V be respectively neighborhoods of the associated projective bundles $\mathbf{P}(AS(f))$ and $\mathbf{P}(AS(f^{-1}))$, then $Df^n(\mathbf{P}(TM) - U) \subset V$, for n sufficiently large.

Ergodic properties. *Let O be an open f-invariant subset and σ a continuous f-invariant function defined in O. Then σ is constant along the leaves of the restriction to O of the two 1-dimensional foliations tangent to $AS(f)^\perp$ and $AS(f^{-1})^\perp$.*

Finally, the topological entropy of f vanishes exactly when the approximately stable and unstable foliations of f are identical.

It is a nuisance that we were not able to exclude, *a priori*, the possibility that the approximately stable and unstable foliations of an isometry, may be different but coincide somewhere!

Generalized dynamical systems. The following terminology will be useful. Consider a sequence (f_n) of diffeomorphisms of a compact manifold M. To emphasize the fact that we are treating such a sequence in a dynamical viewpoint, we shall call it a *generalized dynamical system*. For instance we say that a vector $v \in TM$ is (f_n)-*approximately stable*, if v is a limit of a sequence (v_n) in TM such that the image sequence $(Df_n v_n)$ is bounded. We denote the set of such vectors by $AS((f_n)) \subset TM$.

So, classical dynamical systems correspond to the case: $f_n = f^n$, where f is a diffeomorphism. But, even the less classical case $f_n = f^{k_n}$, for a sequence (k_n) of integers (for example the return times to some subset) is very interesting. Our philosophy here is that, sometimes, one does not need the rich structure of classical dynamical systems, especially the various associated cocycles. Conversely, sometimes (in fact in many situations in dynamical systems), one cannot avoid use (implicitly) of generalized dynamical systems. For instance, in our situation, it might happen that the theorem above is dramatically empty, because every isometry f of M generates a precompact subgroup, although the groupIsom(M) itself is non-compact (for example,Isom(M) might be an infinite discrete torsion group). In fact the proof of the theorem above, will come via the following generalized version.

Theorem 1.2. *Let M be a compact Lorentz manifold and (f_n) a non-equicontinuous sequence of isometries of M (that is (f_n) is not contained in a compact subset of* $\mathrm{Isom}(M)$*).*

Then there is a subsequence (ϕ_n) such that the approximately stable set $AS((\phi_n))$ is a codimension one bundle tangent to a codimension 1 Lipschitz foliation $\mathcal{AS}((\phi_n))$, called the approximately stable foliation of (ϕ_n). The leaves of this foliation are geodesic and lightlike (that is the restriction of the Lorentz metric to the leaves is degenerate).

After passing to a subsequence, we can assume that the same is true for (ϕ_n^{-1}). In this case, if $v \in TM - AS((\phi_n))$, $D\phi_n v$ tends to ∞, and converges projectively (i.e. after normalization) to $AS(\phi_n^{-1})$. The convergence is uniform in compact subsets of $TM - AS((\phi_n))$. That is, if U and V are neighborhoods of the associated projective bundles $\mathbf{P}(AS((\phi_n)))$ and $\mathbf{P}(AS((\phi_n^{-1})))$, respectively, then $D\phi_n(\mathbf{P}(TM) - U) \subset V$, for n sufficiently large.

In addition, we have the following ergodic property. Let $\sigma : O \to \mathbf{R}$ be a continuous function defined in an open subset O. Suppose that σ and O are invariant by each f_n. Then σ is constant along the leaves of the restriction to O of the two 1-dimensional foliations tangent to $AS((\phi_n))^\perp$ and $AS((\phi_n^{-1}))^\perp$.

More details and a non-compact variant of this theorem are in §8.

Previous works (see also §4.2). The idea of the existence of geodesic foliations whenIsom(M) is non-compact appears for the first time in [DGr]. There, they are called *asymptotic foliations* and are introduced as follows. Keeping the notation above, consider the sequence of graphs $Gr(f_n) \subset M \times M$ of (f_n). They are geodesic and isotropic in the sense of a natural pseudo-Riemannian structure on $M \times M$. The non-equicontinuity of (f_n) implies that $Gr(f_n)$ does not converge to a graph, but rather, to a "foliated object", with isotropic geodesic leaves (in $M \times M$) having the same dimension as M.

The projections onto M of these leaves are lightlike geodesic hypersurfaces. These properties (i.e. codimension 1, geodesibility and degeneracy) seem to be characteristic of Lorentz dynamics. They were considered in a similar context in [Ca], but in the special flat case.

There is also in [DGr] a sketch of proof of the fact that, taking a suitable subsequence (ϕ_n), the projected hypersurfaces in M give rise to a foliation on M. However, I do not see how to prove this fact, following the suggestions of [DGr], without passing by the approximate stability interpretation, together with all its subtleties (the distinction between punctual and non-punctual approximate stability, modulus of stability, ..., see §§3 and 4).

2 Results on Isometry Groups

2.1 Compactification. The following compactification is discussed in [DGr]. The space of codimension 1 geodesic foliations of a compact affine manifold M (i.e. M is endowed with a torsion free connection) is naturally compact (this is because such a foliation is uniformly Lipschitz, see §8). The same is true for the space of codimension 1 lightlike geodesic foliations if M is Lorentzian. We denote this space by $\mathcal{FG}$. We construct a topology on $\mathrm{Isom}\, M \cup \mathcal{FG}$, essentially by the following rule: a sequence (ϕ_n) in $\mathrm{Isom}\, M$ converges to $F \in \mathcal{FG}$ if F is the approximately stable foliation of (ϕ_n^{-1}) (that is, if $AS((\phi_n^{-1})) = TF$). One then shows that $\mathrm{Isom}\, M \cup \mathcal{FG}$ is a (metrizable) compact space, and that the action of $\mathrm{Isom}\, M$ on it is continuous. Therefore, by taking its closure we get a natural compactification of $\mathrm{Isom}\, M$. The same construction yields a compactification and a *boundary* $\partial_\infty \Gamma$ for any closed subgroup of $\mathrm{Isom}\, M$.

As an example, in the case of the unit tangent bundle $PSL(2, \mathbf{R})/\Gamma$ of a hyperbolic surface $\mathbf{H}^2/\Gamma$, the isometry group is $PSL(2,\mathbf{R})$. Its boundary is the circle, endowed with the usual projective action, see §15 for further examples.

2.2 Elementary groups. The source-sink property described in the two preceding theorems tends to suggest that the action of $\operatorname{Isom} M$ on its boundary is of *convergence* type, as in the case of Fuschian groups [Tu]. This analogy between Lorentz isometry groups and Fuschian groups may arise in dynamical as well as in geometrical contexts. In §9 we will see how a Lorentz geometry gives rise to a fiberwise hyperbolic geometry, i.e. a family of hyperbolic spaces... This suggests that one could translate and update notions on Fuschian groups to Lorentz isometry groups. Let us formulate the simplest one.

DEFINITION 2.1. *A compact Lorentz manifold M is called* polarized *if the boundary of its isometry group is one point.*

As an example, an important geometrical class of polarized Lorentz manifolds (that we will not consider here) consists of manifolds for which the holonomy group is reducible, but non-decomposable (this cannot happen in Riemannian geometry). In this case the polarizing foliation is parallel, i.e. its tangent bundle is invariant by parallel transport. (In this case the term "polarization" may be justified by physical, symplectical or optical considerations.)

The Fuschian notion which corresponds to polarization is that of elementary parabolic groups. Now, elementary hyperbolic Fuschian groups are those with limit sets of cardinality 2. In the Lorentz case, it is not obvious (at least at this stage) how to check that parabolic and hyperbolic behaviors do not coexist. So, we were able to define elementary groups only via the following somewhat technical notion.

DEFINITION 2.2. *A codimension 1 lightlike geodesic* bifoliation *is a map $x \in M \to L(x)$ = a pair of hyperplanes of T_xM, such that there exist two codimension 1 lightlike geodesic foliations $\mathcal{F}^1$ and $\mathcal{F}^2$, with $L(x) = \{T_x\mathcal{F}^1_x, T_x\mathcal{F}^2_x\}$. We will denote such a bifoliation by $\mathcal{F}^1 \cup \mathcal{F}^2$.*

For instance, a pair of two (perhaps identical) foliations gives a bifoliation. Notice, however, that in our definition, the foliations $\mathcal{F}^1$ and $\mathcal{F}^2$ are not part of the data. Any foliation $\mathcal{F}$ such that $T_x\mathcal{F} \in L(x)$, for any $x \in M$, is called tangent to the given bifoliation. The nuisance here is that the coincidence locus C where the bifoliation is one-valued, i.e. where the two defining foliations are tangent, may be non-empty, or even worse,

$M - C$ may have infinitely many connected components (because one has not assumed things are analytic). In this case the space of tangent foliations (to the given bifoliation) is an infinite compact space! Here, by analogy, not with Fuschian groups, but with higher rank groups, one may call this space, the apartment generated by the bifoliation, and denote it by $\mathrm{Apa}(\mathcal{F}^1, \mathcal{F}^2)$ (in despite of the reference in notation to the foliations $\mathcal{F}^1$ and $\mathcal{F}^2$, the apartment is clearly related to the bifoliation, only).

DEFINITION 2.3. *We say that a closed non-compact subgroup of* Isom M *is* elementary *if it preserves a codimension 1 lightlike geodesic bifoliation. We say that M is* bipolarized *if its isometry group is elementary.*

The following is a characterization of elementary groups.

Theorem 2.4. *Let Γ be a closed non-compact subgroup of* Isom M. *Then Γ is elementary if and only if its boundary has cardinality 1 or 2.*

In the case $\partial_\infty \Gamma = 1$ point, there is no other 1-codimension geodesic lightlike foliation, preserved by Γ. All the elements of Γ have vanishing topological entropy.

In the case $\partial_\infty \Gamma = 2$ points, up to a finite index subgroup, Γ is a direct product of a compact group by **Z** *or* **R**. *Any element of the* **Z** *or the* **R** *factor has positive entropy.*

REMARK 2.5. In the case where the boundary of Γ is is one point, we obtain nothing but the Alexendroff compactification. In fact the group Γ may be very large in this case. For instance, there are many compact homogeneous polarized Lorentz manifolds. Their isometry groups contain solvable groups obtained as semi-direct products of the circle S^1 with Heisenberg groups (see [ASt1] and [Z1]).

2.3 Amenable groups. Dynamical structure of one parameter groups. We also have the following convergence aspect of the action of Isom M on its boundary.

Theorem 2.6. *A closed non-compact amenable subgroup of* Isom M *is elementary.*

Theorem 1.1 concerning groups isomorphic to **Z** will be a consequence of the two above theorems. Similarly, the following result describes the nice dynamical structure of non-equicontinuous isometric flows.

Theorem 2.7. *Let X be a Killing field on a compact Lorentz manifold M, generating a non-equicontinuous flow f^t.*

Then X is everywhere isotropic or spacelike, that is $\langle X(x), X(x)\rangle \geq 0$. Furthermore, X is contained in (and hence f_t preserves) two codimension

1 geodesic lightlike foliations, called the approximately stable and approximately unstable foliations of f_t.

These two foliations coincide on the (invariant) set of points where X is isotropic (but the coincidence may occur in a larger set). (In particular, if X is everywhere isotropic, then $X^\perp$ is integrable. This is false for equicontinuous isotropic Killing fields).

Remark 2.8. It may happen that the approximately stable and unstable foliations are identical, as in the case of the horocycle flow. They might also, a priori, coincide in some proper subset of M.

2.4 Ergodic properties of non-bipolarized manifolds. Let M be a compact non-bipolarized manifold, that is by definition, it has a non-compact and non-elementary isometry group. Let S be the set of isotropic vector fields X, which are tangent to foliations $\mathcal{F}$ belonging to the boundary ofIsom(M) (note that because X is isotropic, then X tangent to $\mathcal{F}$ is equivalent to X normal to $\mathcal{F}$). We denote $S_x = \{X(x)/X \in S\}$ the evaluation of S at x, and $TAS_x \subset T_xM$ the linear space that it generates. The "bundle" TAS, obtained in this way will be called the *total approximately stable* bundle. In a straightforward way, one sees that somewhere the dimension of TAS is ≥ 3, because $\mathrm{Isom}(M)$ is not elementary. In the open subset where its dimension is locally constant, TAS is a Lipschitz bundle. The same is true for its orthogonal $TAS^\perp$. Observe that this last bundle is integrable. Indeed the geodesic foliation obtained as the intersection of all the foliations belonging to $\partial_\infty \mathrm{Isom}(M)$ is a natural candidate to be tangent to $TAS^\perp$. The starting point of Part II of this work, will be the claim that TAS is also integrable, in fact with umbilical leaves, which yields a local warped product structure for M. Here we will not deal with these geometric details but instead with the following ergodic property behind their proofs.

Theorem 2.9. *Let E be a measurable linear subbundle of TM, containing TAS and invariant by $\mathrm{Isom}(M)$. Let $t : E \times E \to TM$ be a measurable invariant bilinear bundle map. Let $\mathcal{U}$ be the open set $\{x{\in}M/ \dim TAS_x \geq 3\}$. Then there is a vector field n defined almost everywhere in $\mathcal{U}$, such that $t(X,Y) = \langle X, Y\rangle n$, when $X \in TAS$ and $Y \in E$. In addition, we have the following ergodic property. Let $\sigma : O \to \mathbf{R}$ be a C^1 function defined in an open set O, and invariant by $\mathrm{Isom}(M)$. Then σ is constant along TAS, that is TAS is contained in $\mathrm{Ker}(d\sigma)$.*

2.5 Approximate isometries. Questions on isometry groups of (fixed) Lorentz metrics on a manifold M, may be translated in a more stable way to questions on the dynamics of the action of the group of diffeomorphisms $\mathrm{Diff}(M)$ on the space of Lorentz metrics $\mathrm{Lor}(M)$. So, the compactness of $\mathrm{Isom}(M,g)$ for any Lorentz metric g on M, means that the stabilizers of the $\mathrm{Diff}(M)$-action are compact. A well known result of D'Ambra [D] states this is true, for compact simply connected manifolds, but, only in the analytic category. In fact one could ask for the more stable:

QUESTION 2.10 ([DGr]). *Let M be a compact simply connected manifold. Is the action of $\mathrm{Diff}^2(M)$ on $\mathrm{Lor}^2(M)$ proper? (The exponent 2 stands for the C^2 differentiability and the C^2 topology.)*

Also here the language of generalized dynamical systems is pertinent.

DEFINITION 2.11. *Let g and h be two elements of $\mathrm{Lor}^2(M)$ and $(f_i) \subset \mathrm{Diff}^2(M)$ a generalized dynamical system. We say that (f_i) is approximately isometric for (M,g,h) if $f_i^*g \to h$ in $\mathrm{Lor}^2(M)$.*

Such dynamical systems occur exactly when the action of $\mathrm{Diff}^2(M)$ on $\mathrm{Lor}^2(M)$ is not proper. In this article, we will not treat approximately isometric dynamical systems (to avoid interference with the approximate stability notion, which is more central for us). However, most of the results, in particular Theorem 1.2, generalizes, with the same proofs, to this context. As a corollary we have:

Theorem 2.12. *Let M be a compact manifold such that $\mathrm{Diff}^2(M)$ acts non-properly on $\mathrm{Lor}^2(M)$. Then, M admits a codimension 1 lightlike geodesic foliation, in the sense of some metric $g \in \mathrm{Lor}^2(M)$.*

From a result of [Z3], a 3-manifold admitting such a foliation is covered by $\mathbf{R}^3$. So, we have the following partial positive answer to the previous question asked in [DGr].

Theorem 2.13. *Let M be a compact 3-manifold not covered by $\mathbf{R}^3$. Then $\mathrm{Diff}^2(M)$ acts properly on $\mathrm{Lor}^2(M)$.*

2.6 Organization of the article. In the sequel, in order to simplify notation, we shall restrict our presentation to compact manifolds. At certain points, we shall remark on the differences in definitions and statements, for the non-compact case. Our investigation of approximate stability starts with the linear case and is contained in §4. The general affine (i.e. connection preserving) appears in §§5, 6 and 7, and is followed by the Lorentz case in §8. Next, we introduce a bundle compactification (§9), a first step towards the (foliation) compactification of isometry groups of Lorentz manifolds (§10). Theorem 1.2 is proved in §8 and §9. In §§11 and 12 we study elementary groups. This part is somewhat technical due to the fact of the possible coexistence of parabolic and hyperbolic behavior for elementary groups. The last sections are devoted to proofs of theorems and exposition of illustrative examples.

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3 Approximate Stability

Here we give precise definitions of approximately stable objects associated to a *generalized dynamical system* $(M, (f_n))$ (that is (f_n) is just a sequence of diffeomorphisms on M).

DEFINITION 3.1. *Let (f_n) be a generalized dynamical system.*

1. *Stability. A sequence of vectors (v_n) in TM is* stable *if $(f_n(v_n))$ is bounded. Similarly, a sequence $P_n \subset T_{x_n}M$ of tangent linear subspaces (of some fixed dimension) is stable, if $D_{x_n}f_n|P_n$ is bounded.*
2. *Approximate stability. A vector $v \in T_xM$ is* approximately stable *if v is a limit of a stable sequence of vectors, that is there exists a convergent sequence $v_n \to v$, such that Df_nv_n is bounded. We denote by $AS((f_n))$ the set of approximately stable vectors of (f_n) and by $AS(x, (f_n)) \subset T_xM$ its intersection with T_xM.*
   We say that v is approximately strongly stable *if in addition $Df_nv_n \to 0$ (in TM).*
3. *Punctual approximate stability. A vector v is* punctually approximately stable, *if it is a limit of stable sequence in the same tangent space T_xM: $v_n \in T_xM$, $v_n \to v$ and $D_xf_nv_n$ bounded. As above we introduce the notation $PAS((f_n))$ and $PAS(x, (f_n))$. We define analogously a notion of* punctually approximately strongly stable.

Similarly, we define the same notions for tangent subspaces in TM.

REMARK 3.2 (Punctual and non-punctual). Observe that $PAS(x, (f_n))$ is a linear space but $AS(x, (f_n))$ is not. It is also important to observe that if $P \subset T_xM$ is such that each $v \in P$ is approximately stable (i.e. $P \subset AS(x, (f_n))$), then one cannot infer that P is an approximately stable subspace. The vectors of P may be approximated by stable sequences of vectors with different base-points, and there is no way to sum (or even approximately sum) them (since we have no control on the distance between base-points). In the punctual case, it is clear that $P \subset PAS(x, (f_n))$ is equivalent to P being an approximately stable subspace.

REMARK 3.3 (The non-compact case). If M is not compact, then in the definition of stable sequence of vectors, we require that the sequence $(f_n(x_n))$ lie in a compact subset of M.

3.1 Sizes and modulus of stability. Although, when considering approximate stability, we are far from uniform estimates, we will sometimes need to specify the sizes of objects. Since we are working on a compact manifold M, we choose a fixed continuous norm $|.|$ on TM.

Let $(P_n) \subset T_{x_n}M$ be a stable sequence with respect to a generalized dynamical system $(M, (f_n)))$. The *modulus of stability* of (P_n) is the inverse of the supremum of the norms of the restrictions $D_{x_n}f_n|P_n$, that is, mod $(P_n) = (\sup_n |D_{x_n}f_n|P_n|)^{-1}$. So, by definition, (P_n) stable means exactly that $\mathrm{mod}(P_n) > 0$.

Now, consider a family of stable sequences $(P_n(\xi))_{\xi\in\Lambda}$ (generally Λ is a subset of M). The modulus of stability of the family is $\mathrm{mod}(P_n(\xi))_{\xi\in\Lambda} = \inf_{\xi\in\Lambda}\mathrm{mod}(P_n(\xi))$, and we say that $(P_n(\xi))_{\xi\in\Lambda}$ has a *uniform modulus of stability* if its modulus of stability is > 0.

Similarly, a family $P(\xi)_{\xi\in\Lambda}$ of stable subspaces is said to have a uniform modulus of stability if each $P(\xi)$ is a limit of a sequence $(P_n(\xi))$ such that the family of sequences $(P_n(\xi))_{\xi\in\Lambda}$ has a uniform modulus of stability.

Let $(P_n(\xi))_{\xi\in\Lambda}$ be a family of sequences of subspaces, say, $P_n(\xi)$ is a subspace of $T_{x_n(\xi)}M$, then, we have the following obvious inclusion:

$$Df_n\big(P_n(\xi) \cap B_{x_n(\xi)}(\epsilon)\big) \subset B_{f_n(x_n(\xi))}(1)\,,$$

where $B_x(r)$ denotes the ball of radius r in T_xM, and ϵ is the modulus of stability of $(P_n(\xi))_{\xi\in\Lambda}$.

However, to avoid explicit use of a particular norm, which would make the statements rather technical, and which has no natural relation with data, we prefer to think of sizes in a "rough" sense, that is up to multiplicative constants, given by the geometric context.

For example, one may express the previous inclusion in a qualitative way, by saying that there is a family $(V_x)_{x\in M}$ where V_x is a neighborhood of 0 in T_xM, depending *continuously* on x, and with a *size proportional* to the modulus of stability of $(P_n(\xi))_{\xi\in\Lambda}$, such that $Df_n(P_n(\xi) \cap V_{x_n(\xi)})$ is contained in a neighborhood of 0 in $T_{f_n(x_n(\xi))}M$, having a *fixed size*.

For instance, in the affine context of §§5, 6 and 8, we insist that the image by the exponential map of $Df_n(P_n(\xi) \cap V_{x_n(\xi)})$, is contained in a convex neighborhood of $f_n(x_n(\xi))$ in M.

4 Examples. First Calculations

The fundamental example. We consider here (classical) linear dynamical systems on $\mathbf{R}^d$ of the form $f_n = A^n$, where $A \in GL(d, \mathbf{R})$. The fundamental example is the following. Let $d = 3$, endow $\mathbf{R}^3$ with its canonical basis $\{e_1, e_2, e_3\}$, and take $f_n = A^n = \begin{pmatrix} 1 & n & n^2/2 \\ 0 & 1 & -n \\ 0 & 0 & 1 \end{pmatrix} = \exp n \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Here approximate stability and pointed approximate stability coincide. At $x = 0$, we have $AS(0) = PAS(0) = \mathbf{R}e_1 \oplus \mathbf{R}e_2$. For instance to see that e_2 is approximately stable, observe that the vector sequence $v_n = (0, 1, -2/n)$ is stable. Indeed, $A^n(v_n) = (0, 3, -2/n)$ is bounded.

Observe that e_1 which is obviously stable, is in fact strongly approximately stable. To check this consider $v_n = (1, 1/n^2, -2/n^2)$, and note that $A^n(v_n) = (0, 1/n^2 + 2/n, -2/n^2)$.

Note, however, that in the case of $\mathbf{R}^2$ and $A_n = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) = \exp n \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, the approximately stable space is reduced to $\mathbf{R}e_1$.

In general, consider on $\mathbf{R}^d$, $A_n = \exp nB$, where B is a Jordan block, $B = \begin{pmatrix} 0 & 1 & 0 & 0 \dots \\ 0 & 0 & 1 & 0 \dots \\ \dots & 0 & 0 & 1 \dots \\ \dots & \dots & 0 & 0 \end{pmatrix}$. Then $AS(0) = PAS(0) = \mathbf{R}e_1 \oplus \mathbf{R}e_2$, if $d \geq 3$.

Finally, for $f_n = A^n$, with A semi-simple, approximate stability coincides with stability.

4.1 Generalized linear systems of $\mathbf{R}^d$. In this section, we consider a general sequence $(A_n) \in GL(d, \mathbf{R})$ of linear transformations of $\mathbf{R}^d$.

Let $A^+ \subset GL(d, \mathbf{R})$ denote the semi-group of diagonal matrices $\mathrm{diag}\{\lambda_1, \dots, \lambda_d\}$ where $0 < \lambda_1 \leq \cdots \leq \lambda_d$. Then we can write $A_n = L_n D_n R_n$ where $D_n \in A^+$ and L_n and R_n belong to $SO(d)$. Therefore, a sequence of vectors (v_n) is stable in the sense of (A_n) if and only if $(R_n^{-1} v_n)$ is stable in the sense of (D_n).

Now, it is straightforward to calculate the approximately stable space for a diagonal sequence. Indeed, as in the classical case above, approximate stability coincides with stability. We point out the following special case.

Fact 4.1. *Let $A_n = L_n D_n R_n$ be such that D_n has only one eigenvalue (with multiplicity) > 1, that is, there exists i such that $\lambda_i \leq 1 < \lambda_{i+1} = \cdots = \lambda_d$. Assume that A_n is divergent (i.e. has no convergent subsequence), then a subspace P is approximately stable if and only there is a subsequence*

Q_n of subspaces of $\mathbf{R}^i \times \{0\}$, such that $P = \lim_{n\to\infty} R_n^{-1} Q_n$. In particular, if $\lim_{n\to\infty} R_n^{-1}(\mathbf{R}^i \times \{0\})$ exists, then it equals $AS(0,(A_n))$. Furthermore the moduli of stability are always uniform (they do not depend on (A_n)). In any case, we can always extract a subsequence of (A_n), which has an approximately stable space of dimension i, and with uniform modulus of stability.

The linear Lorentz case. Here we consider sequences (A_n) in $SO(1,d-1)$, i.e. the orthogonal group of the Lorentz quadratic form on $\mathbf{R}^d : q = -x_1^2 + x_2^2 + \cdots + x_d^2$. We note firstly that the above observation holds.

FACT 4.2. *An element $A \in SO(1,d-1)$ can be written as $A = LDR$, with $L, R \in SO(d)$ and $D = \operatorname{diag}\{\lambda, 1, \ldots, 1, \lambda^{-1}\}$, with $\lambda \leq 1$.*

Proof. The KA^+K-decomposition for semi-simple Lie groups yields in our case, $K = SO(1,d-1) \cap SO(d) = SO(d-1)$, and A^+ any one parameter group of symmetric matrices that belongs to $SO(1,d-1)$. So, after conjugation by a rotation $r \in SO(d)$, rA^+r^{-1} becomes diagonal. The eigenspace associated to an eigenvalue < 1 (resp. eigenvalues > 1) is isotropic (for the form q). Hence, it is one dimensional (because q is Lorentz). This proves the fact. □

COROLLARY 4.3. *Let (A_n) be a divergent sequence in $SO(1,d-1)$. Then, there is a subsequence (B_n) such that the approximately stable space $AS(0,(B_n))$ is a lightlike hyperplane. The strongly approximately stable space of (B_n) is the orthogonal of $AS(0,(B_n))$. It is an isotropic one-dimensional space. The approximately stable space of (A_n) is the intersection of the hyperplanes obtained from all the subsequences (B_n). In particular, if all the approximately stable hyperplanes involved coincide, then, this equals the approximately stable space of (A_n) itself.*

Finally, all the moduli of stability are uniform.

4.2 Some comments.

Discompactness. In what follows we give an equivalent definition of the approximately stable space, reminiscent to the Carrière's notion of discompactness [Ca]. The codimension 1 fact in the Lorentz case translates to that $SO(1,d-1)$ has discompactness 1. This fact is as crucial for our work, as it was for Carrière's.

Consider a sequence (A_n) in $GL(d,\mathbf{R})$, and let U be the unit ball of R^d. Then $E_n = U \cap A_n U$ is a d-dimensional ellipse. A limit (in the sense of

Hausdorff) of (E_n) is an ellipse of dimension $\leq d$. Let U' be the intersection of all the limits of all convergent subsequences of (E_n). It is an ellipse of certain dimension. Observe that $AS(0, (A_n))$ is the linear space generated by U'.

Graphs. The approach of [DGr] consists of taking graphs. Keeping the notation above, consider the Graphs $Gr(A_n) = \{(x, A_n(x)/x \in \mathbf{R}^d\} \subset \mathbf{R}^d \times \mathbf{R}^d$. Then $AS(0, (A_n))$ is the intersection of all the projections of all the limits of subsequences of $(Gr(A_n))$. In the Lorentz case, we endow $\mathbf{R}^d \times \mathbf{R}^d$ with the product $\langle .,. \rangle \oplus -\langle .,. \rangle$. Thus, an element A of $GL(d, \mathbf{R})$ belongs to $SO(1, d-1)$ iff $Gr(A)$ is isotropic. Observe that if for some sequence (A_n), $(Gr(A_n))$ converges to (a d-plane) $E \subset \mathbf{R}^d \times \mathbf{R}^d$, then the intersection of E with each of the factors $\mathbf{R}^d \times \{0\}$ and $\{0\} \times \mathbf{R}^d$ are isotropic and hence of dimensions ≤ 1. This is the content of discompactness 1.

Rank one groups. The analogue of the fact above is valid for all the simple groups of non-compact type and of rank one, but now we allow diagonal matrices of the form: $\mathrm{diag}\{\lambda, \dots, \lambda, 1, \dots, 1, \lambda^{-1}, \dots, \lambda^{-1}\}$. The multiplicity of λ (or λ^{-1}) may then be 1, 2, 4 or 8 (thanks to the classification of simple Lie groups).

Chaos. Consider the two Lorentz linear systems: $B_n = \begin{pmatrix} 1 & b_n & b_n^2/2 \\ 0 & 1 & -b_n \\ 0 & 0 & 1 \end{pmatrix}$ and $C_n = \begin{pmatrix} c_n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c_n^{-1} \end{pmatrix}$. They are orthogonal for the form: $q = x_1x_3 + x_2^2$. We suppose that both (b_n) and (c_n) go to ∞ when $n \to \infty$. So we have as above, $AS(0, (B_n)) = \mathbf{R}e_1 \oplus \mathbf{R}e_2$, and $AS(0, (C_n)) = \mathbf{R}e_2 \oplus \mathbf{R}e_3$. Let now $A_n = C_nB_n$ and observe that as above, we have $AS(0, (A_n)) = \mathbf{R}e_1 \oplus \mathbf{R}e_2$ and that e_1 is strongly approximately stable, that is, there is a sequence $u_n \to e_1$ such that $A_n(u_n) \to 0$. Of course, $A_n(e_1) = c_ne_1 \to \infty$!

4.3 The derivative cocycle. In the (Liapunov) measurable theory one uses measurable trivializations of TM, with respect to which the derivative of a diffeomorphism f, is written as a map $C_f : M \to GL(d)$, where $d = \dim M$. In consideration of approximate stability, one needs some control of the "continuity" of the trivialization. So, to treat the punctual approximate stability, it suffices to consider a bounded trivialization. That is a frame-field with image in a compact set of the frame bundle. To handle approximate stability at some point, we further assume that the frame-field is continuous at this point. So we will always suppose that the trivializations satisfy the needed requisitions.

Let now $(M,(f_n))$ be a generalized dynamical system on a compact manifold M. We denote $C_n = C_{f_n}$. In the classical case, i.e. $f_n = f^n$, we obtain a cocycle $C : M \times \mathbf{Z} \to GL(d)$. In the generalized case, we just obtain a collection of linear systems $(C_n(x))$, for x running over M. So one can relate the punctually approximately stable space of (f_n) at a point x with the analogous one of $(C_n(x))$ at 0.

FACT 4.4. *We have:* $PAS(x,(f_n)) = PAS(0,(C_n(x))) = AS(0,(C_n(x)))$.

The Lorentz case. From the above facts we deduce:

PROPOSITION 4.5. *Let (f_n) be a non-equicontinuous sequence of isometries of a compact Lorentz manifold M. Let M' be a countable subset of M. Then there is a subsequence (ϕ_n) of (f_n) such that, for $x \in M'$, $PAS(x,(\phi_n))$ is a lightlike hyperplane, and $PAS(x,(\phi_n))^\perp = SPAS(x,(\phi_n))$ (= the strongly punctually approximately stable space of (ϕ_n) at x). All these hyperplanes have a uniform modulus of stability.*

Proof. Keeping the notation above and using a Lorentz trivialization, we have at each x, a derivative sequence $(C_n(x))$. We shall see in 5.2, since (f_n) is not equicontinuous, that for any x, $(C_n(x))$ is not equicontinuous. The proof follows by using a diagonal procedure and Corollary 4.3. □

5 Affine Dynamics: Uniformity

Henceforth, we will only consider affine generalized dynamical systems, that is M is endowed with a linear torsion free connection ∇ and (f_n) is a sequence of connection preserving transformations.

Equicontinuity. Divergent sequences. The first fundamental property of affine dynamics is the following:

PROPOSITION 5.1 (Uniformity). *Let (f_n) be an affine generalized dynamical system on a compact manifold M. Suppose that there is a sequence (x_n) such that $(D_{x_n} f_n)$ is equicontinuous (that is $(D_{x_n} f_n)$ and $(D_{x_n} f_n^{-1} = D_{f_n(x_n)} f_n^{-1})$ are bounded). Then (f_n) is equicontinuous, that is, (f_n) lies in a compact subset of the affine group* Affin(M).

The proof follows easily from the fact that Affin(M) acts properly on the frame bundle of M [K].

COROLLARY 5.2 (The unimodular case). *Let (f_n) be an affine unimodular generalized dynamical system on a compact manifold M, that is all the f_n preserve a volume form. (This is for example the case if there is a parallel* *volume form, e.g. the connection derives from a pseudo-Riemannian metric). Suppose that there is a sequence* (x_n) *such that* $(D_{x_n} f_n)$ *is bounded. Then* (f_n) *is equicontinuous.*

Proof. The unimodularity and the boundedness of $(D_{x_n} f_n)$, imply that $(D_{x_n} f_n^{-1})$ is also bounded, and hence $(D_{x_n} f_n)$ is equicontinuous. □

In the sequel we will be only interested in the opposite situation of equicontinuity. Specifically, we say that a sequence (f_n) is *divergent* if $\{f_n / n \in \mathbf{N}\}$ is a closed discrete subset of the group of homeomorphisms of M. So (f_n) is not divergent if it contains a convergent subsequence (in the group of homeomorphisms of M).

The codimension 0 case. We use the notation of the proposition above. By definition $D_{x_n} f_n$ is bounded if and only if $(T_{x_n} M)$ is a (f_n)-stable sequence of subspaces (of codimension 0). In particular if $x_n \to x$, then $T_x M$ is an approximately stable subspace. So the corollary above translates to the fact that an unimodular affine generalized dynamical system is equicontinuous whenever $T_x M$ is an approximately subspace, for some $x \in M$. So, in particular a divergent sequence (f_n) can only have approximately stable subspaces of codimension > 0.

Modulus of stability. One may then ask whether or not the unimodularity condition is necessary. The answer is yes, as we shall see below in the case of Hopf manifolds. These examples also explain why the phenomena of "propagation of stability" (expressed in 6.1), which generalizes the uniformity fact 5.1, is only local (more precisely, proportional to modulus of stability). In fact, the pathology of Hopf manifolds is due to their non-completeness. In affine flat dynamics, non-unimodularity and non-completeness are generally thought of as being equivalent phenomena.

An example: Hopf manifolds. Recall that an affine (flat) Hopf manifold is the quotient of $\mathbf{R}^d - \{0\}$ by a linear contraction. The simplest case is when this contraction is given by a multiplication map: $x \to \alpha x$, $0 < \alpha < 1$. The quotient H_α is thus endowed with an affine action of $GL(d, \mathbf{R})$. This action does not preserve any (non-trivial) measure. One can see this in the following way. For the sake of simplicity, we only consider the the case $d = 2$, so that the Hopf manifold is topologically a torus. Let A^t be a non-compact one parameter group of $SL(2, \mathbf{R})$. Then, its orbits determine a Reeb foliation, and hence has only finitely many recurrent leaves.

Let f be the diffeomorphism of H_α corresponding to a matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right)$

with $\lambda > 1$. Then f^n corresponds to any of the matrices $\begin{pmatrix} \alpha^{-m} & 0 \\ 0 & \lambda^n \alpha^{-m} \end{pmatrix}$, for any integer m. Let $x \in H_\alpha$, be the projection of a point $(a, b) \in \mathbf{R}^2 - \{0\}$. For calculation, one chooses a fundamental domain containing (a, b) and for each n, one chooses m such that $\begin{pmatrix} \alpha^{-m} & 0 \\ 0 & \lambda^n \alpha^{-m} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$ belongs to the same domain. If $b = 0$, x is a fixed point of f and so $D_x f^n$ is identified with $\begin{pmatrix} 1 & 0 \\ 0 & \lambda^n \end{pmatrix}$. Therefore, the approximately stable space is the x-axis. For $a \neq 0$, in order to return to the fundamental domain, we will need an integer m increasing with n and such that $\lambda^n \alpha^{-m} b$ is proportional to b. Therefore, $D_x f^n$ equals $\begin{pmatrix} \alpha^{-m} & 0 \\ 0 & \lambda^n \alpha^{-m} \end{pmatrix}$ and is thus equivalent to $\begin{pmatrix} \alpha^{-m} & 0 \\ 0 & 1/b \end{pmatrix}$ modulo multiplication by α. In particular, $(D_x f^n)$ is bounded (but not equicontinuous). Despite the fact the modulus of stability is not uniform.

6 Approximate Stability in Affine Dynamics: Partial Uniformity

In the present section, we prove some integrability and geodesibility properties for the approximately stable objects, due to a propagation of stability phenomenon, which holds in affine dynamics, and generalizing the uniformity Proposition 5.1.

For $x \in M$, the exponential map $\exp_x$ is defined in an open subset of $\mathcal{D}ef_x \subset T_x M$. We recall that a submanifold V of M is geodesic if whenever a geodesic $c : [a, b] \to M$ is somewhere tangent to M : $c'(t_0) \in T_{c(t_0)} V$ then $c(t) \in M$, when t belongs to some neighborhood of t_0. Notice that in general, if P is a linear subspace of $T_x M$, then $\exp_x P \cap \mathcal{D}ef_x$ is not geodesic except when $\dim P = 1$ or for specific nice manifolds (see Part II).

Notation. In the sequel, for any $y \in M$, we choose $V_y \subset T_y M$ a neighborhood of 0, with size proportional to the modulus of stability of a given (f_n)-stable sequence of linear subspaces (P_n).

6.1 Propagation of stability.

Proposition 6.1 (Propagation of stability). *Let (P_n) be a (f_n)-stable sequence of linear spaces, with $P_n \subset T_{x_n} M$. Let $\mathcal{P}_n = \exp_{x_n}(P_n \cap V_{x_n})$ and $y_n \in \mathcal{P}_n$. Consider the restrictions $h_n = f_n | \mathcal{P}_n$. Then, the derivatives $D_{y_n} h_n$ are uniformly bounded by the size of (P_n).*

In particular, let $v_n \in P_n$ be a convergent sequence of vectors, $y_n = \exp_{x_n} v_n$ and $P'_n = T_{y_n}(\exp_{x_n} P_n \cap V_{y_n})$. Then (P'_n) is a (f_n)-stable sequence, with size controlled by means of that of P_n.

Proof. Firstly, observe that the claim is obvious if f_n are linear transformations of an Euclidean space $\mathbf{R}^d$, and $x_n = 0$ (here, without size restriction).

The proof of the general case follows by linearization. Assume to begin with that the sequence (x_n) is stationary: $x_n = x_0$, and furthermore, x_0 is fixed by all the f_n, $f_n(x_0) = x_0$, so that the problem becomes linear after conjugation by $\exp_{x_0}$. More precisely, let $g_n = \exp_{x_0}^{-1} f_n \exp_{x_0}$ be defined on some neighborhood U of 0 in $T_{x_0}M$. It follows from the stability of (P_n), we may choose U so that $g_n(P_n \cap U)$ is contained in some fixed small neighborhood U' of 0. Hence, the derivatives of $f_n = \exp_{x_0} g_n \exp_{x_0}^{-1}$ along points of $\exp_{x_0}(P_n \cap U)$ are comparable to the corresponding ones for g_n, because of the fact that all things stay in a compact set, where the derivatives of $\exp_{x_0}$ and $\exp_{x_0}^{-1}$ are controlled.

When (x_n) is not stationary, we consider the family of derivatives $g_n = D_{x_n} f_n : T_{x_n}M \to T_{f_n(x_n)}M$. Inasmuch as these spaces are equipped with norms induced from a metric on TM, with respect to which we define stability, these norms are defined up to a bounded distortion. Now, since by definition x_n and $f_n(x_n)$ stay in a compact set, we can find identifications of bounded distortion of all our linear tangent spaces with a fixed Euclidean space. Therefore the stability notions are preserved and the proof goes as in the previous case. □

PROPOSITION 6.2 (Compatibility with parallel transport). *Keeping the notation of the proposition above, let (c_n) be a sequence of curves, such that the image of $c_n : [0,1] \to M$ is contained in $\exp_{x_n} P_n \cap V_{x_n}$ with $c_n(0) = x_n$. Consider P''_n the parallel transport of P_n along c_n. Suppose that (c_n) is bounded in the C^1 topology. Then P''_n is a stable sequence.*

Proof. From the proposition above the image curves $d_n = f_n(c_n)$ are bounded in the C^1 topology. Let $\tau_n = T_{c_n(0)}M \to T_{c_n(1)}M$ be the parallel transport along c_n and let τ'_n be the analogous parallel transport along d_n. Then, τ_n and τ'_n are uniformly bounded (since c_n and d_n are C^1 bounded). Now because f_n are affine, they commute with parallel transport, in particular: $D_{c_n(1)} f_n = (\tau'_n)^D_{c_n(0)} f_n \tau_n^{-1}$. Therefore, $(\tau_n(P_n))$ is a stable sequence. □

6.2 Geodesibility properties.

PROPOSITION 6.3. *Suppose that the stable sequence (P_n) converges to an approximately stable linear subspace $P \subset T_xM$, which is maximal among such subspaces (that is not strictly contained in another approximately stable subspace). Then $\exp_x P \cap V_x$ is geodesic in M. The same result is true in the punctual approximately stable case, that is, for any $x \in M$,*

$\exp_x(PAS(x,(f_n)) \cap V_x)$ *is geodesic.*

Proof. Let $c_n : [0,1] \to \exp_{x_n} P_n \cap V_{x_n}$ be a sequence of curves converging in the C^1 topology to a curve $c : [0,1] \to \exp_x P \cap V_x$, and such that $c_n(0) = x_n$. Let $y_n = c_n(1)$, $P'_n \subset T_{y_n}M$ be the tangent space of $\exp_{x_n} P_n \cap V_{x_n}$ at y_n, and $P''_n \subset T_{y_n}M$ the parallel transport of P_n along c_n. By the above propositions, both (P'_n) and (P''_n) are stable sequences of linear subspaces.

Denote the analogous objects at $y = c(1)$ by P' and P'', which are obviously the limits of P'_n and P''_n, respectively.

To prove that $\exp_x P \cap V_x$ is geodesic, it suffices to check the equality: $P' = P''$. Indeed, if this is true for arbitrary c, then the tangent space of $\exp_x P \cap V_x$ is parallel (along itself), which is equivalent to the geodesibility (for torsion free connections).

But, if $P' \neq P''$, then $P' \oplus P''$ is an approximately stable linear subspace. Indeed, let $\{e^i\}$ be a basis of this later subspace, and choose $\{e^i_n\}$ vectors of $P'_n \oplus P''_n$, such that $e^i_n \to e^i$. Denote by E_n the vector subspace generated by the $\{e^i_n\}$. Then (E_n) is a stable sequence converging to $P' \oplus P''$. This contradicts the fact that P is maximal. Therefore $P' = P''$. □

6.3 The codimension 1 case: lamination properties.

FACT 6.4. *Assume that (f_n) has no codimension 0 approximately stable subspace (that is for no $x \in M$, T_xM is an approximately stable space). Let $P \subset T_xM$ be an approximately stable hyperplane. Then $AS(x,(f_n)) = P$. In particular, in this case P is maximal and hence from the proposition above,* $\exp_x P \cap V_x$ *is a geodesic hypersurface.*

Proof. Assume the contrary, that is there exists an approximately stable vector $v \in T_xM$ which is transverse to P. Thus, P and v are respectively limits of stable sequences $P_n \subset T_{x_n}M$, and $v_n \in T_{y_n}M$. By transversality (of P and v), $\exp_{x_n} P_n \cap V_{x_n}$ and $\exp_{y_n} \mathbf{R}v_n \cap V_{y_n}$ intersect, in an uniform transverse meaner at a point z_n (near x, for n large). Moreover by Proposition 6.1, $D_{z_n}f_n$ is uniformly bounded along the tangent spaces at z_n, of each of the submanifolds $\exp_{x_n} P_n \cap V_{x_n}$ and $\exp_{y_n} \mathbf{R}v_n \cap V_{y_n}$. By definite transversality, $D_{z_n}f_n$ is bounded in $T_{z_n}M$, that is $(T_{z_n}M)$ is a (f_n)-stable sequence, a fact that is excluded by hypothesis. □

COROLLARY 6.5. *With the same hypothesis, if $y \in \exp_x P \cap V_x$, then $AS(y) = T_y(\exp_x P \cap V_x)$.*

Proof. It follows from Proposition 6.1, that $T_y(\exp_x P \cap V_x) \subset AS(y)$, and we infer from the fact above, the equality: $AS(y) = T_y(\exp_x P)$, as desired. □

By the same argument we get:

COROLLARY 6.6. *If $P_x \subset T_xM$ and $P_y \subset T_xM$ are approximately stable hyperplanes, then the two geodesic hypersurfaces $\exp_x P_x \cap V_x$ and $\exp_y P_y \cap V_y$ are either disjoint or tangent, in which case their intersection is open in each of them.*

The method of Graphs. As in the linear case (§4), there is a graph approach leading to some geometric proofs of the above properties of the approximately stable submanifolds $\exp_x P \cap V_x$. To see this, note that if we endow the product $M \times M$ with the product connection. Then $(\mathrm{Graph}(f_n))$ is a sequence of geodesic submanifolds in $M \times M$. Any approximately stable submanifold $\exp_x P \cap V_x$ is obtained as a projection of a limit of a sequence of connected components of $U \cap \mathrm{Graph}(f_n)$, where U is an open subset of $M \times M$.

The geodesic character of $\exp_x P \cap V_x$ is thus obvious. Nevertheless, neither the lamination properties (6.4), nor the control of sizes by modulus of stability, seems to be easy to treat, via this approach.

7 The Approximately Stable Foliation Theorem

Here follows a fundamental existence and regularity result:

Theorem 7.1. *Suppose that there is a dense subset $M' \subset M$, in which $PAS(x, (f_n))$ is a hyperplane, with uniform modulus of stability. Then $AS((f_n))$ is a Lipschitz codimension 1 subbundle of TM, tangent to a geodesic foliation $\mathcal{AS}((f_n))$, called the approximately stable foliation of (f_n).*

For the proof we need:

Digression: Codimension 1 geodesic laminations. Geodesic laminations (and in particular foliations) of codimension 1, enjoy some remarkable regularity properties. These properties are well known for laminations of hyperbolic surfaces [T], but are in fact valid in the general context of codimension 1 geodesic laminations, in the sense of a connection (and also some codimension 1 foliations with other geometric origins). For proofs and related questions, see [Z3] and [S]. Behind all of the regularity results for geodesic laminations, is the following fundamental Lipschitz regularity fact.

LEMMA 7.2. *Let M be a compact manifold endowed with a torsion free connection and an auxiliary norm $|.|$ on TM. Let M' be a subset of M*

and suppose given a real r and for $x \in M'$ a hyperplane $P_x \subset T_xM$ and let $\mathcal{L}_{x,r} = \exp_x P_x \cap B_x(r)$ where $B_x(r)$ is the ball of T_xM centered at 0 and with radius r. Also, suppose that $\mathcal{L}_{x,r}$ is geodesic and that if two plaques $\mathcal{L}_{x,r}$ and $\mathcal{L}_{y,r}$ intersect at some point, then they are tangent at that point (and hence by geodesibility, the intersection $\mathcal{L}_{x,r} \cap \mathcal{L}_{y,r}$ is open in both $\mathcal{L}_{x,r}$ and $\mathcal{L}_{y,r}$). Then, along M', the map $x \to P_x$ is Lipschitz, with Lipschitz constant depending only on the geometry of the connection, the auxiliary norm and r.

We infer from the above lemma the following corollary:

COROLLARY 7.3. (i) *A codimension 1 geodesic lamination on M is Lipschitz, with Lipschitz constant depending only on the geometry of the connection and the auxiliary norm. It then follows that the space of codimension 1 geodesic foliations of M, endowed with the C^0 topology (or equivalently the Lipschitz topology) on hyperplane fields, is compact.*

(ii) *With the same hypothesis as in the lemma above, suppose that the set M' is dense. Then, the geodesic plaques $\mathcal{L}_{x,r}$ extend to a geodesic foliation of M.*

Proof of Theorem 7.1. Since the modulus of stability is uniform, the geodesic plaques $\exp_x PAS(x,(f_n)) \cap V_x$ given by 6.4, satisfy the conditions of 7.2. Therefore, we have a geodesic foliation $\mathcal{F}$ of M such that $T_x\mathcal{F} = PAS(x,(f_n))$ for $x \in M'$. Again by uniformity of the modulus of stability, and the assumption that M' is dense, for any $x \in M$, $T_x\mathcal{F}$ is an approximately stable hyperplane at x. Therefore, it follows from Fact 6.4 that for any $x \in M$, $AS(x,(f_n)) = T_x\mathcal{F}$. □

8 Lorentz Dynamics. Proof of the Existence Part in Theorem 1.2

Henceforth, we will only deal with transformations preserving a Lorentz structure. So M is now a compact Lorentz manifold and $f_n \in \mathrm{Isom}\, M$. In the present section, we shall prove the existence part of Theorem 1.2 that is the existence of approximately stable foliations for subsequences (cf. Fact 8.1). The remaining part of Theorem 1.2, will be proved in §9.3. To begin with, we recall some facts about lightlike geodesic foliations.

Lightlike geodesic foliations. The compact space $\mathcal{FG}$. A (codimension 1) foliation $\mathcal{F}$ is *lightlike* if the restriction of the metric to $T\mathcal{F}$ is degenerate. This metric is thus positive non-definite and its Kernel is a

1-dimensional sub-foliation of $\mathcal{F}$, denoted by $\mathrm{Nor}\,\mathcal{F}$ and called the *normal foliation* of $\mathcal{F}$. This 1-dimensional foliation is isotropic (i.e. the metric vanishes along it) and completely determines $\mathcal{F}$, since $T\mathcal{F}$ is just the orthogonal of $T(\mathrm{Nor}\,\mathcal{F})$.

Here, we consider lightlike geodesic foliations [Z3]. One of their basic properties is that their (1-dimensional) normal foliations are also geodesic.

We denote by $\mathcal{FG}$ the space of all lightlike geodesic foliations on M. It is a closed subset in the space of all the geodesic foliations, and so it is compact whenever M is.

Existence of approximately stable lightlike geodesic foliations. The existence part in Theorem 1.2 is a consequence of:

FACT 8.1. *Let M be a compact Lorentz manifold and (f_n) a non-equicontinuous sequence of isometries. Then, there is a subsequence ϕ_n admitting an approximately stable lightlike (codimension 1) geodesic foliation $\mathcal{AS}((\phi_n))$, that is the approximately stable set $AS((\phi_n))$ is a codimension one bundle tangent to a codimension 1 geodesic lightlike foliation.*

In fact, $AS((\phi_n))$ coincide with $PAS((\phi_n))$, the punctually approximately stable bundle. Furthermore, the 1-dimensional normal foliation of $\mathcal{AS}((\phi_n))$ equals the (punctually or not) approximately strongly stable bundle of (ϕ_n).

Proof. Choose a countable dense subset M' of M. Then, by 4.5, there is a subsequence (ϕ_n) such that $PAS(x,(\phi_n))$ is a lightlike hyperplane, with uniform modulus of stability, when $x \in M'$. We may infer from Theorem 7.1, the existence on M of a codimension 1 lightlike geodesic foliation $\mathcal{AS}((\phi_n))$ tangent to $AS((\phi_n))$.

On M', we have the equality: $AS(x,(\phi_n)) = PAS(x,(\phi_n))$. To prove the equality for an arbitrary $x \in M$, we apply 4.5 for (ϕ_n) (instead of (f_n)) and for $M' = \{x\}$. We obtain a subsequence (ψ_n) for which $PAS(x,(\psi_n))$ is a hyperplane and hence equals $AS(x,(\phi_n))$ (since obviously $AS(x,(\phi_n)) \subset AS(x,(\psi_n))$ and $AS(x,(\psi_n)) = PAS(x,(\psi_n))$ from 6.4). In particular $PAS(x,(\psi_n))$ does not depend on the subsequence (ψ_n) (provided it is a hyperplane). By 4.3, this implies that $PAS(x,(\phi_n))$ is itself a hyperplane, and thus equals $AS(x,(\phi_n))$.

The fact that the normal direction of $\mathcal{AS}((\phi_n)$ is strongly approximately stable follows from the analogous statement in the linear case, 4.3. In the same fashion, one can prove the coincidence between strong approximate stability and strong punctual approximate stability. □

The non-compact case. If M is not compact, it may happen that for some x, $AS(x, (\phi_n)) = \{0\}$ for any subsequence (ϕ_n) of (f_n). Indeed, from our Definition 3.1, this happens when x satisfies the uniform escaping property: for any sequence $x_n \to x$ and a subsequence (ϕ_n) of (f_n), $(\phi_n(x_n))$ tends to ∞ (i.e. leaves all compact subsets of M).

"Recurrence" for generalized dynamical systems. There is no natural way to define recurrence notions for generalized dynamical systems, so that they satisfy a kind of Poincaré recurrence lemma. For our purpose, the following notion seems interesting:

DEFINITION 8.2. *Let K be a compact subset of M. Define its non-escaping subset $NE(K, (f_n))$ as the (compact) subset of points $x \in K$ such that there is $x_n \in K$, $x_n \to x$ and $f_n x_n \in K$. In other words $NE(K, (f_n))$ is the Hausdorff limit of the sequence of compact sets $K \cap f_n^{-1}K$. In particular* $\mathrm{Vol}(NE(K, (f_n)) \geq \mathrm{Vol}\, K \cap f_n^{-1}K) \geq 2\,\mathrm{Vol}(K) - \mathrm{Vol}(M)$ *(because the f_n are volume preserving). (In particular, letting K having a large relative volume, we see that almost every $x \in M$ is non-escaping for some compact K).*

We lose uniformity when M is not compact, and so to estimate sizes, we will choose a norm $|.|$ on TM. As sizes may depend upon the choice of such a norm, the previous uniformity of modulus of stability (4.5) fails. A straightforward localization of the previous arguments yields:

PROPOSITION 8.3. *Let M be a (not necessarily compact) Lorentz manifold, (f_n) a sequence in* $\mathrm{Isom}(M)$ *and K a compact subset of M. Then there is a subsequence (ϕ_n) such that, on $NE(K, (f_n))$, $AS((\phi_n))$ is a Lipschitz codimension 1 bundle, and there is $r = r(K)$ (not depending on (f_n)), such that the family of plaques $\mathcal{L}_{x,r} = \exp_x AS(\phi_n)) \cap B_x(r)$ determine a codimension 1 lightlike geodesic lamination of a neighborhood of $NE(K, (f_n))$, tangent to $AS((\phi_n))$ in $NE(K, (f_n))$.*

This applies at least in the finite volume case, because for instance if $\mathrm{Vol}(K) > \mathrm{Vol}(M) - \epsilon$, then, $\mathrm{Vol}(NE(K, (f_n))) > \mathrm{Vol}(M) - 2\epsilon$. Therefore, approximately stable bundles of some subsequences of (f_n), give rise to laminations, along big volume subsets of M. However, in order to obtain a foliation, on the whole of M, we must find a volume exhausting sequence of compact sets (K_j) with non-collapsing radii $(r(K_j))$. This is generally impossible because of non-compatibility of the auxiliary metric $|.|$ with the natural data.

Therefore, to avoid use of such a norm, we introduce for $x \in M$,

$\mathcal{D}ef_x \subset T_xM$ to be the domain of definition of the exponential map $\exp_x$. It is open (by definition) and star shaped in T_xM, that is, if $u \in \mathcal{D}ef_x$, then $tu \in \mathcal{D}ef_x$, for $t \in [0,1]$). Consider also the regular domain of definition $\mathcal{D}ef_x^*$ defined as the (open) set of vectors $u \in \mathcal{D}ef_x$, such that tu is a regular point of $\exp_x$ for any $t \in [0,1]$. Thus, if E is a linear subspace of T_xM, then $\exp_x(E \cap \mathcal{D}ef_x^*)$ is an immersed submanifold of M. This defines open subsets $\mathcal{D}ef$ and $\mathcal{D}ef^*$ of TM, which are invariant by $\mathrm{Isom}(M)$.

The propagation of stability 6.1 is valid a priori only in a domain having a size proportional to the modulus of stability. The obstruction lies in fact essentially in non-completeness, as this was shown in the examples of Hopf manifolds. Therefore, one may hope for a propagation of stability in $\mathcal{D}ef$. Indeed, this was shown in [Z5] in the context of (strict) stability (instead of approximate stability here), but only generically. The point is that we just need to ensure continuity (by using Lusin's Theorem) of the map $x \to \mathcal{D}ef_x^*$, and also the continuity of its variants obtained by intersection with stable (here approximately stable) bundles. The same proof yields the following fact:

Proposition 8.4. *Let M be a (not necessarily compact) Lorentz manifold, (f_n) a sequence of* $\mathrm{Isom}(M)$. *Let μ be a Borel finite measure (not related to data, but serves in applying the Lusin almost everywhere continuity theorem). Given a positive real ϵ, there is a compact K in M, with $\mu(K) > \mu(M) - \epsilon$ and there is a subsequence (ϕ_n) such that, the family of plaques $\mathcal{L}_x = \exp_x AS((\phi_n)) \cap \mathcal{D}ef_x^*$, for $x \in NE(K,(f_n))$, determines a codimension 1 lightlike geodesic lamination of a neighborhood of $NE(K,(f_n))$, tangent to $AS((\phi_n))$ in $NE(K,(f_n))$ (that is if $y \in NE(K,(f_n)) \cap \mathcal{L}_x$, then $AS(y,(\phi_n)) = T_y\mathcal{L}_x$).*

In the finite volume case, we use this proposition for the Lorentz measure. As we remarked earlier, we get $\mathrm{Vol}(NE(K,(f_n)) > \mathrm{Vol}(M) - 2\epsilon$. By letting K get larger and larger, and applying a diagonal process, we obtain a foliation:

Theorem 8.5 (The finite volume case)**.** *Theorem 1.2, extends to the case of M of finite volume: a divergent sequence (f_n) of* $\mathrm{Isom}(M)$ *possesses a subsequence (ϕ_n) admitting an approximately stable foliation $\mathcal{AS}((\phi_n))$. Furthermore, almost everywhere: $AS(x,(\phi_n)) = PAS(x,(\phi_n))$* (*this equality is everywhere true in the compact case*).

A relative version. Here we give a relative version:

Theorem 8.6. *Let M be a finite volume Lorentz manifold, (f_n) a diver-*

gent sequence of isometries that leaves invariant a finite volume Lorentz submanifold N. Let (ϕ_n) be a subsequence of (f_n) having an approximately stable foliation $\mathcal{AS}((\phi))$. Then, the restriction $(\phi_n|N)$ admits an approximately stable foliation, which is just the trace of $\mathcal{AS}((\phi_n))$ on N. In particular, the intersection of a leaf of $\mathcal{AS}((\phi_n))$ with N is geodesic in N.

This result follows from the fact that: if $x \in N$, then $PAS(x, (\phi_n|N)) = PAS(x, (\phi_n)) \cap T_x N$

A foliated version. We also have the following foliated version:

Theorem 8.7. *Let M be a finite volume Lorentz manifold and (f_n) a divergent sequence of isometries. Let U be an open subset of M and $L \subset TM|U$ a C^0 subbundle over U, invariant by each Df_n (in particular U itself is invariant by f_n). Let M_0 be the subset of U where L is of Lorentzian type (i.e. the restriction of the metric has signature $- + \cdots +$). Let (ϕ_n) be a subsequence of (f_n) having an approximately stable foliation $\mathcal{AS}((\phi))$. Then in $U - M_0$, $L \subset AS((\phi_n))$ and in M_0, $L^\perp \subset AS((\phi_n))$ (where $L^\perp$ is the orthogonal of L).*

Suppose that L is integrable. Then, in M_0, the leaves of $\mathcal{AS}((\phi_n)) \cap L$ are geodesic inside the leaves of L.

9 Bundle Compactification

Let M be a manifold. One may naturally construct a fiber metric on its (principal) frame bundle $P_M \to M$, so that any C^1 diffeomorphism acts isometrically on the fibers. In fact, instead of P_M, it is more convenient to consider the associated bundle $S_M \to M$ whose typical fiber the universal symmetric Riemannian space $S_k = SL(k)/SO(k)$, where $k = \dim M$. So, S_M may be interpreted as the bundle of conformal structures on (the fibers of) TM.

One may then naturally compactify the fibers to get a bundle $\overline{S_M} \to M$ with a compact fiber $\overline{S_k}$, the Hadamard compactification of S_k which is topologically a closed ball.

It is also interesting to interpret recurrence properties of the action of f, or more generally, of a generalized dynamical system (f_n) on $\overline{S_M}$. The philosophy is that recurrence conditions may be related to classical notions, such as Oseldec's decomposition, invariant metrics (of some regularity)... In the sequel, we will rather use this construction to interpret the approximate stability.

Notice that these constructions generalize to fiber dynamical systems on a linear bundle $E \to M$. It is also possible to consider the case of bundles with a G-structure (i.e. a reduction of the structural group of the principal bundle $P_M \to M$ to G) where G is a semi-simple subgroup of $SL(k, \mathbf{R})$ of non-compact type. In which case, one can build the same constructions with the symmetric space associated to G. In fact, instead of developing the general situation, we shall henceforth restrict ourselves to the case where G is $SO(1, k-1)$, and E is the tangent bundle of M. This exactly means that M has a Lorentz structure. We mention at this stage that some but not all of the next results are valid in the general case.

Fiberwise hyperbolic geometry. Let $(M^{1+d}, \langle .,. \rangle)$ be a Lorentz manifold. Note $T^rM = \{v \in TM / \langle v, v \rangle = r\}$. After passing if necessary to a double cover of M, we can choose a sheet of $T^{-1}M$ that we note $\mathbf{H}M$. It is a bundle over M with type fiber the hyperbolic space $\mathbf{H}^d$ (recall that the Lorentz metric has signature $- + \cdots +$). It is compactified by adding the projectivization $\mathbf{S}^\infty M$ of the isotropic cone T^0M. We denote it by $\overline{\mathbf{H}}M$ the bundle over M with type fiber a topological closed ball of dimension d.

The group $\mathrm{Isom}(M)$ acts on $\mathbf{H}M$ and on $\overline{\mathbf{H}}M$, by preserving the hyperbolic metric and the conformal structure on the fibers, respectively. This is reminiscent, as the following analogies will confirm, to a Kleinian group acting on the Riemann sphere.

9.1 Limit sets.

DEFINITION 9.1. *Consider $s : M \to \mathbf{H}M$ a continuous section (this always exists since $\mathbf{H}^d$ is contractible). Let A be a* closed non-compact *subset of* $\mathrm{Isom}(M)$. *The limit set L_A of A is the set of the limits in $\overline{\mathbf{H}}M$ of the sequences $(\gamma_n(s(x_n)))$, for (x_n) a sequence in M and (γ_n) a divergent sequence of A (i.e. $\{\gamma_n / n \in \mathbf{N}\}$ is a closed discrete subset of* $\mathrm{Isom}(M)$).

FACT 9.2. *The definition above does not depend on the choice of the section s. The limit set L_A is a non-empty closed subset of $\mathbf{S}^\infty M$.*

Proof. L_A is non-empty since $\overline{\mathbf{H}}M$ is compact. One easily see that if for a sequence (γ_n), there is $s_0 \in \mathbf{H}M$, such that the elements $\{\gamma_n(s_0)/n \in \mathbf{N}\}$ remain in a compact subset of $\mathbf{H}M$, then (γ_n) is equicontinuous. This shows that L_A is contained in $\mathbf{S}^\infty$. To see that L_A does not depend on the choice of the section s, consider a distance d on $\mathbf{H}M$ inducing the hyperbolic metric on the fibers. If s' is another section, then $d(f_n s(x), f_n s'(x)) = d(s(x), s'(x))$. As usual, this implies that if $(\gamma_n(s(x)))$ converges in $\overline{\mathbf{H}}M$, then, $(\gamma_n(s'(x)))$ also tends to the same limit. □

PROPOSITION 9.3. *Let* Γ *be a closed non-compact subgroup of* $\mathrm{Isom}(M)$. *Then the limit set* L_Γ *is the smallest* Γ*-invariant closed subset of* $\mathbf{S}^\infty M$ *which projects surjectively onto* M.

Proof. The proof is the same as for Kleinian groups. Let F be a Γ-invariant subset of $\mathbf{S}^\infty M$ projecting onto M. One naturally construct $H(F) \subset \overline{\mathbf{H}}M$, the fiberwise convex hull of F. We can choose the section s (in the definition of limit sets) to have image in $H(F) \cap \mathbf{H}M$ (since the fibers of $H(F)$ are still contractible and since F projects onto M). Therefore $L_\Gamma \subset H(F)$ since $H(F)$ is closed and invariant. In fact $L_\Gamma \subset H(F) \cap \mathbf{S}^\infty M = F$. □

In the definition of limit sets, we used continuous sections. In fact, less regular sections may be equally useful. We will need later the particular following statement

FACT 9.4 (Measurable sections). *Suppose that a closed subgroup* Γ *of* $\mathrm{Isom}(M)$ *preserves a measurable section* $N \to \mathbf{H}M$, *defined over a measurable* Γ*-invariant subset* N *with a positive volume. Then,* Γ *is compact.*

Proof. Let $s : M \to \mathbf{H}M$ be a measurable Γ-invariant section. Let K be a compact subset of M, along which s is continuous and such that $\mathrm{Vol}(K) > 1/2\ \mathrm{Vol}(M)$. Let (f_n) be a sequence in Γ and $x_n \in f_n^{-1}K \cap K$, this exists because of the volume condition. By continuity of $s|K$, the set $\{f_n(s(x_n)), n \in \mathbf{N}\} \subset \mathbf{H}M$ is precompact. This implies that (f_n) is equicontinuous. Therefore, Γ is compact. □

9.2 Limit sets and approximately stable foliations.

PROPOSITION 9.5. *Let* (f_n) *be a sequence of* $\mathrm{Isom}(M)$ *such that* (f_n^{-1}) *admits an approximately stable foliation* $\mathcal{AS}((f_n^{-1}))$. *Then the limit set* $L_{(f_n)}$ *is the image of a section* $M \to \mathbf{S}^\infty M$, *i.e. a field of isotropic lines, which is in fact the normal direction of* $\mathcal{AS}((f_n^{-1}))$.

Assume furthermore that (f_n) *admits an approximately stable foliation. Then if* $v \in TM - AS((f_n))$, $Df_n v$ *tends to* ∞, *and converges projectively (i.e. after normalization) to* $L_{(f_n)}$. *The convergence is uniform in compact subsets of* $TM - AS((f_n))$.

This convergence is in particular valid in $\mathbf{H}M$ *(since it is missed by lightlike hyperplanes and hence* $\mathbf{H}M \subset TM - AS((f_n))$. *More precisely, let* U *and* V *be neighborhoods of respectively* $AS((f_n^{-1}))^\perp$ *and* $AS((f_n))^\perp$ *in* $\mathbf{S}^\infty$. *Then there is* N, *such for,* $f_n(\mathbf{S}^\infty M - V) \subset U$, *for* $n > N$.

Proof. We argue by contradiction to prove that the direction $L_{(f_n)}$ is the normal direction of $\mathcal{AS}((f_n))$. Let $s : M \to \mathbf{H}M$ be a section, that is a vector field $\langle s(x), s(x)\rangle = -1$ for $x \in M$. We have to prove that $Df_n s$ converges projectively (i.e. in direction) to $AS((f_n^{-1}))^{\perp}$. If this were false, there would exist a sequence x_n such that $D_{x_n} f_n(s(x_n))$ converges projectively to a vector $u \notin AS((f_n^{-1}))^{\perp}$. Hence,, $D_{x_n} f_n(s(x_n)) = \alpha_n u_n$, with $u_n \to u$ and $\alpha_n \to \infty$ (because if not (f_n) would be equicontinuous). Thus $Df_n^{-1} u_n = \alpha_n^{-1} s(x_n)$. Thus, by definition u is strongly approximately stable for (f_n^{-1}), and hence by 4.5, belongs to $AS((f_n^{-1}))^{\perp}$, which yields a contradiction. The last statement may be checked in a similar fashion. □

From the above theorem, one can conclude the two following "continuity" corollaries:

Corollary 9.6. *Let (f_n) be a sequence of* $\mathrm{Isom}(M)$ *such that (f_n) and (f_n^{-1}) admit approximately stable foliations $\mathcal{AS}((f_n))$ and $\mathcal{AS}((f_n^{-1}))$. Let X (resp. Y) be a continuous vector field tangent to $AS((f_n))^{\perp}$ (resp. $AS((f_n)))$. Then there is sequence (X_n) (resp. (Y_n)) of continuous vector fields converging in the C^0 topology to X (resp. Y), and such that $(Df_n X_n)$ converges in the C^0 topology to 0 and $(Df_n Y_n)$ is bounded.*

Proof. We will only prove the claim for X, as the same argument works for Y. To simplify notation, we will do the proof for $g_n = f_n^{-1}$ instead of (f_n). By the proposition above, after passing to subsequences, we may find sequences of neighborhoods (U_n) and (V_n), collapsing to $AS((f_n))^{\perp}$ and $AS((f_n^{-1}))^{\perp}$, respectively, and such that $Df_n(TM - U_n) \subset V_n$. Now, let (X'_n) be a sequence of vector fields such that $X'_n \notin U_n$, but (X'_n) converges to some non-singular vector field tangent to $AS((f_n))^{\perp}$. Thus, from the proposition above, we infer that $X''_n = Df_n X'_n$ is a vector field with a big norm and converging in direction in $AS((f_n^{-1}))^{\perp}$. Therefore, for suitable distortion functions λ_n, going uniformly to ∞, we may get $X_n = \lambda_n X''_n \to X$. But $Dg_n X_n = (1/\lambda_n) X'_n \to 0$. □

Corollary 9.7. *Let A be a closed non-compact subset of* $\mathrm{Isom}\, M$ *and $u \in L_A \cap \mathbf{S}^{\infty}_x M$. Then, there is a continuous (in fact Lipschitz) section $\sigma : M \to L_A$ with $\sigma(x) = u$.*

Proof. u is a (projective) limit of a sequence $(f_n(s(x_n)))$ where (f_n) is a sequence in A and $x_n \to x$ (and s is fixed section of $\mathbf{H}M$). After passing to a subsequence we may assume that (f_n^{-1}) admits an approximately stable foliation. We set $\sigma(x) = AS((f_n^{-1}))^{\perp}$. From the proposition above, we conclude that $\sigma(x) = u$. □

9.3 End of the proof of Theorem 1.2. The statement in Theorem 1.2, about the dynamics of $(D\phi_n)$ on TM, follows from Proposition 9.5.

To finish the proof of Theorem 1.1, it remains to check its ergodicity statement. For this end, to simplify notation, suppose that $\phi_n = f_n$. let $\sigma : O \to \mathbf{R}$ be a continuous function, invariant by all the f_n.

Consider a vector field X_n as in 9.6. Let α be a real such that $\exp_x \alpha X_n(x)$ exist for any $x \in M$. Then, $\sigma(\exp_x \alpha X_n(x)) - \sigma(x) = \sigma(\exp_{f_n x} \alpha D f_n X_n(x)) - \sigma(f_n x)$. Denote this difference by $\Delta_n(x)$.

Let (x_n) be sequence such that $f_n x_n$ belong to the same compact K. Then $\Delta_n(x_n) \to 0$, as $D f_n X_n \to 0$, in the C^0 topology and because σ is uniformly continuous on K.

Suppose that $x_n \to x$ and $X_n(x_n) \to u \in AS(f)^\perp$. Then $\sigma(\exp_x \alpha u) - \sigma(x) = \lim(\sigma(\exp_{x_n} \alpha u) - \sigma(x_n)) = \lim \Delta_n(x_n) = 0$, for any α such that $\exp_x \alpha u \in O$. That is σ is locally constant along the leaf of $AS(f)^\perp$ passing through x. Remember finally that, from 8.2, almost every x is a limit of a sequence such as (x_n), for some compact K.

10 Foliation Compactification

Here we define a topology on $\mathrm{Isom}(M) \cup \mathcal{FG}$ and then an ideal boundary $\partial_\infty \Gamma$ for Γ a closed non-compact subgroup of $\mathrm{Isom}(M)$.

We fix a section $s : M \to \mathbf{H}M$ and we choose a distance d on $\overline{\mathbf{H}}M$ (in fact the choice of s permit to construct a natural Euclidean fiberwise distance, we then tensorize by a distance on M and thus we get a semi-canonical distance d). Now we embed $\mathrm{Isom}\, M \cup \mathcal{FG}$ in the $\mathrm{Sect}\, \overline{\mathbf{H}}M$, the space of sections of $\overline{\mathbf{H}}M$. To $f \in \mathrm{Isom}\, M$, we associate the section fs, and to a foliation $\mathcal{F}$ we associate its normal direction field $\mathcal{F}^\perp$. We then endow $\mathrm{Isom}\, M \cup \mathcal{FG}$ with the C^0 topology defined by the distance $\rho(\sigma_1, \sigma_2) = \sup_{x \in M} d(\sigma_1(x), \sigma_2(x))$.

From 9.5, we deduce:

Fact 10.1. *If a sequence (f_n) of $\mathrm{Isom}\, M$ converges to $\mathcal{F} \in \mathcal{FG}$, then $\mathcal{F} = \mathcal{AS}((f_n^{-1}))$.*

Since $\mathcal{FG}$ is compact, we have:

Corollary 10.2. *$\mathrm{Isom}\, M \cup \mathcal{FG}$ is a (metrizable) compact space endowed with a natural continuous action of $\mathrm{Isom}\, M$.*

Definition 10.3. *Let Γ be a closed non compact subgroup of $\mathrm{Isom}\, M$. Its boundary $\partial_\infty \Gamma$ is the intersection with $\mathcal{FG}$ of the closure of Γ in $\mathrm{Isom}\, M \cup \mathcal{FG}$.*

11 Preliminaries on Elementary Groups. Proof of Theorem 2.6

Let $\mathcal{F}^1 \cup \mathcal{F}^2$ be a bifoliation (Definition 2.2). Its set of tangent foliations $\mathrm{Apa}(\mathcal{F}^1, \mathcal{F}^2)$ has the following description. Let C be the coincidence set $\{x \in M / T_x\mathcal{F}^1 = T_x\mathcal{F}^2\}$. Let $\pi_0(M - C)$ be the set of connected components of $M - C$, and consider a map: $c : \pi_0(M - C) \to \{1, 2\}$. This allows to construct a foliation, $\mathcal{F}_c$, by the rule: $T_x\mathcal{F}_c = T_x\mathcal{F}^{c(U)}$, in the connected component U of $M - C$, and $T_x\mathcal{F}_c = T_x\mathcal{F}^1 = T_x\mathcal{F}^2$ in C. It is straightforward to verify that $\mathcal{F}_c \in \mathrm{Apa}(\mathcal{F}^1, \mathcal{F}^2)$ (in particular there is no loss of continuity or Lipschitz character in this construction). In fact we have a topological identification of $\mathrm{Apa}(\mathcal{F}^1, \mathcal{F}^2)$ with $\{1, 2\}^{\pi_0(M-C)}$.

Observe that we have a natural distance in $\mathrm{Apa}(\mathcal{F}^1, \mathcal{F}^2)$ defined by $d(\mathcal{F}_c, \mathcal{F}_{c'}) = \Sigma_{U \in \pi_0(M-C)}(c(U) - c'(U)) \,\mathrm{Vol}(U)$. Any group preserving the volume and the bifoliation, preserves this distance.

FACT 11.1. *Let $\Gamma \subset \mathrm{Isom}(M)$ be a closed non-compact subgroup with the property that for any $x \in M$, $L_\Gamma(x) = L_\Gamma \cap \overline{\mathbf{H}}M = \mathbf{S}^\infty M$, has exactly 1 or 2 points. Then, Γ is elementary.*

Proof. Let $u \in L_\Gamma(x)$. From the corollary above there is a foliation $\mathcal{F}(u)$, such that the normal direction $\mathcal{F}^\perp$ is contained in L_Γ and equals u at x. Let C be the coincidence set of all these foliations. Obviously, it equals the set of points x where $L_\Gamma(x)$ has cardinality 1. Let U be a component of $M - C$, then the restrictions to U of two foliations $\mathcal{F}(u)$ and $\mathcal{F}(u')$ are identical or everywhere transverse. Therefore, as above, one may construct two foliations $\mathcal{F}^1$ and $\mathcal{F}^2$, which are transverse in each component U, and each of them equals a foliation of the form $\mathcal{F}(u)$ (in the component U). Therefore, L_Γ can be defined by a bifoliation. □

Proof of Theorem 2.6. Let $\Gamma \subset \mathrm{Isom}(M)$ be a closed non-compact amenable subgroup. To prove that it is elementary, we will check that it satisfies the condition of the fact above. By amenability, there is a Γ-invariant probability measure μ onto $\mathbf{S}^\infty M$, projecting onto the Lorentz measure of M. Let μ_x the conditionals of μ on the fibers $\mathbf{S}^\infty_x M$.

Let us show that for almost all $x \in M$ (in the sense of the Lorentz measure), the support of μ_x has exactly 1 or 2 points. Indeed, if not a fiberwise barycenter construction (see, for instance, [BCoG] for the classical one) yields a measurable Γ-invariant section $\sigma : N \subset M \to \mathbf{H}M$ over a Γ invariant subset N. This is impossible by Fact 9.4. □

The bifoliation of an elementary group. Observe that an elementary group may preserve many bifoliations. However we have the following fact allowing to define the bifoliation associated to an elementary group:

FACT 11.2. *Let Γ be an elementary group. Then Γ preserves a unique bifoliation $\mathcal{F}^1 \cup \mathcal{F}^2$ determining a maximal apartment among that determined by all the bifoliations preserved by Γ. It is characterized by: $L_\Gamma(x) = \{(T_x\mathcal{F}^1)^\perp, (T_x\mathcal{F}^2)^\perp\}$ for any $x \in M$. We call it* the bifoliation of Γ.

Proof. For a bifoliation $\mathcal{G}^1 \cup \mathcal{G}^2$, denote by $T(\mathcal{G}^1, \mathcal{G}^2) = \{x \in M / T_x\mathcal{G}^1 \neq T_x\mathcal{G}^2\}$ its transversality (open) set. Now, we will consider only bifoliations preserved by Γ. Let $\mathcal{G}^1 \cup \mathcal{G}^2$ and $\mathcal{L}^1 \cup \mathcal{L}^2$ be two of them. Observe then that along $T(\mathcal{G}^1, \mathcal{G}^2)$, $\mathcal{L}^1$ and $\mathcal{L}^2$ are tangent to $\mathcal{G}^1 \cup \mathcal{G}^2$. Indeed, the opposite situation, would give a Γ-invariant $3-$ or $4-$valued section of $\mathbf{S}^\infty M$ along some open subset of $T(\mathcal{G}^1, \mathcal{G}^2)$. As above, the barycenter construction (the classical one in this case), yields a Γ-invariant section of $\mathbf{H}M$, over some open subset of M. This implies Γ is compact, by Fact 9.4.

Observe now that by the above gluing process, one may construct a bifoliation $\mathcal{M}^1 \cup \mathcal{M}^2$ such that $T(\mathcal{M}^1, \mathcal{M}^2) = T(\mathcal{G}^1, \mathcal{G}^2) \cup T(\mathcal{L}^1, \mathcal{L}^2)$. We construct the wished maximal foliation as one having a maximal transversality set. This exists by compactness of the space of bifoliations. □

REMARK 11.3. We close this section by making some remarks on the general set-up of the "bundle compactification" (§9), when we let a generalized dynamical system of C^1 diffeomorphisms acting on the bundle $\overline{S_k}M \to M$. One considers a given measure ν preserved by the dynamical system, and choose an invariant measure μ on $\overline{S_k}M$ of maximal support and projecting onto ν. It may happen (in fact generically) that ν has a full measure in $\overline{S_k}M$, or even worse, ν can be ergodic. If not, i.e. ν has support inside $\mathbf{S}_kM$ (that is the support of ν does not interest $\partial_\infty\overline{\mathbf{S}}_kM$) then, the barycenter construction (in general Hadamard spaces) yields an invariant measurable metric. However, if $k > 2$, the universal symmetric space $S_k = SL(k, \mathbf{R})/SO(k)$ is not of negative curvature (i.e. it has a higher rank), and there is no way to construct barycenters for measures supported in the Hadamard boundary. In fact, there are alternative boundaries which may be efficient in this matter. Depending on the interpretation of such a boundary, that is by modeling it as a kind of flag spaces, the construction of limit sets yields, roughly speaking, flag-fields on M. For example, Oseldec's decomposition for a diffeomorphism f may be handled by looking to points in the limit set (of the group $\{f^n, n \in \mathbf{Z}\}$), with special "approach". For instance, the conical and horospherical limit sets may be interesting in this regard (see [Su]).

12 Partial Hyperbolicity

Definition 12.1. *An elementary group* Γ *is called* partially hyperbolic *if its associated bifoliation* $\mathcal{F}^1 \cup \mathcal{F}^2$ *is non-trivial, that is the (open) transversality locus* $T = \{x \in M \,/\, T_x\mathcal{F}^1 \neq T_x\mathcal{F}^2\}$ *is non-empty, or equivalently, the (closed) coincidence locus* $C = \{x \in M \,/\, T_x\mathcal{F}^1 = T_x\mathcal{F}^2\}$ *is a proper subset of* M.

Now, we will justify the word "partially hyperbolic" in a dynamical viewpoint (of course, partially hyperbolic, is also reminiscent to the term "elementary hyperbolic" in the theory of Fuschian groups). We keep the notation above. Denote by $\mathcal{N}^1$ and $\mathcal{N}^2$ the two (one dimensional) normal directions of $\mathcal{F}^1$ and $\mathcal{F}^2$, respectively. We may assume that they are orientable, after passing to a finite covering. Let X^1 and X^2 be two Lipschitz non-singular vector-fields orienting $\mathcal{N}^1$ and $\mathcal{N}^2$, respectively.

The group Γ respects the set of (open) connected components of the transversality locus T. Since these components have positive Lorentz volume and since Γ preserves the volume, then each component U is preserved by a finite index subgroup of Γ, say Γ itself. It then follows that Γ preserves the closure $\bar{U}$ and the directions of X^1 and X^2, along it. Therefore we get two derivative cocycles: λ^1 and $\lambda^2 : \bar{U} \times \Gamma \to \mathbf{R}$, defined by $D_x f(X^i(x)) = \lambda^i(f,x) X^i(f(x))$, for $i \in \{1,2\}$. This section is devoted to the proof of:

Theorem 12.2. *Let* Γ *be a partially hyperbolic elementary group. Then, up to a subgroup of finite index,* Γ *is a direct product of a compact group by* $\mathbf{Z}$ *or* $\mathbf{R}$. *Furthermore, we can find two foliations* $\mathcal{L}^1$ *and* $\mathcal{L}^2$ *generating the same bifoliation* $\mathcal{F}^1 \cup \mathcal{F}^2$, *and defining two cocycles* c^1 *and* c^2 *satisfying the following condition. Let* g *be a non-trivial element of the* $\mathbf{Z}$ *or the* $\mathbf{R}$ *part of* Γ *and* K *a compact subset in the transversality set of the bifoliation. Then, there is an integer* $p = p(K) > 0$, *such that for* $f = g^p$, *we have:*

$c^1(f^n, x) < 1/2$, *if* $n > 0$, $x \in K$ *and* $f^n x \in K$, *and*

$c^2(f^n, x) > 2$, *if* $n > 0$, $x \in K$ *and* $f^n x \in K$.

Finally, in the transversality set T, *the normal foliation of* $\mathcal{L}^1$ *and* $\mathcal{L}^2$ *are respectively the negative and positive Liapunov spaces of* g. *In particular,* $\mathcal{L}^1$ *and* $\mathcal{L}^2$ *are preserved by* g.

Beginning of the proof. Consider an auxiliary complete distance ρ on Γ. For example embed Γ is the frame bundle P_M, and take ρ to be the restriction of a distance on P_M induced by a complete Riemannian metric. We denote open balls around the identity by $B^\rho(1, r)$.

FACT 12.3. *With the notation above let K be a compact of U. There are:*

(i) *a function: $r \in \mathbf{R}^+ \to c_r^K \in \mathbf{R}^+$ such that $\lim_{r\to\infty} c_r^K = \infty$, and*
(ii) *a map $s : K \times \Gamma \to \{1, 2\}$, defined for (f, x) such that $f(x) \in K$. We denote $u(f, x) = s(f, x) + 1 \bmod 2$.*

These maps satisfy the condition that whenever, $f \notin B^\rho(1, r)$, then:

$$\lambda^{s(f,x)}(f, x) < (c_r^K)^{-1} \text{ and } \lambda^{u(f,x)}(f, x) > c_r^K\,.$$

Proof. In U, Γ preserves the directions $\mathcal{N}^1$ and $\mathcal{N}^2$, and hence also the orthogonal $(\mathcal{N}^1 \oplus \mathcal{N}^2)^\perp$. This last space is spacelike, i.e. the Lorentz metric restricted to it is positive definite. Therefore, in the compact K of U, we have an uniform bound of the restriction $D_xf|(\mathcal{N}^1 \oplus \mathcal{N}^2)^\perp$ and $(D_xf)^{-1}|(\mathcal{N}^1 \oplus \mathcal{N}^2)^\perp$, for $x \in K$ and $f \in \Gamma$, such that $f(x) \in K$.

It then follows by the volume preservation property that the product $\lambda^1(f, x)\lambda^2(f, x)$ belongs to some fixed compact interval (around 1) in $]0, \infty[$, for $x \in K$, and $f \in \Gamma$, with $f(x) \in K$. That is the cocycles λ^1 and λ^2 are almost one the inverse of the other, provided we restrict ourself to K.

To prove the estimates contained in the statement of the fact, we argue by contradiction. Suppose that for a divergent sequence (f_n) in Γ, there is a sequence (x_n) of points of K, such that $f_n(x_n) \in K$ and such both $\lambda^1(f_n, x_n)$ and $\lambda^2(f_n, x_n)$ remain bounded. Hence, $(D_{x_n}f_n)$ is equicontinuous and thus it follows from 5.2,that (f_n) is not divergent! Therefore, there is a function c_r^K, satisfying property (i) of the fact, and such that for any x and f as in the fact, there is some $s = s(f, x) \in \{1, 2\}$, such that the following inequalities hold: $\lambda^s(f, x) < (c_r^K)^{-1}$ and $\lambda^u(f, x) > c_r^K$ with $u = s + 1 \bmod 2$. □

FACT 12.4. *There is a partition $f = A_1 \cup A_2$ satisfying the following conditions. Let U_1 (resp. U_2) be a neighborhood in (the projective isotropic cone) $\mathbf{S}^\infty M$ of $\mathcal{N}^1$ (resp. $\mathcal{N}^2$) along the closure $\bar{U}$.*

There is r such for $f \in \Gamma - B^\rho(1, r)$, we have:

$$\text{if } f \in A_1, \text{ then} : Df(\mathbf{S}^\infty M - U_1) \subset U_2, \text{ and}$$
$$\text{if } f \in A_2, \text{ then} : Df(\mathbf{S}^\infty M - U_2) \subset U_1\,.$$

Proof. It is easily seen that one can localize the compactification of isometry groups of M to that of $\bar{U}$. So, here the boundary of Γ (acting on $\bar{U}$) consists of the two foliations $\mathcal{F}^1|\bar{U}$ and $\mathcal{F}^2|\bar{U}$. Let B_1 and B_2 be two disjoint neighborhoods of these last foliations in the compact $\overline{\mathrm{Isom}(\bar{U})}$. So, for r big enough, every element f, with $\rho(1, f) > r$, belongs to exactly one of the neighborhoods B_1 or B_2. Suppose that we cannot find r (big enough) satisfying the claim. Then there is a divergent sequence (f_n) contained in B_1 or B_2 (say B_1) and not satisfying the desired inclusions. Thus the approximately stable foliation of (f_n) is $\mathcal{F}^1|\bar{U}$. Now we apply the expanding property of the approximately stable foliation 9.5. This would lead to a contradiction if we check that for some subsequence (ϕ_n) of (f_n), the approximately stable foliation $\mathcal{AS}(\phi_n^{-1})$ is $\mathcal{F}^2$. For this last fact, we just remark that the opposite situation, i.e. $\mathcal{AS}((f_n^{-1})) = \mathcal{F}^1$ is impossible, indeed this would imply that the complementary of a small neighborhood of $\mathcal{N}^1$ is mapped by Df_n to a small neighborhood of $\mathcal{N}^2$. This contradicts the fact that Df_n preserves $\mathcal{N}^2$.

Finally, extend arbitrarily the partial partition $B_1 \cup B_2$ to a partition $\Gamma = A_1 \cup A_2$. □

Fact 12.5. *$s(f,x)$ and $u(f,x)$ are independent of x.*

Proof. It is clear that by choosing U_1 and U_2 small enough that for $f \in A_2$ far from the identity map, $s(f,x)$ cannot be 2 for any $x \in K$. Therefore, $s(f) = 1$, and thus does not depend on x. □

Liapunov exponents. Henceforth, we will assume that the following hypothesis holds:

Hypothesis. $\mathrm{Vol}(K) > (1/2)\mathrm{Vol}(U)$ and choose r_0 such that $c_{r_0}^K > 2$.

Lemma 12.6. *Let f and g be two elements of Γ outside the ball $B^\rho(1,r_0)$ such that $s(f) = s(g)$. Then there is $x \in K$ such that $f(x)$ and $gf(x)$ belong to K and $\lambda^{s(f)}(gf,x) < 1/4$. In particular, if $gf \notin B^\rho(1,r_0)$, then $s(gf) = s(f)$ $(= s(g))$. Furthermore, if $f^n \notin B^\rho(1,r_0)$ for all $n > 0$, then $s(f^n) = s(f)$ for all $n > 0$.*

Proof. Observe that if three subsets A, B and C of U have volume $>$ $(1/2)\mathrm{Vol}(U)$, then $A \cap B \cap C \neq \emptyset$. Apply this to K, $f^{-1}K$ and $gf^{-1}K$, we get a point $x \in K$ such that $f(x) \in K$, and $gf(x) \in K$. Thus $\lambda^1(gf,x) = \lambda^1(g,f(x))\lambda^1(f,x)$, and hence $\lambda^1(gf,x) < 1/4$, if f and g satisfy the conditions of the lemma. Of course if $gf \notin B^\rho(1,r_0)$, then $s(gf) \neq 2$, and thus $s(gf) = 1$. Using this, the last part of the lemma is proved by induction. □

Consider $l^1(f,x) = \lim_{n\to+\infty}(\log \lambda^1(f^n,x))/n$. Define analogously $l^2(f,x)$. They are Liapunov exponents and thus exist almost everywhere.

Fact 12.7. *Let f be such that $f^n \notin B^\rho(1,r_0)$ for all $n > 0$. Suppose that $s(f) = 1$. Then, for almost every $x \in U$, $l^1(f,x) < 0$, that is $\mathcal{N}^1$ is the negative Liapunov space of $f|U$. Furthermore, $\int_U l^1(f,x)dx < (-\log 2)\,\mathrm{Vol}(K)$.*

Proof. To $x \in K$, associate its sequence of positive return times $(n_i(x))_{i\in\mathbf{N}}$. To simplify notation, fix x and denote the sequence by (n_i). Thus, $f^{n_i}x{\in}K$. From Lemma 12.6, $s(f^n) = 1$ for all $n > 0$, and hence $\lambda^1(f^{n_{i+1}-n_i}, f^{n_i}x) < 1/2$. Thus, by the cocycle property of λ^1, $\lambda^1(f^{n_i}, x) < (1/2)^i$, and hence, $\log \lambda^1(f^{n_i}, x)/n_i < i/n_i(-\log 2)$. Let χ_K denote the characteristic function of K. Observe that i/n_i equals the partial Birkhoff sum $(\chi_K(x)+\chi_K(fx)+ \dots \chi_K(f^{n_i}x))/n_i$. So, we have proved $l^1(f,x) < (-\log 2)\chi^*_K(x)$, where χ^*_K stands for the Birkhoff sum of χ_K. In particular $l^1(f,x) < 0$ for almost every $x \in K$. Let K^* be the saturation of K, by f, $l^1(f,x) < (-\log 2)\chi^*_K(x)$, for $x \in K^*$, because both of the two functions $l^1(f,x)$ and χ^*_K are f-invariant.

In particular: $\int_{K^*} l^1(f,x)dx(-\log 2)\int_{K^*}\chi^* = \int_U \chi^*_K = \mathrm{Vol}(K)$.

It remains to prove that $l^1(f,x) < 0$, almost everywhere in U. To this end, observe that the sequence generated by f is divergent because f has non-vanishing exponents. Hence if we replace K by a bigger compact K', then for some $g = f^p$, $p > 0$, the powers $\{g^n, n > 0\}$ lie outside the ball analogous to $B^\rho(1,r_0)$ associated to K'. Obviously, the index $s^{K'}(g)$ is the same as $s(f)$ and thus equals 1. Therefore, almost everywhere in K', $l^1(f^p,x) < 0$. But $l^1(f,x) = l^1(f^p,x)/p$. □

Consider now the map $\Lambda^1 : \Gamma \to \mathbf{R}$, $\Lambda^1(f) = \int_U \log \lambda^1(f,x)dx$.

FACT 12.8. *Λ^1 is a homomorphism and for f such that $f^n \notin B^\rho(1,r_0)$ for all $n > 0$, we have $|\Lambda^1(f)| > \log 2\,\mathrm{Vol}(K)$. Furthermore, $\Lambda^1(f) \neq 0$ if and only if f generates a non-precompact group (that is $\overline{\{f^n/n \in \mathbf{Z}\}}$ is non-compact).*

Proof. We have $\log\lambda^1(fg,x) = \log\lambda^1(f,gx) + \log\lambda^1(f,x)$. Thus Λ^1 is a homomorphism because g is volume preserving and hence: $\int_U \log\lambda^1(f,gx) = \int_U \log\lambda^1(f,x)$.

The remaining parts of the fact are obvious or follow from the preceding fact. □

COROLLARY 12.9. *Suppose that $\mathrm{Ker}\,\Lambda^1$ is compact. Then, up to a subgroup of finite index, Γ is a direct product of a compact group by $\mathbf{Z}$ or $\mathbf{R}$.*

Proof. Suppose to start with that $\mathrm{Ker}\,\Lambda^1$ is trivial, that is every element $f \in \Gamma$ generates a non-precompact group. Therefore, from Fact 12.8, Λ^1 is injective, and thus Γ is abelian and torsion free. Let G be a closed subgroup of Γ isomorphic to $\mathbf{Z}^2$. It is obvious that for a big enough radius r, $f \in G - B^\rho(1,r) \implies f^n \notin B^\rho(1,r_0)$. But, from Lemma 12.6, $|\Lambda|$ is bounded from below for elements satisfying this condition. This means that $\Lambda^1 : G \to \mathbf{R}$ is proper, which is impossible. Thus, because it cannot contain a closed copy of $\mathbf{Z}^2$, Γ must be isomorphic to $\mathbf{Z}$ or $\mathbf{R}$ (remember that it is torsion free).

Now when $\operatorname{Ker}\Lambda^1$ is merely compact, we may argue with the quotient $\Gamma/\operatorname{Ker}\Lambda^1$ which enjoys the same properties as Γ. So, we obtain that Γ is a semi-direct product of $\mathbf{Z}$ or $\mathbf{R}$ by a (normal) compact group. We may find in a standard way a subgroup of finite index which is a direct product of $\mathbf{Z}$ or $\mathbf{R}$ by a compact group. □

Torsion. We now check that $\operatorname{Ker}\Lambda^1$ is compact. Equivalently, we suppose that $\Lambda^1 = 0$ and thus show that Γ is compact. This is standard for the identity component of Γ. Indeed, a Lie group for which every element generates a precompact subgroup, is compact (see for instance [D]). Hence, without loss of generality, we may restrict ourselves to the case where Γ is discrete. It is thus a torsion group, and we have to check it is finite.

Let f be an element of order k. Consider $I_1(f) = \{i \in \{1,\dots,k-1\} \mid f^i \notin B^\rho(1,r_0), s(f)=1\}$. Define analogously $I_2(f)$.

From Lemma 12.6, we have the following "semi-group" property:

FACT 12.10. *Let $\alpha,\beta \in I_1(f)$, then $\alpha+\beta \in I_1(f)$, unless, $f^{\alpha+\beta} \in B^\rho(1,r_0)$. The same statement holds for $I_2(f)$.*

Consider $P = \{f^n \mid f \in B^\rho(1,r_0), n \in \mathbf{Z}\}$. Since $B^\rho(1,r_0)$ is finite and any element has finite order, it follows that P is finite.

We deduce from the fact above that every (finite cyclic) group intersects non-trivially the ball $B^\rho(1,r_0)$. Indeed, if not, we would obtain a partition $\{1,\dots,k-1\} = I_1(f) \cup I_2(f)$, where f is a generator of order k of the given cyclic subgroup. The fact above implies that only one part, say $I_1(f)$ is non-empty. So, we apply, Lemma 12.6 to f and f^{k-1} and we obtain that $\lambda^1(\mathrm{Identity}, x) < 1/4$ which is obviously impossible.

If f is of prime order, then, all its non- trivial powers generate the same group, and hence f is a power of some element of $B^\rho(1,r_0)$, that is $f \in P$.

To treat the general case, represent the congruence group $\mathbf{Z}/k\mathbf{Z}$ as $\{\dot{0}, \dot{1}, \dots, (k \dot{-} 1)\}$. Let $\mathcal{G} = \{\dot{\alpha}/(\alpha,k) = 1\}$ be the set of generators of $\mathbf{Z}/k\mathbf{Z}$.

Let f be an element of order k that does not belong to P, then $f^\alpha \notin B^\rho(1,r_0)$, for $\dot{\alpha} \in \mathcal{G}$. Hence, we have a partition $\mathcal{G} = (\mathcal{G} \cap I_1(f)) \cup (\mathcal{G} \cap I_2(f))$, as above.

Suppose that $k = p^m$ with p prime and $\neq 2$. Then $\dot{\alpha} \in \mathcal{G}$ if and only α is not a multiple of p. Therefore, we cannot have $\dot{\alpha}$ and $\alpha \dot{+} 1$ in $\mathcal{G}$, noting that $p \neq 2$. Also, suppose that $\dot{1} \in I_1(f)$, then $\dot{2} \in I_2(f)$. Let α be the smallest number such that $\dot{\alpha} + \dot{1} \notin I_1(f)$. Then necessarily, $\dot{\alpha} \in I_1(f)$ and $\dot{\alpha} + \dot{1} \notin \mathcal{G}$, and hence $\dot{\alpha} + \dot{2} \in I_1(f)$. By recurrence, we get $I_2(f) \cap \mathcal{G} = \emptyset$. In particular f and f^{k-1} belong to $I_1(f)$. We get a contradiction as in the above case where k was prime.

To treat the case $p = 2$, the previous combinatorial approach fails. However, this may be adapted, if we suppose a stronger " semi-group" property of $I_1(f)$ and $I_2(f)$, involving three instead of two elements. That is, given three elements $\dot{\alpha}, \dot{\beta}$ and $\dot{\gamma}$ of $I_1(f)$, then each of the elements $\dot{\alpha} + \dot{\beta}$, and $\dot{\alpha} + \dot{\beta} + \dot{\gamma}$ belongs to $I_1(f)$, unless it belongs to $B^\rho(1, r_0)$. To have this, we choose K in Lemma 12.6, with relative big volume (that is $\mathrm{Vol}(K) > 3/4\,\mathrm{Vol}(U)$), and we obtain a statement (of the lemma) involving three elements of Γ.

We have thus proved that P contains all the elements having order of the form p^m with p prime. One may push forward the combinatorial argument to prove that $P = \Gamma$. Instead, we prefer to argue as follows. Since every cyclic group is generated by groups with order of the form p^m, we deduce that every subgroup Γ' of Γ is generated by its intersection with P. In particular, Γ' is a finitely generated torsion group. Consider the adjoint action of Γ on itself. It preserves the finite set P. Its Kernel Γ' is of finite index in Γ, and centralizes Γ because it centralizes the generating set P. Therefore, Γ' is finite as it is a finitely generated abelian torsion group. It then follows that Γ is finite. □

End of the proof of Theorem 12.2. Let $g \in \Gamma$ be as in Theorem 12.2. From the previous development, for each component U of the transversality set T, we can associate $s(U) \in \{1, 2\}$, such that the normal direction of $\mathcal{F}^{s(U)}|U$ is the negative Liapunov space of the restriction $g|U$. As in §11, this allows us to construct two foliations $\mathcal{L}^1$ and $\mathcal{L}^2$, elements of $\mathrm{Apa}(\mathcal{F}^1, \mathcal{F}^2)$ whose normal directions are the negative and positive spaces of $g|T$, respectively. Now let K be a compact subset of T, then it meets only finitely many components U, and hence for a sufficiently big positive integer p, we have the estimates stated in the theorem for the power $f = g^p$. This ends the proof of the theorem.

13 Proofs of Theorems 2.4, 1.1 and 2.7

The case where the bifoliation of Γ is trivial. This means there is a foliation $\mathcal{F}^1$ such that everywhere $L_\Gamma(x) = T_x\mathcal{F}^\perp$. It then follows from

Proposition 9.5, that the approximately stable foliation of any sequence of of Γ is $\mathcal{F}^1$ and therefore the boundary of Γ consists exactly of $\mathcal{F}^1$.

To chow the vanishing of the entropy of the elements of Γ, we argue by contradiction. Suppose that some $f \in \Gamma$ has positive entropy with respect to some invariant measure. Then f must have somewhere non-trivial negative and positive Liapunov spaces E^1 and E^2. Observe then that these spaces must be isotropic (in the sense of the Lorentz metric) and hence are 1-dimensional. At most only one of these directions, say E^1, is contained in $\mathcal{F}$. However, by the uniform attraction of $\mathcal{F}$ (Proposition 9.5), the direction E^2 is mapped by powers of f, near $\mathcal{F}$, which contradicts the fact that it is preserved by f.

The case of partially hyperbolic groups. In order to estimate its boundary, it is easy to see that we may replace Γ by a subgroup $\{g^n/n \in \mathbf{Z}\}$ generated by an element g as in Theorem 12.2. We will thus prove: $\partial_\infty \Gamma = \{\mathcal{L}^1, \mathcal{L}^2\}$ (following the notation of Theorem 12.2). More precisely, we will prove that: $\lim_{n \to +\infty} g^n = \mathcal{L}^2$ and $\lim_{n \to -\infty} g^n = \mathcal{L}^1$.

Let (f_n) be a sequence of Γ of the form $f_n = g^{k_n}$, for some sequence of integers (k_n). Suppose that (f_n) has an approximately stable foliation, that is, (f_n^{-1}) converges in $\Gamma \cup \partial_\infty \Gamma$ to some foliation $\mathcal{F}$.

Suppose that $k_n \to +\infty$, when $n \to +\infty$. Then, by localization to larger and larger compact subsets, and using Theorem 12.2, one deduces that the limit of (g^{k_n}) cannot be different from $\mathcal{L}^2$. Therefore, by compactness, $g^n \to \mathcal{L}^1$, when $n \to +\infty$ (because $\mathcal{L}^2$ is the unique limit of its convergent subsequences). By the same argument $\lim_{n \to -\infty} g^n = \mathcal{L}^1$.

It then follows that any sequence $(f_n = g^{k_n})$ for (k_n) oscillating between $-\infty$ and $+\infty$ is not convergent. Hence, the boundary of Γ is $\{\mathcal{L}^1, \mathcal{L}^2\}$.

Proof of Theorems 1.1 and 2.7. Let f be an isometry of M generating a non-equicontinuous subgroup Γ. Firstly, observe that Γ is closed. Indeed, the closure of Γ is an abelian Lie group and hence, up to finite index, it can be written as a product $\overline{\Gamma} = \mathbf{T}^i \times \mathbf{R}^j \times \mathbf{Z}^k$, where $\mathbf{T}$ is the torus part. But, because $\overline{\Gamma}$ is non-compact and has $\mathbf{Z}$ as a dense subgroup, then we must have $\overline{\Gamma} = \mathbf{Z}$. Therefore, Γ is closed and amenable, and hence elementary.

We apply Theorem 2.4 and we get, with the help of the previous notation, $\mathcal{L}^1$ (resp. $\mathcal{L}^2$) as an approximately stable (resp. unstable) foliation for f.

The weak partial ergodicity part of Theorem 1.1 follows from the analogous statement in Theorem 1.2. This ends the proof of Theorem 1.1.

The same argument yields approximately stable and unstable foliations for isometric flows, as stated in Theorem 2.7. The statement concerning the causal character of non-equicontinuous flows, that is, their infinitesimal generators must be non-timelike, was noticed in [Z4].

14 Non-bipolarized Manifolds. Proof of Theorem 2.9

Observe that, over $\mathcal{U}$, TAS is Lorentzian, that is the restriction of the metric along TAS is of Lorentzian type. Indeed, for $x \in \mathcal{U}$, TAS_x contains at least two different isotropic directions.

Let K be a compact subset of $\mathcal{U}$ over which E is continuous. We will consider as in 8.2, the non-escaping set subset $NE(K,(f_n))$, where (f_n) is sequence in $\mathrm{Isom}(M)$. We firstly present the following relative version of Corollary 9.6:

FACT 14.1. *Let K be a compact subset of $\mathcal{U}$ over which E is continuous. Moreover, let $\mathcal{F} = \lim f_n \in \partial_\infty \mathrm{Isom}(M)$, and X (resp. Y) a vector field tangent to $\mathcal{F}^\perp$ (resp. $\mathcal{F}$). Then, there is a sequence of continuous sections (X_n) (resp. (Y_n)) of E over K, such that $X_n \to X$ (resp. $Y_n \to Y$) in the C^0 topology, and satisfying the following. Choose an auxiliary norm $|.|$ on TM. Then, there is a real sequence (a_n) converging to 0 (resp. a bounded real sequence (b_n)) such that $|Df_nX_n(x_n)| < a_n$ (resp. $|Df_nY_n(x_n)| < b_n$) whenever $x_n \in K \cap f_n^{-1}K$.*

FACT 14.2. *Let $\mathcal{F} \in \partial_\infty \mathrm{Isom}\, M$. Then $t(u,v) = 0$ for $u \in T_x\mathcal{F}^\perp$, $v \in T_x\mathcal{F} \cap E_x$, and almost every $x \in \mathcal{U}$.*

Proof. Let K be a compact subset over which E and t are continuous. Let $x \in NE(K,(f_n))$ and u, v as in the fact. Extend u and v to local vector fields X and Y tangent to $\mathcal{F}^\perp$ and $\mathcal{F}$, respectively. Approximate X and Y as in Corollary 9.6, and consider a sequence $x_n \to x$ in K, such that $f_nx_x \in K$. Thus, by continuity, $t(u,v) = \lim_{n\to\infty} t(X(x_n), Y(x_n))$ which equals 0, from the properties of X_n and Y_n in Corollary 9.6.

Finally, recall that (see 8.2) $\mathrm{Vol}\, NE(K,(f_n)) > \mathrm{Vol}(\mathcal{U}) - 2\epsilon$, whenever $\mathrm{Vol}(K) > \mathrm{Vol}(\mathcal{U})$. Therefore, the property holds is almost everywhere because we can choose the volume of K arbitrarily approaching that of $\mathcal{U}$. □

It then follows that $t(u,v) = 0$ if $u \in S_x$ and $v \in u^\perp \cap E_x$, for almost every $x \in \mathcal{U}$.

Let $\{e_i\}$ be a basis of T_xM, and write $t(X,Y) = \Sigma b_i(X,Y)e_i$ where the b_i are bilinear scalar forms. To prove Theorem 2.9, it suffices to show that for each b_i there is α_i such that $b_i(u,v) = \alpha_i\langle u,v\rangle$, whenever $u \in TAS_x$.

So let b be one of this forms and write it $b(u,v) = \langle Au, v\rangle$ for some linear endomorphism of E_x. From the above, one sees that every $u \in S_x$, is an eigenvector for A, $Au = \lambda_u u$. Now, if λ_u does not depend on u, A is a homothety on TAS_x and we are done. If not, A induces an eigenspace decomposition on E_x.

By considering all the endomorphisms corresponding to the b_i, and by letting x varying over $\mathcal{U}$, we get an invariant measurable decomposition $E = E^1 \oplus \cdots \oplus E^k$ of TAS such that $S_x \subset E^1_x \cup \cdots \cup E^k_x$. To show that this is impossible (and hence finishing the proof of the theorem) we use the following irreducibility fact:

FACT 14.3. *There are no* Isom(M)*-invariant measurable subbundles* E^1, ...,E^k *of* TAS, *with* $k \geq 2$, *such that* $S_x \subset E^1_x \cup ... \cup E^k_x$ *in some subset of* $\mathcal{U}$ *with positive volume* (*it is here where we use that* Isom(M) *is non-elementary*).

Proof. We may suppose $k = 2$ and that E^1 is Lorentzian in some subset of positive volume. Since we are dealing with measurable bundles, we may suppose this is everywhere true, just restricting domains of definition. So, we have an orthogonal decomposition $TM = E^1 \oplus {E^1}^{\perp}$.

Let $\mathcal{F} = \lim f_n \in \partial_\infty \operatorname{Isom}(M)$ and apply Fact 14.1, to a compact K over which E^1 and E^2 are continuous. Observe that if $u_n \in {E^1}^{\perp}_{x_n} \to u \neq 0$, and $f_n x_n \in K$, then $(Df_n u_n)$ cannot tend to 0 because the metric on ${E^1}^{\perp}$ is Riemannian (i.e. positive definite). This implies as in the proof of the fact above, that, over $NE(K,(f_n))$, $\mathcal{F}^{\perp}$ is contained in E^1. Choosing K with larger volume as necessary, we may conclude that this is always true. □

To finish the proof of Theorem 2.9, it remains to check its last partial ergodicity statement. Let σ be a function as in the theorem. From the ergodicity property in Theorem 1.2, σ is constant along any (1-dimensional) foliation defined by a vector field $X \in S$ (see the notation of §2.4). Therefore, by definition $d\sigma | TAS = 0$.

15 Examples

Here we will give some examples of boundaries of isometry groups of compact Lorentz manifolds. As we will see in the part II of this work, the non-trivial (i.e. non-bipolarized) cases, involve constant curvature manifolds. In what follows, we therefore investigate the structure of isometry groups of manifolds which are locally isometric to a product of a Riemannian manifold by a constant curvature Lorentz manifold. The following theorem summarizes the non-trivial cases.

Theorem 15.1. *Let M be a compact Lorentz manifold whose universal cover is a product of a simply connected Riemannian manifold $\tilde{N}$ and a complete simply connected space X_c of constant curvature c. Let $\pi \subset \operatorname{Isom} \tilde{N} \times \operatorname{Isom}(X_c)$. Then,*

(i) *Such a manifold does not exist if $c > 0$.*

(ii) *If $c > 0$ and M is not bipolarized, then $\dim M = 3$, and M is a quotient of $\tilde{N} \times \widetilde{SL(2,\mathbf{R})}$ by a subgroup of $\operatorname{Isom} \tilde{N} \times \widetilde{SL(2,\mathbf{R})}$. The isometry group of M is a product of a compact group by a finite cover of $PSL(2,\mathbf{R})$. Its boundary is the circle S^1 endowed with the usual action of this latter group.*

(iii) *If $c = 0$ and M is not bipolarized, then there is a metric decomposition $\tilde{M} = \tilde{N}' \times \mathbf{R}^d$, where $\tilde{N}'$ is a Riemannian manifold, and $\mathbf{R}^d$ is a Minkowski space, and such that $\pi = \pi_1(M) \subset \operatorname{Isom} \tilde{N}' \times \mathbf{R}^d$. The action of $\operatorname{Isom}(M)$ on its boundary factors through the action of a (arithmetic) lattice of a $(d-1)$-dimensional hyperbolic space on its sphere at infinity.*

Remark 15.2 (Erratum). The case (iii), i.e. the flat non-bipolarized case, was forgotten in [Z6]. So, the statement of the principal result in this reference has to be modified to take into account this case, see part II of the present article.

Observe that that we do not address completeness questions, which will be treated in detail in part II of this article. The theorem will be proved through the present section, which contains further details. Let us start with the following general fact about de Rham decomposition of Lorentz manifolds, which follows from the foliated version 8.7.

Proposition 15.3. *Let M be a compact Lorentz manifold whose universal cover admits a de Rham decomposition $\tilde{M} = \tilde{N} \times X$, where $\tilde{N}$ is Riemannian and X is Lorentzian. Then any foliation $\mathcal{F}$, element of the boundary of $\operatorname{Isom}(M)$ lifts to a foliation $\tilde{\mathcal{F}}$ containing the factor $\tilde{N}$, that is its leaves have the form $\tilde{N} \times \tilde{\mathcal{L}}_x$, where $\tilde{\mathcal{L}}$ is a codimension one lightlike geodesic foliation of X, invariant under the action of the projection of $\pi_1(M)$ in $\operatorname{Isom}(X)$.*

15.1 De Sitter space. Let $\mathbf{R}^{p,q}$ be the space $\mathbf{R}^{p+q}$ endowed with a non-degenerate quadratic form of signature $(-p, q)$ (for example $\mathbf{R}^{1,q}$ is endowed with a form of signature $- + \cdots +$). For a real r, we denote by $\mathbf{S}^{p,q}(r)$ the level r in $\mathbf{R}^{p,q}$. The (universal) de Sitter space of dimension q is $\mathbf{S}^{1,q}(+1)$. We only consider the case $q > 2$, and hence, the de Sitter space is simply connected. It inherits from $\mathbf{R}^{1,q}$ a Lorentz metric of positive constant curvature, and has isometry group $O(1,q)$.

The well-known Calabi–Markus phenomena states that a Lorentz manifold covered by $\mathbf{S}^{1,q}(+1)$ has a finite fundamental group. Let $\tilde{N}$ be a complete simply connected manifold and consider the product $\tilde{M}=\tilde{N}\times\mathbf{S}^{1,q}(+1)$. This may have quotients with large fundamental group, but no compact ones. The proof of this claim resembles that of the Calabi–Markus phenomena and goes as follows. Suppose that $\pi \subset \mathrm{Isom}\,\tilde{N} \times O(1,q)$ is the fundamental group of a compact manifold M. Then, the projection of π on $\mathrm{Isom}\,\tilde{N}$ is not discrete, otherwise, M would fiber over a quotient of $\tilde{N}$, with fiber a compact quotient of $\mathbf{S}^{1,q}(+1)$, which does not exist. Hence, there is a divergent sequence $\gamma_n = (g_n, h_n) \in \pi$ such that $g_n \to 1$ in $\mathrm{Isom}\,\tilde{N}$. The basic fact behind the Calabi–Markus phenomena is that if $K = \mathbf{S}^{1,q}(+1) \cap \mathbf{R}^{0,q}$, then $h(K) \cap K \neq \emptyset$ for any $h \in O(1,q)$ (because both K and $h(K)$ are traces in $\mathbf{S}^{1,q}(+1)$ of linear hyperplanes in $\mathbf{R}^{1+q}$). Therefore, for any open set $U \subset \tilde{N}$, $\gamma_n(U \times K) \cap (U \times K) \neq \emptyset$, because $g_n \to 1$. This means that π does not act properly on $\tilde{N} \times \mathbf{S}^{1,q}(+1)$.

15.2 Anti-de Sitter manifolds. The anti-de Sitter space of dimension $1+q$ corresponds to the level -1 in $\mathbf{R}^{2,q}$. More precisely, the (universal) anti-de Sitter space $\mathbf{H}^{1,q}$ is the universal cover of $\mathbf{S}^{2,q}(-1)$. However, it is more convenient to work in the "linear model" $\mathbf{S}^{2,q}(-1)$ and then, translate into $\mathbf{H}^{1,q}$. In fact one would hope that, one needs to pass to $\mathbf{H}^{1,q}$ only in pathological situations. Indeed, as we will recall below compact quotients of $\mathbf{H}^{1,q}$ are in fact quotients of $\mathbf{S}^{2,q}(-1)$. So let us work in this latter space. Let $Q = -x_1^2 - x_2^2 + x_3^2 + \dots x_{q+2}^2$ be a quadratic form defining $\mathbf{R}^{2,q}$. Then $\mathbf{S}^{2,q}(-1)$ inherits a Lorentz metric of negative constant curvature and its isometry group is the orthogonal group of Q, that is $O(2,q)$. The totally geodesic subspaces of $\mathbf{S}^{2,q}(-1)$ are exactly the traces on it of linear subspaces of $\mathbf{R}^{2+q}$. Furthermore, the lightlike geodesic hypersurfaces of $\mathbf{S}^{2,q}(-1)$ have the form: $H_u = \mathbf{R}u^\perp \cap \mathbf{S}^{2,q}(-1)$ for $u \in \mathbf{R}^{2,q}$ an isotropic vector.

Two hypersurfaces H_u and H_v (for u and v isotropic) are disjoint if and only if u and v are orthogonal but not collinear. It follows that a codimension one lightlike geodesic foliation $\mathcal{F}$ of $\mathbf{S}^{2,q}(-1)$ is determined by hypersurfaces H_u, for u running over an isotropic 2-plane P. The group $O(2,q)$ acts transitively on the space of isotropic 2-planes. Its action on the space of pairs (P_1, P_2) of isotropic 2-planes has exactly 3 orbits, one for $P_1 = P_2$, one for $\dim(P_1 \cap P_2) = 1$, and the last for $P_1 \cap P_2 = 0$.

We can write $\mathbf{R}^{2,q} = \mathbf{R}^{2,2} \oplus \mathbf{R}^{0,q}$. Because the factor $\mathbf{R}^{0,2}$ is Riemannian, most of the dynamics happens on $\mathbf{R}^{2,2}$. Note that the decomposition above is not canonical, but we will see this does not matter. So, we now decorticate the case $q = 2$, i.e. the 3-dimensional anti-de Sitter space. Instead of the standard form Q, one consider the form Q' on $\mathbf{R}^4 = \mathbf{R}^2 \times \mathbf{R}^2$ defined by $Q'(u, v) = \omega(u, v)$, where ω is the volume form on $\mathbf{R}^2$. An element $A \in SL(2, \mathbf{R})$ acts diagonally: $A(u, v) = (Au, Av)$, by preserving Q'. The group $SL(2, \mathbf{R})$ acts freely on the non-vanishing levels of Q'. Hence, the $SL(2, \mathbf{R})$ orbits coincide (for dimension reasons) with the components of non-vanishing levels. It turns out that metrics defined in that way on $SL(2, \mathbf{R})$ are multiples of its Killing form. Indeed, one verifies that the identity component of $O(2, 2)$ contains another copy of $SL(2, \mathbf{R})$ commuting with the given $SL(2, \mathbf{R})$-action. More precisely, this identity component is isomorphic to $SL(2, \mathbf{R}) \times Sl(2, \mathbf{R})$, which turns out to be the identity component of the isometry group of the Killing form of $SL(2, \mathbf{R})$, acting by $(g, h)x = gxh^{-1}$, where g, h, and x belong to $SL(2, \mathbf{R})$.

A lightlike geodesic foliation is determined by the orbits of a subgroup of the form $A \times \{1\}$ or $\{1\} \times A$ where A is conjugate to the affine group $AG \subset SL(2, \mathbf{R})$. The case of two different subgroups lying in the same factor $\{1\} \times SL(2, \mathbf{R})$ or $SL(2, \mathbf{R}) \times \{1\}$ corresponds to two isotropic 2-planes such that $P_1 \cap P_2 = 0$. The case of different factors corresponds to two intersecting 2-planes. It then follows that the symmetry group of a pair of isotropic 2-planes (P_1, P_2) is conjugate up to switch of factors to either $SL(2, \mathbf{R}) \times \{1\}$ or $AG \times AG$.

For example, the group $SL(2, \mathbf{R})$ acting as above on $(\mathbf{R}^4, Q')$ preserves each isotropic 2-plane of the form $\{(u, \alpha u), u \in \mathbf{R}^2\}$ for α a real number, together with the plane $\{0\} \times \mathbf{R}^2$. The set of the so defined foliations is a circle, on which the other factor $SL(2, \mathbf{R})$ acts as usually.

From this we deduce that the symmetry group of 3 distinguish isotropic 2-planes is contained up to switch of factors in $SL(2, \mathbf{R}) \times \{1\}$. In particular, this symmetry group is centralized by the other factor $\{1\} \times SL(2, \mathbf{R})$. Therefore, if a compact quotient M of a product $\tilde{N} \times \mathbf{S}^{2,q}(-1)$, is not bipolarized and has non-compact isometry group, then $\mathrm{Isom}(M)$ contains $SL(2, \mathbf{R})$. From [Gr] (see also [ASt1] and [Z1]), $\mathrm{Isom}(M)$ is a product of a compact group by a finite cover of $PSL(2, \mathbf{R})$.

In higher dimensions the symmetry group of 3 isotropic 2-planes is contained in a product of $SL(2, \mathbf{R})$ by a compact group. This does not act cocompactly on $\mathbf{S}^{2,q}(-1)$. By translating the argumentation to the universal cover $\mathbf{H}^{1,q}$, one checks that a cocompact subgroup cannot preserve 3 distinct codimension one lightlike geodesic foliations. This proves the claim that if a compact Lorentz manifold has a non-compact isometry group and is not bipolarized, then the anti-de Sitter factor of its universal cover has dimension 3.

Finiteness of levels. As mentioned above, a compact Lorentz manifold M, which is a quotient of the universal anti-de Sitter space, is in fact up to finite covers, a quotient of the more concrete one $\mathbf{S}^{2,q}(-1)$. This property was called in [KuR] the finiteness of levels of compact anti-de Sitter manifolds. It is related to isometry groups as follows. The fundamental group of $\mathbf{S}^{2,q}(-1)$ is cyclic, generated by an element σ. The statement is that some power of σ belongs to $\pi_1(M)$. Because σ is central in $\mathrm{Isom}(\mathbf{H}^{1,q})$, it defines an isometry f of M. The finiteness of the level of M is equivalent to f having finite order. Thus the opposite situation implies $\mathrm{Isom}(M)$ is infinite. If the identity component of $\mathrm{Isom}(M)$ is not trivial, we get a connected subgroup of $\mathrm{Isom}(\mathbf{H}^{1,q})$ centralizing $\pi_1(M)$. One can thus completely understand this latter group, and in particular get a contradiction to the hypothesis that f is of infinite order. If $\mathrm{Isom}(M)$ is discrete, then it is non-compact, and therefore, it preserves a foliation, and thus $\pi_1(M)$ is contained in the symmetry group of a foliation as described above. Again here, by working algebraically, one gets a contradiction to the fact that f has an infinite order.

15.3 The flat case. Let g be a Lorentz quadratic form on $\mathbf{R}^k$. It is called rational, if some multiple αg has rational coefficients when expressed in the canonical basis. The isometry group of the Lorentz space $(\mathbf{R}^k, g)$ is the semi-direct product $\mathbf{R}^k \rtimes O(g)$ where $O(g) \subset GL(n, \mathbf{R})$ is the orthogonal group of g. Of course all these spaces are isometric to the Minkowski space. However, this representation may help to understand the modulus of flat Lorentz structures. For instance, consider a topological torus $\mathcal{T}^k = \mathbf{R}^k/\mathbf{Z}^k$, then the family $(\mathbf{T}^k, g)$, for g as above, exhausts the space of Lorentz flat structures on $\mathbf{T}^k$.

A lightlike geodesic foliation in $\mathbf{R}^k$ consists obviously of parallel lightlike affine hyperplanes, and hence it is determined by specifying an isotropic direction of g. Let $C(g)$ be the space of such directions. It is a sphere of dimension $k-2$. The action of $\mathbf{R}^k \rtimes O(g)$ on it, factors via the usual conformal action of $O(g)$ (i.e. it is isomorphic to the conformal action of $O(1, k-1)$ on S^{k-2}).

Let M be a flat Lorentz manifold obtained as a quotient of $(\mathbf{R}^k, g)$ by a subgroup $\pi \subset \mathbf{R}^k \rtimes O(g)$. The space $\mathcal{FG}(M)$ of codimension 1 lightlike geodesic foliations of M, is identified with the fixed points of the action of π (via its linear part) on $C(g)$. It follows in particular, that in the torus case, this space is identified with $C(g)$, for any g. Conversely, if $\mathrm{card}(\mathcal{FG}(M)) \geq 3$, then up to a subgroup of finite index, π is a lattice in the translation part $\mathbf{R}^k$ of $\mathbf{R}^k \rtimes O(g)$. To see this, let E be the linear space generated by the isotopic directions determined by the elements of $\mathcal{FG}(M)$. Because of the dimension, the quadratic form g is of Lorentz type on E and is positive on $E^\perp$.

Observe now that if an element $A \in O(g)$ fixes 3 isotropic directions X_1, X_2 and X_3, then $A = \pm$ Identity on the linear space F that they generate. Indeed, write $AX_i = \lambda_i X_i$, then $\lambda_i \lambda_j = 1$ for $i \neq j$, because $\langle X_i, X_j \rangle \neq 0$, and hence $\lambda_i = \pm 1$.

It then follows that Γ preserves a positive scalar product. Therefore, by the Bieberbach theorem, π is virtually a lattice in $\mathbf{R}^k$. Equivalently M is covered by a torus.

As a corollary, we get that a compact flat manifold is bipolarized, unless it is, up to a finite cover, a torus. Now for a torus $(\mathbf{T}^k, g)$, its isometry group is $O(g, \mathbf{Z}) \rtimes \mathbf{R}^k$, where $O(g, \mathbf{Z}) = O(g) \cap GL(k, \mathbf{Z})$.

The action of $\mathrm{Isom}(\mathbf{T}^k, g)$ on its boundary can be identified with that of $O(g, \mathbf{Z})$ on its limit set in $C(g)$. So, we are not leaving Kleinian groups, but what kind of group could $O(g, \mathbf{Z})$ be and what might it have as a limit set?

A classical theorem of Harish-Chandra and Borel states that $O(g, \mathbf{Z})$ is a lattice in $O(g)$ if g is rational. Hence its limit set in this case is the whole of $C(g)$. Let us now treat the general case (that is when g may be partially rational).

Fact 15.4. *Let G be a non-compact connected Lie subgroup of $O(g)$. Then either G fixes a point of $C(g)$ (that is G is elementary) or G is reductive. In this last case, there is a Lorentzian plane E in $\mathbf{R}^k$ (that is g restricted to E is of Lorentzian type) which is preserved by G and such that $G = O(g|E) \times K$, where K is a compact subgroup of $O(g|E^\perp)$. (Here for a non-degenerate plane F, we denote $O(g|F) = \{A \in O(g)/A(F) = F$ and $A|F^\perp = $ Identity$\}$).*

Proof. The proof is standard and briefly the idea is as follows. The radical R of G fixes a point of $C(g)$. If it is non compact, R contains a parabolic or hyperbolic one parameter group. It then follows that R fixes at most two points, and in particular, G also fixes these points, and is thus elementary.

Therefore, if G is not elementary, then its radical is compact. Hence, it is by definition reductive. The remaining part of the Fact is standard. $\square$

Let G be the Zariski closure of $O(g, \mathbf{Z})$. It is defined over $\mathbf{Q}$ (see for instance [GrP]). Therefore, if non-elementary, $O(g, \mathbf{Z})$ is a lattice in G, by the Harish-Chandra–Borel theorem, for general reductive $\mathbf{Q}$-groups. Thus $O(g, \mathbf{Z})$ may be thought of as a lattice of $O(g|E)$, and so its limit set is the projective isotopic cone of $g|E$.

Partial rationality. In fact, it seems that we have more precise partial rationality when $O(g, \mathbf{Z})$ is non-elementary, that is E and $E^{\perp}$ seem to be rational. We were able to check it, assuming $O(g, \mathbf{Z})$ non-cocompact in G. Indeed in this case, $O(g, \mathbf{Z})$ possesses parabolic elements, that is there exist an element of $O(g, \mathbf{Z})$ of the form $A = (B, C) \in O(g|E) \times K$ such that B is unipotent, i.e. $B - 1$ is nilpotent. In fact unipotent elements of orthogonal groups of Lorentz forms, have degree of nilpotency 3. Let $\mathcal{A}$ be the set of $A \in O(g, \mathbf{Z})$ such that B is unipotent.

Then E is contained in the vector subspace $F = \bigcap_{A \in \mathcal{A}} \ker(A - 1)^3$, which is rational.

Observe that if $A \in \mathcal{A}$, then its projection C in K, acts trivially on $F' = F \cap E^{\perp}$, because there it detemines an unipotent element in a compact group. In other words, $F^{\perp} \subset \bigcap_{A \in \mathcal{A}}(\ker(A - 1))$. In fact, we have equality. Otherwise, the intersection $E \cap \left(\bigcap_{A \in \mathcal{A}} \ker(A - 1)\right)$ would give a proper subspace of E invariant by $O(g, \mathbf{Z})$; which is easily seen to be impossible. It then follows that F' is rational. By similar arguments, one sees that $E = \bigoplus_{A \in \mathcal{A}}(A - 1)(F)$, and is hence rational. Again, consider $F'' = \bigoplus_{A \in \mathbf{A}} \mathrm{Image}(A - 1)^3$. This is a rational space, which is (by a similar argument) a supplementary of F' in $E^{\perp}$. Hence $E^{\perp}$ is rational.

In conclusion, at least when $O(g, \mathbf{Z})$ is non-elementary and non-cocompact, then up to a finite cover, M is a metric product of a Lorentz rational torus with a Riemannian torus.

The product case. Suppose now that M is a quotient of a product $\tilde{N} \times \mathbf{R}^k$, where $\tilde{N}$ is Riemannian and $\mathbf{R}^k$ is a Minkowski space. As above, assuming M is not bipolarized, we construct another invariant metric decomposition $\tilde{M} = \tilde{N}' \times \mathbf{R}^d$, where the projection of $\pi_1(M)$ on the isometry group of the Minkowski space $\mathbf{R}^d$ consists of translations, only. Here $\mathbf{R}^d$ corresponds to the space E defined above and $\tilde{N}' = \tilde{N} \times E^{\perp}$. Now we can write $\mathrm{Isom}(M) = \mathrm{Nor}(\pi_1(M))/\pi_1(M)$ where $\mathrm{Nor}(\pi_1(M))$ is the normalizer of $\pi_1(M)$ in $\mathrm{Isom}(\tilde{N}') \times \mathrm{Isom}(\mathbf{R}^d)$. Nevertheless, it is not obvious how to exploit further such a formula in an algebraic way, and so we argue as follows. Observe that the translations along $\mathbf{R}^d$ centralize $\pi_1(M)$ and thus determine an isometric action of $\mathbf{R}^d$ on M. This action is contained in a compact group because of its obvious equicontinuity. Therefore, we get a torus $\mathbf{T}^k$ in $\mathrm{Isom}(M)$. This torus inherits a metric defined on its Lie algebra $\mathcal{T}^k$, by means of the formula: $\langle X, Y\rangle = \int \langle X(x), Y(x)\rangle$, where X and Y are Killing fields generating flows in $\mathbf{T}^k$. Parallel fields tangent to $\mathbf{R}^d$, allow us to embed isometrically the Minkowski space $\mathbf{R}^d$ in $(\mathcal{T}^k, \langle .,. \rangle)$. In fact, as $\tilde{M}$ itself, $\mathcal{T}^k$ admits an orthogonal decomposition $A \oplus \mathbf{R}^d$, such that the scalar product $\langle .,. \rangle$ on A is positive definite, and hence the metric on $\mathbf{T}^k$ is Lorentzian. This construction is natural in all its steps, and therefore we have succeeded to essentially incorporate questions about the isometry group of M into ones concerning $\mathbf{T}^k$. In particular one sees that the action of $\mathrm{Isom}(M)$ on its boundary factors through the action of an arithmetic lattice of the hyperbolic space $\mathbf{H}^{d-1}$ on its boundary $\mathbf{S}^{d-2}$.

References

[ASt1] S. Adams, G. Stuck, The isometry group of a compact Lorentz manifold, I, Invent. Math. 129 (1997), 239–261.

[ASt1] S. Adams, G. Stuck, The isometry group of a compact Lorentz manifold, II, Invent. Math. 129 (1997), 263–287.

[BCoG] G. Besson, G. Courtois, S. Gallot, Entropies et rigidités des espaces localement symétriques de courbure strictement négative, GAFA 5:5 (1995), 731–799.

[CM] E. Calabi, L. Markus, Relativistic space forms, Ann. of Math. 75 (1962), 63–76.

[Ca] Y. Carrière, Autour de la conjecture de L. Markus sur les variétès affines, Invent. Math. 95 (1989), 615–628.

[D] G. D'Ambra, Isometry groups of Lorentz manifolds, Invent. Math. 92 (1988), 555–565.

[DGr] G. D'Ambra, M. Gromov, Lectures on transformation groups: geometry and dynamics, Surveys in Differential Geometry (Supplement to the Journal of Differential Geometry) 1 (1991), 19–111.

[Gr] M. Gromov, Rigid transformation groups, in "Géometrie différentielle" (D. Bernard, Choquet-Bruhat, ed.), Travaux encours 33, Hermann, Paris (1988).

[GrP] M. Gromov, I. Piatetski-Shapiro, Non-arithmetic groups in Lobatchevsky spaces, Publ. Math. IHES 66 (1988), 93–103.

[K] S. Kobayashi, Transformation Groups in Differential Geometry, Springer-Verlag, 1972.

[Ko] N. Kowalsky, Noncompact simple automorphism groups of Lorentz manifolds, Ann. Math. 144 (1997), 611–640.

[KuR] R. Kulkarni, F. Raymond, 3-dimensional Lorentz space-forms and Seifert fiber spaces, J. Diff. Geom. 21 (1985), 231–268.

[S] B. Solomon, On foliations of $\mathbf{R}^{n+1}$ by minimal hypersurfaces, Comment. Math. Helvetici 61 (1986), 67–83.

[Su] D. Sullivan, On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions, in "Riemannian Surfaces and Related Topics", Proceedings of the 1978 Stony brook Conference, Ann. of Math. Studies 97 (1981).

[T] W. Thurston, Geometry and Topology of 3-manifolds, Princeton University, 1978.

[Tu] P. Tukia, Homeomorphic conjugates of Fuschian groups, J. reine. angew. Math. 391 (1988), 1–54.

[Z1] A. Zeghib, Sur les espaces-temps homogènes, Geometry and Topology Monographs 1 (1998), 531–556.

[Z2] A. Zeghib, The identity component of the isometry group of a compact Lorentz manifold, Duke Math. J. 92 (1998), 321–333.

[Z3] A. Zeghib, Geodesic foliations in Lorentz 3-manifolds, Comm. Math. Helv. 74:1 (1999), 1–21.

[Z4] A. Zeghib, Killing fields in compact Lorentz 3-manifolds, J. Diff. Geom. 43 (1996), 859–894.

[Z5] A. Zeghib, Sur les actions affines des groupes discrets, Ann. Inst. Fourier 47 (1997), 641–685.

[Z6] A. Zeghib, Sur le bord du groupe d'isométries d'une variété lorentzienne compacte, C.R. Acad. Sci., Série I, 322 (1996), 149–154.

[Zi] R. Zimmer, On the automorphism group of a compact Lorentz manifold and other geometric manifolds, Invent. Math. 83 (1986), 411–426.

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