Ergod. Th. & Dynam. Sys. (2005), **25**, 503–516
doi:10.1017/S0143385704000793

Dynamics on the space of harmonic functions and the foliated Liouville problem

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(*Received* 11 *October* 2002 *and accepted in revised form* 5 *June* 2003)

Abstract. We study here the action of subgroups of $PSL(2, \mathbb{R})$ on the space of harmonic functions on the unit disc bounded by a common constant, as well as the relationship this action has with the foliated Liouville problem. Given a foliation of a compact manifold by Riemannian leaves and a leafwise harmonic continuous function on the manifold, is the function leafwise constant? We give a number of positive results and also show a general class of examples for which the Liouville property does not hold. The connection between the Liouville property and the dynamics on the space of harmonic functions as well as general properties of this dynamical system are explored. It is shown among other properties that the $\mathbb{Z}$-action generated by hyperbolic or parabolic elements of $PSL(2, \mathbb{R})$ is chaotic.

1. *Introduction*

Let $\text{Har}(\mathbb{D})$ denote the space of complex-valued harmonic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ bounded by a common constant, which will be taken without loss of generality to be 1. Endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$, $\text{Har}(\mathbb{D})$ is a compact metrizable space. The group $G = PSL(2, \mathbb{R})$ of hyperbolic isometries of $\mathbb{D}$ acts on $\text{Har}(\mathbb{D})$ by composition: $(g, f) \mapsto f \circ g^{-1}$, where $f \in \text{Har}(\mathbb{D})$ and $g \in G$. A central concern of the present paper is the dynamics of the action on $\text{Har}(\mathbb{D})$ of G and its subgroups, in particular, of lattices in G.

We are led to study the dynamics of lattices in G on $\text{Har}(\mathbb{D})$ by what will be called here the *foliated Liouville problem*. The general setting for this problem is a foliated space $(M, \mathcal{F})$ (as defined in [**6**]), where M is a compact topological space and the leaves of $\mathcal{F}$ are smooth Riemannian manifolds. The Riemannian metric and all its derivatives are assumed to vary continuously on M.

The foliated space $(M, \mathcal{F})$ will be said to have the *Liouville property* if continuous leafwise harmonic functions on M are leafwise constant. More precisely, let Δ denote the tangential Laplace operator associated to the Riemannian metric on leaves of $\mathcal{F}$. A continuous function f which is smooth along leaves and satisfies $\Delta f = 0$ will be called *leafwise harmonic*. Then, the Liouville property holds if for any such function its restriction to each leaf of $\mathcal{F}$ is constant.

If the leaves of $\mathcal{F}$ are complex rather than Riemannian manifolds, the related problem of deciding when leafwise *holomorphic* continuous functions on M are leafwise constant is studied in our paper [**8**]. Clearly, these two problems are related. For example, if the leaves are Kähler manifolds (as in the main class of examples of foliated bundles over Riemann surfaces), the real and imaginary parts of holomorphic functions are harmonic, so that some of the results obtained in our other paper [**8**] have immediate implications to the present setting. On the other hand, some tools used in that paper are unique to holomorphic functions (such as the open mapping principle), which makes the harmonic case more difficult and more interesting. Since both the harmonic and holomorphic versions of the problem will arise in the course of the present paper, it will be convenient at times to use the alternative terminology: a foliated space with the Liouville property will also be called *harmonically simple*, while its holomorphic counterpart will be called *holomorphically simple*. (The reader should note that the latter is called in [**8**] *holomorphically plain*.)

This property obviously holds for the trivial foliation, consisting of a single leaf (M itself). Less trivial, but for our concerns equally uninteresting, examples are provided by foliations whose leaves are in a certain sense parabolic, so that Liouville theorems extensively studied in geometry and complex analysis can be applied to each leaf separately.

When the foliation has leaves that individually admit non-constant bounded harmonic functions, dynamical considerations must come into play in order to decide whether or not the Liouville property holds. To give a rather simple illustration of this point, note that a minimal foliation (that is, one that does not contain proper closed saturated sets) has the Liouville property regardless of the geometry of the leaves, as an application of the maximum principle for harmonic functions immediately shows. More interesting related results will be offered later. On the other hand, simply having complicated transversal dynamics is not by itself an obstacle to the existence of non-constant, leafwise harmonic continuous functions on M, as will be seen later with an example of a codimension-two ergodic foliation for which the Liouville property fails.

In this paper we describe results and examples connecting the Liouville property with the foliation's transverse dynamics. We will consider in some detail foliated bundles over compact Riemann surfaces of genus 2 or greater. This is a particularly well-suited class of examples for this study since typical leaves often admit non-constant bounded harmonic functions, and such examples can be constructed so as to possess a wide range of dynamical properties.

It will be shown that constructing foliations without the Liouville property, in the class of foliated bundles over compact Riemann surfaces, naturally leads to the study of the dynamics of uniform lattices in G on $\mathrm{Har}(\mathbb{D})$. Among other properties, it will be seen that this action is topologically transitive, and that the $\mathbb{Z}$-action generated by any hyperbolic element in the lattice is chaotic in the sense of [**7**].

The plan of the paper is as follows. After setting some notation, we describe a number of results, topological and measure-theoretic, proving the Liouville property for certain classes of foliations. The topological results are harmonic counterparts to results proven in [**8**] in the holomorphic setting, and most proofs, although not all, are similar to those in our other paper. The measure-theoretic results are mainly aimed at connecting our problem with L. Garnett's theory of harmonic measures for foliations.

Next, we describe a class of examples of harmonically non-simple foliations. These are foliated bundles over a compact Riemann surface, of codimension two, and ergodic. It is also shown how foliated bundles over a compact Riemann surface for which the Liouville property does not hold (and non-trivial leafwise harmonic functions) can be obtained from a universal foliated bundle construction.

We then compare the harmonic problem with the corresponding holomorphic problem studied in [**8**]. It is remarked that each continuous leafwise harmonic function defines a cohomology class in the first tangential de Rham cohomology space of a leafwise Kähler foliation, and that this class vanishes if and only if the function is the real part of a leafwise holomorphic function. The example of a harmonically non-simple foliation referred to above has, in particular, non-trivial first cohomology, as it is shown that the example is holomorphically simple.

Finally, we prove a number of results about the dynamics of actions of subgroups of $PSL(2, \mathbb{R})$ on $\mathrm{Har}(\mathbb{D})$. One of the main results is that the $\mathbb{Z}$-action of hyperbolic or parabolic elements in $PSL(2, \mathbb{R})$ defines chaotic dynamical systems in the sense of [**7**].

2. *Notations and general facts*

We first set some notation. The unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ can be viewed alternatively as a Riemannian manifold with the metric of constant curvature -1, or as a complex manifold. In the first case, the set of all complex-valued harmonic functions f on $\mathbb{D}$ such that $f(\mathbb{D}) \subset \overline{\mathbb{D}}$ will be written $\mathrm{Har}(\mathbb{D})$, while $\mathrm{Hol}(\mathbb{D})$ will denote the set of all holomorphic functions from $\mathbb{D}$ into $\overline{\mathbb{D}}$. Notice that $\mathrm{Har}(\mathbb{D}) \subset \mathrm{Hol}(\mathbb{D})$. On the other hand, a holomorphic map whose real part lies in $\mathrm{Har}(\mathbb{D})$ may fail to be bounded.

The subset of $\mathrm{Har}(\mathbb{D})$ (respectively $\mathrm{Hol}(\mathbb{D})$) of non-constant functions will be denoted by $\mathrm{Har}(\mathbb{D})^*$ (respectively $\mathrm{Hol}(\mathbb{D})^*$).

Endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$, both $\mathrm{Har}(\mathbb{D})$ and $\mathrm{Hol}(\mathbb{D})$ are compact metrizable spaces. Recall that $\mathrm{Har}(\mathbb{D})$ can be identified, by means of the Poisson representation formula, with the unit ball of $L^\infty(S^1)$ endowed with the weak*-topology, where $S^1 = \partial\mathbb{D}$. Thus, denoting the Poisson kernel by $K(z, \theta) = (1 - |z|^2)/(|e^{2\pi i\theta} - z|^2)$, convergence in $\mathrm{Har}(\mathbb{D})$

$$\varphi_n(\cdot) = \int_0^1 K(\cdot, \theta) f_n(\theta)\, d\theta \to \varphi(\cdot) = \int_0^1 K(\cdot, \theta) f(\theta)\, d\theta$$

holds if for every $g \in L^1(S^1)$ (equivalently, for each g in a dense subset of $L^1(S^1)$ of continuous functions), one has

$$\int_0^1 (f_n(\theta) - f(\theta)) g(\theta)\, d\theta \to 0$$

as $n \to \infty$.

It is well known that $PSL(2, \mathbb{R})$, acting on $\mathbb{H} = \{z = x + iy : y > 0\}$ by partial linear transformations, $gz = (az+b)/(cz+d)$, is the group of isometries of the hyperbolic plane.

The conformal isomorphism $\phi : \mathbb{D} \to \mathbb{H}$ given by $\phi(z) = i(1-z)/(1+z)$, conjugates the action on the upper half-plane to an action of $PSL(2, \mathbb{R})$ on the unit disc, according to the expression $gz = (\alpha z + \beta)/(\bar{\beta} z + \bar{\alpha})$, where $2\alpha = a + d + (b-c)i$, $2\beta = -a + d + (b+c)i$ and $z \in \mathbb{D}$. Conjugation by ϕ gives, in fact, an isomorphism between $PSL(2, \mathbb{R})$ and $PSU(1, 1)$. Elements of the latter group are represented by complex matrices $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ of determinant 1, modulo the center. The group $PSL(2, \mathbb{R})$, which consists of harmonic (respectively holomorphic) self-maps of $\mathbb{D}$, naturally acts on $\mathrm{Har}(\mathbb{D})$ (respectively $\mathrm{Hol}(\mathbb{D})$) by: $(g, f) \mapsto f \circ g^{-1}$, where $g^{-1}(z) = (\bar{\alpha} z - \beta)/(\alpha - \bar{\beta} z)$.

We now recall the construction of a foliated bundle over a Riemann surface. Let S be a compact connected Riemann surface, $\tilde{S}$ its universal covering space, and Γ the fundamental group of S. The covering action of $\gamma \in \Gamma$ on $p \in \tilde{S}$ will be written $p\gamma$. Let X be a compact connected space on which Γ acts by homeomorphisms, for a given homomorphism $\rho : \Gamma \to \mathrm{Homeo}(X)$ from Γ into the group of homeomorphisms of X. Given $\gamma \in \Gamma$ and $x \in X$, $\rho(\gamma)(x)$ will be written simply as $\gamma(x)$. Γ naturally acts on the product $\tilde{S} \times X$, properly discontinuously, by $(p, x) \cdot \gamma := (p\gamma, \gamma^{-1}(x))$. The space of orbits for this action, $M = (\tilde{S} \times X)/\Gamma$, is a foliated space whose leaves are transverse to the fibers of the natural projection $\pi : M \to S$. The restriction of the projection map to individual leaves is covering maps. The resulting foliated space will be written $(M_\rho, \mathcal{F}_\rho)$.

3. *Harmonically simple foliations*

We describe in this section conditions under which a foliated space has the Liouville property. These are mostly results that are given in [**8**] for the holomorphic Liouville problem, but which also hold in the harmonic setting. The proofs are essentially the same as for the corresponding results in that paper, except for Theorem 3.6 below. On the other hand, not all that is shown in our other paper seems to have an easy translation to the present setting. For example, in [**8**, Theorem 1.15], it is shown that $(M, \mathcal{F})$ is holomorphically simple whenever it has codimension one. We do not know whether the counterpart for harmonic functions holds.

Some of the results in this section are of a measure-theoretic nature while others are purely topological. In all cases, $(M, \mathcal{F})$ is a compact foliated space with leafwise Riemannian metric.

3.1. *Topological results.*

Let f be a continuous leafwise harmonic function on M. It is clear that the set of leaves where f is constant is a compact non-empty $\mathcal{F}$-saturated set. Notice that it is non-empty by the maximum-value property of harmonic functions, since on a leaf containing a point where f attains a maximum or a minimum value, f must be constant. This remark immediately yields the following proposition.

PROPOSITION 3.1. *If the closure of every leaf of $\mathcal{F}$ contains a unique minimal set, then the leafwise Riemannian compact foliated space $(M, \mathcal{F})$ is harmonically simple.*

Proof. Under these assumptions, the maximum and minimum values of the function must coincide on each leaf closure. □

COROLLARY 3.2. *Let $(M_\rho, \mathcal{F}_\rho)$ be a foliated bundle over a compact Riemannian manifold S with compact fiber a differentiable manifold V, where $\rho : \Gamma \to H$ is a homomorphism from the fundamental group of S into a compact group of diffeomorphisms of V. Then $(M_\rho, \mathcal{F}_\rho)$ is harmonically simple.*

Proof. Since H is compact, there exists an H-invariant Riemannian metric on V, making $(M_\rho, \mathcal{F}_\rho)$ a Riemannian foliation. By [**13**], the closure of each leaf is a minimal set. □

Denote by $P(W)$ the projective space associated to a finite-dimensional real or complex vector space W. The general linear group $GL(W)$ naturally acts on $P(W)$. We consider, next, foliated bundles with fiber $P(W)$ over a compact Riemannian manifold S, associated to homomorphisms $\rho : \Gamma \to GL(W)$, where Γ is the fundamental group of S. An element $\gamma \in \Gamma$ is called *proximal* if the maximal characteristic exponent of $\rho(\gamma)$ is simple. Proposition 3.1 and [**12**, 3.4 and 3.6, Ch. VI] yields the next proposition.

PROPOSITION 3.3. *Let S be a connected, compact, Riemannian manifold with fundamental group Γ, let W be a finite-dimensional vector space and suppose that $\rho : \Gamma \to GL(W)$ is a homomorphism such that Γ contains a proximal element. Let $(M_\rho, \mathcal{F}_\rho)$ be the foliated bundle over S with fiber $P(W)$ and Γ-action on $P(W)$ given by ρ. Then $(M_\rho, \mathcal{F}_\rho)$ is harmonically simple.*

COROLLARY 3.4. *Let S, Γ, W and $(M_\rho, \mathcal{F}_\rho)$ be as in Proposition 3.3, where the representation $\rho : \Gamma \to GL(W)$ is now such that the projection of $\rho(\Gamma)$ into $PGL(W)$ is Zariski dense. Then $(M_\rho, \mathcal{F}_\rho)$ is harmonically simple.*

Proof. If the image of $\rho(\Gamma)$ in $PGL(W)$ is not precompact, the hypothesis of Proposition 3.3 holds by [**12**, Theorem 4.3(i)]. Otherwise, the conclusion follows from Corollary 3.2. □

The proof of the next theorem is, with obvious changes, the same as for [**8**, Theorem 1.11]. It uses Corollary 3.4, as well as [**14**] and [**4**, Proposition 8.2].

THEOREM 3.5. *Let $S = \mathbb{D}/\Gamma$ be a compact Riemann surface of genus $g \geq 2$ and write $G = GL(n, \mathbb{C})$. Then there is a Zariski dense subset U of the representation variety* $\mathrm{Hom}(\Gamma, G)$ *such that for each $\rho \in U$, the foliated space $(M_\rho, \mathcal{F}_\rho)$ with fiber $P(\mathbb{C}^n)$ is harmonically simple.*

Another class of foliations to which the ideas of this section apply are obtained from actions of Gromov-hyperbolic groups on their boundary. (See [**10**] for the general definitions and facts concerning such groups.) The proof of the next theorem requires some modifications of the proof of its holomorphic counterpart [**8**, Proposition 1.13].

THEOREM 3.6. *Let Λ be a Gromov-hyperbolic group, X the boundary of Λ, and S a compact connected Riemannian manifold with fundamental group Γ. Suppose that Γ acts on X via a homomorphism $\rho : \Gamma \to \Lambda$ and let $(M_\rho, \mathcal{F}_\rho)$ be the corresponding foliated bundle over S. Then $(M_\rho, \mathcal{F}_\rho)$ is harmonically simple.*

Proof. Let L denote the limit set for the action of $\rho(\Lambda)$ on X. There are only three possibilities for the cardinality of L: $1, 2, \infty$. If the cardinality is either 1 or ∞, the action of Λ on X has a unique minimal set (see [**11**]), so the conclusion of the theorem follows from Proposition 3.1.

So suppose that L consists of exactly two points. In this case, $\rho(\Gamma)$ fixes the geodesic joining them. But the stabilizer of a geodesic in a (countable) hyperbolic group is virtually cyclic. If $\rho(\Gamma)$ is finite, all leaves of $\mathcal{F}$ are compact, so the conclusion of the theorem holds. If $\rho(\Gamma)$ is not finite, any non-compact leaf is a finitely generated Abelian Galois covering of a common compact Riemann surface. But, it is known that such surfaces, individually, satisfy the Liouville property (see [**2**]). Therefore, the theorem follows. □

We note that, in the next theorem, a non-discrete $\rho(\Gamma)$ is not excluded.

THEOREM 3.7. *The previous theorem still holds after replacing* Λ *by* $SL(2, \mathbb{C})$.

Proof. The proof is essentially the same as for Theorem 3.6. Of the three possibilities for the limit set of the action of $\rho(\Gamma)$ on the boundary of the symmetric space, which is now $\mathbb{H}^3$, the difficult case corresponds to a two-point limit set. So, it can be assumed that Γ fixes a geodesic in $\mathbb{H}^3$. The stabilizer in $SL(2, \mathbb{C})$ of this geodesic is $\mathbb{R} \times \mathrm{SO}(2)$, an Abelian group. Consequently, non-compact leaves are (virtually) finitely generated Abelian Galois coverings of the same compact Riemann surface. As before, [**2**] finishes the proof. □

It seems plausible to expect the statement of Theorem 3.7 still to be valid if one replaces $SL(2, \mathbb{C})$ with a general rank-one semisimple Lie group. The proof given above, however, breaks down since the stabilizer of a geodesic in the corresponding symmetric space is $\mathbb{R} \times K$, for a compact K which may be non-Abelian. Nevertheless, we expect that a more detailed analysis should yield the result in this more general case.

3.2. *Measure-theoretic results.* The next theorem is from [**9**]. We recall that a measure m on M is said to be *harmonic* if for all continuous leafwise smooth functions $h : M \to \mathbb{R}$,

$$\int_M (\Delta h)(x)\, dm(x) = 0.$$

A harmonic measure on $(M, \mathcal{F})$ is called *totally invariant* if, together with the leafwise Riemannian measure, it yields a transverse invariant measure for $\mathcal{F}$. We refer to [**9**] or [**5**] for the general properties and results concerning harmonic measures.

THEOREM 3.8. (Garnett) *Let* m *be a harmonic measure on* $(M, \mathcal{F})$ *and* f *a measurable,* m*-integrable, leafwise harmonic function on* M. *Then* f *is constant on* m*-a.e. leaf.*

Some generalizations of Garnett's result are obtained by S. Adams in [**1**].

COROLLARY 3.9. *If the union of the supports of harmonic measures is all of* M, *then* $(M, \mathcal{F})$ *is harmonically simple.*

COROLLARY 3.10. $(M, \mathcal{F})$ *is harmonically simple whenever it admits a transverse invariant measure of full support.*

The last corollary shows that it is easy to obtain examples of harmonically simple foliations. In fact, for every volume-preserving action of a cocompact lattice $\Gamma \in PSL(2, \mathbb{R})$ on a compact manifold X, the corresponding foliated bundle $(\mathbb{D} \times X)/\Gamma$ has that property.

4. *Harmonically non-simple foliations*

We give in [**8**] the following example of a foliated bundle $(M, \mathcal{F})$ that is not holomorphically or harmonically simple. Here $M = (\mathbb{D} \times \mathcal{C})/\Gamma$, where

$$\mathcal{C} = \{[z_1, z_2, t] \in \mathbb{R}P^4 : |z_1|^2 - |z_2|^2 = t^2\}$$

is an $SU(1, 1)$-invariant submanifold of projective space for an action of $SU(1, 1)$ on $\mathbb{R}P^4$ defined as follows:

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \cdot [z_1, z_2, t] = [\alpha z_1 + \beta \bar{z}_2, \alpha z_2 + \beta \bar{z}_1, t].$$

The function $f : \mathbb{D} \times \mathcal{C} \to \mathbb{C}$ given by

$$f(z, [u_1, u_2, t]) = \frac{\bar{u}_1 z - u_2}{-\bar{u}_2 z + u_1}$$

is easily shown to pass to the quotient by Γ, yielding a real-analytic leafwise holomorphic function.

The transverse dynamics of the above example 'essentially' corresponds to the action of Γ on a 3-sphere obtained by compactifying $PSU(1, 1)$ (on which Γ acts by translations) by adding a circle at infinity. It is not a particularly complicated dynamics, topologically or measure-theoretically.

It is natural to ask whether dynamical properties such as recurrence and ergodicity might somehow prevent examples of this kind. The purpose of this section is to construct a *harmonically* non-simple foliated space which is ergodic with respect to a transverse measure class that is positive on open sets. (It will be shown later that the example is holomorphically simple.) More precisely, we have the following theorem.

THEOREM 4.1. *Let $S = \mathbb{D}/\Gamma$ be a compact Riemann surface, where Γ is a cocompact lattice in $PSL(2, \mathbb{R})$. Then there exists a foliated bundle $(M, \mathcal{F})$ over S with fiber S^2 such that:*

(1) *$(M, \mathcal{F})$ is real-analytic in the complement of a pair of compact leaves, S_1, S_2 homeomorphic to S;*
(2) *$(M, \mathcal{F})$ is ergodic with respect to the smooth measure class;*
(3) *the complement of $S_1 \cup S_2$ has a real-analytic compactification, which is an ergodic foliated bundle over S with fiber $S^1 \times [0, 2\pi]$;*
(4) *for both $(M, \mathcal{F})$ and its analytic compactification the Liouville property does not hold. Moreover, a continuous leafwise harmonic, not leafwise constant, can be found that is real-analytic in the complement of $S_1 \cup S_2$.*

We need the following definitions. By an *arc* in S^1 we refer to a set of the form $\{\zeta e^{it\theta} : 0 \leq t \leq 1\}$, where $\zeta \in S^1$ and $\theta \in (0, 2\pi)$. Therefore, the space of arcs is parametrized by the cylinder $S^1 \times (0, 2\pi)$ and has a natural completion, $S^1 \times [0, 2\pi]$, in which $S^1 \times \{0\}$ consists of trivial (single-point) arcs and $S^1 \times \{2\pi\}$ consists of full-circle arcs with the initial point specified. Since $PSL(2, \mathbb{R})$ acts on the boundary of the unit disc, it also acts on the space of arcs $S^1 \times (0, 2\pi)$. The action can be written explicitly as follows: $g(\zeta, \theta) = (g(\zeta), \Theta_g(\zeta, \theta))$, where $g(\zeta) = (\alpha\zeta + \beta)/(\bar{\beta}\zeta + \bar{\alpha})$, $|\alpha|^2 - |\beta|^2 = 1$, and

$$\Theta_g(\zeta, \theta) = \int_0^1 \frac{dt}{|\bar{\alpha} + \zeta\bar{\beta}e^{it\theta}|^2}.$$

This is a real-analytic action. It extends an action on $S^1 \times [0, 2\pi]$ which is also real-analytic. The boundary components $S^1 \times \{0\}$ and $S^1 \times \{2\pi\}$ are invariant subsets.

LEMMA 4.2. *Let Γ be an arbitrary lattice in $PSL(2,\mathbb{R})$. Then the action of Γ on $S^1 \times [0, 2\pi]$ defined above is ergodic with respect to the smooth measure class on the cylinder.*

Proof. It is easily seen that $PSL(2,\mathbb{R})$ acts transitively on the space of (non-trivial) arcs and that the isotropy group of the arc specified by $\zeta = 1$ and $\theta = \pi$ is the subgroup $A \subset PSL(2,\mathbb{R})$ represented by diagonal matrices in $SL(2,\mathbb{R})$. Thus, the action of Γ on the space of arcs is analytically conjugate to the action of Γ by left translations on the quotient $PSL(2,\mathbb{R})/A$. But it is well known that this action is ergodic with respect to the smooth measure class on the quotient. In fact, its dual action, of A on $PSL(2,\mathbb{R})/\Gamma$, is the geodesic flow of a compact negatively curved surface. □

Write $X_0 = S^1 \times [0, 2\pi]$ and define $F : \mathbb{D} \times X_0 \to \mathbb{R}$ according to

$$F(z, \zeta, \theta) = \frac{\theta}{2\pi} \int_0^1 \frac{1 - |z|^2}{|\zeta e^{it\theta} - z|^2}\, dt.$$

Observe that F is real-analytic and $F(z, \zeta, 0) = 0$, $F(z, \zeta, 2\pi) = 1$, for all $z \in \mathbb{D}$ and $\zeta \in S^1$.

LEMMA 4.3. *We have $F(g(z), g(x)) = F(z, x)$ for all $g \in PSL(2,\mathbb{R})$, $z \in \mathbb{D}$, and $x \in X_0$.*

Proof. Denote by χ_I the indicator function of the arc I determined by ζ and θ. By the Poisson formula, $z \mapsto \varphi(z) = F(z, \zeta, \theta)$ is the unique harmonic function on the unit disc with boundary value χ_I. Since the action of $PSL(2,\mathbb{R})$ on the unit disc is conformal, $\varphi \circ g$ is the unique harmonic function on $\mathbb{D}$ with boundary value $\chi_I \circ g = \chi_{g^{-1}(I)}$, from which the claim follows. □

Denote by $(M_0, \mathcal{F}_0)$ the foliated bundle for which $M_0 = (\mathbb{D} \times X_0)/\Gamma$ and $\mathcal{F}_0$ is the resulting foliation by coverings of $\mathbb{D}/\Gamma$. Notice that M_0 is a compact manifold with boundary, whose boundary components are the two compact leaves of $\mathcal{F}_0$ associated to the invariant boundary circles of X_0. Due to Lemma 7.7, $\mathcal{F}_0$ is ergodic.

Due to Lemma 7.8, F yields on the quotient M_0 a leafwise harmonic continuous function, $F_0 : M_0 \to \mathbb{R}$. Notice that F_0 is constant on the boundary leaves.

Denote by X_1 the sphere S^2 obtained as the quotient of $S^1 \times [0, 2\pi]$ by collapsing the two boundary circles to the north and south poles. By repeating the above construction, now with X_1, we obtain a continuous foliated bundle over $\mathbb{D}/\Gamma$ with transversal fiber S^2, which is ergodic and admits a continuous leafwise harmonic function.

Notice that the Γ-action on $S^1 \times [0, 2\pi]$ that was used to define $(M, \mathcal{F})$ passes to the quotient so as to define an action on the torus $S^1 \times S^1$. The latter action is real-analytic and ergodic with respect to the smooth measure class, so it induces an ergodic real-analytic foliated bundle. This foliated bundle, however, is harmonically simple.

We end the section with the following two propositions. The first one is an immediate consequence of Garnett's theorem.

PROPOSITION 4.4. *The support of any harmonic measure for the foliation $(M, \mathcal{F})$ of Theorem 4.1 is contained in the union of the compact leaves S_1 and S_2.*

PROPOSITION 4.5. *$(M, \mathcal{F})$, of Theorem 4.1, is holomorphically simple.*

Proof. The claim is a corollary of [**8**, Proposition 1.5]. We give below a simple direct proof. Let $f : M \to \mathbb{C}$ be a leafwise holomorphic function. Clearly, f is constant on the compact leaves S_1 and S_2. Let K be the union of leaves on which f is constant. The transversal dynamics is such that any leaf which is not S_1 or S_2 approaches S_1 or S_2 along the direction of a hyperbolic element of Γ. Therefore, f can take at most two values on K. On the other hand, the image of the complement of K under f is an open subset of $\mathbb{C}$, due to the open mapping principle for holomorphic functions. Since the union of this open set with the (at most two points of the) image of K is compact, it follows that the complement of K is empty. □

5. *A universal non-Liouville foliation*

Foliated bundles over a compact Riemann surface of genus at least 2 for which the Liouville property fails afford a general description, which is explained in this section. A similar construction for the holomorphic case is given in [**8**].

Write $X_0 = \text{Har}(\mathbb{D})$, and let $S = \mathbb{D}/\Gamma$ be a compact Riemann surface. As already noted, X_0 is a compact metrizable space with the topology of uniform convergence on compact subsets of the unit disc, and supports a continuous action of $PSL(2, \mathbb{R})$ given by $(g, f) \mapsto f \circ g^{-1}$. Thus, it makes sense to form the compact foliated space $M_0 = (\mathbb{D} \times X_0)/\Gamma$ over S. We denote this foliated bundle by $(M_0, \mathcal{F}_0)$.

A leafwise harmonic function on M_0 is given by the following tautological construction. Define $\bar{\phi} : \mathbb{D} \times X_0 \to \mathbb{C}$ by $\bar{\phi}(z, f) := f(z)$. It is easily checked that ϕ passes to the quotient and defines a continuous, leafwise harmonic function on M_0.

Given foliated bundles $(M', \mathcal{F}')$ and $(M, \mathcal{F})$ over S, we define a *harmonic morphism* between them as a continuous, fiber-preserving map $F : M' \to M$ that sends leaves of $\mathcal{F}'$ to leaves of $\mathcal{F}$ and the restriction to each leaf of $\mathcal{F}'$ is a harmonic map.

In trying to construct examples of harmonically non-simple foliations, it is useful to have in mind the following easy fact.

PROPOSITION 5.1. *Let $(M, \mathcal{F})$ be a foliated bundle over $S = \mathbb{D}/\Gamma$ with fiber V. Then, there is a one-to-one correspondence between continuous leafwise harmonic functions $\psi : M \to \mathbb{R}$ and Γ-equivariant continuous maps $\hat{\psi} : V \to$ Har$(\mathbb{D})$. Furthermore, if $\Psi : M \to M_0$ is the harmonic morphism induced from $\hat{\psi}$, then $\psi = \phi \circ \Psi$, and Ψ is the unique morphism from M to M_0 that satisfies this last equality.*

Therefore, if V is any Γ-invariant compact subset of X_0 that contains non-constant functions, the foliated bundle with fiber V will be an example of foliation for which the Liouville property fails, and any example for which the property fails which is a foliated bundle over S is obtained in this way.

As an example, we note the following alternative construction of the foliation of Theorem 4.1. Consider the element of X_0 whose boundary value is the function f_0 on

S^1 such that

$$f_0(z) = \begin{cases} 0 & \text{if } \mathrm{Im}(z) > 0, \\ 1 & \text{if } \mathrm{Im}(z) \leq 0. \end{cases}$$

Any $g \in PSL(2, \mathbb{R})$ that fixes f_0 must also fix the points $1, -1 \in S^1$, which forces g to be diagonal, as a simple calculation shows. Therefore, the orbit $X = PSL(2, \mathbb{R}) \cdot f_0 \in X_0$ is identified with the quotient $PSL(2, \mathbb{R})/A$, where A is the diagonal subgroup of $PSL(2, \mathbb{R})$. (This quotient is the space of geodesics of the Poincaré disc.) The closure of the orbit of f_0 is a topological sphere. Let V be this orbit closure and $(M, \mathcal{F})$ the corresponding foliated bundle. By noting that f_0 is the indicator function of the arc $t \mapsto -e^{i\pi t}$, it is easy to check that the two descriptions of the foliation of Theorem 4.1 are isomorphic.

6. *Relation with the holomorphic case*

If the leaves of $(M, \mathcal{F})$ are Kähler manifolds, it makes sense to ask whether a given continuous leafwise harmonic function corresponds to the real part of a continuous leafwise holomorphic function. This question is of interest since the holomorphically simple property seems to be much easier to obtain than its harmonic counterpart. In fact, we show in [**8**] a number of results which prove that a foliation is holomorphically simple, whose harmonic versions are either not true, or we do not yet know to hold.

For example, if the leaves of $\mathcal{F}$ are complex manifolds (not necessarily Kähler) and the codimension is 1, then $(M, \mathcal{F})$ is holomorphically simple. It is still open whether a codimension-one leafwise Riemannian $(M, \mathcal{F})$ is harmonically simple. It was also shown in [**8**] that if the closure of each leaf contains no more than countably many minimal sets, then $\mathcal{F}$ is holomorphically simple. But the harmonic counterpart is not true, as the foliation of Theorem 4.1 shows.

Regarding the relationship between holomorphic and harmonic functions, we limit ourselves here to making the following remark. Recall that the (continuous) tangential de Rham cohomology of a foliated space is the cohomology of the complex $\Omega^*(M, \mathcal{F})$ of (continuous) tangentially smooth differential forms, with the tangential exterior derivative. We denote the cohomology spaces by $H^*_{\mathrm{dR}}(M, \mathcal{F})$. If f is a continuous leafwise harmonic function on M, then we can obtain, on foliation boxes, continuous functions $\hat{f}$ which are harmonic conjugates of f. These are well defined up to an additive constant, so the various locally defined $d\hat{f}$ piece together to make a closed 1-form ω_f in $\Omega^1(M, \mathcal{F})$. If the corresponding cohomology class, $[\omega_f]$, is zero, then f admits a global harmonic conjugate, $\hat{f}$, and $f + i\hat{f}$ is a continuous leafwise holomorphic function. This remark yields the following proposition.

PROPOSITION 6.1. *Let $(M, \mathcal{F})$ be a foliation of a compact manifold M by Kähler manifolds. To each leafwise harmonic continuous function $f : M \to \mathbb{R}$ is associated a cohomology class $[\omega_f] \in H^1_{\mathrm{dR}}(M, \mathcal{F})$ having the property that $[\omega_f] = 0$ if and only if f is the real part of a leafwise holomorphic continuous function.*

COROLLARY 6.2. *Suppose that $H^1_{\mathrm{dR}}(M, \mathcal{F}) = \{0\}$, for $(M, \mathcal{F})$ as in the previous proposition. Then $(M, \mathcal{F})$ is harmonically simple if and only if it is holomorphically simple.*

We noted in Proposition 4.5 that the example given earlier of a codimension-two foliated bundle which is not harmonically simple is holomorphically simple. In particular, it follows that it has non-trivial first-tangential cohomology. It is also interesting to observe that in codimension-one only in very special cases does $H^1_{\mathrm{dR}}(M, \mathcal{F}) = \{0\}$ hold. For this reason, if it is at all true in general that codimension-one foliations of a compact manifold by Riemann surfaces are harmonically simple, we expect the proof to be much harder than the related holomorphic result shown in [**8**].

7. *Dynamics on* Har($\mathbb{D}$)

This section discusses some of the dynamical properties of the action of subgroups of $G = PSL(2, \mathbb{R})$ on Har($\mathbb{D}$). Unless further hypotheses are explicitly assumed, Γ will denote an arbitrary subgroup of G.

7.1. *General properties of* Γ*-actions on* Har($\mathbb{D}$).

We begin with a few general facts showing that limit sets of orbits in Har($\mathbb{D}$) often contain constant functions.

PROPOSITION 7.1. *Let g_n be any unbounded sequence in $PSL(2, \mathbb{R})$. Then, for any $\varphi \in$ Har($\mathbb{D}$) whose boundary value is a continuous function on S^1, all limit points of $\{g_n \cdot \varphi : n = 1, 2, \dots\}$ are constant functions.*

Proof. Write $g_n(z) = (\alpha_n z + \beta_n)/(\bar{\beta}_n z + \bar{\alpha}_n)$, where $|\alpha_n|^2 - |\beta_n|^2 = 1$, be an unbounded sequence of elements of $PSU(1, 1)$. By taking a subsequence it can be assumed that $\beta_n \neq 0$, and that α_n/β_n and $\alpha_n/\bar{\beta}_n$ converge to points on the unit circle. Let η be the limit of the latter sequence and observe that $\varphi(g_n(z)) = \varphi((\alpha_n/\bar{\beta}_n)(z+\beta_n/\alpha_n)/(z+\bar{\alpha}_n/\bar{\beta}_n)) \to \varphi(\eta)$, uniformly on z in any compact subset of $\mathbb{D}$. □

Since $PSL(2, \mathbb{R})$ has dimension three, we have the following.

COROLLARY 7.2. *Let $f \in C^0(S^1)$ and φ the harmonic function on $\mathbb{D}$ with boundary value f. Let X denote the closure of the orbit $\Gamma \cdot \varphi$. Then X has topological dimension at most* 3 *and the complement of $PSL(2, \mathbb{R}) \cdot \varphi$ in X consists of constant functions.*

As the previous proposition indicates, orbits in Har($\mathbb{D}$) with interesting dynamical properties are associated to harmonic functions whose boundary values are not continuous functions on S^1.

Recall that Har($\mathbb{D}$)* denotes the non-constant elements of Har($\mathbb{D}$).

PROPOSITION 7.3. *Γ fixes a point in* Har($\mathbb{D}$)* *if and only if the Γ-action on S^1 is not ergodic.*

Proof. This is an immediate consequence of the Poisson representation formula and the fact that the action of Γ on S^1 is not ergodic if and only if there exists a measurable, Γ-invariant, bounded function on S^1. □

COROLLARY 7.4. *A lattice in $PSL(2, \mathbb{R})$ cannot have fixed points in* Har($\mathbb{D}$)*.

More generally, we may ask whether a subgroup $\Gamma \subset PSL(2, \mathbb{R})$ leaves invariant a compact subset of Har($\mathbb{D}$)*. For cocompact lattices the answer is no.

PROPOSITION 7.5. *A cocompact lattice in* $PSL(2,\mathbb{R})$ *does not leave invariant a compact subset of* $\mathrm{Har}(\mathbb{D})^*$.

Proof. If such a V existed, it would be possible to construct a foliated space $(M, \mathcal{F})$ and a leafwise harmonic continuous function on M which is not constant on any leaf. But this is clearly impossible. □

7.2. *Chaos.* Although there is no universally accepted mathematical definition of chaos, one popular definition was proposed by Devaney in [**7**]. He isolates three properties as being the essential features of a chaotic dynamical system: (i) topological transitivity, (ii) a dense set of periodic points, and (iii) sensitive dependence on initial conditions. The third condition means the following. Let $f : X \to X$ be a continuous map of a metric space X. Then f (or the $\mathbb{Z}$-action it generates on X) is said to have *sensitive dependence on initial conditions* if there exist $\delta > 0$ such that for every $x \in X$ and every neighborhood N of x, there exists $y \in N$ and a positive integer n such that $f^n(x)$ and $f^n(y)$ are more than δ apart.

We show in this subsection that the $\mathbb{Z}$-action on $\mathrm{Har}(\mathbb{D})$ generated by parabolic or hyperbolic elements of $PSL(2,\mathbb{R})$ is chaotic.

THEOREM 7.6. *Let* γ *be a hyperbolic or parabolic element of* $PSL(2,\mathbb{R})$*, regarded as a transformation on* $\mathrm{Har}(\mathbb{D})$*. Then* γ *defines a chaotic dynamical system.*

It is proven in [**3**] that the first two conditions in the definition of chaos imply the third, so we only need to verify that these $\mathbb{Z}$-actions are topologically transitive and have a dense set of periodic points.

Let $K_n = \{z \in \mathbb{C} : |z| \le 1 - 1/n\}$ and define for $\phi \in \mathrm{Har}(\mathbb{D})$,

$$\|\phi\| = \sum_{n=1}^{\infty} \frac{\sup_{z\in K_n} |\phi(z)|}{n^2 2^n}.$$

This norm induces a metric on $\mathrm{Har}(\mathbb{D})$ compatible with the topology of uniform convergence on compact sets. Let $f : S^1 \to \overline{\mathbb{D}}$ be the boundary value of ϕ, which means that

$$\phi(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} f(e^{i\theta})\, d\theta.$$

It is a simple calculation to check that

$$\|\phi\| \le \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|\, d\theta.$$

LEMMA 7.7. *The* $\mathbb{Z}$*-action on* $\mathrm{Har}(\mathbb{D})$ *generated by any hyperbolic element* γ *of* $PSL(2,\mathbb{R})$ *is topologically transitive.*

Proof. Without loss of generality, it can be assumed that γ fixes 1 and -1 in S^1, and that 1 is an expanding fixed point, while -1 is contracting. Let f_n, $n = 1, 2, \ldots$, be a weak*-dense sequence in the unit ball of $L^\infty(S^1)$. It will be convenient to regard S^1 as the compactified real line, $\overline{\mathbb{R}}$, by means of the map $\eta : z \mapsto i(1-z)/(1+z)$. Therefore, the f_n

are now regarded as functions on the real line and γ takes the form $\gamma(x) = \lambda x$, for some $\lambda > 1$.

Choose a sequence of positive integers $k_1 < k_2 < \cdots$ having the property

$$n(n+1) < \lambda^{k_{n+1}-k_n},$$

for $n = 1, 2, \ldots$, and define $a_n = \lambda^{-k_n}/n$, $b_n = n\lambda^{-k_n}$. Notice that the above inequality implies that $b_{n+1} < a_n$, so the intervals $[a_n, b_n]$ are disjoint and accumulate at 0. Define now a function f_∞ on $\overline{\mathbb{R}}$ by:

$$f_\infty(x) = \begin{cases} f_n(\lambda^{k_n} x) & \text{for } x \in [-b_n, -a_n] \cup [a_n, b_n], \\ 0 & \text{for } x \in (-a_n, -b_{n+1}) \cup (b_{n+1}, a_n). \end{cases}$$

Thus, we have by construction that $\gamma^{k_n} \cdot f_\infty := f_\infty \circ \gamma^{-k_n} = f_n$ on the set

$$A_n := [-n, -1/n] \cup [1/n, n] = \gamma^{k_n}([-b_n, -a_n] \cup [a_n, b_n]).$$

If $\varphi_\infty, \varphi_n$ are the elements of $\mathrm{Har}(\mathbb{D})$ associated to f_∞, f_n via the Poisson representation formula, then it follows that $\gamma^{k_n} \cdot \varphi_\infty$, $n = 1, 2, \ldots$ is arbitrarily close to φ_n for large n. In fact, let $B_n \subset S^1$ be the image of A_n under η^{-1}. Note that $f_n \circ \eta$ and $\gamma^{k_n} \cdot \varphi_\infty$ coincide on B_n, whereas B_n^c has length that goes to zero like $1/n$. But $\|\gamma^{k_n} \cdot \varphi_\infty - \varphi_n\| \leq l(B_n^c)/\pi$ so that $\mathbb{Z} \cdot \varphi_\infty$ is dense in $\mathrm{Har}(\mathbb{D})$, as claimed. □

LEMMA 7.8. *The set of periodic points in* $\mathrm{Har}(\mathbb{D})$ *for the* $\mathbb{Z}$*-action generated by any hyperbolic element* γ *of* $PSL(2, \mathbb{R})$ *is dense.*

Proof. For each $\varphi \in \mathrm{Har}(\mathbb{D})$ and for each $\epsilon > 0$, we should find $\varphi_\epsilon \in \mathrm{Har}(\mathbb{D})$ and an integer $k \geq 2$ such that $\varphi_\epsilon \circ \gamma^k = \varphi_\epsilon$ and $\|\varphi - \varphi_\epsilon\| \leq \epsilon$. Let $f : S^1 \to \overline{\mathbb{D}}$ be the boundary value of φ.

Let $B = B_n$ be as in the proof of the previous lemma, where n is large enough that the length of the complement of B satisfies $l(B^c) \leq \epsilon\pi$. Choose a positive integer k such that $\gamma^k(B)$ and B are disjoint, and define

$$f_\epsilon = \sum_{m=-\infty}^{\infty} (\chi_B f) \circ \gamma^{mk}.$$

Notice that $f_\epsilon \circ \gamma^k = f_\epsilon$ and that f_ϵ still takes values into $\overline{\mathbb{D}}$. Furthermore,

$$\begin{aligned} \int_0^{2\pi} |f(e^{i\theta}) - f_\epsilon(e^{i\theta})|\, d\theta &= \int_B |f(e^{i\theta}) - f_\epsilon(e^{i\theta})|\, d\theta + \int_{B^c} |f(e^{i\theta}) - f_\epsilon(e^{i\theta})|\, d\theta \\ &= \int_{B^c} |f(e^{i\theta}) - f_\epsilon(e^{i\theta})|\, d\theta \\ &\leq 2l(B^c). \end{aligned}$$

Consequently, if φ_ϵ is the element of $\mathrm{Har}(\mathbb{D})$ with boundary value f_ϵ, then $\varphi_\epsilon \circ \gamma^k = \varphi_\epsilon$ and $\|\varphi - \varphi_\epsilon\| \leq \epsilon$. □

If γ is a parabolic element of $PSL(2, \mathbb{R})$, it can be assumed (after conjugation) that its action on the extended real line fixes ∞ so that $\gamma(x) = x + a$ for some $a > 0$. The proofs of Lemmas 7.7 and 7.8 are easily modified so as to apply to γ parabolic. Notice that now it is convenient to take $A_n = [-n, n]$.

7.3. *The conjugacy problem.* Given two distinct elements γ_1, γ_2 of $PSL(2,\mathbb{R})$, it is natural to ask whether or not they define dynamical systems on $\mathrm{Har}(\mathbb{D})$ that have the same topological dynamics. More precisely, can we find a homeomorphism $\Phi : \mathrm{Har}(\mathbb{D}) \to \mathrm{Har}(\mathbb{D})$ that conjugates (intertwines) the two transformations?

If γ_1 and γ_2 are both hyperbolic (respectively parabolic) elements, then they are topologically conjugate. The conjugating homeomorphism can be taken to be the map that associates to each element of $\mathrm{Har}(\mathbb{D})$ with boundary value f the element of $\mathrm{Har}(\mathbb{D})$ with boundary value $f \circ h$, where h is a homeomorphism of S^1 that conjugates γ_1 and γ_2, these two now regarded as transformations of the circle. (It is easy to show that h exists.)

It would be interesting to know, for example, whether a hyperbolic and a parabolic element can be topologically conjugate. We do not yet know the answer to this question.

There are many other topics about the dynamics of the action of $PSL(2,\mathbb{R})$ and its subgroups on $\mathrm{Har}(\mathbb{D})$ that we have not considered here. Some of these topics, particularly those about the ergodic theory of these actions, will be taken up elsewhere.

Acknowledgement. The first author would like to thank the École Normale Supérieure de Lyon, UMPA, for its support and hospitality while this paper was being written.

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