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On Gromov's theory of rigid transformation groups: a dual approach

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Abstract. Geometric problems are usually formulated by means of (exterior) differential systems. In this theory, one enriches the system by adding algebraic and differential constraints, and then looks for regular solutions. Here we adopt a dual approach, which consists of enriching a plane field, as this is often practised in control theory, by adding brackets of the vector fields tangent to it and, then, looking for singular solutions of the obtained distribution. We apply this to the isometry problem of rigid geometric structures.

0. *Content*

In §1 we exhibit a natural class of plane fields for which the accessibility behaviour, as studied in control theory, possesses, essentially, the same nice properties as in the analytic case.

In §2, we observe that there is a control theory approach to the local isometry problem of affine manifolds (e.g. pseudo-Riemannian manifolds), which is dually equivalent, to the usual differential systems (i.e. partial differential relations) approach. We then apply the results of §1 to deduce a celebrated corollary of Gromov's theory on rigid transformation groups.

In fact the developments of §1 suggest how to proceed in order to recover essentially most of the part of Gromov's theory related to Corollary 2.1, together with some independent results. However, we will not follow this because our primary goal here is to be as elementary as possible. Further extensions and applications of our approach will be developed elsewhere.

1. *Control theory*

Let P be a smooth plane field of dimension d on a manifold N. From an integrability viewpoint there are two extremal cases, described by the classical Frobenius and Chow's theorems, which concern the completely integrable and absolutely non-integrable cases, respectively.

Let $\chi(N)$ be the Lie algebra of C^∞ vector fields of N, and denote by $\mathcal{G}$ the Lie subalgebra generated by smooth vector fields everywhere tangent to P. Let G be the 'evaluation plane field', $G(x) = \{X(x) : X \in \mathcal{G}\}$ (this is not necessarily a continuous plane field).

Frobenius' theorem states that in the 'degenerate case' where P is involutive, that is $G = P$, then through each point of N passes a *leaf* of P that is a submanifold of dimension d (the same as that of P) which is (everywhere) tangent to P. In contrast, in the 'generic case', when $G = TM$, Chow's theorem says that any pair of points can be joined by a curve tangent to P. However (unfortunately), it is the intermediate (non-generic and non-degenerate) situation that one usually meets in geometric and differential problems.

Integrability and infinitesimal integrability domains. In searching leaves, let us 'naively' introduce the *integrability domain* $\mathcal{D}$ as the set of points of N, through which passes a (germ of a) leaf of P. This set may behave very badly, for instance, it is not *a priori* closed. For this, let us introduce its infinitesimal variant, the 'involutivity domain', $\mathcal{D}^\infty = \{x \in N / G(x) = P(x)\}$. We call $\mathcal{D}^\infty$ the *infinitesimal integrability domain* of P. Clearly, $\mathcal{D}^\infty$ contains $\mathcal{D}$, and it is closed since it is the set of points where the dimension of G equals that of P, that is the dimension of G is minimal (obviously, the dimension of G is lower semi-continuous).

Along $\mathcal{D}^\infty$, the Frobenius condition is satisfied, and so, one may hope to find leaves through each of its points, that is $\mathcal{D} = \mathcal{D}^\infty$. However, $\mathcal{D}^\infty$ is not, *a priori*, a manifold, and we do not yet know a fractal Frobenius' theorem. Worse, even if we assume $\mathcal{D}^\infty$ is a submanifold it is not clear that P is tangent to it!

In the analytic case everything works well and there are many ways leading to the equality $\mathcal{D} = \mathcal{D}^\infty$ [**7**]. For instance $\mathcal{D}^\infty$ is an analytic set, and may be thought out as being a submanifold, and so in order to apply Frobenius' Theorem to the restriction $P|\mathcal{D}^\infty$, one just has to show that P is tangent to $\mathcal{D}^\infty$.

Distributions. However, the most consistent approach to this problem is a generalization of Frobenius' theorem in another direction, that of (singular) *distributions* (the singularity is topological and not differential). Recall that a C^∞ distribution Δ on N is a $C^\infty(N)$-submodule of $\chi(N)$, the space of C^∞ vector fields on N. For example, a smooth plane field P is associated with distribution of vector fields tangent to it. Conversely, to a distribution Δ, one defines its 'evaluation plane field' by $\Delta(x) = \{X(x)/X \in \Delta\}$. In general, this determines a discontinuous plane field (i.e. a plane field with non-constant dimension). One calls a distribution *regular* if its 'evaluation plane field' has a constant dimension.

A distribution is called *involutive* if it is a Lie subalgebra of $\chi(N)$. Any distribution generates an involutive distribution, this is the advantage of generalizing plane fields to distributions, since the involutive distributions generated by plane fields, are not plane fields in general, i.e. they are not necessarily regular.

The integrability problem. A *leaf* of a distribution Δ is a submanifold S such that along S the tangent space of S *coincides* with the evaluation of Δ. The distribution is called *integrable* if leaves exist everywhere, i.e. any $x \in N$ belongs to a leaf. In particular, if the involutive distribution $\mathcal{G}$ generated by a plane field P is integrable, then we have in particular the equality $\mathcal{D} = \mathcal{D}^\infty$. Indeed, if $x \in \mathcal{D}^\infty$, then its $\mathcal{G}$-leaf, is a leaf of P. (In the generic, but non-interesting case, $\mathcal{G}$ is integrable, and $\mathcal{D}^\infty = \emptyset$).

Obviously, an integrable distribution has the same leaves as its generated involutive distribution. However, an involutive distribution is not integrable in general (see §2.1 for counter-examples). The integrability problem consists of finding conditions so that involutive implies integrable. For instance, Frobenius' theorem says nothing but regular involutive distributions are integrable.

Finitely generated distributions. A distribution Δ is called *locally finitely generated*, if for any $x \in N$, there is a neighbourhood U of x, and a finite family $V_1, \ldots, V_l$ of vector fields of Δ, such that, on U, any $V \in \Delta$ can be written as $V = \sum g_i V_i$, where $g_i \in C^\infty(U)$. Regular distributions are locally finitely *freely* generated. Conversely, there are many pieces of evidence leading to speculation that there exists a suitable 'blowing up' manipulation transforming a locally finitely generated distribution to a regular finitely generated distribution. In any case, Frobenius' theorem is valid in this context.

THEOREM 1.1. (Frobenius' theorem for finitely generated distributions; R. Hermann [**7**], see §3.1 for an outline of the proof.) *A locally finitely generated involutive distribution is integrable. In particular, let P be a smooth plane field such that its associated involutive distribution is finitely generated. Then $\mathcal{D} = \mathcal{D}^\infty$.*

Partially algebraic vector fields. The theorem above applies in the analytic case, due to standard Noetherian facts. We are now going to extend the applicability of the above theorem to a partially analytic, in fact partially algebraic, situation. The starting point is to consider *partially algebraic* vector fields on $\mathbb{R}^n \times \mathbb{R}^m$. These are C^∞ vector fields of the form: $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \to (R(x, u), Q(x, u))$ such that, for x fixed, $R(x, u)$ and $Q(x, u)$ are polynomials. In other words, partially algebraic vector fields are mapping: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$, with co-ordinates in the ring $C^\infty(\mathbb{R}^n)[X_1, \ldots, X_m]$. Observe that the bracket of two partially algebraic vector fields is a partially algebraic vector field. That is, partially algebraic vector fields form a Lie subalgebra.

Let Φ be a partially linear (local) diffeomorphism of $\mathbb{R}^n \times \mathbb{R}^m$, that is Φ has the form $\Phi(x, u) = (f(x), A_x(u))$, where $f : U \to U'$ is a local diffeomorphism of $\mathbb{R}^n$, and $A : x \in U \to A_x \in GL(m)$ is a C^∞ mapping.

Observe that partially linear diffeomorphisms preserve the space of partially algebraic vector fields. (Here one can also consider partially polynomial diffeomorphisms, but for the sake of simplicity we restrict ourselves to the partially linear case).

Fiberwise algebraic vector fields on vector bundles. Suppose that $N \to B$ is a vector bundle. The above, allows us to define *fiberwise algebraic vector fields* on N. They form a Lie subalgebra. One can also define *fiberwise algebraic plane fields* and *fiberwise algebraic distributions*. The involutive distribution generated by a fiberwise algebraic distribution is fiberwise algebraic. One can also define fiberwise algebraic functions, and then *fiberwise algebraic sets*, as zero loci of systems of fiberwise algebraic functions.

THEOREM 1.2. (Integrability) *Let P be a fiberwise algebraic plane field on a vector bundle $\pi : N \to B$. Then, there is an open dense set $U \subset B$ over which $\mathcal{D} = \mathcal{D}^\infty$. More precisely, the involutive distribution generated by P is integrable on $\pi^{-1}(U)$.*

Proof. Let $\mathcal{G}$ be the involutive distribution generated by P. From the previous discussion, it can be described locally as an R-submodule I of R^{n+m}, where $R = C^\infty(\mathbb{R}^n)[X_1, \dots, X_m]$.

Following Theorem 1.1, it suffices to show that over an open dense set U of $\mathbb{R}^n$, I is locally finitely generated. This will follow from the Noetherian Theorem 3.2. The intuitive proof of it is that we have a family $\{I_x : x \in \mathbb{R}^n\}$ of $\mathbb{R}[X_1, \dots, X_m]$-submodules of $(\mathbb{R}[X_1, \dots, X_m])^{n+m}$. Each I_x is finitely generated and, in a dense open subset of $\mathbb{R}^n$, the cardinality of the generating family of I_x is locally bounded. □

Differential structure of $\mathcal{D}$. The infinitesimal integrability domain $\mathcal{D}^\infty$ (and hence the integrability domain $\mathcal{D}$, if we restrict over U) is a fiberwise algebraic set. Indeed, $\mathcal{D}^\infty$ is the set of points where the involutive distribution $\mathcal{G}$ generated by P, has dimension d (that is the dimension of P). Thus, $\mathcal{D}^\infty = \{x \in N / V_1 \wedge \cdots \wedge V_{d+1} = 0$ for all $V_1, \dots, V_{d+1}$ elements of $\mathcal{G}\}$. Locally $\mathcal{D}^\infty$ is the zero locus of a family of elements of $C^\infty(\mathbb{R}^n)[X_1, \dots, X_m]$. The fibers $\mathcal{D}^\infty_x$ are thus algebraic sets of $\mathbb{R}^m$. In fact, fiberwise algebraic sets enjoy, in addition, many regularity properties regarding the dependence on $x \in B$. In local co-ordinates, around a point where the distribution $\mathcal{G}$ is locally finitely generated, $\mathcal{D}^\infty$ is the common zero locus of a finite set $f_1, \dots, f_l$ of elements of $C^\infty(\mathbb{R}^n)[X_1, \dots, X_m]$. However, because we reason here over $\mathbb{R}$ (and not $\mathbb{C}$), $\mathcal{D}^\infty$ equals the zero locus of a single element $g = \sum f_i^2$. This element g may be seen as a map $f : \mathbb{R}^n \to \mathbb{R}[X_1, \dots, X_m]_{\leq k}$, the space of polynomials of degree $\leq k$ ($f(x)$ is the restriction of g to $\{x\} \times \mathbb{R}^m$). More concretely, by definition of $C^\infty(\mathbb{R}^n)[X_1, \dots, X_m]$, we have a representation $g(x, X_1, \dots X_m) = \sum_{|I| \leq k} g_I(x) X^I$, where I is a multi-index, then $f(x)$ is the polynomial with coefficients $(g_I(x))_{|I| \leq k}$.

Suppose for example that $f(x)$ has a unique (real) root $z(x) \in \mathbb{R}^m$, and thus $\mathcal{D}^\infty$ is the graph of z. Then, $z(x)$ is expressed 'algebraically' from the coefficients of $f(x)$. Therefore, $\mathcal{D}^\infty$ is the graph of a very 'tame' function.

The same idea may be adapted when $f(x)$ has infinitely many roots. This may lead to a stratified structure of $\mathcal{D}^\infty$, after removing singular fibers. We will restrict our investigation here to a weak regularity aspect, which will follow from the following general fact.

LEMMA 1.3. *Let B be a topological space and $f : B \to \mathbb{R}[X_1, \dots, X_m]_{\leq k}$, a continuous map which associates a polynomial $f(x)$ of degree $\leq k$, to each $x \in B$. Let $Y = \{x \in B : f(x)$ has a (real) root $\}$. Then Y contains an open dense set of its closure.*

Proof. Consider the 'universal' polynomial

$$\Phi : (X_1, \dots, X_m, p) \in \mathbb{R}^m \times \mathbb{R}[X_1, \dots, X_m]_{\leq k} \to p(X_1, \dots, X_m) \in \mathbb{R}$$

(p is a polynomial of degree $\leq k$ on $(X_1, \dots, X_m)$).

Consider the 'universal' algebraic set $\Phi^{-1}(0)$ determined by Φ. Let Z be the projection of $\Phi^{-1}(0)$ on $\mathbb{R}[X_1, \dots, X_m]_{\leq k}$. It is not *a priori* an algebraic set, but, almost by definition, a *semi-algebraic set.*

One fundamental fact about semi-algebraic sets is that they admit good stratification (see, for example, [**1**]). In particular, Z is a finite disjoint union $Z = \cup Z_i$, where Z_i are *locally closed* sets, that is, there are open sets O_i in $\mathbb{R}[X_1, \dots, X_m]_{\leq k}$, such that $Z_i = \overline{Z_i} \cap O_i$.

For the lemma, we may assume that Y is dense in B, we have then to show that Y contains an open dense set of B. By continuity, $f(B)$ is contained in $\overline{Z}$ (which also equals $\cup \overline{Z_i}$). We have, $Y = f^{-1}(Z)$.

If Z itself were locally closed (for example for $m = 1$), then $f^{-1}(Z)$ would be open in B, and we would be done.

We argue as follows in the general case. Let $F_i = f^{-1}(\overline{Z_i})$ and $A_i = f^{-1}(Z_i)$. Then A_i is open in F_i (because Z_i is locally closed). We have, $B = \cup F_i$. One firstly observes that $\cup \operatorname{int}(F_i)$ is dense in B, where int stands for the interior (this is Baire's theorem, for the *finite* union of closed sets, which is true for all topological spaces). Next, since $Y = U A_i$ is dense in B, it follows that $U = \cup(A_i \cap \operatorname{int}(F_i))$ is dense in B. Moreover U is open in B (since A_i is open in F_i) and is contained in Y. □

The discussion before the lemma applies to any fiberwise algebraic set (such as $\mathcal{D}^\infty$), and therefore leads to the following result.

COROLLARY 1.4. *Let $\mathcal{S}$ be a fiberwise algebraic set of N and Y its projection on B. Then Y contains an open dense subset of its closure $\overline{Y}$.*

Fiberwise constructible sets. In view of further applications, we need the following slight generalization of fiberwise algebraic sets. A subset $\mathcal{S}$ of N is called *fiberwise constructible* if it can be written as a difference $\mathcal{S}_1 - \mathcal{S}_2$ of two fiberwise algebraic sets $\mathcal{S}_1$ and $\mathcal{S}_2$.

Such a set has a structure as nice as that of a fiberwise algebraic set. Indeed, locally, suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are respectively defined by f and g elements of $C^\infty(\mathbb{R}^n)[X_1, \dots, X_m]$. Then, consider the mapping $\phi : (x, X) \in \mathbb{R}^n \times \mathbb{R}^m - \mathcal{S}_2 \to (x, X, 1/g(x, X)) \in \mathbb{R}^n \times \mathbb{R}^{m+1}$ (here $X = (X_1, \dots, X_m)$). Then, the image $\phi(\mathcal{S}_1 - \mathcal{S}_2)$ becomes fiberwise algebraic, since it is defined by the equations $X_{m+1} g(x, X) - 1 = 0$ and $f(x, X) = 0$.

The above corollary is therefore valid for fiberwise constructible sets.

THEOREM 1.5. (Rough structure) *Let $\mathcal{S}$ be a fiberwise constructible set of N and Y its projection on B. Then Y contains an open dense subset of its closure $\overline{Y}$. In particular if Y is dense in B, then Y contains an open dense subset of B.*

Integrability with constraints. One is sometimes interested in leaves through points in a given subset $\mathcal{S} \subset N$ (the plane field P is not assumed to be tangent to $\mathcal{S}$, although this usually happens in practice). The following result unifies Theorems 1.1 and1.5 above.

THEOREM 1.6. *Let P be a fiberwise algebraic plane field on a vector bundle $\pi : N \to B$, and $\mathcal{S}$ a fiberwise constructible subset of N. There is an open dense set $U \subset B$, over which, the sets of integrability and infinitesimal integrability points of P in $\mathcal{S}$ are equal, that is, $\mathcal{D} \cap \mathcal{S}|U = \mathcal{D}^\infty \cap \mathcal{S}|U$.*

In addition, the projection of $\mathcal{D} \cap \mathcal{S}|U$ is a closed (posisibly empty) subset of U.

Proof. Let U_1 be an open dense set given by the integrabilty theorem, Theorem 1.1, that is $\mathcal{D}|U_1 = \mathcal{D}^\infty|U_1$. Over U_1, $\mathcal{D} \cap \mathcal{S}$ is fiberwise constructible. Let $Y_1 \subset U_1$ be its projection, and let $\overline{Y_1}$ be its closure in U_1. From the structure theorem, Theorem 1.5, there is an open subset U_2 of U_1, such that Y_1 contains $\overline{Y_1} \cap U_2$, which is in addition dense in $\overline{Y_1}$. In particular $Y_1 \cap U_2$ is closed in U_2.

We claim that $U = U_2 \cup (U_1 - \overline{Y_1})$ satisfies the conditions of the theorem. Indeed, U is open, and it is dense in U_1 (and hence in B), since $\overline{Y_1} \cap U_2$ is dense in $\overline{Y_1}$. Furthermore, $U \cap Y_1 = U \cap \overline{Y_1}$ and hence, over U, the projection of $\mathcal{D} \cap \mathcal{S}$ is closed. □

2. *The isometry pseudo-group of an affine connection*

Fiberwise algebraic objects are abundant in geometry. For instance, a fiberwise algebraic function on the cotangent bundle of a smooth manifold generates a fiberwise algebraic Hamiltonian vector field. In particular the geodesic flow of a Riemannian metric is fiberwise algebraic (being seen on the cotangent as well as on the tangent bundles).

The tautological geodesic plane field of an affine manifold. More generally, let (M, ∇) be an affine manifold, that is, ∇ is a torsion free connection on M (not necessarily flat). Its geodesic flow is generated by a fiberwise algebraic vector field. Indeed, locally, this vector field has the form: $V : (x, p) = (x_1, \ldots, x_n, p_1, \ldots, p_n) \in U \times \mathbb{R}^n \to (p_1, \ldots, p_n, \sum_{ij} \Gamma^1_{ij}(x) p_i p_j, \ldots, \sum_{ij} \Gamma^n_{ij}(x) p_i p_j)$. (The Γ^k_{ij} are the Christoffel symbols).

Now, we introduce a generalization of geodesic flows as plane fields on Grassmann bundles. Let $\pi : Gr^d(M) \to M$ be the Grassman bundle of d-planes tangent to M. The connection determines a splitting $TGr^d(M) = V \bigoplus H$, where V is the vertical and H is the horizontal space (given by ∇). For $p \in Gr^d_x(M)$, $d_p\pi$ maps isomorphically H_p onto T_xM. Let $\tau^d(p)$ be the d-plane contained in H_p which is mapped by $d_p\pi$ to p (as a subspace of T_xM). Thus τ^d is a d-plane field on $Gr^d(M)$, called the *tautological geodesic* plane field on $Gr^d(M)$. (We think that this construction must be known, although we have not found any reference where it is explicitly mentioned, see [**9**] for more details and a systematic study).

The tautological character of τ^d is clear. The geodesic adjective is justified by the fact that, the projection of a leaf of τ^d is a (totally) geodesic submanifold of dimension d in M. Conversely, if S is a d-dimensional geodesic submanifold of M, then its Gauss lift $x \in S \to T_xS \in Gr^d(M)$ is a leaf of τ^d.

The fiberwise algebraic discussion on vector bundles, extends in a straightforward way, to projective bundles (i.e. fiber bundles whose fibers are projective spaces …). In particular, here, as in the case of the geodesic flow, the tautological plane fields τ^d are fiberwise algebraic. In fact, for the following application, we will immediately come back to a vector bundle situation.

The pseudo-group of local isometries. A (local) isometry or a (local) affine diffeomorphism is a local diffeomorphism of M, which preserves ∇. Equivalently, an affine diffeomophism is a diffeomorphism which sends (parametrized) geodesics to (parameterized) geodesics. One may also define *affine mappings* as, not necessarily diffeomorphic mappings, sending geodesics to geodesics.

One may naturally construct a product connection $\nabla \bigoplus \nabla$ on the product $M \times M$. If ∇ is the Levi-Civita connection of a pseudo-Riemannian metric g, then $\nabla \bigoplus \nabla$ is the Levi-Civita connection of the product metric $g \bigoplus g$ (which is the same as the Levi-Civita of the product $g \bigoplus -g$). A curve $t \to (c(t), d(t))$ is (a parameterized) geodesic iff both of its projections $t \to c(t)$, and $t \to d(t)$ are geodesic in M.

Let $f : U \to V$ be a smooth map. Its graph Graph(f) is a n-submanifold of $M \times M$. One easily sees, from the characterization of geodesics in $M \times M$, that f is an affine mapping, iff Graph(f) is a (totally) geodesic submanifold in $M \times M$ (the proof works as in the case of $\mathbb{R}^n$). In particular, local affine mappings give rise to leaves of the tautological geodesic plane field τ^n on $Gr^n(M \times M)$.

Let $Gr^*(M \times M)$ consist of n-planes which are graphs, that is $p \in Gr^*_{(x,y)}(M \times M)$ iff p is a graph of a linear map $T_xM \to T_yM$ (or equivalently p projects injectively on T_xM). Then, a leaf of τ^n trough an element $p \in Gr^*(M \times M)$, determines a local affine mapping.

Observe that $Gr^*(M \times M)$ is a vector bundle on $M \times M$, the fiber over (x, y) being Hom$(T_xM \to T_yM)$.

To get local affine diffeomorphisms, one considers $Gr^{**}(M \times M)$, the set of n-planes transverse to each of the factors $M \times \{\cdot\}$ and $\{\cdot\} \times M$, that is, $p \in Gr^{**}_{(x,y)}(M \times M)$ iff p is the graph of an isomorphism $T_xM \to T_yM$.

We have the following interpretation: (x, y) belongs to the projection of the integrability domain of τ^n on $Gr^{**}(M \times M)$ iff there is a local affine diffeomorphism sending x to y, that is x and y have the same orbit under the pseudo-group of local affine diffeomorphisms.

It is easy to see $Gr^{**}(M \times M)$ as the complement in $Gr^*(M \times M)$ of a fiberwise algebraic set, and hence in particular, it is an (open) fiberwise constructible set.

COROLLARY 2.1. (Gromov [**6**], see also [**3**]) *Let M be an affine manifold. Suppose that its pseudo-group of local affine diffeomorphisms admits a dense orbit, then, it has an open dense orbit (that is there is an open dense homogeneous set in M).*

Proof. Apply Theorem 1.6 to $P = \tau^n$ on $Gr^*(M \times M)$, with a constraint set $\mathcal{S} = Gr^{**}(M \times M)$.

Let $x_0 \in M$, be a point with a dense orbit $\mathcal{O}_0$ under the affine pseudo-group. The projection of $\mathcal{S} \cap \mathcal{D}$ contains $\mathcal{O}_0 \times \mathcal{O}_0$. From Theorem 1.6, the projection of $\mathcal{S} \cap \mathcal{D}$ contains an open dense set U in $M \times M$. Let $(x, y) \in U$, then the orbit $\mathcal{O}_x$ of x under the affine pseudo-group contains the open (non-empty) set $(\{x\} \times M) \cap U$ of $\{x\} \times M$. Since the orbit $\mathcal{O}_0$ is dense, we have $\mathcal{O}_0 \cap \mathcal{O}_x \neq \emptyset$, and hence $\mathcal{O}_0 = \mathcal{O}_x$, but obviously, an orbit with a non-empty interior is open, therefore $\mathcal{O}_0$ is open and dense. □

2.1. *Some comments*

Example. Consider on $\mathbb{R}^n$ a connection $\nabla = \nabla^0 + T$, where ∇^0 is the usual flat connection (that is $\nabla^0_X Y = D_X Y$), and $T = (T^k_{ij})$ is a symmetric tensor $T\mathbb{R}^n \times T\mathbb{R}^n \to T\mathbb{R}^n$. Suppose that T is flat at zero, that is, all the partial derivatives of all orders of the functions T^k_{ij}, vanish at zero.

Consider the tautological geodesic plane field τ^n of $\mathbb{R}^n \times \mathbb{R}^n$. It is easy to see, that $\mathcal{D}^{\infty}_{(0,0)} = Gr^n_{(0,0)}(\mathbb{R}^n \times \mathbb{R}^n)$. If the integrability domain $\mathcal{D}$ contains $\mathcal{D}^{\infty}$, or more precisely, $\mathcal{D}^{\infty}_{(0,0)}$, then, in particular, every linear map $A : \mathbb{R}^n \to \mathbb{R}^n$, will be the derivative of a local affine (for $(\mathbb{R}^n, \nabla)$) map F_A fixing zero. It is easy to see that this implies that T is very special. Indeed, the existence of non-diffeomorphic affine maps leads to vanishing relations of the curvature, not only at zero, but also near it.

Other constraints. In the proof of the above corollary, one may add further constraints of an algebraic nature. For example, if M is endowed with a pseudo-Riemannian metric g, then one considers n-planes of $M \times M$, which are isotropic with respect to the pseudo-Riemannian metric $g \bigoplus -g$ on $M \times M$. The obtained solutions correspond then to local isometries of (M, g). Observe that the constraint set here is tangent to τ^n, and it is in fact fiberwise algebraic (not only fiberwise constructible) in $Gr^*(M \times M)$, since an isotropic plane which belongs to $Gr^*(M \times M)$ must belong to $Gr^{**}(M \times M)$.

Similarly, one may treat the isometry pseudo-group of a unimodular affine structure, and in general, any algebraic enrichment of the affine structure.

The full Gromov theorem. It is the above corollary of Gromov's theory that was used in the celebrated work [**2**].

The full Gromov theorem, that is for non-necessarily topologicaly transitive isometry pseudo-groups and for general rigid geometric structures was utilized in [**4**], in the analytic case. As we have said above, in our approach, there are no integrability or structure difficulties in the analytic case. In fact, [**5**] contains a direct approach in the analytic case.

It is generally admitted that there are no serious difficulties to pass from affine structures to general rigid (algebraic) structures.

Observe that here, just the idea of affine structures enriched with algebraic constraints allows us to generalize Corollary 2.1 to a large class of rigid structures (for example that utilized in the proof of the main result of [**4**]).

Now, for affine structures with non-necessarily topologicaly transitive isometry pseudo-group, the idea of the proof of Gromov's theorem, is to find a submanifold in $M \times M$, which, 'essentially', contains as an open subset, the projection of the infinitesimal integrability domain of τ^n.

Compactification. Singular isometries. We hope that our approach here provides elements leading to the analysis of the non-completeness of the locally homogeneous open dense set U in M (here, by non-completeness, we mean the fact that $U \neq M$). Indeed, $\mathcal{D}$ is naturally compact by $\mathcal{D}^{\infty}$, and there is sometimes strong evidence (as in the Anosov case of [**2**]) that the set $\mathcal{D}^{\infty} - \mathcal{D}$ must be empty.

Moreover, $Gr^{**}(M \times M)$ is naturally compacted by $Gr^n(M \times M)$. The (new) leaves of τ^n in this latter space may be interpreted as singular affine mappings, and from another point of view, as 'stable laminations' of (regular) affine mappings.

In fact, compactifications may be defined in the general set-up of control theory of §1. Indeed, as $\mathbb{R}^n$ is projectively compactified by $\mathbb{R}P^n$ (and not $\mathbb{R}P^{n-1}$), any vector bundle $N \to B$ with fiber type $\mathbb{R}^n$ can be (fiberwise) compactified by a 'projective' bundle $\bar{N} \to B$, with fiber type $\mathbb{R}P^n$. Fiberwise objects on N extend to $\bar{N}$, and it seems interesting to interpret them there.

Fiberwise algebraic closure. Let us try to see, among the integrabilty Theorem 1.1 and the structure Theorem 1.5, what is the most important for the proof of Corollary 2.1, or, more exactly, what is the contribution of each of them in this proof. For this, let us consider the following situation. Take G, a group of (global) affine diffeomorphisms of M. We have a proper embedding $(g, x) \in G \times M \to \text{Graph}(D_x g) \in Gr^{**}(M \times M)$. Denote its image by L. The projection of L in $M \times M$ is the union of the graphs of all the elements of G, and L itself is nothing but the union of the Gauss lifts of these graphs. For this, let us call L the graph of G.

For example, if G is discrete and infinite, then the projection of L is a countable union of graphs. Therefore, from the structure theorem, L is far from being a fiberwise algebraic set (although it is closed). It is thus natural to take the *fiberwise algebraic closure* $\overline{L}^{\text{fib,alg}}$ of L. The structure theorem ensures that $\overline{L}^{\text{fib,alg}}$ has a nice projection. However, one needs to interpret elements of $\overline{L}^{\text{fib,alg}}$; in other words, one asks, what properties of elements of L pass to its fiberwise algebraic closure? It is the integrability theorem which answers this question by stating that, away from a nowhere dense set, the new elements of $\overline{L}^{\text{fib,alg}}$ are local isometries.

In other words, the integrability theorem states, essentially, that in contrast to L the graph of the local isometry pseudo-group is a fiberwise algebraic set. The structure theorem says that one has won much from the statement of the integrability theorem.

Remark 2.2. Similarly to the above embedding, there is a classical way of breaking the dynamics of G (i.e. killing its recurrence, in such a way that the action becomes proper), by letting it act on the frame bundle $P \to M$. To keep everything elementary, compactify P by seeing it as an open set in N, the vector bundle with fibers, $N_x = \text{Hom}(\mathbb{R}^n \to T_x M)$ ($n = \dim M$). It is endowed with a principal $GL(n, \mathbb{R})$-action.

Suppose that the G-action on M is topologically transitive, that is, it has a dense orbit. Then, there is an open dense set U of M, such that for all $p \in P$, over U, the fiberwise-algebraic closure of $\overline{G.p}^{\text{fib,alg}}$ projects onto U. Of course, $GL(n, \mathbb{R})$ permutes these fiberwise algebraic closures. The stabilizer in $GL(n, \mathbb{R})$ of any closure $\overline{G.p}^{\text{fib,alg}}$, may be identified to the C^∞*-algebraic hull* of G, as introduced in [**10**]. One may define in a natural way, C^s-fiberwise algebraic sets, for any $s \geq 0$, and find C^s-algebraic hulls as defined by Zimmer, for all $s \geq 0$.

3. *Proofs*

3.1. *Sketch of proof of Theorem 1.1.* Let Δ be an involutive locally finitely generated distribution on N. At $x \in N$, we denote $\Delta(x)$ the evaluation of Δ at x.

Let $x_0 \in N$. To construct a leaf of x_0, start with a vector field V_0 of Δ, non-singular at x_0, and let ϕ^t be its flow. Suppose that $(\phi^t)^*$ preserves the evaluation of Δ along the orbit $\phi^t(x_0)$, that is $D_{x_0}\phi^t(\Delta(x_0)) = \Delta(\phi^t(x_0))$.

Take another vector field V_1 linearly independent of V_0, and let ψ^t be its flow. Suppose that, like ϕ^t, the flow ψ^t preserves the evaluation of Δ, then the surface obtained by saturating the ϕ^t-orbit by the flow ψ^t is tangent to Δ.

Reiterating the construction, by taking a maximal family of linearly independent similar vector fields, $V_2, \ldots$, we would obtain a leaf, provided that we check that each of the flows of these vector fields preserves the evaluation of Δ. So, let us show this property for V_0 (the vector fields are indiscernible).

Locally, in some co-ordinate system, we may assume $N = \mathbb{R} \times \mathbb{R}^{n-1}$, and $V_0 = \partial/\partial t$. So, V_0 generates a translation flow.

Let $V_1, \ldots, V_k$ a finite set of generating vector fields of Δ near x_0. Since Δ is involutive, $[\partial/\partial t, V_i] = \partial V_i/\partial t \in \Delta$ (here we see V_i as vectorial maps on $\mathbb{R}^n$). Write: $\partial V_i/\partial t = \sum_{1 \leq j \leq k} a_{ij} V_j$.

So, the problem becomes the following, along the t-axis $(t, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we are given vector fields, $V_1(t), \ldots, V_k(t)$, and there are smooth functions $a_{ij}(t)$, such that $\partial V_i/\partial t = \sum a_{ij} V_j$. Does this imply that the space generated by $\{V_1(t), \ldots, V_k(t)\}$ is independent of t (i.e. it is parallel along the t-axis)?

This is clear in the case $k = 1$, that is, if a vector field $V(t)$ satisfies a relation $\partial V/\partial t = a(t)V(t)$, for $a(t)$ continuous, then $V(t)$ has a parallel direction, and if $V(t)$ vanishes somewhere then it vanishes everywhere. Indeed, $e(t) = V/|V(t)|$ is parallel where it is defined, that is, where $V(t) \neq 0$. Now, if for example zero is a boundary point of the set where $V(t) = 0$, then, on a semi-open interval, say $[0, \epsilon[$, we have $V(t) = f(t)e$ ($e = e(t)$), and thus, $f(t)$ is a C^0 non-trivial solution of the equation $f' = a(t)f(t)$, with $f(0) = 0$, which is impossible.

Next, in the general case, that is $k > 1$, near a generic t, it is possible to write all of the vector fields $V_1, \ldots, V_k$ as smooth combinations of r-linearly independent elements, say $V_1, \ldots, V_r$. One then considers the exterior product $V(t) = V_1(t) \wedge \cdots \wedge V_r(t)$. It satisfies (near a generic point) a relation $\partial V/\partial t = aV$, and is therefore parallel by the first step.

To finish the proof, it suffices to show that the dimension of the space generated by the $V_i(t)$ is constant. This dimension equals the rank of the matrix $X = (x_{ij})_{i \leq k, j \leq n}$, defined by $V_i = \sum_j x_{ij} e_j$, where $(e_i)_{1 \leq i \leq n}$ is the canonical basis of $\mathbb{R}^n$. We have, $\partial V_i/\partial t = \sum_j (\partial x_{ij}/\partial t) e_j$. On the other hand, $\partial V_i/\partial t = \sum_l a_{il} V_l = \sum_{lj} a_{il} x_{lj} e_j$. Thus X satisfies the equation (on $k \times n$ matrices) $X' = A(t)X$, where A is the $k \times k$ matrix (a_{ij}). Thus $X(t) = R(t)X(0)$, where $R(t)$ is the $k \times k$ matrix, resolvent of the equation on $\mathbb{R}^k$, $Y' = A(t)Y$ ($Y \in \mathbb{R}^k$). In particular, the rank of $X(t)$ does not depend on t.

3.2. *Noetherian properties.* We will deal here with polynomials (with many indeterminates) on a ring R which is $C^0(Y)$, the ring of continuous functions on a topological space Y, or $C^k(Y)$, $0 \leq k \leq \infty$, the ring of k-differentiable functions on a subset Y of a smooth manifold B. (Recall that $f \in C^k(Y)$, means that f extends locally to an element of $C^k(B)$).

If Y' is a subset of Y, there is a restriction homomorphism $C^k(Y) \to C^k(Y')$, and by the same way restriction homomorphisms $C^k(Y)[X_1, \ldots, X_m] \to C^k(Y')[X_1, \ldots, X_m]$.

This allows us to restrict other associated objects, for example, if I is an ideal of $C^k(Y)[X_1, \ldots, X_m]$, then its restriction $I|Y'$ is the ideal of $C^k(Y')[X_1, \ldots, X_m]$ generated by the restriction to Y' of all the elements of I.

An ideal I of $C^k(Y)$ (or $C^k(Y)[X_1, \ldots, X_m]$) is locally finitely generated if every $x \in Y$ admits a neighbourhood U_x such that $I|U_x$ is finitely generated.

LEMMA 3.1. *Let $I_1 \subset \cdots I_j \subset \cdots$ be an increasing sequence of ideals in $C^k(Y)$. Then, there is an open dense set $U \subset Y$ over which all of the ideals are locally finitely generated (that is $I_i|U$ is locally finitely generated, for all i), and the sequence of ideals is locally stationary (on U).*

Proof. Let U_x be a neighbourhood of x, and I an ideal. We say that $I|U_x$ is trivial, if either $I|U_x = 0$, or $I|U_x = C^k(U_x)$.

Observe that to ensure the existence of U_x, such that $I|U_x$ equals $C^k(U_x)$, it suffices that I contains an element f such that $f(x) \neq 0$.

Let $U = \{x \in Y/\text{there is a neighbourhood } U_x \text{ of } x\text{, such that, for all } j,\ I_j|U_x \text{ is trivial}\}$.

By definition, U is open. It is clear, that, over U, the sequence of ideals satisfies the requirements of the lemma. Therefore, it suffices to show that U is dense.

Firstly, U is non-empty. Indeed, let j be the first integer such that $I_j \neq 0$. Then, there is $x \in Y$ and $f \in I_j$ such that $f(x) \neq 0$, and hence there is a neighbourhood U_x such that $I_j|U_x = C^k(U_x)$ and, thus (by definition of j), we have $0 = I_1|U_x = \cdots I_{j-1}|U_x$, and $C^k(U_x) = I_i|U_x$, for all $i \geq j$, that is $x \in U$.

To see that U is dense, suppose the contrary, and consider the open (non-empty set) $Y - \overline{U}$. Restrict everything to it, and conclude, as we have just proved, that its corresponding U, is non-empty. Therefore, there is x in $Y - \overline{U}$, having a neighbourhood U_x (relative to $Y - \overline{U}$), such that all the restrictions $I_j|U_x$ are trivial, but, since $Y - \overline{U}$ is open in Y, U_x is a neighbourhood of x in Y and, therefore, by definition, $x \in U$, which contradicts our hypothesis. □

THEOREM 3.2. (Noetherian theorem) *Let $A = C^k(Y)[X_1, \ldots, X_m]$ and let $\mathfrak{a}$ be an A-submodule of A^l (l is an integer). Then, there is an open dense set $U \subset Y$, over which, $\mathfrak{a}$ is locally finitely generated.*

Proof. Firstly, as in the classical case, it suffices to consider the case $l = 1$, that is $\mathfrak{a}$ is an ideal of A. The proof (in this case), then follows, as for Hilbert's basis theorem, that is, if a ring R is Noetherian, then $R[X_1, \ldots, X_m]$ is also Noetherian.

The (classical) proof of this theorem is achieved by induction on m (see for example [**8**]). Let us recall how the reduction from $R[X_1]$ to R works. One associates to the ideal $\mathfrak{a}$ of $R[X_1]$, an increasing sequence I_i of ideals of R, where I_i is the set of elements appearing as a leading coefficient of an element of $\mathfrak{a}$ of degree $\leq i$. One then arranges a finitely generating set for $\mathfrak{a}$, if one knows that the sequence is stationary, and has at one's disposal finite generating sets for each I_i (the number of i's in account is finite).

In our case, from Lemma 3.1, the sequence of ideals I_i, satisfies the finiteness requirements, after restricting to an open dense set Y'. Therefore $\mathfrak{a}|Y'$ is finitely generated. □

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