Phase transitions in self-gravitating systems

Hamiltonian and Brownian systems

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Plan of the presentation

1. Statistical mechanics of classical self-gravitating systems
   - The deterministic and stochastic N-body problems
   - Maximum entropy state
   - The series of equilibria of isothermal spheres
2. Self-gravitating Brownian particles
   - The Smoluchowski-Poisson system
   - Pre-collapse : self-similar solution
   - Post-collapse : the growth of a Dirac peak
3. Statistial mechanics of quantum particles : fermions
   - Self-gravitating fermions
   - Degeneracy parameter
   - Dependence of the series of equilibria on the degeneracy parameter

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The deterministic and stochastic N-body problems

We consider N classical point mass particles in gravitational interaction

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i,$$

$$\frac{d\mathbf{v}_i}{dt} = -Gm \sum_{j\neq i} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} - \xi \mathbf{v}_i + \sqrt{2D}\,\mathbf{R}_i(t),$$

associated with the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{1}{2} m v_i^2 - Gm^2 \sum_{i<j} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}.$$

We assume that the friction and diffusion coefficients satisfy the Einstein relation (fluctuation-dissipation theorem) :

$$\xi = \frac{Dm}{k_B T}.$$

- If $\xi = D = 0$: Hamiltonian system (Newton)
- If $\xi > 0$: Self-gravitating Brownian particles

Maximum entropy state

We use a mean field approximation and look for the most probable distribution of self-gravitating particles at statistical equilibrium.

- Microcanonical ensemble (Hamiltonian systems)

$$\max_{f} \{S[f] \quad | \quad E[f]=E, M[f]=M\}.$$

- Canonical ensemble (Brownian systems)

$$\min_{f} \{F[f]=E[f]-TS[f] \quad | \quad M[f]=M\}.$$

where

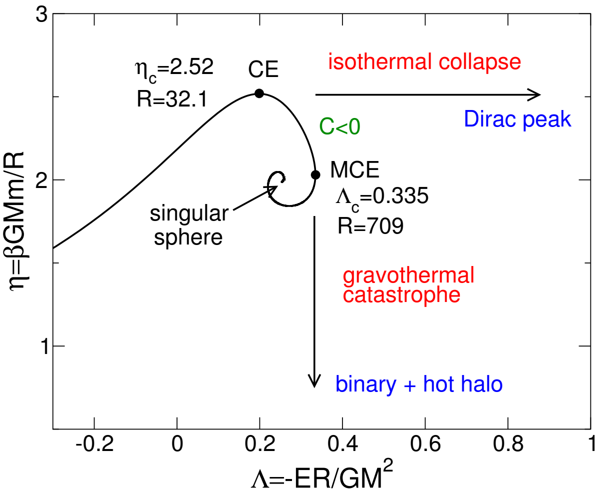
$$S_B[f]=-\int \frac{f}{m}\ln\frac{f}{m}\, d\mathbf{r}\, d\mathbf{v} \qquad \text{(entropy)}$$

$$E[f]=\int f\frac{v^2}{2}\, d\mathbf{r}\, d\mathbf{v}+\frac{1}{2}\int \rho\Phi\, d\mathbf{r} \qquad \text{(energy)}$$

$$M[f]=\int f\, d\mathbf{r}\, d\mathbf{v} \qquad \text{(mass)}$$

The series of equilibria of isothermal spheres

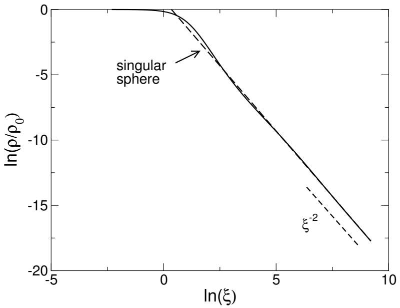
Following Antonov (1962) we consider the statistical mechanics of self-gravitating systems confined within a spherical box of radius R.



The isothermal density profile

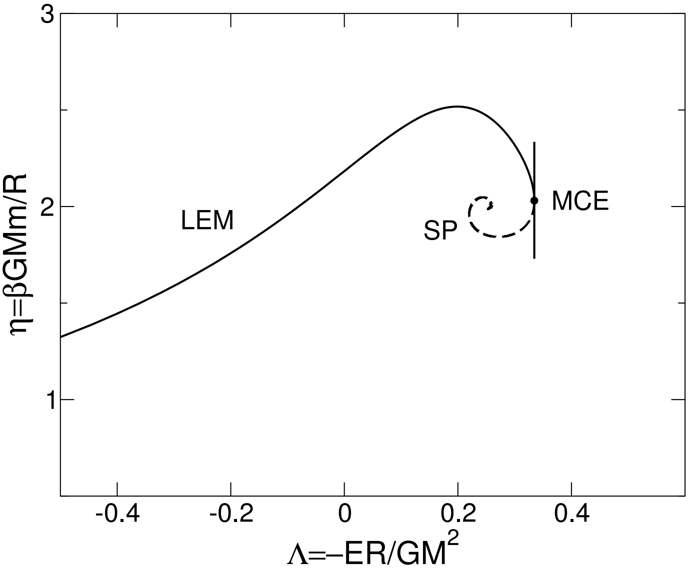
Boltzmann-Poisson equation :

$$\Delta\Phi = 4\pi G\rho,$$
$$\rho(\mathbf{r}) = Ae^{-\beta m\Phi(\mathbf{r})}.$$

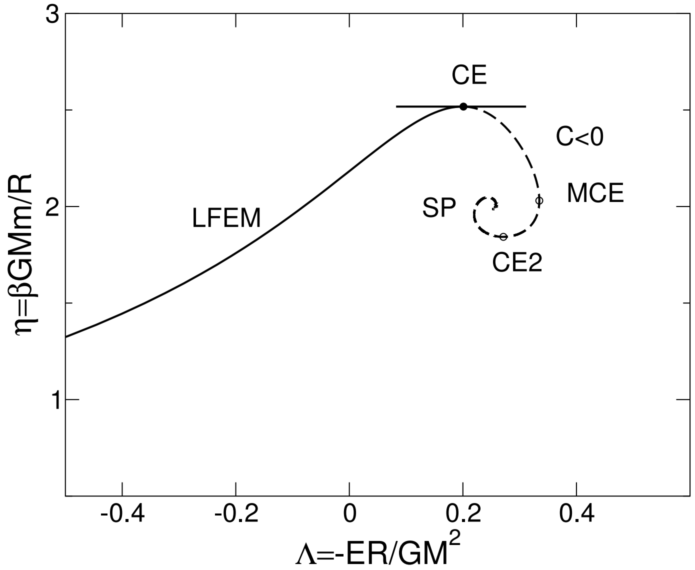


For $r \to +\infty$, the density profile decreases as $\rho \sim 1/(2\pi G\beta m r^2)$.

The microcanonical caloric curve



The canonical caloric curve



Plan of the presentation

The Smoluchowski-Poisson system

In the mean-field approximation, the dynamical evolution of self-gravitating Brownian particles is governed by the Vlasov-Kramers-Poisson system

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} = \xi\frac{\partial}{\partial \mathbf{v}}\cdot\left(\frac{k_B T}{m}\frac{\partial f}{\partial \mathbf{v}} + f\mathbf{v}\right),$$

$$\Delta\Phi = 4\pi G\int f\, d\mathbf{v}.$$

In the strong friction limit $\xi\to+\infty$, one gets the Smoluchowski-Poisson system

$$\frac{\partial \rho}{\partial t} = \frac{1}{\xi}\nabla\cdot\left(\frac{k_B T}{m}\nabla\rho + \rho\nabla\Phi\right),$$

$$\Delta\Phi = 4\pi G\rho.$$

Pre-collapse : self-similar solution

When $T < T_c$, the system undergoes an isothermal collapse. The SP system admits an analytical self-similar solution

$$\rho(\mathbf{r},t) = \rho_0(t) f\left(\frac{r}{r_0(t)}\right), \qquad f(x) = \frac{1}{\pi}\frac{3+x^2}{(1+x^2)^2}$$

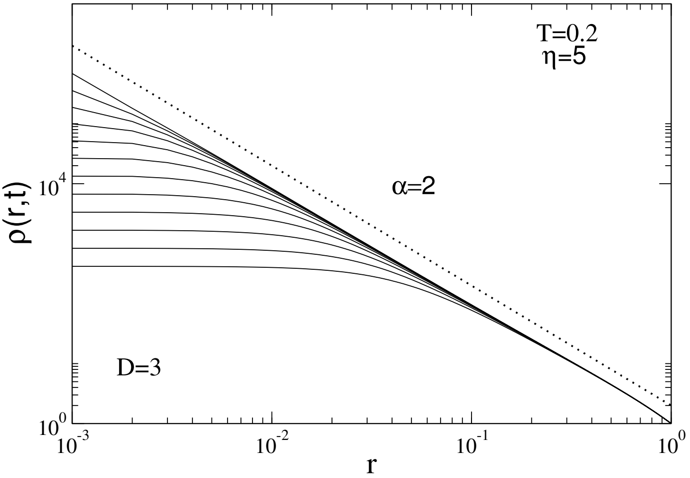
$$\rho_0(t) = \frac{1}{2}(t_{coll}-t)^{-1}, \qquad r_0(t) = (2T)^{1/2}(t_{coll}-t)^{1/2}$$

This solution generates a finite time singularity. At $t = t_{coll}$, we get the singular profile

$$\rho(r, t = t_{coll}) = \frac{T}{\pi r^2}.$$

This singular profile is *not* a Dirac peak since the mass contained in the core vanishes : $M_0(t) \sim \rho_0(t) r_0^3(t) \sim T^{3/2}(t_{coll}-t)^{1/2}$.

Pre-collapse : self-similar solution

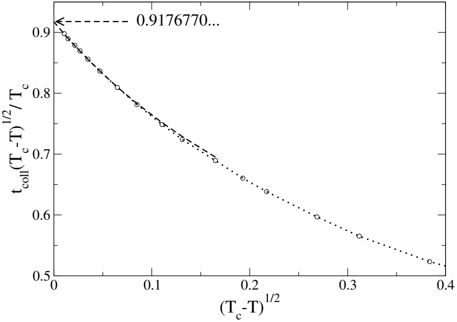


The collapse time

The collapse time $t_{coll}(T)$ depends on the temperature and diverges as $T \to T_c$. We have developed a perturbative theory that gives analytically the value of the collapse time. We find

$$t_{coll}(T) = 0.91767702...T_c(T_c - T)^{-1/2}$$

The exponent $-1/2$ is the same as the one arising in the expression of the relaxation time when $T > T_c$.



Post-collapse : the growth of a Dirac peak

The evolution continues in the post-collapse regime with the formation of a Dirac peak surrounded by a "halo" which undergoes a reversed self-similar solution

$$\rho(\mathbf{r},t) = N_0(t)\delta(\mathbf{r}) + \rho_0(t) g\left(\frac{r}{r_0(t)}\right)$$

$$\rho_0(t) = \frac{1}{2}(t - t_{coll})^{-1}, \qquad r_0(t) = (2T)^{1/2}(t - t_{coll})^{1/2}$$

For $t \gtrsim t_{coll}$, the mass contained in the Dirac peak increases as

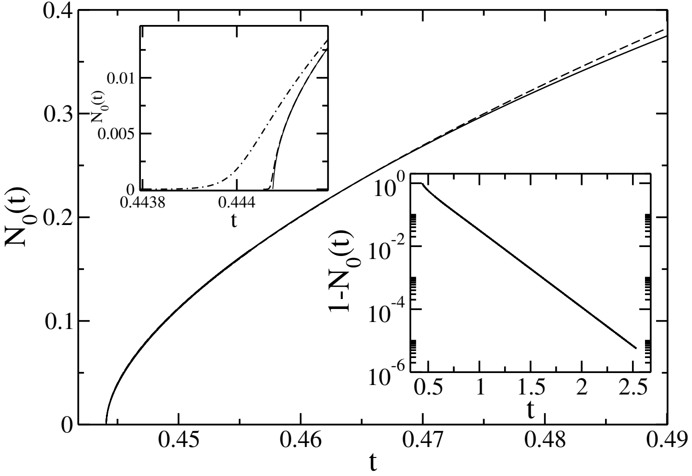
$$N_0(t) = 8.38917147...\sqrt{2}\,T^{3/2}(t - t_{coll})^{1/2}$$

For $t \to +\infty$, we find $1 - N_0 \sim e^{-\lambda(T)t}$. For $T \to 0$, using a semi-classical approach ($\hbar \leftrightarrow T$), we find

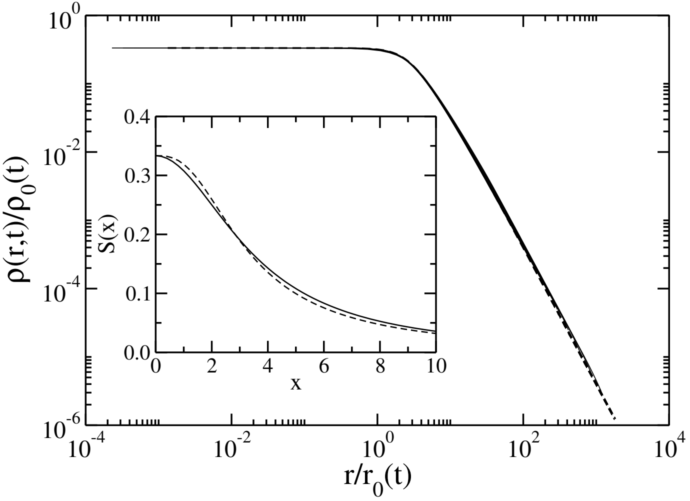
$$\lambda = \frac{1}{4T} + \frac{2.33810741...}{T^{1/3}}$$

For $T > 0$, the system develops a Dirac peak containing all the mass in infinite time. At $T = 0$ (deterministic motion), the Dirac peak containing all the mass is formed in a finite time.

Post-collapse : the growth of a Dirac peak



Post-collapse : the growth of a Dirac peak



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Self-gravitating fermions

We use a mean field approximation and look for the most probable distribution of self-gravitating fermions at statistical equilibrium.

- Microcanonical ensemble

$$\max_f \{S[f] \quad | \quad E[f] = E, M[f] = M\}.$$

- Canonical ensemble

$$\min_f \{F[f] = E[f] - TS[f] \quad | \quad M[f] = M\}.$$

where

$$S_{FD}[f] = -\int \left\{ \frac{f}{\eta_0} \ln \frac{f}{\eta_0} + \left(1 - \frac{f}{\eta_0}\right) \ln \left(1 - \frac{f}{\eta_0}\right) \right\} d\mathbf{r}\, d\mathbf{v}$$

Pauli exclusion principle : $\eta_0 = 2m^4/h^3$ represents the maximum value of the distribution function.

Degeneracy parameter

In the dimensionless equations appears the parameter :

$$\mu = \eta_0 \sqrt{512\pi^4 G^3 M R^3}$$

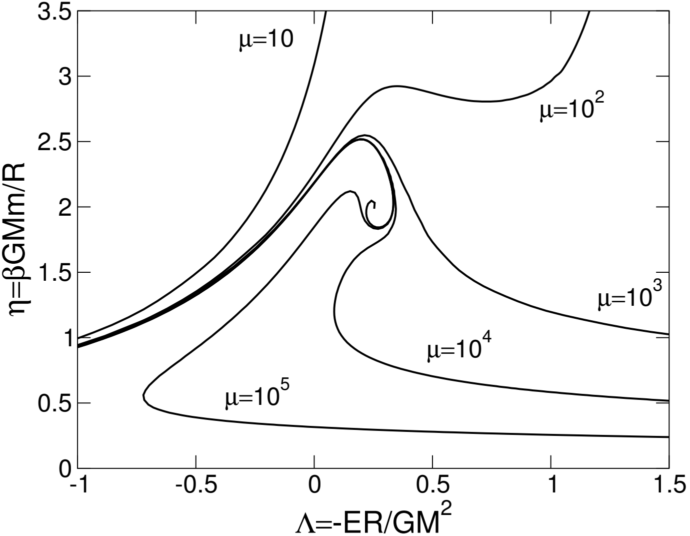
It can be written as

$$\mu \sim \frac{\eta_0}{\langle f \rangle} \sim \left(\frac{R}{R_*}\right)^{3/2} \sim \frac{1}{h^3}$$

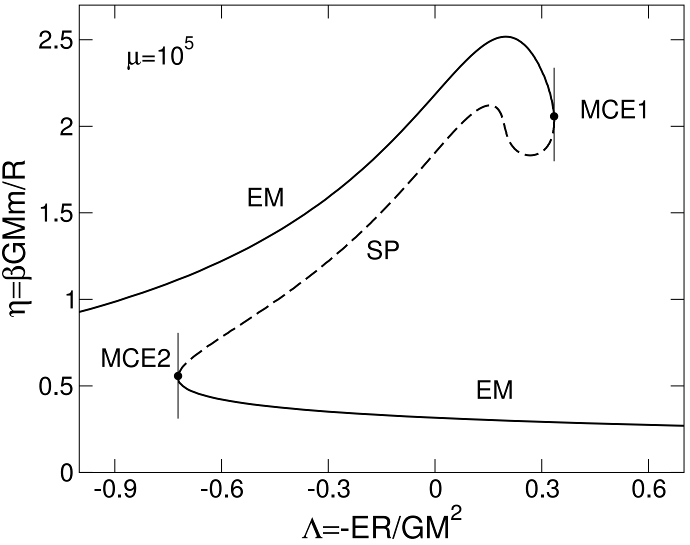
where $R_* = 0.181433 h^2 G^{-1} m^{-8/3} g^{-2/3} M^{-1/3}$ is the radius of a completely degenerate fermion ball (e.g. white dwarf star at $T = 0$) with mass M.

The classical limit correponds to $\mu \to +\infty$.

Dependence of the series of equilibria on the degeneracy parameter



The case of large systems : Z-shape caloric curve

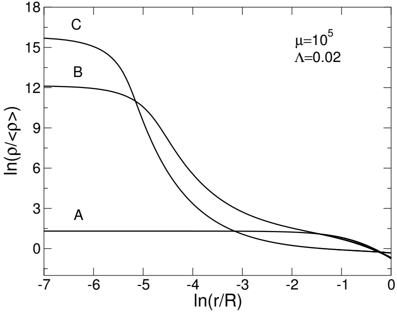


The case of large systems : density profiles

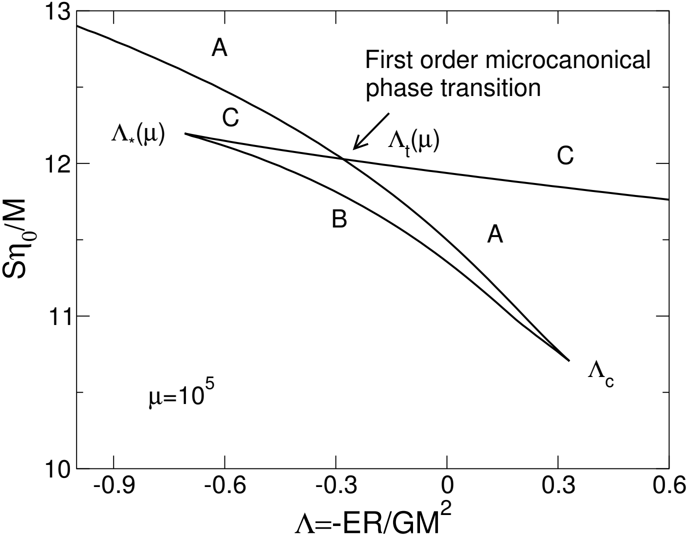
Fermi-Dirac-Poisson equation :

$$f = \frac{\eta_0}{1 + \lambda e^{\beta m \left(\frac{v^2}{2} + \Phi(\mathbf{r})\right)}},$$

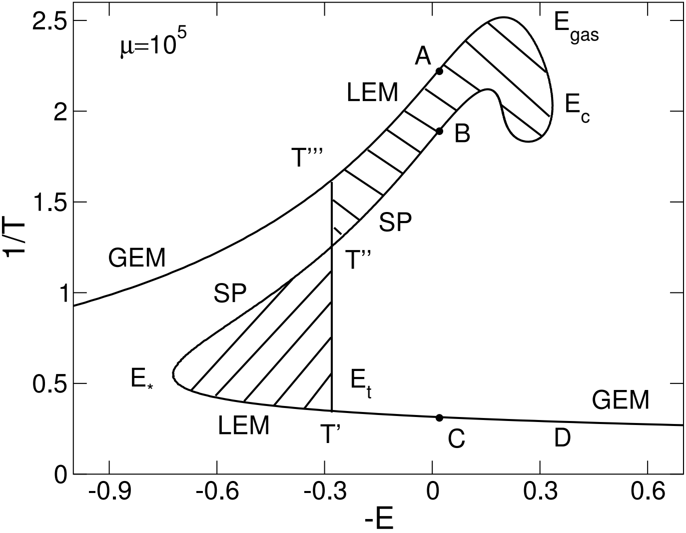
$$\Delta\Phi = 4\pi G \int f \, d\mathbf{v}.$$



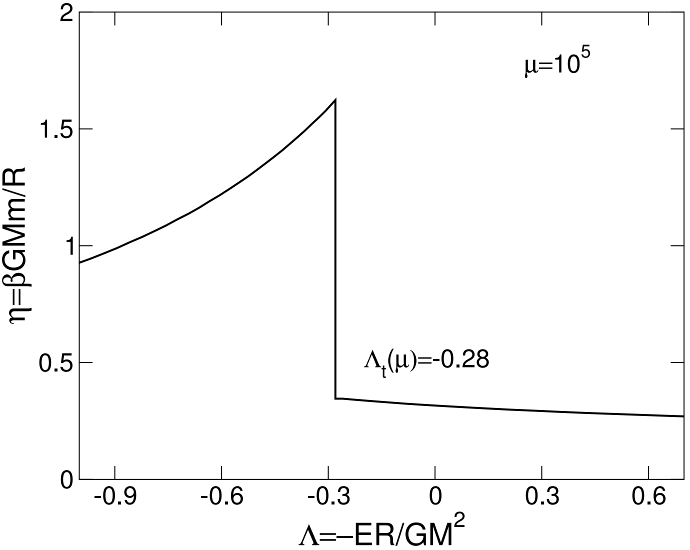
The case of large systems : microcanonical first order phase transition



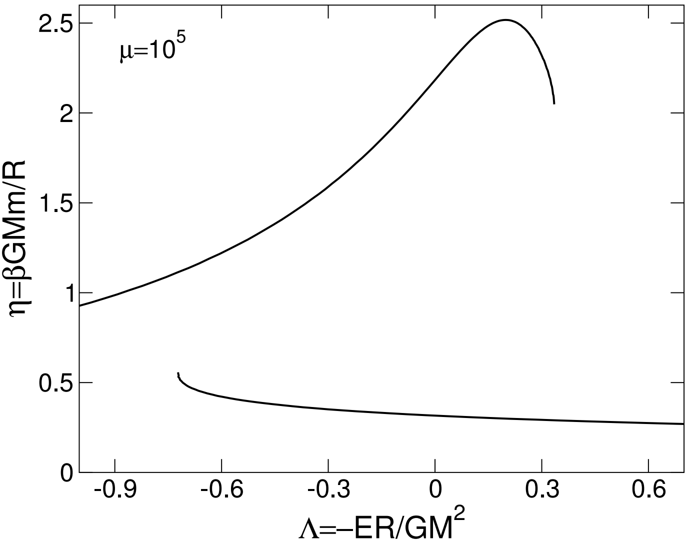
The case of large systems : vertical Maxwell construction



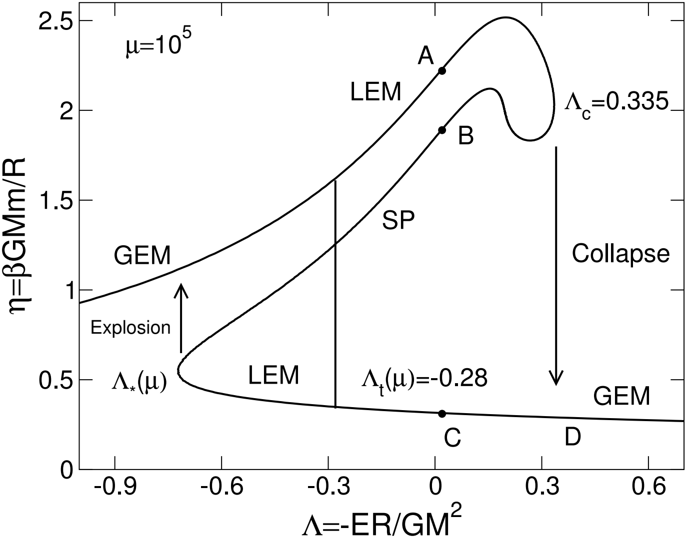
The case of large systems : strict caloric curve



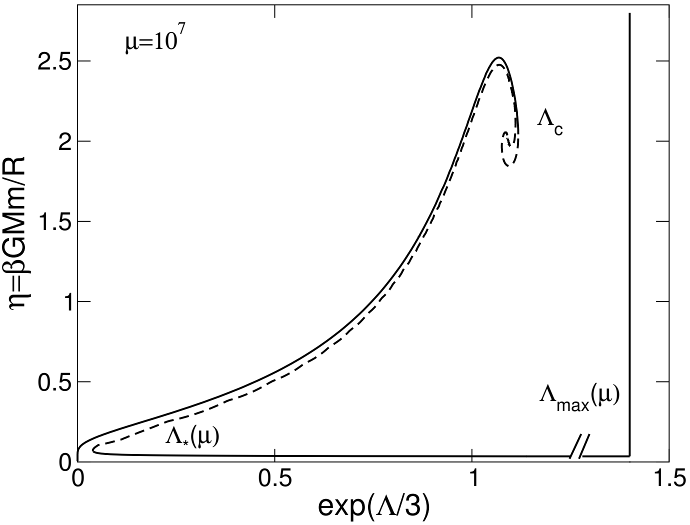
The case of large systems : physical caloric curve



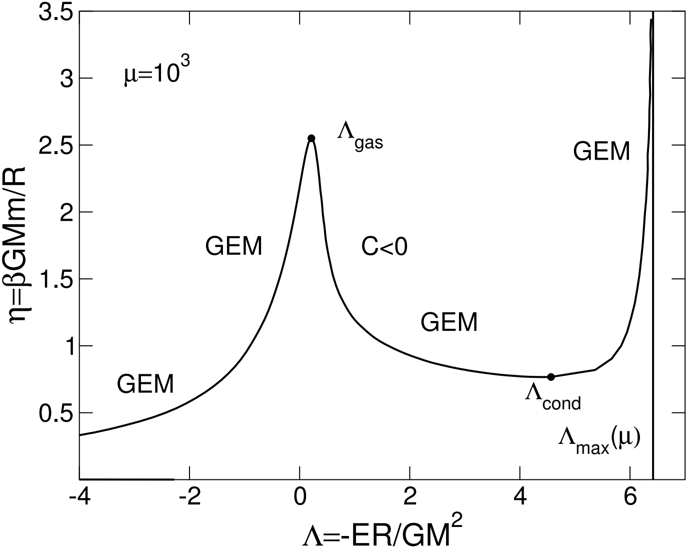
The case of large systems : summary



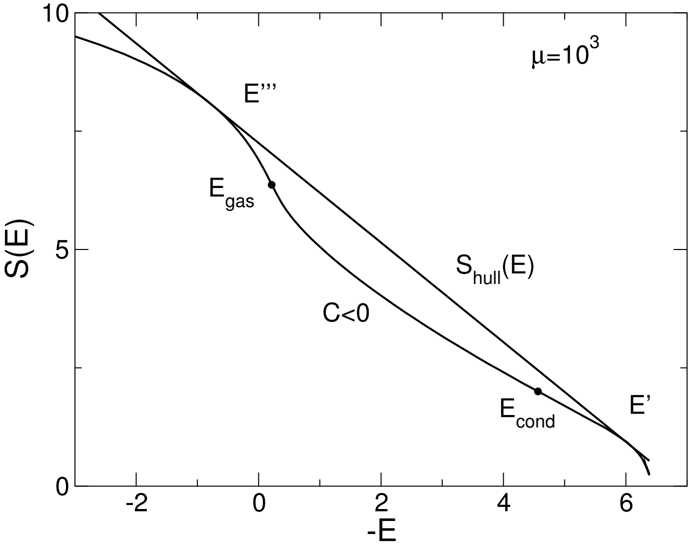
The classical limit $\mu \to +\infty$



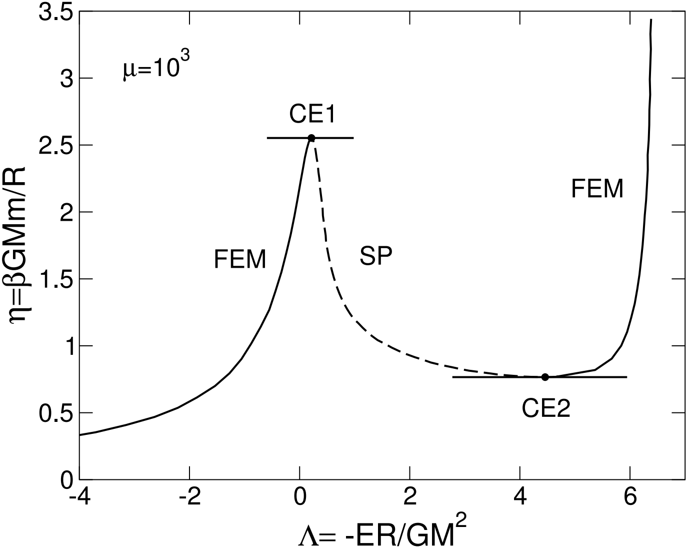
The case of small systems : microcanonical caloric curve



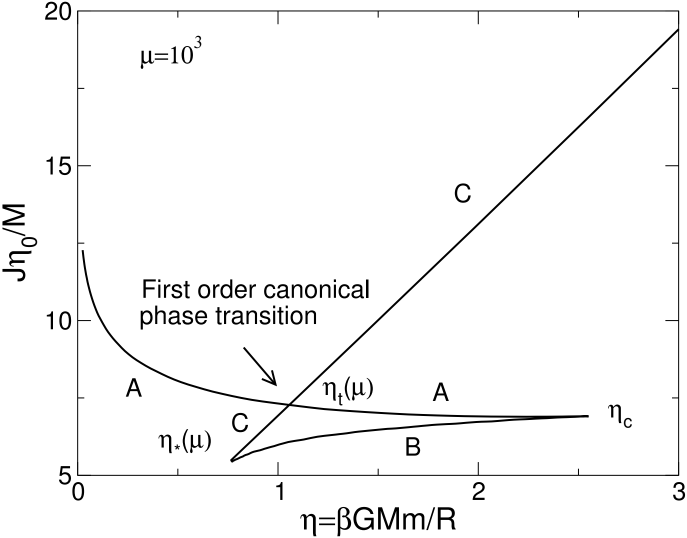
The case of small systems : convex dip



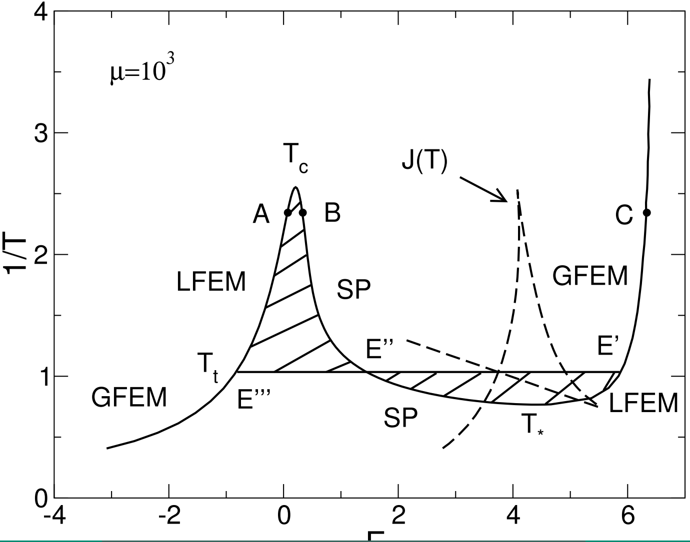
The case of small systems : N-shape caloric curve



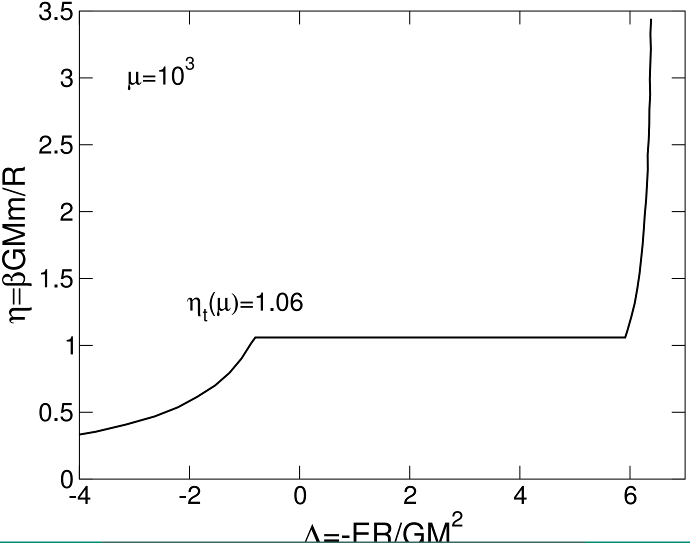
The case of small systems : canonical first order phase transition



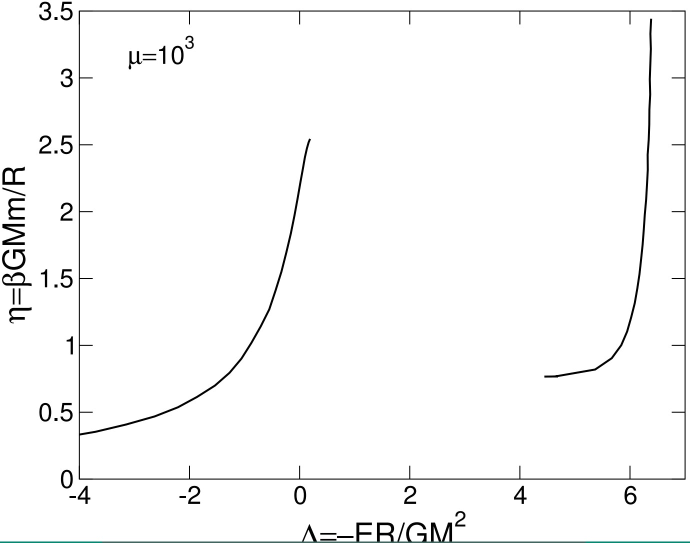
The case of small systems : horizontal Maxwell construction



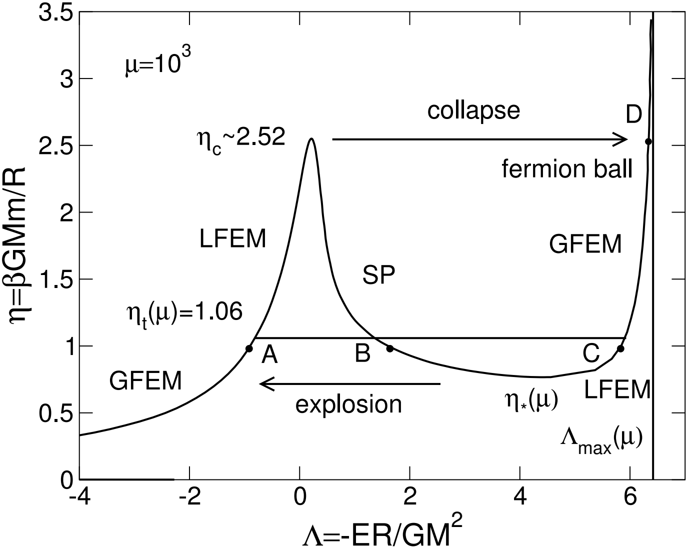
The case of small systems : strict canonical caloric curve



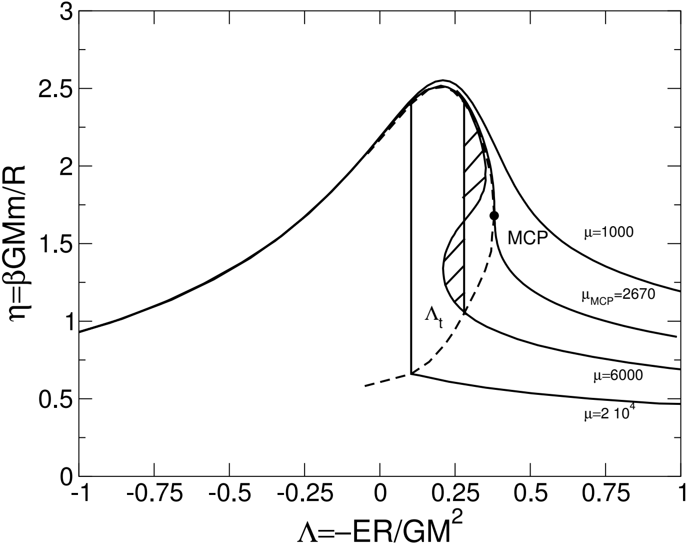
The case of small systems : physical canonical caloric curve



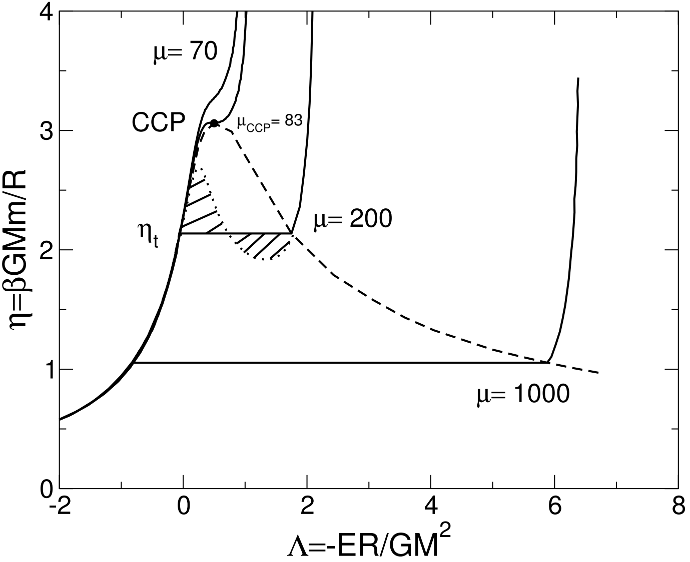
The case of small systems : summary



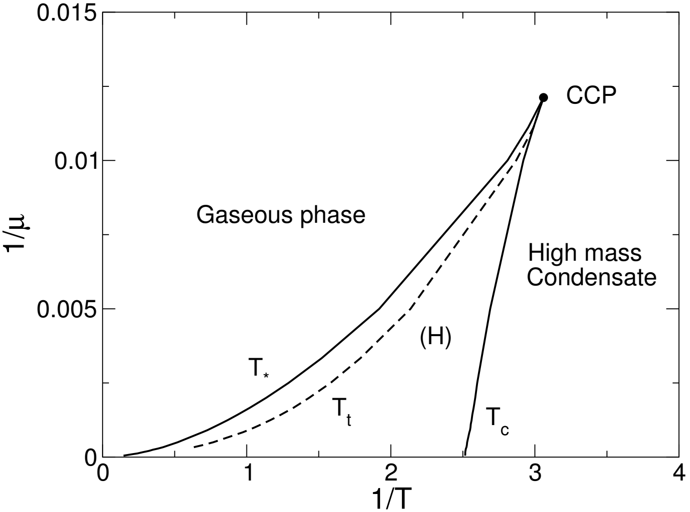
The case of large systems : microcanonical critical point



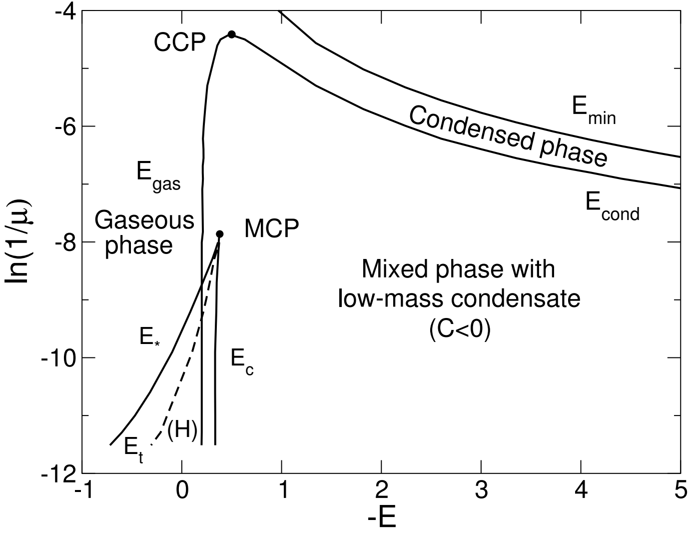
The case of small systems : canonical critical point



Canonical phase diagram



Microcanonical phase diagram



References

See a detailed list of references in :

P.-H. Chavanis, *Phase transitions in self-gravitating systems*, Int. J. Mod. Phys. B **20**, 3113 (2006)