Probability and Statistical Physics in Two (and More) Dimensions

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The sunny Brazilian peninsula of Buzios made for a perfect location for the 2010 Clay Mathematics Institute Summer School. The goal of the school was to provide a complete picture of a number of recent and groundbreaking developments in the study of probability and statistical physics in two and more dimensions. In the past ten to fifteen years, various areas of probability theory related to rigorous statistical mechanics, disordered systems, and combinatorics have enjoyed an intensive development with regards to two-dimensional random structures. Progress has come mainly in two forms: understanding large-scale properties of lattice-based models (on a periodic, deterministic lattice or in the case where the lattice is itself random), and directly constructing and manipulating continuous objects that describe these scaling limits. These themes guided the three foundational courses around which the first two weeks centered:

- Large random planar maps and their scaling limits by Jean-Francois Le Gall and Gregory Miermont
- SLE and other conformally invariant objects by Vincent Beffara
- Noise-sensitivity and percolation by Jeffrey Steif and Christophe Garban

Building on the foundations of the first two weeks, a variety of mini-courses covered very exciting and recent research:

- Random geometry and Gaussian free field by Scott Sheffield
- Conformal invariance of lattice models by Stanislav Smirnov
- Integrable combinatorics by Philippe Di Francesco
- Fractal and multifractal properties of SLE by Gregory Lawler
- The double-dimer model by Rick Kenyon

The fourth week of the school was held jointly with the XIV Brazilian School on Probability and focused on two main courses:

- Random polymers by Frank den Hollander
- Self-avoiding walks by Gordon Slade

Tutorials were organized for all courses and enabled the students to do hands-on work on the proofs of results mentioned in the lectures as well as to get familiar with numerous explicit examples. Teaching assistants were Curien, Duminil-Copin, and Freij for the fundamental courses and Alberts, Bauerschmidt, Caravenna, Goodman, Hongler, Pétrélis, and Werness for the mini-courses. Evening research talks supplemented the courses and mini-courses and were interspersed among the four weeks. The speakers included Adams, Benjamini, Biskup, Dubédat, Duplantier, Garcia, Ioffe, Koenig, Kozma, Le Jan, Maas, Mountford, Mytnik, Nolin, Peres, Sidoravicius, Turova, and van der Hofstadt. Students also organized a lunch-time seminar in which they could present their own work.

Much of the school was concerned with statistical physics models on lattices. Such models are random processes indexed by the vertices or the edges of a lattice, often considered to be a planar periodic graph (such as $\mathbf{Z}^2$). Each index point has a *spin* that takes values in a finite alphabet, typically {0,1}. The energy of a configuration of spins σ is given by a *Hamiltonian* $H(\sigma)$ and the probability of seeing σ is proportional to a *Gibbs factor* $e^{-\beta H(\sigma)}$ where β is called the *inverse temperature*. Different Hamiltonians give rise to different Gibbs measures and in particular to different behaviors for various natural *observables* such as interfaces between regions of 0's

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Frank den Hollander (Leiden University)
Rick Kenyon (Brown University)
Jean-François Le Gall (Universite Paris-Sud)
Gregory Lawler (University of Chicago)
Gregory Miermont (Universite Paris-Sud)
Scott Sheffield (MIT)
Gordon Slade (University of British Columbia)
Stanislav Smirnov (Université de Genève)
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and 1's. A variety of such models was introduced in the last century by physicists to study the properties of matter. The *Ising model* is perhaps the most famous lattice model (and received ample attention during the school).

Bernoulli percolation is another important lattice model with a particularly simple Gibbs measure, namely, the product measure such that each index point has spin 0 or 1 independently with probability 1-p and p. All observables of this process can thus be expressed as Boolean functions of these spins. Garban and Steif's course focused on an innovative approach to the study of this model via the Fourier transform of these Boolean functions. Using results of theoretical computer scientists on the stability of Boolean functions, they studied the sensitivity of critical percolation to small perturbations. This *noise sensitivity* is measured in terms of the spectrum of the Boolean functions and gives precise estimates on the *influence* of different spins—essentially a measure of the contribution of the spin at a particular index point to the probability of an event. Key concepts of *pivotality* and *revealment* were introduced as was a dynamical version of percolation. In addition, Garban and Steif showed how randomized algorithms may be used to approximate percolation interfaces at low computation cost.

Complementing the discrete approach of Garban and Steif, much of the rest of the school focused on studying lattice models from the perspective of determining their scaling limits and deducing properties of the discrete models from these continuum limits. A long-held belief among statistical physicists is that scaling limits of critically tuned lattice models will display a great deal of universality with respect to perturbations of the lattice or model. For instance, it is believed that regardless of the lattice, the scaling limit of the interfaces between 0's and 1's in percolation with critically tuned probability p will converge (in law) to the same random collection of curves and that this limit will be *conformally invariant* (i.e., invariant in law under the action of conformal maps). Similar beliefs exist for other lattice models (like Ising). Significant progress was made about ten years ago with the introduction of the *Schramm-Loewner evolution* (SLE) which is a one-parameter family of measures on curves that should serve as the basis for the critical scaling limits of a variety of lattice models.

Beffara's course focused on rigorously defining these random curves and using them to describe the scaling limits and critical exponents (governing, for instance, correlation length and crossing probabilities) of a variety of lattice models. As an illustration of the power of these techniques, Beffara presented a proof (adapted from Smirnov's work) of Cardy's formula for the probability that there exists a connected path of 1's between two opposite sides of a large

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rectangle (in a particular lattice called the honeycomb lattice). Beffara also presented results on the geometry of the random curves, showing that the Hausdorff dimension of the SLE_κ is $1+\kappa/8$ (κ is the aforementioned parameter for this family of measures). Werner built upon this with some further techniques necessary to translate these continuum results into analogous statements about lattice models. Lawler went into more technical details about the path properties of SLE including the rigorous proof of their existence and Hölder continuity, as well as their natural time parameterization and the reverse Loewner flow.

In his mini-course, Smirnov explained the theory he has developed to prove scaling limits for a variety of models, which now includes both percolation and the Ising model (and more generally the *random-cluster* model). His approach emphasized the link between statistical physics in two dimensions and discrete complex analysis. In particular he presented a number of observables of models (such as his *parafermionic* observable) that can be shown to be discrete holomorphic (or preholomorphic). In certain cases these observables have been shown to converge to continuous holomorphic functions (as expected by the physics belief of conformal invariance of scaling limits). As Beffara had explained in his course, using this result as well as methods involving martingales, it is then possible to prove convergence of the entire interface of the lattice model to an SLE. To further emphasize the deep link between lattice models and complex analysis, Smirnov also gave a beautiful constructive proof of the Riemann mapping theorem via a discrete approximation using an appropriate measure on uniform spanning trees.

To round out the study of lattice models, Kenyon's mini-course and Dubédat's evening talk focused on the *dimer* model (related to perfect matchings in graph theory). In particular, Kenyon lectured on the *double-dimer* model, which provides a natural measure on non-intersecting loops. He gave evidence for the conformal invariance of the scaling limit of this model that is conjectured to be given by a variant of SLE called CLE_4 (the conformal loop ensemble that looks locally like SLE_4). Di Francesco studied a variety of other models in statistical physics using techniques from integrable—exactly solvable systems such as the Yang-Baxter equations, the transfer matrix approach, and formulas coming from representation theory.

Random discrete surfaces and Riemannian manifolds play essential roles in combinatorics and statistical physics, as does the study of lattice models on these surfaces. For example, significant progress in theoretical physics has been made in the last thirty years from the understanding that in string theory and gauge theory one should sum over random surfaces (as opposed to over random paths as in Feynman's formulation of quantum mechanics). Le Gall and Miermont approached this subject from the discrete side in their course on large random planar maps and their scaling limits. A *random map* provides a very natural approach to defining a discrete random surface and its accompanying metric. They explained an important line of recent progress in understanding the large limit of these random surfaces and metrics. Under an appropriate re-scaling of distances there exist (sub-sequential) limits of these discrete metrics that are called the *Brownian map*. Figuring prominently during the course was the so-called Bijective approach that emphasizes how these maps can be encoded and studied in terms of a correspondence with certain decorated plane trees.

On the continuum side, there exists another formulation (believed to be equivalent to the limit of the large planar maps) for a random geometry, which is called *Liouville quantum gravity*. This random geometry was the subject of the mini-course by Scott Sheffield and the evening talk of Bertrand Duplantier. Sheffield's course built on the foundational courses of both Le Gall and Miermont, as well as Beffara. Based on very recent work, Sheffield showed that SLE arises when gluing (via conformal welding) two random geometries together and then conformally mapping the result to the plane. Duplantier spoke of other exciting connections between statistical physics models on deterministic lattices and on random geometries. In particular he showed how to prove the KPZ formula that relates scaling exponents and fractal dimensions between the two types of geometries. This shows how, by studying lattice models in a random geometry, one can gain information about the geometry itself. In fact, in his evening talk, Benjamini emphasized this perspective by explaining how the study of two basic lattice models (percolation and random walks) on deterministic and random graphs provides a great deal of information in describing properties of the geometry itself.

The fourth week of the school (held jointly with the XIV Brazilian School on Probability) shifted focus to the study of random path measures on $\mathbf{Z}^d$. These paths are commonly called *polymers* and den Hollander's mini-course focused on the general theory of polymers, while Slade's mini-course delved deep into a particularly important polymer, called the *self-avoiding walk* (SAW). The SAW is a measure on paths of a fixed length that assigns equal probability to each nearest neighbor path starting at the origin and never intersecting itself. More generally, a random polymer is usually defined by a Gibbs measure (on the set of paths of a fixed length) with a Hamiltonian that may take into account the self-interactions, self-avoidance, and possibly interactions with a (possibly random) environment.

Many of the important questions about polymers and the SAW are expressed in terms of their asymptotic behavior, as their length goes to infinity. Important questions include studying the growth in the number of such collections of paths, the behavior of the mean square displacement, and the possibility of critical scaling limits. This last point drew us back to the subjects covered in the first three weeks of the school. In fact, Beffara had visited this subject in his course and explained that, while not rigorously established, if the scaling limit for the SAW in two dimensions exists and is conformally invariant, then it ought to be given by $SLE_{8/3}$, due to the fact that it naturally would inherit a certain *restriction property* from the discrete SAW and that this property is only verified by SLE_κ for the value $\kappa = 8/3$.

Much of Slade's course focused, however, on a large number of rigorous results about the critical behavior of the SAW in dimensions $d = 4$ and $d \geq 5$. He introduced a few critical exponents (governing how the SAW scales and behaves when very long) and explained the universality of these exponents, relations between them, and how they change according to the dimension. Slade then introduced the *Lace Expansion* and showed its convergence for sufficiently high dimension (which can be reduced to $d \geq 5$) which, in turn, implies that the number of length n SAWs grows purely exponentially, just as with the simple random walk. For $d = 4$, Slade proved exact functional integral representation formulas for the two-point function of the continuous weakly SAW and showed how, via a renormalization-group analysis, these formulas prove that the two-point function decays as the inverse square of distance.

Rather than totally excluding self-intersection, one may consider polymers whose Hamiltonian is a function of the number of self-intersections. In fact, a variety of other energetic rewards (or punishments) are important and studied. For example, the path can interact with a linear surface with energy dependent on random charges along the surface. Alternatively, the entire lattice may have charges, providing a random potential in which the path arranges itself. A critical simplification to these polymers used in den Hollander's course was to consider directed, or semi-directed, versions of polymers (thus avoiding some of the complexities of the SAW). In his course, large deviations served as a central tool to prove several recent results related to the existence of phase transitions in the behavior of polymers. These included results about collapse, localization, and pinning of polymers interacting with a linear surface, and about the diffusivity of the polymer endpoint.

Probability and statistical physics in two and more dimensions have recently benefited from the introduction of a variety of important and powerful new tools and techniques. The summer school was held at a perfect time: a few important problems in the field have recently been solved, but many other important open problems remain unsolved. It is likely that some of these will yield eventually to variants of these new tools and techniques. Thanks to the school, a new generation of mathematicians has been made aware of these problems and new approaches. Perhaps the main theme underlying this school was that there exist certain universal classes of continuum scaling limits that underlie and unite many discrete lattice models and random geometries. This theme will likely echo for many years in the work of those who participated in the 2010 Clay Mathematics Institute Summer School.

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