Matrix models with singular potentials

Yuri Makeenko (ITEP, Moscow)

Based on:

- J. Ambjørn, T. Budd, Y. M.
  Generalized multicritical one-matrix models
  Nucl. Phys. B913 (2016) 357 [arXiv:1604.04522 [hep-th]]
- J. Ambjørn, L. Chekhov, Y. M.
  Perturbed generalized multicritical one-matrix models
  Nucl. Phys. B928 (2018) 1 [arXiv:1712.03879]
- Y. M. unpublished notes

Talk at Lyon June 28, 2018

Content of the talk

- 1-MAMO: from regular to singular potential
  - genus zero

- Scaling limit and critical indexes
  - generalized Kazakov's potential

- The moments and higher genera

- Generalization of KdV hierarchy

1. Introduction

Hermitian one-matrix model

Ensemble of random matrices.

$$Z_{1\mathrm{h}} = \int \mathrm{d}\varphi \, \mathrm{e}^{-N \mathrm{tr} V(\varphi)} \qquad V(\varphi) = \sum_i g_i \, \mathrm{tr} \varphi^i$$

where $\mathrm{d}\varphi$ is the measure for integrating over Hermitian $N \times N$ matrices. Representing $\varphi = U P U^\dagger$ with unitary U and diagonal $P = \mathrm{diag}\,\{p_1, \ldots, p_N\}$, $\mathrm{d}\varphi$ can be written in a standard Weyl form

$$\mathrm{d}\varphi = \mathrm{d}U \prod_{i=1}^{N} \mathrm{d}p_i \, \Delta^2(P) \, ,$$

where $\Delta(P) = \prod_{i<j} \left(p_i - p_j \right)$ is the Vandermonde determinant.

Angular degrees of freedom residing in U factorize, so Z_{1h} depends on the N eigenvalues of φ. Thus the saddle point applies at large N: N integrals but the action

$$N \mathrm{tr} V(\varphi) = N \sum_{i=1}^{N} V(p_i) \sim N^2$$

Hermitian one-matrix model (cont.)

Large N addle-point equation Brézin, Itzykson, Parisi, Zuber (1978)

$$V'(p) = 2 \fint d\lambda \frac{\rho(\lambda)}{p-\lambda} \qquad \boxed{p \in \text{support of } \rho}$$

for the (continuous nonnegative normalized) spectral density

$$\rho(p) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta^{(1)}(p-p_i)$$

which describes the distribution of eigenvalues of φ.

For polynomial $V(p)$ the simplest is one-cut solution when $\rho(p)$ has support on a single interval $[a,b]$ like Wigner's semicircle law

$$\rho(p) = \frac{M(p)}{2\pi}\sqrt{(p-a)\,(b-p)}$$

Here a and b are the ends of the cut and $M(p)$ is a polynomial of degree $K{-}2$ if $V(p)$ is a polynomial of degree K.

One-cut solution works if $M(p) \geq 0$ for $p \in [a,b]$ which always happens for small couplings g_3, g_4, *etc.* With increasing couplings a more complicated multi-cut solution is realized.

Logarithmic singular potential

Y.M. (1993, 1995)

Logarithmic potential (V has two cuts while V' has two poles)

$$V(\phi) = (\alpha + 1)\ln(\beta + \phi) - \alpha \ln(\beta - \phi) - 2\beta\phi$$

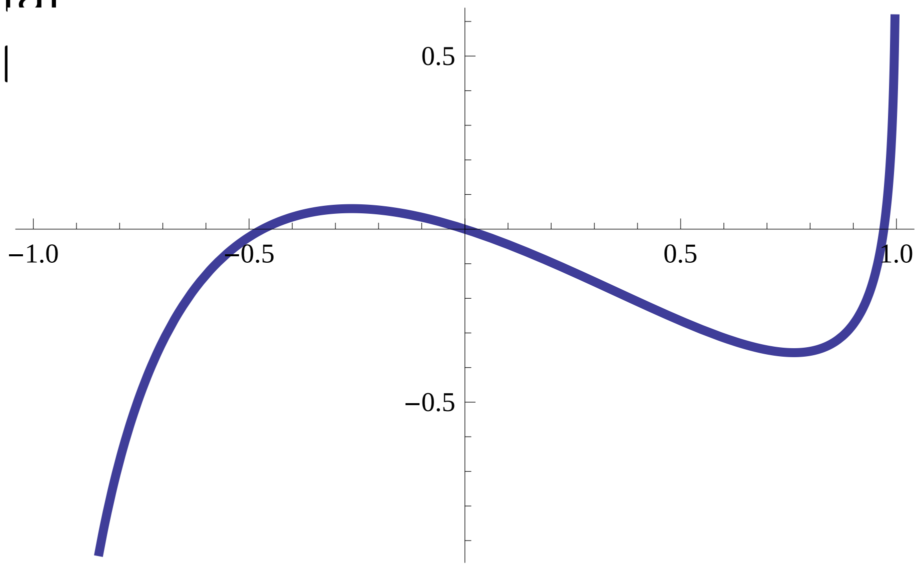
and we can set $\beta = 1$ without loss of generality

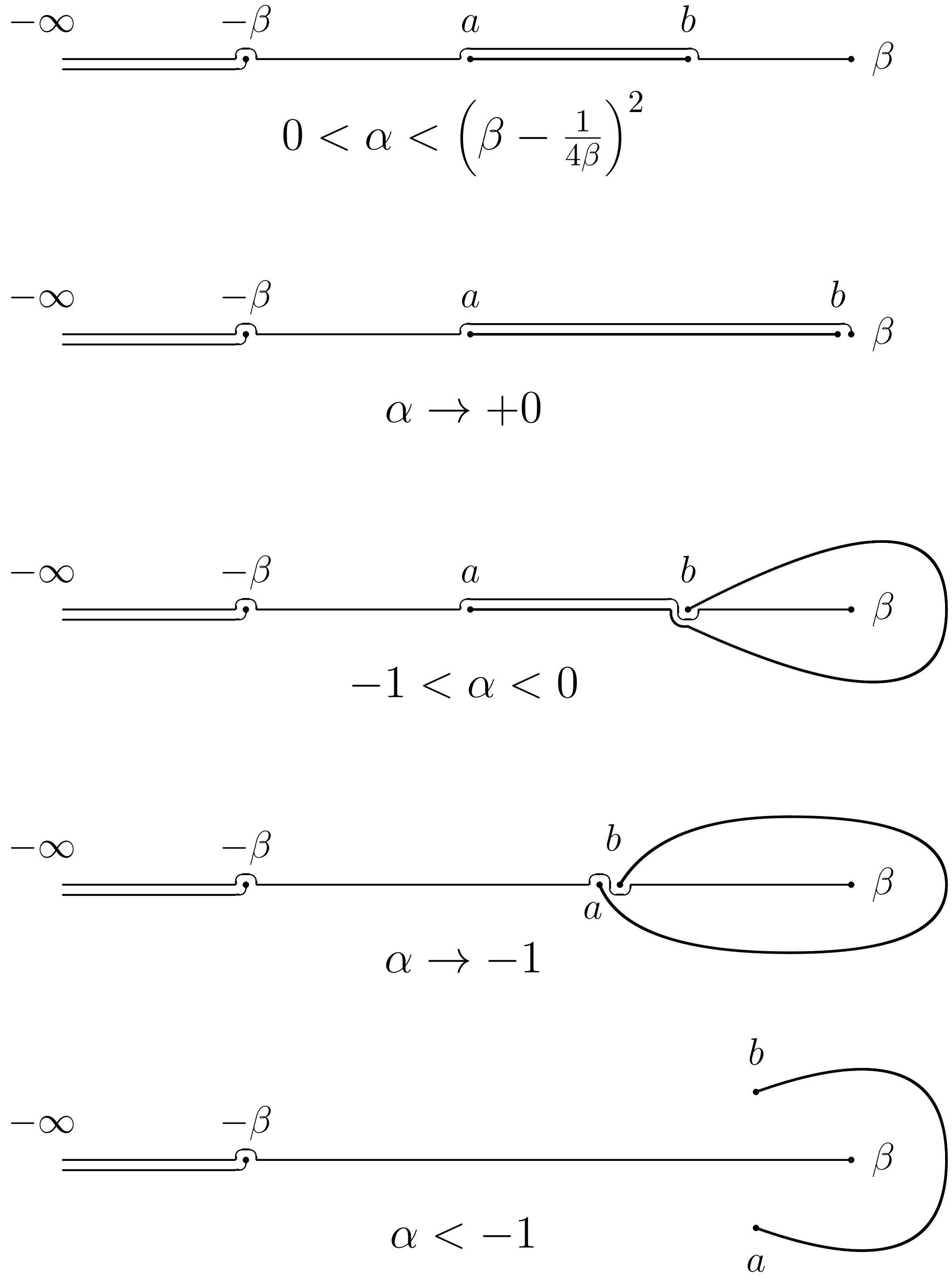
The cut from a to b always avoids singularities of V

Limiting cases:

- quadratic potential
- Penner potential
- cubic potential

$\alpha = 0.3$
$\beta = 1$





$-\infty$
$-\beta$
a
b
β
$0 < \alpha < \left(\beta - \frac{1}{4\beta}\right)^2$
$-\infty$
$-\beta$
a
b
β
$\alpha \to +0$
$-\infty$
$-\beta$
a
b
β
$-1 < \alpha < 0$
$-\infty$
$-\beta$
b
β
a
$\alpha \to -1$
b
$-\infty$
$-\beta$
β
$\alpha < -1$
a

2. From Riemann-Hilbert method to Loop equation

Riemann-Hilbert at work

Inspired by Le Gall, Miermont (2011), Borot, Bouttier, Guitter (2012)

Let V' has cuts and poles at the real axis. To solve the integral saddle-point equation

$$V'(x) = 2 \fint \mathrm{d}y \frac{\rho(y)}{x-y} \qquad \boxed{x \in \text{support of } \rho}$$

we introduce the analytic function

$$W(z) = \int \mathrm{d}y \frac{\rho(y)}{z-y} \qquad W(z) \stackrel{z\to\infty}{\to} \frac{1}{z}$$

and rewrite the equation on the real axis as

$$\Im\left(W^2 - V'W\right) + \Im V' \Re W = \Im W \left(2\Re W - \Re V'\right) = 0$$

Usually, the term with $\Im V'$ is missing since V' is real at the real axis

This implies the following equation in the whole complex plane:

$$W^2(z) - V'(z)W(z) + \int_{C_2} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}} \frac{V'(\omega)W(\omega)}{(z-\omega)} = Q(z),$$

where C_2 encircles possible cuts and poles of $V'(\omega)$ at the real axis, leaving outside z and the cut(s) of $W(\omega)$. $Q(z)$ is an entire function

Riemann-Hilbert at work (cont.)

Deforming the contour, we can rewrite the latter equation as

$$W^2(z) - \int_{C_1} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}} \frac{V'(\omega)W(\omega)}{(z-\omega)} = 0,$$

where C_1 encircles (clockwise) the cut(s) of $W(\omega)$, leaving outside z and possible cuts and poles of $V'(\omega)$. The difference between C_1 and C_2 is the residuals at $\omega = z$ and $\omega = \infty$ which equals $Q(z)$.

We got the usual loop equation of the one-matrix model at $N = \infty$ whose standard derivation by an infinitesimal shift of φ works for all potentials, including the ones with cuts at the real axis
$\Longrightarrow$ the usual (Migdal's) formula for the one-cut solution

$$W(z) = \int_a^b \frac{\mathrm{d}x}{2\pi} \frac{V'(x)}{(z-x)} \frac{\sqrt{(z-a)(z-b)}}{\sqrt{(x-a)(b-x)}}, \qquad W(z) \stackrel{z\to\infty}{=} \frac{1}{z}$$

For even $V(x) = V(-x)$ we have $a = -b$ and

$$W(z) = \int_0^b \frac{\mathrm{d}x}{\pi} \frac{xV'(x)}{(z^2-x^2)} \frac{\sqrt{z^2-b^2}}{\sqrt{a^2-x^2}}, \qquad \int_0^b \frac{\mathrm{d}x}{\pi} \frac{xV'(x)}{\sqrt{b^2-x^2}} = 1$$

Simplest example: logarithmic potential

Simplest potential with V' having a cut at the real axis from 1 to ∞

$$V(x) = \frac{1}{g}\left[(1-x)\log(1-x) + x\right] = \frac{1}{g}\sum_{n=1}^{\infty}\frac{x^{n+1}}{n(n+1)},$$

$$V'(x) = -\frac{1}{g}\log(1-x) = \frac{1}{g}\sum_{n=1}^{\infty}\frac{x^n}{n}$$

The one-cut solution

$$W(z) = \frac{1}{g}\left[\operatorname{actanh}\sqrt{\frac{(z-b)}{(z-a)}} - \operatorname{actanh}\sqrt{\frac{(1-a)(z-b)}{(1-b)(z-a)}} - \frac{1}{2}\log(1-z)\right]$$

$$a = b - 4\left(1-\sqrt{1-b}\right), \qquad g = \frac{(b-a)^2}{16} = \left(1-\sqrt{1-b}\right)^2.$$

Simplest example: logarithmic potential (cont.)

The solution has all required properties: analytic outside of $[a, b]$, reproduces Wigner's law as $g \to 0$ etc.. The discontinuity across the cut determines the (normalized) spectral density

$$\rho(x) = \frac{1}{\pi g}\left[\arctan\sqrt{\frac{(1-a)(b-x)}{(1-b)(x-a)}} - \arctan\sqrt{\frac{(b-x)}{(x-a)}}\right]$$

which indeed obeys the saddle-point equation as can be checked.

The spectral density is positive for $b < 1$, vanishes at the ends of the cut, but looks pretty different from the previously known cases, where V' has no cut at the real axis. In those cases ρ had a square root singularity, which is now hidden under the arctan.

Critical behavior is reached as $b \to 1$, when

$$g \to g_* - 2\sqrt{1-b}, \qquad g_* = 1$$

3. Generalized Kazakov's potentials

Multi-critical long-tail potential

Ambjørn, Budd, Y.M. (2016)

Singular potential with a cut for $x \geq 1$

$$V(x) = \sum_{n=1}^{\infty} \frac{1}{4g} \frac{\Gamma(n+\frac{1}{2}-s)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2}-s)\Gamma(n+\frac{1}{2})\, n} x^{2n} = \frac{1}{g}\, {}_3F_2\left(1,1,\frac{3}{2}-s;2,\frac{3}{2};x^2\right)\frac{x^2}{2}$$

For $s \neq m+1/2$ the coefficients behave as x^{2n}/n^{s+1} for $n \to \infty$.
For $s = m+1/2$ the infinite sum terminates at $n = m$ giving the m^{th} multi-critical Kazakov potential.

We have

$$xV'(x) = \frac{1}{g}\, {}_2F_1\left(1,\frac{3}{2}-s,\frac{3}{2},x^2\right) x^2 \qquad \sim (1-x)^{s-1} \text{ as } x \to 1$$

$$g(a^2) = \frac{1-(1-a^2)^{s-1/2}}{4(s-1/2)}, \qquad \frac{\mathrm{d}g}{\mathrm{d}a^2} = \frac{1}{4}(1-a^2)^{s-3/2}, \qquad g_* = \frac{1}{4(s-1/2)}$$

which is the most obvious generalization to $s \neq m+1/2$

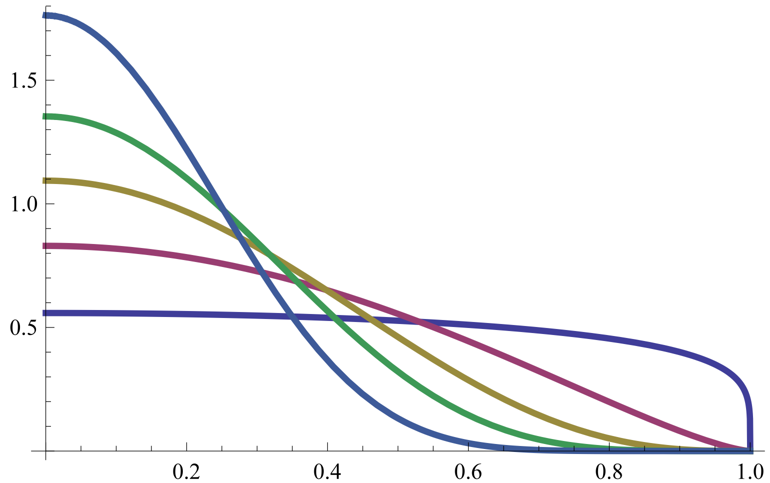
Multi-critical long-tail potential (cont.)

One-cut solution

$$W(z) = \frac{1}{2}V'(z) - \frac{1}{2g}\,{}_2F_1\left(1, \frac{3}{2} - s, \frac{3}{2}, \frac{z^2 - a^2}{1 - a^2}\right)(1 - a^2)^{s-\frac{3}{2}}\sqrt{z^2 - a^2}$$

becomes the standard one for the polynomial Kazakov potential.
Several equivalent forms possible because of Kummer's relations.
The discontinuity of $W(z)$ across the cut

$$\rho(x) = \frac{{}_2F_1\left(1, s, \frac{3}{2}, \frac{a^2-x^2}{1-x^2}\right)(1 - a^2)^{s-\frac{1}{2}}\sqrt{a^2 - x^2}}{2\pi g(1 - x^2)} \overset{a\to 1}{\to} \frac{\Gamma(s + \frac{1}{2})}{\sqrt{\pi}\Gamma(s)}(1-x^2)^{s-1}$$



Plot of $\rho(x)$ versus x for $s = 1.2,\ 2.4,\ 4,\ 6,\ 10$ from bottom to top:
$\rho(x) > 0$ in $x \in]-a, a[$, $\rho(x) = 0$ at $x = \pm a$ $\rho(x) \to \delta^{(1)}(x)$ as $s \to \infty$

Scaling limit

Let $g \to g_* = \dfrac{1}{4(s-1/2)}$ so that $a^2 = 1 - \left(1 - \dfrac{g}{g_*}\right)^{\frac{1}{s-\frac{1}{2}}} \to 1$

Susceptibility index γ_s:

$$\chi(g) = \left(g\frac{\mathrm{d}}{\mathrm{d}g}\right)^2 \frac{1}{N^2} \log Z = \chi(g_*) + c(g_* - g)^{-\gamma_s} + \text{less singular.}$$

Expanding Z in $(1-a^2)$ we find

$$\gamma_s = -\frac{1}{s-\frac{1}{2}}$$

For $s \in]m-1/2, m+1/2[$ the potential has many features of the $s = m+1/2$ multicritical potential: the first m terms have oscillating signs, the signs of terms x^{2n}, $n \geq m$ are the same.
γ_s interpolates between the values $-1/m$ of the multicritical points.
For $3/2 < s < 5/2$ the coefficients of the Taylor expansion of $V(x)$ beyond quadratic are negative (unitarity)
Otherwise, the same relation to the minimal conformal models (KPZ) as for Kazakov's multicritical potentials

4. Generalized moments

The moments

Introduced in Ambjørn, Chekhov, Y.M. (1992)
Inspired by Itzykson, Zuber (1992) for the Kontsevich model
Inspired by Y.M., Semenoff (1991) the Kontsevich model at genus zero
Elaborated in Ambjørn, Chekhov, Kristjansen, Y.M. (1993)

Moments or an even potential $V(x) = V(-x)$

$$M_n = \int_{C_2} \frac{\mathrm{d}z}{4\pi \mathrm{i}} \frac{zV'(z)}{(z^2 - a^2)^{n+1/2}}$$

The partition function Z to genus h depends on only $n \leq 3h - 2$ lower moments ($n \leq 3h - 2 + N$ for N-loop correlators)

$$F_1 = -\frac{1}{12} \log \left(M_1 a^2 \right), \qquad \text{etc.}$$

$$F_2 = -\frac{53}{120 M_1^2 \cdot 16 a^4} - \frac{181 M_2^2}{320 M_1^4 \cdot 4a^2} + \frac{43 M_3}{96 M_1^3 \cdot 4a^2}$$

(for slightly different moments).
Very nice algebraic structure to the next orders

But the scaling limit of M_n is not well defined for singular potentials

New moments for singular potential

Well-defined for the singular potential new moments ($\alpha = s - [s]$)

$$(-1)^{\alpha-1/2}\,\widetilde{M}_n = \int_{C_2} \frac{\mathrm{d}z}{4\pi\mathrm{i}} \frac{zV'(z)}{(z^2-a^2)^{n+1/2}(z^2-1)^{\alpha-1/2}},$$

where the extra factor compensates the singularity of the potential. The old moments M_n and the new moments are related by

$$M_n = \sum_{k=0}^{n-1} (-1)^k \frac{\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha-k-\frac{1}{2})k!}(1-a^2)^{\alpha-k-1/2}\widetilde{M}_{n-k}.$$

only the new moments with lower indexes enter this relation
$\alpha = 1/2$ for Kazakov's potential $\Longrightarrow$ $\widetilde{M}_n = M_n$

Continuum moments

$$\widetilde{\mu}_n \equiv 4\varepsilon^{n+1/2-s}\widetilde{M}_n = (-1)^{1/2-\alpha}\int_{C_2} \frac{\mathrm{d}Z}{2\pi\mathrm{i}} \frac{\widetilde{V}'(Z)}{(Z+\sqrt{\Lambda})^{n+1/2}Z^{\alpha-1/2}},$$

are finite in the scaling limit (ε has canceled)

$$a^2 = 1 - \varepsilon\sqrt{\Lambda}, \qquad z^2 = 1 + \varepsilon Z \qquad\qquad \varepsilon \to 0$$

where $\widetilde{V}(Z) = (-1)^{\alpha-1/2} Z^s$

Interpolating model

Sum over the generalized multicritical potentials

$$V'(z) = \sum_{n=1}^{\infty} \widetilde{T}_n \frac{\varepsilon^{-n-\alpha}}{\sqrt{G}} \frac{2\Gamma(n+\alpha+1)}{\sqrt{\pi}\Gamma(n+\alpha-\frac{1}{2})} {}_2F_1(1, 3/2-n-\alpha, 3/2, z^2) z.$$

$$(-1)^{\alpha-1/2} \widetilde{T}_n = \frac{2\sqrt{G}\varepsilon^{n+\alpha}}{n+\alpha} \int_{C_2} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{zV'(z)}{(z^2-1)^{n+\alpha}}$$

Continuum interpolating potential

$$\widetilde{V}(Z) = (-1)^{\alpha-1/2} \sum_{n=1}^{\infty} \widetilde{T}_n Z^{n+\alpha} \qquad \alpha = s - [s]$$

For Kazakov's multicritical points $s = m + 1/2$, so $\alpha = 1/2$.
Before we have $\widetilde{T}_n = \delta_{nm}$ with $m = [s - 1/2]$ Critical behavior is governed by the boundary equation

$$g = \sum_{n=1}^{\infty} \frac{\widetilde{T}_n[1-(1-a^2)^{n+\alpha-1/2}]}{4(n+\alpha-1/2)} \frac{\varepsilon^{-n-\alpha}}{\sqrt{G}} \frac{2\Gamma(n+\alpha+1)}{\sqrt{\pi}\Gamma(n+\alpha-\frac{1}{2})}$$

Interpolating model (cont.)

We tune $\widetilde{T}_n$'s for the critical behavior to be again

$$(g_* - g) \propto (1 - a^2)^{s-1/2} = (\varepsilon\sqrt{\Lambda})^{s-1/2}$$

with the normalization constant

$$T_0 = 4\sqrt{G\varepsilon}\left[\int_{C_2} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{zV'(z)}{\sqrt{z^2-1}} - 1\right]$$

(-1 comes from the normalization $W(z) \to 1/z$ as $z \to \infty$) to be finite for $\sqrt{G} = 1/\varepsilon^s$ that determines the associated double scaling limit, because the genus expansion goes in

$$\mathcal{G} = \frac{G}{N^2} = \frac{1}{N^2\varepsilon^{2s}}.$$

Equation for T_0 gives for the interpolating potential

$$T_0 = 4\varepsilon^{1/2-s}[\frac{g_*}{g} - 1], \qquad g_* = \sum_{n=1}^{\infty} \frac{\widetilde{T}_n \varepsilon^{s-n-\alpha}}{4(n+\alpha-1/2)} \frac{2\Gamma(n+\alpha+1)}{\sqrt{\pi}\Gamma(n+\alpha-\frac{1}{2})}$$

It is now clear that $\widetilde{T}_n$ with $n > m$ (m is the integer part of $s - 1/2$) must vanish, while the ones with $n \le m$ are allowed. This determines a critical hypersurface of the same universality class.

5. Generalized KdV hierarchy

Gel'fand-Dikii at work

Gel'fand-Dikii differential polynomials for analytic $V'(z)$

$$R_n[u] = \left(\mathcal{G}D^2 + \frac{u + D^{-1}uD}{2}\right)^n \cdot \frac{1}{2}$$

where $D = -\mathrm{d}/\mathrm{d}T_0$ and explicitly

$$R_0 = \frac{1}{2}, \qquad R_1 = \frac{u}{4}, \qquad R_2 = \frac{\mathcal{G}}{4}D^2u + \frac{3}{16}u^2$$

$\mathcal{G}$ is the string coupling that enters the string equation

$$\sum_{n=0}^{\infty}(n+\tfrac{1}{2})T_n R_n[u] = 0$$

which expresses u through T_n's. Introducing the resolvent

$$R(Z) = \left\langle T_0 \left| \frac{1}{-\mathcal{G}D^2 - u + Z} \right| T_0 \right\rangle = \sum_{n=0}^{\infty} \frac{R_n[u]}{Z^{n+1/2}},$$

we write the string equation as

$$T_0 = \int_{C_1} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}} \omega \widetilde{V}'(\omega^2)\left[R(\omega^2) - \frac{1}{2\omega}\right],$$

This form applies for singular potential as well

Genus zero

To genus zero

$$R^{(0)}(Z) = \frac{1}{2\sqrt{Z-u}}$$

and the string equation gives

$$\frac{T_0}{4} = \sum_{n=1}^{\infty} \frac{\widetilde{T}_n u^{n+\alpha-1/2}}{4(n+\alpha-1/2)}$$

Above scaling limit is reproduced if $\widetilde{T}_n = 0$ $(n \geq 1)$ for $n \neq [s-1/2]$.

Genus expansion

Ambjørn, Chekhov, Y.M. (2018)

Genus expansion of the Gel'fand-Dikii resolvent

$$R(\omega^2) = \sum_{k=0}^{\infty} G^k R^{(k)}(\omega^2)$$

From the third-order (linear) equation on the GD resolvent

$$\partial\left(\sqrt{\omega^2-u}\,R(\omega^2)\right) = \frac{G}{\sqrt{\omega^2-u}}\partial^3 R(\omega^2)$$

we get the recurrence relation

$$R^{(n+1)}(\omega^2) = \frac{1}{\sqrt{\omega^2-u}}\partial^{-1}\frac{1}{\sqrt{\omega^2-u}}\partial^3 R^{(n)}(\omega^2), \qquad R^{(0)}(\omega^2) = \frac{1}{2\sqrt{\omega^2-u}}$$

or

$$R^{(n)}(\omega^2) = \left(\frac{1}{\sqrt{\omega^2-u}}\partial^{-1}\frac{1}{\sqrt{\omega^2-u}}\partial^3\right)^n R^{(0)}(\omega^2)$$

$$R^{(0)} = \frac{1}{2\sqrt{\omega^2-u}}, \qquad R^{(1)} = \frac{5(u')^2}{16\left(\omega^2-u\right)^{7/2}} + \frac{u''}{4\left(\omega^2-u\right)^{5/2}}$$

Genus expansion (cont).

$$R^{(2)} = \frac{1155(u')^4}{256\left(\omega^2-u\right)^{13/2}} + \frac{231(u')^2u''}{32\left(\omega^2-u\right)^{11/2}} + \frac{21(u'')^2}{16\left(\omega^2-u\right)^{9/2}} + \frac{7u'u'''}{4\left(\omega^2-u\right)^{9/2}} + \frac{u''''}{4\left(\omega^2-u\right)^{7/2}}.$$

$$R^{(3)} = \frac{425425}{2048}\frac{(u')^6}{\left(\omega^2-u\right)^{19/2}} + \frac{255255}{512}\frac{u''(u')^4}{\left(\omega^2-u\right)^{17/2}} + \frac{35607}{128}\frac{(u'')^2(u')^2}{\left(\omega^2-u\right)^{15/2}} + \frac{2145}{16}\frac{u^{(3)}(u')^3}{\left(\omega^2-u\right)^{15/2}} + \frac{825}{32}\frac{u^{(4)}(u')^2}{\left(\omega^2-u\right)^{13/2}} + \frac{1419}{16}\frac{u^{(3)}u''u'}{\left(\omega^2-u\right)^{13/2}} + \frac{671}{32}\frac{(u'')^3}{\left(\omega^2-u\right)^{13/2}} + \frac{69}{16}\frac{(u^{(3)})^2}{\left(\omega^2-u\right)^{11/2}} + \frac{57}{8}\frac{u^{(4)}u''}{\left(\omega^2-u\right)^{11/2}} + \frac{27}{8}\frac{u^{(5)}u'}{\left(\omega^2-u\right)^{11/2}} + \frac{1}{4}\frac{u^{(6)}}{\left(\omega^2-u\right)^{9/2}}$$

Pseudo-differential polynomials?

Integrating we find to order G^2

$$R_{s-1/2}[u] \propto \Big[u^{s-1/2} + Gu^{s-7/2}(s-1/2)(s-3/2)[\frac{1}{2}u''u + \frac{1}{3}(u')^2(s-5/2)]$$
$$+G^2u^{s-13/2}(s-1/2)(s-3/2)(s-5/2)[\frac{4}{15}u^{(4)}u^3$$
$$+(\frac{8}{15}u^{(3)}u'u^2 + \frac{2}{5}(u'')^2u^2)(s-7/2) + \frac{22}{45}u''(u')^2u(s-7/2)(s-9/2)$$
$$+\frac{1}{18}(u')^4(s-7/2)(s-9/2)(s-11/2)\,]\} + \mathcal{O}(G^3)$$

It is possible to go to arbitrary genus n.

Recurrently

$$R_{s+1/2} = \left(G\partial^2 + \frac{u + \partial^{-1}u\partial}{2}\right) R_{s-1/2}$$

which for $s = m + 1/2$ reproduces the recurrence relation between the GD differential polynomials. For an arbitrary s we can also write

$$R_{s-1/2} = \left(G\partial^2 + \frac{u + \partial^{-1}u\partial}{2}\right)^{s-1/2} \cdot \frac{1}{2}$$

Conclusion

- Singular potential are interesting for applications
- Standard methods apply for singular potentials to genus zero
- Generalized multicritical potentials given by the hypergeometric functions are very convenient
- String susceptibility index $\gamma_s = -1/(s-1/2)$ interpolates between that for minimal models
- The double scaling limit applies for higher genera like usually
- Gelfand-Dikii technique is useful in the continuum but is to be extended for noninteger $s-1/2$