# Doubly Nonstationary Blind Source Separation

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**Context:**

- Blind source separation problem
- Nonstationary sources
- Time varying mixing matrix

**Goal:**

- Estimate sources from the observations of their linear mixture ⇒ BSS
- Estimate deformation functions (time warping) which characterize the nonstationarity of the sources } JEFAS
- Estimate spectra of the underlying stationary sources } JEFAS

## I. Model

### A class of nonstationary signals

- $x$ is a realization of a stationary random process $X$ with power spectrum $\mathcal{S}_X$.

Acting on $x$ with a time warping operator yields a nonstationary signal $y$ given by:

$$y(t) = \sqrt{\gamma'(t)}\, x(\gamma(t)) \ .$$

- Wavelet transform:

$$\mathcal{W}_x(s,\tau) = \int_{\mathbb{R}} x(t) q^{-\frac{s}{2}} \overline{\psi}\left(\frac{t-\tau}{q^s}\right) dt \quad \text{with} \quad q > 1 \ .$$

⇒ The wavelet transforms of the nonstationary signal $y$ and the underlying stationary signal $x$ are approximately related as follows:

$$\mathcal{W}_y(s,\tau) \approx \mathcal{W}_x(s + \log_q(\gamma'(\tau)), \gamma(\tau)) \ .$$

It can be shown [1] that the approximation error is zero-mean and its variance is controlled by the decay properties of the wavelet $\psi$, and the variations of $\gamma$.

- JEFAS [2] is an algorithm which relies on the approximation equation for the joint estimation of the time warping function $\gamma'$ and the spectrum $\mathcal{S}_X$.

### Doubly nonstationary instantaneous mixture

- Mixture model:
  - Sources $\mathbf{y}(t) \in \mathbb{R}^N$: nonstationary independent signals (time warping model).
  - Observations $\mathbf{z}(t) \in \mathbb{R}^N$: instantaneous linear mixtures of the sources.

$$\mathbf{z}(t) = \mathbf{A}(t)\mathbf{y}(t) \ .$$

- **Goal:** determine jointly the mixing matrix $\mathbf{A}(t)$, the time warping functions $\gamma_i(t)$, and the spectra $\mathcal{S}_{X_i}$ of the stationary sources for $i \in \{1,\dots,N\}$ from the observations $\mathbf{z}(t)$.
- Approximation equation:
  - Assumption: the mixing matrix coefficients are slowly varying.
  - ⇒ At fixed time $\tau$: approximate linear relation between the wavelet transforms of the sources and the observations:

$$\mathbf{w}_{\mathbf{z},\tau} \approx \mathbf{A}(\tau)\mathbf{w}_{\mathbf{y},\tau} \ .$$

where $\mathbf{w}_{\mathbf{z},\tau} = \left(\mathbf{w}_{z_1,\tau}^T \cdots \mathbf{w}_{z_N,\tau}^T\right)^T$ with $\mathbf{w}_{z_i,\tau} = \mathcal{W}_{z_i}(\mathbf{s},\tau)$. The same notation is used for the wavelet transform of the sources $\mathbf{w}_{\mathbf{y},\tau}$.

## II. Estimation procedure and algorithm

For each $\tau$, the parameters to estimate are:

- The unmixing matrix: $\mathbf{B}_\tau = \mathbf{A}(\tau)^{-1}$
- The time warping parameters: $\boldsymbol{\theta}_\tau = (\theta_{1,\tau} \cdots \theta_{N,\tau})^T$ where $\theta_{i,\tau} = \log_q(\gamma_i'(\tau))$

### Probabilistic setting

- Assume the stationary sources $X_i$ are Gaussian. Then, we derive the law of the wavelet transforms of the sources $\mathbf{w}_{\mathbf{y},\tau}$ from the approximation equations. And, one can write the approximated likelihood on $\mathbf{w}_{z,\tau}$ given by

$$\ell_\tau(\mathbf{B}_\tau, \boldsymbol{\theta}_\tau) = \log(p_{\mathbf{w}_{\mathbf{z},\tau}|(\mathbf{B}_\tau,\boldsymbol{\theta}_\tau)}(\mathbf{w}_{\mathbf{z},\tau}; \mathbf{B}_\tau, \boldsymbol{\theta}_\tau))$$

- Besides, a regularity prior is given to $\mathbf{B}_\tau$ in order to take into account the smoothness assumption on the mixing matrix.

### t y Estimation algorithm: JEFAS-BSS

- Initialization: Evaluate $\tilde{\mathbf{B}}_\tau^{(0)}$ ⇒ Estimate the sources via $\tilde{\mathbf{y}}^{(0)}(\tau) = \tilde{\mathbf{B}}_\tau^{(0)}\mathbf{z}(\tau)$.
- The following two steps are computed alternatively until convergence:

1. For each source, estimate parameters $\tilde{\theta}_{i,\tau}^{(k)}$, $\forall\tau$ and spectrum $\tilde{\mathcal{S}}_{X_i}^{(k)}$ applying JEFAS to $\tilde{y}_i^{(k-1)}$.
2. For each $\tau$, solve the MAP problem replacing $\boldsymbol{\theta}_\tau$ with its current estimation:

$$\tilde{\mathbf{B}}_\tau^{(k)} = \arg\max_{\mathbf{B}_\tau} \ \ell_\tau(\mathbf{B}_\tau, \tilde{\boldsymbol{\theta}}_\tau^{(k)}) \ \text{ s.t. } \ \|\mathbf{B}_\tau - \tilde{\mathbf{B}}_{\tau-\Delta_\tau}^{(k)}\|_\infty \le \epsilon_B \Delta_\tau .$$

⇒ Estimate the sources via $\tilde{\mathbf{y}}^{(k)}(\tau) = \tilde{\mathbf{B}}_\tau^{(k)}\mathbf{z}(\tau)$

## III. Results

The sharp wavelet $\psi_\sharp$ (with infinitely many vanishing moments) is used, supported in the positive Fourier domain and defined by

$$\hat{\psi}_\sharp(\omega) = \epsilon^{\frac{\delta(\omega,\omega_0)}{\delta(\omega_1,\omega_0)}} , \ \omega > 0 \text{ where } \delta(a,b) = \frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right) - 1 \ .$$

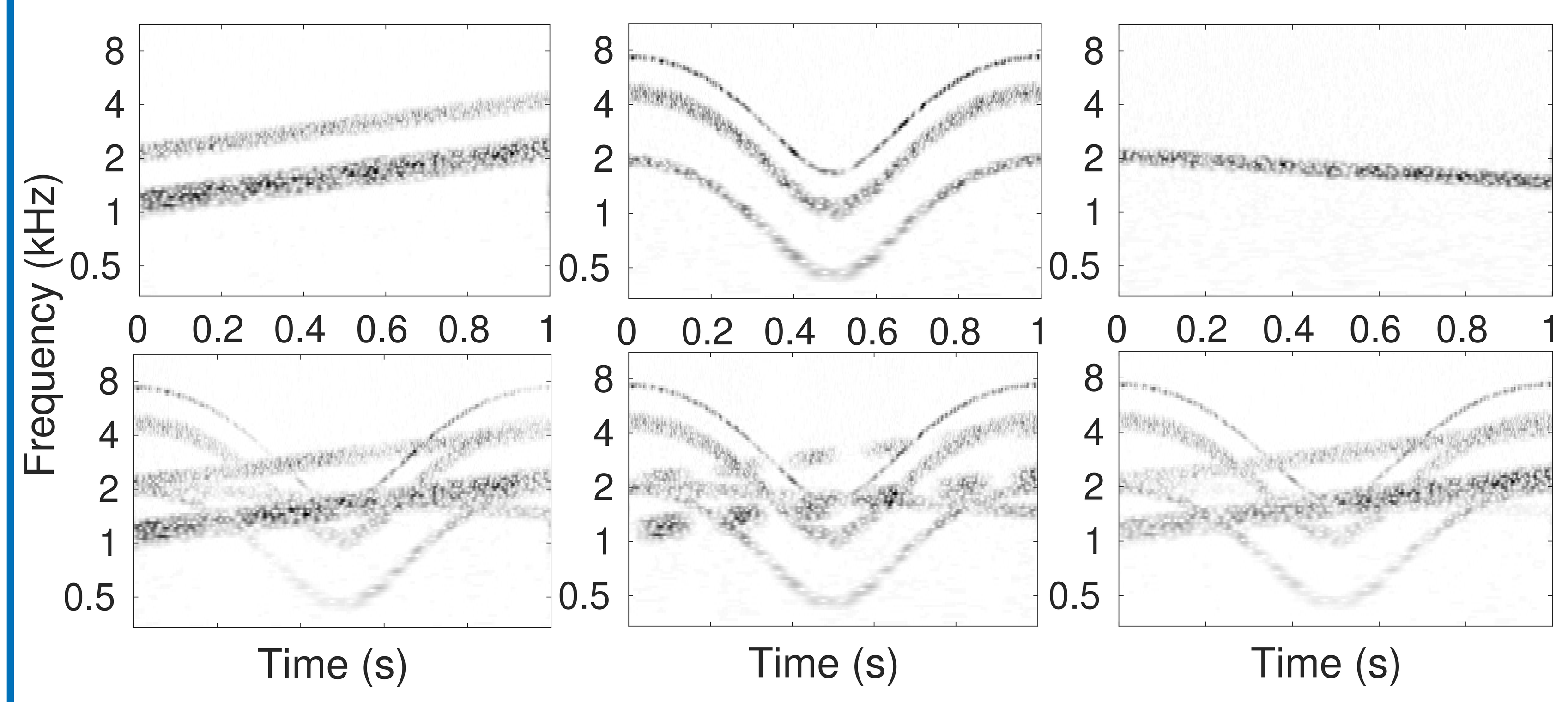
Here $\omega_0$ is the mode of $\hat{\psi}$, $\omega_1$ is chosen so that $\hat{\psi}_\sharp(\omega_1) = \epsilon$.

### Synthetic example

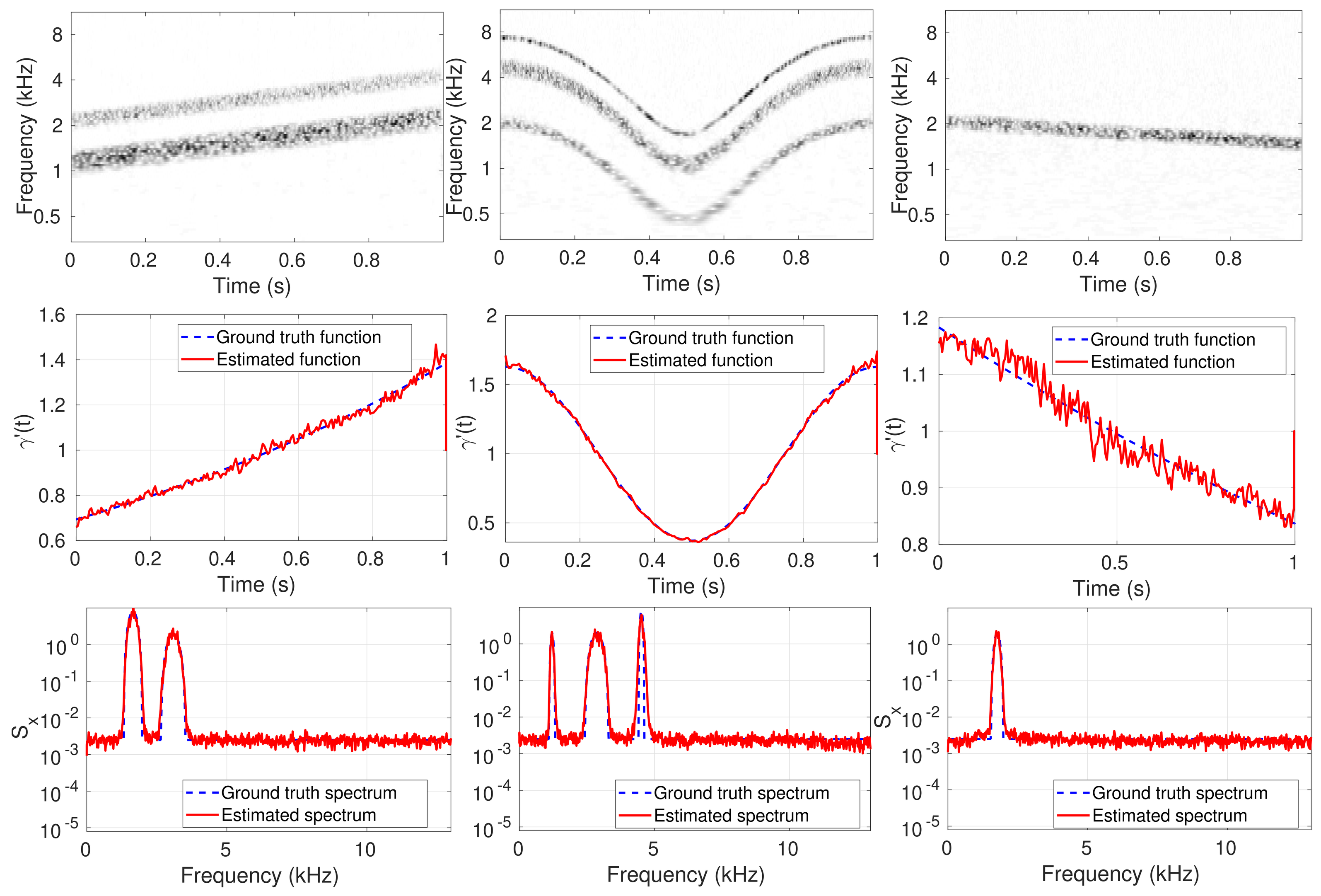
- $N = 3$
- Both nonstationary sources (time warping model) are Gaussian. Their underlying power spectra are made up of overlapping Hann windows.
- The time-varying mixing matrix coefficients are sinusoidally varying over time:

$$\mathbf{A}(t) = \begin{pmatrix} 1 + 0.3\cos(5\pi\frac{t}{T}) & 0.75 + 0.4\cos(3\pi\frac{t}{T}) & 0.6 + 0.1\cos(2\pi\frac{t}{T}) \\ -0.5 + 0.5\cos(11\pi\frac{t}{T}) & 1 + 0.1\cos(8\pi\frac{t}{T}) & 0.8 + 0.3\cos(5\pi\frac{t}{T}) \\ -0.5 + 0.2\cos(\pi\frac{t}{T}) & 0.5 + 0.1\cos(2\pi\frac{t}{T}) & 0.2 + 0.1\cos(3\pi\frac{t}{T}) \end{pmatrix} .$$

⇒ The wavelet transforms of both sources and observations are displayed below.




- JEFAS-BSS converges in 8 iterations. The estimated sources are displayed below, together with the corresponding estimated time warping functions and spectra.




- We compare the performances of JEFAS-BSS with reference BSS algorithms adapted to stationary sources (SOBI [3]) or constant mixing matrix (QTF-BSS [4]).

| Algorithm | SIR (dB) | | Amari index (dB) | |
|---|---|---|---|---|
| | Mean | SD | Mean | SD |
| SOBI | 11.82 | 3.65 | −6.54 | 0.86 |
| p-SOBI | 3.58 | 1.93 | −9.06 | 0.21 |
| QTF-BSS | 0.79 | 3.88 | −3.87 | 0.42 |
| JEFAS-BSS | 30.26 | 2.37 | −15.36 | 0.70 |

Table: Comparison of the Signal to Interference Ratio and the averaged Amari index for four BSS algorithms.

## References

[1] Harold Omer and Bruno Torrésani. Time-frequency and time-scale analysis of deformed stationary processes, with application to non-stationary sound modeling. *Applied and Computational Harmonic Analysis*, 43(1):1 – 22, 2017.

[2] Adrien Meynard and Bruno Torrésani. Spectral Analysis for Nonstationary Audio. *IEEE/ACM Transactions on Audio, Speech and Language Processing*, 26(12):2371 – 2380, December 2018.

[3] Adel Belouchrani, Karim Abed-Meraim, Jean-François Cardoso, and Eric Moulines. A blind source separation technique using second-order statistics. *IEEE Transactions on Signal Processing*, 45(2):434–444, Feb 1997.

[4] Nadège Thirion-Moreau and Moeness G. Amin. Chapter 11 - quadratic time-frequency domain methods. In P. Comon and C. Jutten, editors, *Handbook of Blind Source Separation*, pages 421 – 466. Academic Press, Oxford, 2010.